J. Nonlinear Var. Anal. 6 (2022), No. 1, pp. 1-15
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.6.2022.1.01

STRONG AND TOTAL LAGRANGE DUALITIES FOR QUASICONVEX PROGRAMMING

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Abstract. In this paper, we present the strong and total Lagrange dualities for quasiconvex programming in real locally convex Hausdorff topological vector spaces. By using the epigraphs, subdifferential and generators of involved functions, we introduce some new constraint qualifications. Under these new constraint qualifications, we establish sufficient and necessary conditions to characterize the strong and total Lagrange dualities for quasiconvex programming.

Keywords. Quasiconvex programming; Constraint qualification; Strong duality; Total duality; Generator of a quasiconvex function.

1. INTRODUCTION

Consider the following optimization problem:

$$(P) \qquad \begin{aligned} &\inf f(x) \\ &\text{s.t. } x \in C, g_i(x) \le 0, i \in I, \end{aligned}$$

where I is an arbitrary set, C is a nonempty convex subset of the locally convex Hausdorff topological vector space X, and $f, g_i, i \in I : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ are extended real-valued functions.

It is well known that the dual method is fundamental in mathematical programming and in other fields, such as, game theory, set containment problems, etc. and has played very important roles in solving optimization problem (P) which finds a lot of real applications; see, e.g, [1–4] and the references therein. A main target of the dual method is the establishment of the so-called strong duality, which means that the values of the primal problem and the dual problem coincide, and the dual problem has at least an optimal solution. Usually, the weak duality holds, that is, the value of the primal problem is not less that that of its dual problem. But, a duality gap may occur. A challenge in mathematical programming has been to give

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Received August 1, 2021; Accepted September 8, 2021.

constraint qualifications which guarantee the strong duality. Over the last decade, necessary and sufficient constraint qualifications for duality theorems have been investigated extensively. For example, necessary and sufficient constraint qualifications for Lagrange duality have been given in [4–11] where f and $g_i, i \in I$ are convex functions, and in [12–16] where $g_i, i \in I$ are quasiconvex functions. In the research of these constraint qualifications, epigraph technique and subdifferentials' property play central roles. Recently, based on the property of quasiconvex function, the notion of generator of a quasiconvex function was defined in [17]. By using the notion of generator, necessary and sufficient constraint qualifications for Lagrange-type duality theorems for quasiconvex programming have been investigated (cf. [17–21]).

Motivated by the works mentioned above, we continue to study the optimization problem (P) with the functions $g_i, i \in I$ being quasiconvex functions. Our interest in the above optimization problems in the present paper is focused on two aspects: One is about the strong duality, and the other is about the total duality, that is, the values of the primal problem and the dual problem are equal and both problems have optimal solutions. Our main aim is to use the epigraph technique, properties of the subdifferential and the generator of quasiconvex function to provide new constraint qualifications, which characterize the strong dualities, the stable strong dualities, as well as the total dualities between (P) and its Lagrange dual problem. In general, the subset C is not necessarily closed and the objective function f need not be lower semicontinuous (lsc in brief).

The paper is organized as follows. The next section contains necessary notations and preliminary results. In Section 3, some new constraint qualifications are introduced and several relationships among them are given. By using these constraint qualifications, the strong duality and total Lagrange dualities between (P) and its dual problems are established in Section 4 and 5, respectively.

2. Notations and Preliminary Results

The notations used in the present paper are standard (cf. [12] and [14]). In particular, we assume throughout the whole paper that X is a real locally convex Hausdorff topological vector space, X^* denotes the dual space of X, endowed with the w^*-topology $w^*(X^*,X)$. By $\langle x^*, x\rangle$, we shall denote the value of the functional $x^* \in X^*$ at $x \in X$; i.e., $\langle x^*, x\rangle = x^*(x)$. Let Z be a nonempty set in X. The interior (resp. closure, convex hull, conical hull) of Z is denoted by $\operatorname{int} Z$ (resp., $\operatorname{cl} Z$, $\operatorname{co} Z$, $\operatorname{cone} Z$). By convention, we define $\operatorname{cone}\emptyset = \{0\}$. If $W \subseteq X^*$, then $\operatorname{cl} W$ denotes the w^*-closure of W. The indicator function δ_Z of Z is defined by

$$\delta_Z(x) := \begin{cases} 0, & x \in Z, \\ +\infty, & \text{otherwise}. \end{cases} \tag{2.1}$$

The normal cone of Z at $x_0 \in Z$ denoted by $N_Z(x_0)$ is defined by

$$N_Z(x_0) := \{x^* \in X^* : \langle x^*, x - x_0\rangle \le 0 \ \text{ for each } x \in Z\}. \tag{2.2}$$

Following [9], we use $\mathbb{R}^{(T)}$ to denote the space of real tuples $\lambda = (\lambda_t)_{t\in T}$ with only finitely many λ_t different from zero, and let $\mathbb{R}^{(T)}_+$ be the nonnegative cone in $\mathbb{R}^{(T)}$, that is

$$\mathbb{R}^{(T)}_+ := \{(\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} : \lambda_t \ge 0 \ \text{ for each } t \in T\}.$$

Let $f : X \to \overline{\mathbb{R}}$ be a proper function. The effective domain, the Fenchel conjugate function and the epigraph of f are denoted by $\operatorname{dom} f$, f^* and $\operatorname{epi} f$, respectively; they are defined respectively by

$$\operatorname{dom} f := \{x \in X: \ f(x) < +\infty\},$$

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x): \ x \in X\} \ \text{ for each } x^* \in X^*, \tag{2.3}$$

and

$$\operatorname{epi} f := \{(x,r) \in X \times \mathbb{R}: \ f(x) \le r\}. \tag{2.4}$$

It is well known and easy to verify that $\operatorname{epi} f^*$ is weak*-closed. The subdifferential of f at $x \in \operatorname{dom} f$ is defined by

$$\partial f(x) := \{x^* \in X^* : f(x) + \langle x^*, y - x\rangle \le f(y) \ \text{ for each } y \in X\}. \tag{2.5}$$

We also define

$$\operatorname{dom} \partial f := \{x \in X : \partial f(x) \ne \emptyset\},$$

and

$$\operatorname{im} \partial f := \{x^* \in X^* : x^* \in \partial f(x) \ \text{ for some } x \in X\}. \tag{2.6}$$

By definition of the Fenchel conjugate (2.3), the Young-Fenchel inequality holds:

$$f(x) + f^*(x^*) \ge \langle x, x^*\rangle \ \text{ for each } (x,x^*) \in X \times X^*. \tag{2.7}$$

Moreover, the following Young equality holds:

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle \tag{2.8}$$

and, by definition,

$$\big(x^*, \langle x^*, x\rangle - f(x)\big) \in \operatorname{epi} f^* \ \text{ for each } x^* \in \partial f(x). \tag{2.9}$$

In particular,

$$N_Z(x) = \partial \delta_Z(x) \ \text{ for each } x \in Z. \tag{2.10}$$

If $f, h : X \to \overline{\mathbb{R}}$ are proper convex functions satisfying $\operatorname{dom} f \cap \operatorname{dom} h \ne \emptyset$, then

$$\operatorname{epi} f^* + \operatorname{epi} h^* \subseteq \operatorname{epi} (f+h)^*, \tag{2.11}$$

and

$$\partial f(a) + \partial h(a) \subseteq \partial (f+h)(a) \ \text{ for each a} \in \operatorname{dom} f \cap \operatorname{dom} h. \tag{2.12}$$

Furthermore, for all $p \in X^*$and $a \in \mathbb{R}$, the following facts are clear:

$$(h+p+a)^*(x^*) = h^*(x^* - p) - a \ \text{ for each } x^* \in X^*. \tag{2.13}$$

$$\operatorname{epi}(h+p+a)^* = \operatorname{epi} h^* + (p, -a). \tag{2.14}$$

Recall that a function $f : X \to \overline{\mathbb{R}}$ is said to be quasiconvex if, for all $x, y \in X$ and $\alpha \in [0,1]$, the following inequality holds:

$$f\big(\alpha x + (1-\alpha) y\big) \le \max\{f(x), f(y)\},$$

or equivalently its sublevel set

$$L(f, \le, r) := \{x \in X : f(x) \le r\}$$

is a convex set for each $r \in \mathbb{R}$. Obviously, each convex function is a quasiconvex function, but the opposite is not true. The following lemma is taken from [17].

Lemma 2.1. *Let f be a function from X to $\overline{\mathbb{R}}$. Then f is lsc quasiconvex if and only if there exists $\{(k_i, w_i) : i \in I\} \subseteq Q \times X^*$ such that $f = \sup_{i \in I} k_i \circ w_i$, where $Q := \{k : \mathbb{R} \to \overline{\mathbb{R}} :$ k is lsc and non-decreasing$\}$.*

The following lemma will be used in the sequel (see [12, Theorem 2.8.7]).

Lemma 2.2. *Let $f, h : X \to \overline{\mathbb{R}}$ be proper convex functions such that $\operatorname{dom} f \cap \operatorname{dom} h \neq \emptyset$. If f or h is continuous at some point of $\operatorname{dom} f \cap \operatorname{dom} h$, then*

$$\operatorname{epi}(f+h)^* = \operatorname{epi} f^* + \operatorname{epi} h^*, \tag{2.15}$$

$$\partial(f+h)(x) = \partial f(x) + \partial h(x) \text{ for each } x \in \operatorname{dom} f \cap \operatorname{dom} h. \tag{2.16}$$

Consequently, if $h \in X^*$, then

$$\operatorname{epi}(f+h)^* = \operatorname{epi} f^* + \{h\} \times [0, +\infty),$$

$$\partial(f+h)(x) = \partial f(x) + h \text{ for each } x \in \operatorname{dom} f.$$

3. Constraint Qualifications

Throughout this paper, let X be a locally convex Hausdorff topological vector space, $\{g_i : i \in I\}$ a family of lsc quasiconvex functions from X to $\overline{\mathbb{R}}$, $\{(k_{(i,j)}, w_{(i,j)}) : j \in J_i\}$ a generator of g_i for each $i \in I$, $T = \{t = (i, j) : i \in I, j \in J_i\}$, $S = \{x \in X : g_i(x) \leq 0 \text{ for each } i \in I\}$, $C \subseteq X$ a convex set, $f : X \to \overline{\mathbb{R}}$ a proper convex function, and $A \neq \emptyset$ the solution set of the following system:

$$x \in C; \; g_i(x) \leq 0 \;\text{ for each } i \in I. \tag{3.1}$$

Then, by the notion of generator, $\{(k_t, w_t) : t \in T\}$ is a generator of $\sup_{i \in I} g_i$. For each $x \in X$, let $T(x)$ be the active index set of the system (3.1) relative to $\{(k_t, w_t) : t \in T\}$; that is,

$$T(x) := \{t \in T : \langle w_t, x \rangle = k_t^{-1}(0)\}. \tag{3.2}$$

Recall from [18, 19] that the function k^{-1} is said to be the hypo-epi-inverse function of non-decreasing function k:

$$k^{-1}(a) := \inf\{b \in \mathbb{R} : a < k(b)\} = \sup\{b \in \mathbb{R} : k(b) \leq a\}.$$

It is known that if k has the inverse function, then the inverse and the hypo-epi-inverse of k are the same. Let $x \in A$. Following [18], define K and $N'_A(x)$ respectively by

$$K := \operatorname{epi} \delta_C^* + \operatorname{cone\ co} \bigcup_{t \in T} \{(w_t, \delta) \in X^* \times \mathbb{R} : k_t^{-1}(0) \leq \delta\}, \tag{3.3}$$

$$N'_A(x) := N_C(x) + \operatorname{cone\ co} \bigcup_{t \in T(x)} \{w_t\}. \tag{3.4}$$

Then the following inclusions hold:

$$\operatorname{cone\ co} \bigcup_{t \in T} \{(w_t, \delta) \in X^* \times \mathbb{R} : k_t^{-1}(0) \leq \delta\} \subseteq \operatorname{epi} \delta_S^* \tag{3.5}$$

and

$$\operatorname{cone\ co} \bigcup_{t \in T(x)} \{w_t\} \subseteq N_S(x). \tag{3.6}$$

Thus, by (2.11), we have

$$K \subseteq \operatorname{epi} \delta_{C \cap S}^* = \operatorname{epi} \delta_A^*, \tag{3.7}$$

and hence

$$\mathrm{epi} f^* + K \subseteq \mathrm{epi}(f+\delta_A)^*. \tag{3.8}$$

Moreover, by (2.10) and (2.12), for each $x \in \mathrm{dom} f \cap A$,

$$N'_A(x) \subseteq N_C(x) + N_S(x) \subseteq N_A(x), \tag{3.9}$$

and hence

$$\partial f(x) + N'_A(x) \subseteq \partial (f+\delta_A)(x). \tag{3.10}$$

Regarding some possible reversed inclusions in (3.8) and (3.10), we introduce the following definition(for parts (iv), see [18]).

Definition 3.1. It is said that the family $\{\delta_C; g_i : i \in I\}$ satisfies

(i) the conical epigraph hull property (EHP) for quasiconvex programming with respect to (w.r.t.) $\{(k_t, w_t) : t \in T\}$ $\big($denoted by the conical (Q-EHP)$\big)$ if

$$\mathrm{epi}\,\delta_A^* = K; \tag{3.11}$$

(ii) the conical epigraph hull property(EHP) for quasiconvex programming with respect to (w.r.t.) $\{(k_t, w_t) : t \in T\}$ relative to f $\big($denoted by the conical (Q-EHP)$_f\big)$ if

$$\mathrm{epi}(f+\delta_A)^* = \mathrm{epi} f^* + K; \tag{3.12}$$

(iii) the weak conical epigraph hull property(EHP) for quasiconvex programming with respect to (w.r.t.) $\{(k_t, w_t) : t \in T\}$ relative to f $\big($denoted by the conical (Q-WEHP)$_f\big)$ if

$$\mathrm{epi}(f+\delta_A)^* = \mathrm{epi}(f+\delta_C)^* + \mathrm{cone\ co} \bigcup_{t\in T} \{(w_t, \delta) \in X^* \times \mathbb{R} : k_t^{-1}(0) \leq \delta\}; \tag{3.13}$$

(iv) the basic constraint qualification for quasiconvex programming w.r.t. $\{(k_t, w_t) : t \in T\}$ $\big($denoted by (Q-BCQ)$\big)$ at $x \in A$ if

$$N_A(x) = N'_A(x);$$

(v) the basic constraint qualification for quasiconvex programming w.r.t. $\{(k_t, w_t) : t \in T\}$ relative to f $\big($denoted by (Q-BCQ)$_f\big)$ at $x \in \mathrm{dom} f \cap A$ if

$$\partial(f+\delta_A)(x) = \partial f(x) + N'_A(x); \tag{3.14}$$

(vi) the weak basic constraint qualification for quasiconvex programming w.r.t. $\{(k_t, w_t) : t \in T\}$ relative to f $\big($denoted by (Q-WBCQ)$_f\big)$ at $x \in \mathrm{dom} f \cap A$ if

$$\partial(f+\delta_A)(x) = \partial(f+\delta_C)(x) + \mathrm{cone\ co} \bigcup_{t\in T(x)} \{w_t\}; \tag{3.15}$$

(vii) (Q-BCQ) $\big($resp. (Q-BCQ)$_f$, (Q-WBCQ)$_f\big)$ if it has (Q-BCQ) $\big($resp. (Q-BCQ)$_f$, (Q-WBCQ)$_f\big)$ at each point in A $\big($resp. $\mathrm{dom} f \cap A$, $\mathrm{dom} f \cap A\big)$.

Remark 3.1. (i) By (3.7) and (3.9), we see that the conical (Q-EHP) and (Q-BCQ) can be equivalently replaced by

$$\mathrm{epi}\,\delta_A^* \subseteq K \text{ and } N_A(x) \subseteq N'_A(x).$$

While, by (3.7)-(3.10), it is clear that (3.12)-(3.15) can be equivalently replaced by

$$\mathrm{epi}(f+\delta_A)^* \subseteq \mathrm{epi} f^* + K, \tag{3.16}$$

$$\mathrm{epi}(f+\delta_A)^* \subseteq \mathrm{epi}(f+\delta_C)^* + \mathrm{cone\ co}\bigcup_{t\in T}\{(w_t,\delta)\in X^*\times\mathbb{R}: k_t^{-1}(0)\leq\delta\}, \tag{3.17}$$

$$\partial(f+\delta_A)(x)\subseteq\partial f(x)+N'_A(x) \tag{3.18}$$

and

$$\partial(f+\delta_A)(x)\subseteq\partial(f+\delta_C)(x)+\mathrm{cone\ co}\bigcup_{t\in T(x)}\{w_t\}. \tag{3.19}$$

(ii) If f is continuous at some point of A, then, by Lemma 2.2,

$$\text{the conical (Q-EHP)} \Longrightarrow \text{the conical (Q-EHP)}_f,$$

$$\text{(Q-BCQ)} \Longrightarrow \text{(Q-BCQ)}_f.$$

(iii) As the terminologies suggested, by (2.11) and (2.12),

$$\text{the conical (Q-EHP)}_f \Longrightarrow \text{the conical (Q-WEHP)}_f,$$

$$\text{(Q-BCQ)}_f \Longrightarrow \text{(Q-WBCQ)}_f.$$

(iv) If f is continuous at some point of C, it follows from Lemma 2.2 that

$$\text{the conical (Q-EHP)}_f \Longleftrightarrow \text{the conical (Q-WEHP)}_f,$$

$$\text{(Q-BCQ)}_f \Longleftrightarrow \text{(Q-WBCQ)}_f.$$

Recall from [18, 19] that the inequality system $\{g_i\leq 0 : i\in I\}$ satisfies the closed cone constraint qualification for quasiconvex programming (Q-CCCQ) w.r.t. $\{(k_t,w_t): t\in T\}$ relative to C if

$$\mathrm{epi}\delta_C^* + \mathrm{cone\ co}\bigcup_{t\in T}\{(w_t,\delta)\in X^*\times\mathbb{R}: k_t^{-1}(0)\leq\delta\} \text{ is } w^*\text{-closed}. \tag{3.20}$$

The following proposition characterizes the relationship between (Q-CCCQ) and (Q-EHP).

Proposition 3.1. *Assume that g_i is a proper* lsc *quasiconvex function for each $i\in I$ and C is a closed convex set. Then the system $\{g_i\leq 0 : i\in I\}$ satisfies* (Q-CCCQ) *if and only if the family $\{\delta_C; g_i : i\in I\}$ satisfies the conical* (Q-EHP).

Proof. Under the given assumption, by [18, Theorem 3.1], we have that

$$\mathrm{epi}\delta_A^* = \mathrm{cl}(\mathrm{epi}\delta_C^* + \mathrm{cone\ co}\bigcup_{t\in T}\{(w_t,\delta)\in X^*\times\mathbb{R}: k_t^{-1}(0)\leq\delta\}) = \mathrm{cl}K. \tag{3.21}$$

Then the result is seen to hold. □

The following proposition on relationships between notions defined in Definition 3.1 is an extension of [18, Definition 3.2 and Definition 3.4] from $f=0$ to the general function f.

Proposition 3.2. *The following implications hold:*

$$\text{the conical (Q-EHP)}_f \Longrightarrow \text{(Q-BCQ)}_f, \text{the conical (Q-WEHP)}_f \Longrightarrow \text{(Q-WBCQ)}_f, \tag{3.22}$$

Moreover, if $\mathrm{dom}(f+\delta_A)^*\subseteq\mathrm{im}\partial(f+\delta_A)$*, then*

$$\text{the conical (Q-EHP)}_f \Longleftrightarrow \text{(Q-BCQ)}_f, \text{the conical (Q-WEHP)}_f \Longleftrightarrow \text{(Q-WBCQ)}_f. \tag{3.23}$$

Proof. We shall only prove the first assertion in (3.22) and that in (3.23) (the proofs for the second assertions being similar). Suppose that the family $\{\delta_C; g_i : i \in I\}$ satisfies the conical (Q-EHP)$_f$. Let $x \in \operatorname{dom} f \cap A$. By Remark 3.1, it suffices to show that (3.18) holds. To do this, let $x^* \in \partial(f+\delta_A)(x)$. Then,

$$\big(x^*, \langle x^*, x\rangle - (f+\delta_A)(x)\big) \in \operatorname{epi}(f+\delta_A)^* \subseteq \operatorname{epi} f^* + K, \tag{3.24}$$

thanks to (2.9) and (3.16). There exist $(u^*, r_1) \in \operatorname{epi} f^*$, $(v^*, r_2) \in \operatorname{epi}\delta_C^*$ and

$$\Big(\sum_{t_i \in J} \lambda_{t_i} w_{t_i}, \sum_{t_i\in J} \lambda_{t_i}\delta_i\Big) \in \operatorname{cone}\operatorname{co}\bigcup_{t\in T}\{(w_t,\delta) \in X^* \times \mathbb{R} : k_t^{-1}(0) \le \delta\}$$

such that

$$x^* = u^* + v^* + \sum_{t_i\in J} \lambda_{t_i} w_{t_i}, \tag{3.25}$$

$$\langle x^*, x\rangle - (f+\delta_A)(x) = r_1 + r_2 + \sum_{t_i\in J}\lambda_{t_i}\delta_i, \tag{3.26}$$

where $J \subseteq T$ is a finite subset, $\lambda_{t_i} > 0$ for each $t_i \in J$, and $(w_{t_i}, \delta_i) \in \{(w_t, \delta) \in X^* \times \mathbb{R} : k_t^{-1}(0) \le \delta\}$. Then, $\sum_{t_i\in J} \lambda_{t_i} k_{t_i}^{-1}(0) \le \sum_{t_i \in J}\lambda_{t_i}\delta_i$. Noting that $\partial(w_{t_i}(\cdot) - k_{t_i}^{-1}(0))(x) = \{w_{t_i}\}$ for each $t_i \in J$, to show (3.18), we only need to prove that $u^* \in \partial f(x)$, $v^* \in N_C(x)$ and $J \subseteq T(x)$. By the definition of epigraph (2.4),

$$f^*(u^*) \le r_1,\ \ \delta_C^*(v^*) \le r_2. \tag{3.27}$$

By Young-Fenchel inequality (2.7),

$$f^*(u^*) \ge \langle u^*, x\rangle - f(x), \tag{3.28}$$

$$\delta_C^*(v^*) \ge \langle v^*, x\rangle - \delta_C(x). \tag{3.29}$$

Noting that $\delta_A(x) = \delta_C(x) = 0$ and $\langle w_{t_i}, x\rangle - k_{t_i}^{-1}(0) \le 0$ for each $t_i \in J$ as $J \subseteq T$, it follows from (3.25)-(3.29) that

$$\begin{aligned}
\langle x^*, x\rangle - f(x) &\ge f^*(u^*) + \delta_C^*(v^*) + \sum_{t_i\in J}\lambda_{t_i} k_{t_i}^{-1}(0)\\
&\ge \langle u^*, x\rangle - f(x) + \langle v^*, x\rangle - \delta_C(x) + \sum_{t_i\in J}\lambda_{t_i}k_{t_i}^{-1}(0)\\
&= \langle x^* - \sum_{t_i\in J}\lambda_{t_i}w_{t_i}, x\rangle - f(x) - \delta_C(x) + \sum_{t_i\in J}\lambda_{t_i}k_{t_i}^{-1}(0)\\
&= \langle x^*, x\rangle - f(x) - \sum_{t_i\in J}\lambda_{t_i}\big(w_{t_i}(x) - k_{t_i}^{-1}(0)\big)\\
&\ge \langle x^*, x\rangle - f(x).
\end{aligned}$$

This implies that the equalities of (3.28), (3.29) hold and $w_{t_i}(x) = k_{t_i}^{-1}(0)$ for each $t_i \in J$. Therefore, by the Young equality (2.8), $u^* \in \partial f(x)$, $v^* \in N_C(x)$ and $J \subseteq T(x)$.

Below we show that the converse implication holds. Assume that $\operatorname{dom}(f+\delta_A)^* \subseteq \operatorname{im}\partial(f+\delta_A)$ and the family $\{\delta_C; g_i : i \in I\}$ satisfies (Q-BCQ)$_f$. In view of Remark 3.1, to show the conical (Q-EHP)$_f$, we only need to show that (3.16) holds. Let $(x^*, r) \in \operatorname{epi}(f+\delta_A)^*$. Since $x^* \in \operatorname{dom}(f+\delta_A)^* \subseteq \operatorname{im}\partial(f+\delta_A)$, it follows that there exists $x_0 \in \operatorname{dom} f \cap A$ such that $x^* \in \partial(f+\delta_A)(x_0)$. By the assumption of (Q-BCQ)$_f$ (see (3.18)), we have

$$x^* \in \partial f(x_0) + N_C(x_0) + \sum_{t_i\in J}\lambda_{t_i} w_{t_i}$$

for some finite subset $J \subseteq T(x_0)$ and $\{\lambda_{t_i}\} \subseteq \mathbb{R}$ with $\lambda_{t_i} \geq 0$ for each $t_i \in J$. This means that there exist $u^* \in \partial f(x_0)$ and $v^* \in N_C(x_0)$ such that

$$x^* = u^* + v^* + \sum_{t_i \in J} \lambda_{t_i} w_{t_i}. \tag{3.30}$$

Since $t_i \in J \subseteq T(x_0)$, it follows from (3.2) that

$$\langle w_{t_i}, x_0 \rangle = k_{t_i}^{-1}(0). \tag{3.31}$$

By Young equality (2.8),

$$f^*(u^*) = \langle u^*, x_0 \rangle - f(x_0), \tag{3.32}$$

$$\delta_C^*(v^*) = \langle v^*, x_0 \rangle - \delta_C(x_0) \tag{3.33}$$

and

$$(f + \delta_A)^*(x^*) = \langle x^*, x_0 \rangle - (f + \delta_A)(x_0).$$

Moreover, since $(x^*, r) \in \mathrm{epi}(f + \delta_A)^*$, we have

$$\langle x^*, x_0 \rangle \leq r + (f + \delta_A)(x_0) = r + f(x_0). \tag{3.34}$$

Combining (3.30) with (3.34), one has

$$\langle u^*, x_0 \rangle + \langle v^*, x_0 \rangle + \sum_{t_i \in J} \lambda_{t_i} \langle w_{t_i}, x_0 \rangle \leq r + f(x_0).$$

This implies that there exists a set $\{r_1, r_2, \delta_i : i \in \{i : t_i \in J\}\}$ of real numbers such that

$$r = r_1 + r_2 + \sum_{t_i \in J} \lambda_{t_i} \delta_i, \tag{3.35}$$

$$\langle u^*, x_0 \rangle \leq r_1 + f(x_0),\ \langle v^*, x_0 \rangle \leq r_2 \text{ and } \langle w_{t_i}, x_0 \rangle \leq \delta_i \text{ for each } t_i \in J.$$

Consequently, by (3.32), (3.33) and (3.31),

$$f^*(u^*) \leq r_1,\ \delta_C^*(v^*) \leq r_2,\ k_{t_i}^{-1}(0) \leq \delta_i \text{ for each } t_i \in J.$$

That is, $(u^*, r_1) \in \mathrm{epi} f^*$, $(v^*, r_2) \in \mathrm{epi}\delta_C^*$, $(w_{t_i}, \delta_i) \in \{(w_{t_i}, \delta_i) : k_{t_i}^{-1}(0) \leq \delta_i\}$ for each $t_i \in J$. Hence,

$$(x^*, r) = (u^*, r_1) + (v^*, r_2) + (\sum_{t_i \in J} \lambda_{t_i} w_{t_i}, \sum_{t_i \in J} \lambda_{t_i} \delta_i) \in \mathrm{epi} f^* + K,$$

as $t_i \in J \subseteq T(x) \subseteq T$. Therefore, (3.16) holds and the proof is complete. □

Let $h_i : X \to \overline{\mathbb{R}}$ be a proper convex function for each $i \in I$. Recall from [7, Definition 3.1] that the family $\{\delta_C; h_i : i \in I\}$ is said to have the conical epigraph hull property (EHP) if

$$\mathrm{epi}\delta_A^* = \mathrm{epi}\delta_C^* + \mathrm{cone} \bigcup_{i \in I} \mathrm{epi} h_i^*, \tag{3.36}$$

and the conical (EHP) relative to f $\big($denote by the conical $(\mathrm{EHP})_f$ $\big)$ if

$$\mathrm{epi}(f + \delta_A)^* = \mathrm{epi} f^* + \mathrm{epi}\delta_C^* + \mathrm{cone} \bigcup_{i \in I} \mathrm{epi} h_i^*. \tag{3.37}$$

The following proposition shows that the conical (Q-EHP) (resp. the concical $(\text{Q-EHP})_f$) coincides with the conical (EHP) (resp. the conical $(\mathrm{EHP})_f$) in the case when $g_i, i \in I$ is convex.

Proposition 3.3. *Suppose g_i is a proper lsc convex function for each $i \in I$. Then the following equivalences hold:*

$$\text{the conical (Q-EHP)} \Longleftrightarrow \text{the conical (EHP)}, \tag{3.38}$$

$$\text{the conical } (\text{Q-EHP})_f \Longleftrightarrow \text{the conical } (\text{EHP})_f. \tag{3.39}$$

Proof. Since g_i is lsc convex for each $i \in I$, it follows that

$$g_i(x) = g_i^{**}(x) = \sup_{v_i \in \operatorname{dom} g_i^*} \{\langle v_i, x\rangle - g_i^*(v_i)\} \quad \text{for all} \quad x \in X.$$

Then $\{(k_{v_i}, v_i) : v_i \in \operatorname{dom} g_i^*, k_{v_i}(x) = x - g_i^*(v_i) \text{ for each } x \in \mathbb{R}\} \subseteq Q \times X^*$ is a basic generator of g_i. Note that $\cup_{i\in I} \operatorname{epi} g_i^*$ is convex. It follows that

$$\operatorname{epi} g_i^* = \operatorname{co} \bigcup_{v_i \in \operatorname{dom} g_i^*} \{(v_i, \delta) : k_{v_i}^{-1}(0) \le \delta\} \text{ for each } i \in I.$$

Therefore, (3.38) and (3.39) hold. □

4. Strong Lagrange Duality

Let $p \in X^*$. Consider the following problem

$$(P_{f-p}) \qquad \begin{array}{ll} \inf & f(x) - \langle p, x\rangle \\ \text{s.t.} & x \in C, g_i(x) \le 0, i \in I. \end{array} \tag{4.1}$$

Its Lagrange dual problem is defined by

$$(D_{f-p}) \qquad \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} L_{f-p}(x, \lambda), \tag{4.2}$$

where the Lagrange function L_{f-p} on $X \times \mathbb{R}_+^{(T)}$ is defined by

$$L_{f-p}(x, \lambda) := f(x) - \langle p, x\rangle + \sum_{t_i \in T} \lambda_{t_i}\big(w_{t_i}(x) - k_{t_i}^{-1}(0)\big). \tag{4.3}$$

In particular, when $p = 0$, problem (P_{f-p}) as well as its dual problem (D_{f-p}) are reduced to the problem (P_f) and its dual problem (D_f), respectively. The optimal value $v(P_f)$ and the optimal solution set $S(P_f)$ of the problem (P_f) are defined respectively by

$$v(P_f) := \inf\{f(x) : x \in \operatorname{dom} f \cap A\} (\ge -\infty)$$

and

$$S(P_f) := \{x \in \operatorname{dom} f \cap A : f(x) = v(P_f)\}.$$

Similarly, we define

$$v(D_f) := \max_{\lambda \in \mathbb{R}_+^{(T)}} \{\inf_{x \in C} L_f(x, \lambda)\}$$

and

$$S(D_f) := \{\lambda \in \mathbb{R}_+^{(T)} : v(D_f) = \inf_{x \in C} L_f(x, \lambda)\}.$$

Let $a \in \operatorname{dom} f \cap A$. Since $\sum_{t_i \in T} \lambda_{t_i}(w_{t_i}(a) - k_{t_i}^{-1}(0)) \le 0$, it follows that $L_{f-p}(a, \lambda) \le f(a) - \langle p, a\rangle$ for each $\lambda \in \mathbb{R}_+^{(T)}$. Consequently,

$$\inf_{x \in C} L_{f-p}(x, \lambda) \le L_{f-p}(a, \lambda) \le f(a) - \langle p, a\rangle \text{ for each } \lambda \in \mathbb{R}_+^{(T)}.$$

Hence, $v(D_{f-p}) \leq f(a) - \langle p, a\rangle$ and

$$v(D_{f-p}) \leq v(P_{f-p}) \text{ for each } p \in X^*. \tag{4.4}$$

In particular, when $p = 0$,

$$v(D_f) \leq v(P_f). \tag{4.5}$$

Definition 4.1. Between (P_f) and (D_f), it is said that
(i) the weak lagrange duality holds if (4.5) holds;
(ii) the strong lagrange duality holds if $v(D_f) = v(P_f)$ and $S(D_f) \neq \emptyset$;
(iii) the stable weak(resp. stable strong)lagrange duality holds if the weak(resp. strong) lagrange duality between (P_{f-p}) and (D_{f-p}) holds for each $p \in X^*$.

Let $p \in X^*$ and $r \in \mathbb{R}$. By definition, one has that

$$(p, r) \in \text{epi}(f + \delta_A)^* \iff v(P_{f-p}) \geq -r. \tag{4.6}$$

The following theorem gives a sufficient and necessary conditions to ensure the stable strong Lagrange duality between (P_f) and (D_f).

Theorem 4.1. *The family* $\{\delta_C; g_i : i \in I\}$ *has the conical* (Q-WEHP)$_f$ *if and only if the stable strong Lagrange duality between* (P_f) *and* (D_f) *holds.*

Proof. Suppose that the family $\{\delta_C; g_i : i \in I\}$ has the conical (Q-WEHP)$_f$. Let $p \in X^*$. By (4.4), it is clear that $v(D_{f-p}) \leq v(P_{f-p})$. If $v(P_{f-p}) = -\infty$, then the result holds trivially. Below we assume that $-r := v(P_{f-p}) \in \mathbb{R}$. Then, by (4.6) and (3.17), we have $(p, r) \in \text{epi}(f + \delta_A)^* \subseteq \text{epi}(f + \delta_C)^* + \text{cone co} \cup_{t\in T} \{(w_t, \delta) \in X^* \times \mathbb{R} : k_t^{-1}(0) \leq \delta\}$. Hence there exist $\overline{\lambda} \in \mathbb{R}_+^{(T)}$, $(u^*, r_1) \in \text{epi}(f + \delta_C)^*$ and $(\sum_{t_i \in T} \overline{\lambda}_{t_i} w_{t_i}, \sum_{t_i \in T} \overline{\lambda}_{t_i} \delta_i) \in \text{cone co} \bigcup_{t\in T} \{(w_t, \delta) \in X^* \times \mathbb{R} : k_t^{-1}(0) \leq \delta\}$ with $k_{t_i}^{-1}(0) \leq \delta_i$ such that

$$p = u^* + \sum_{t_i \in T} \overline{\lambda}_{t_i} w_{t_i}, \tag{4.7}$$

$$r = r_1 + \sum_{t_i \in T} \overline{\lambda}_{t_i} \delta_i. \tag{4.8}$$

By the definition of epigragh (2.4),

$$(f + \delta_C)^*(u^*) \leq r_1. \tag{4.9}$$

While, by the Young-Fenchel inequality (2.7), we have

$$(f + \delta_C)^*(u^*) \geq \langle u^*, x\rangle - (f + \delta_C)(x) \text{ for each } x \in X. \tag{4.10}$$

It follows from (4.8)-(4.10) that, for each $x \in X$,

$$r \geq (f + \delta_C)^*(u^*) + \sum_{t_i \in T} \overline{\lambda}_{t_i} k_{t_i}^{-1}(0) \geq \langle u^*, x\rangle - (f + \delta_C)(x) + \sum_{t_i \in T} \overline{\lambda}_{t_i} k_{t_i}^{-1}(0).$$

Combine this with (4.7), we see that for each $x \in C$,

$$\begin{aligned} r &\geq \langle p - \sum_{t_i \in T} \overline{\lambda}_{t_i} w_{t_i}, x\rangle - f(x) + \sum_{t_i \in T} \overline{\lambda}_{t_i} k_{t_i}^{-1}(0) \\ &= -f(x) + \langle p, x\rangle + \sum_{t_i \in T} \overline{\lambda}_{t_i} \big(k_{t_i}^{-1}(0) - w_{t_i}(x)\big), \end{aligned}$$

that is,

$$-r \leq f(x) - \langle p,x\rangle + \sum_{t_i\in T} \overline{\lambda}_{t_i}\big(w_{t_i}(x) - k_{t_i}^{-1}(0)\big) \text{ for each } x \in C.$$

Hence,

$$v(P_{f-p}) \leq \inf_{x\in C}\{f(x) - \langle p,x\rangle + \sum_{t_i\in T} \overline{\lambda}_{t_i}\big(w_{t_i}(x) - k_{t_i}^{-1}(0)\big)\} \leq v(D_{f-p}).$$

This together with $v(D_{f-p}) \leq v(P_{f-p})$ shows that $v(D_{f-p}) = v(P_{f-p})$ and $\overline{\lambda} \in S(D_{f-p})$. Therefore, the stable strong Lagrange duality between (P_f) and (D_f) holds.

Conversely, let $p \in X^*$. Suppose that the stable strong Lagrange duality between (P_f) and (D_f) holds. By Remark 3.1 (ii), we only need to show that (3.17) holds. To do this, let $(p,r) \in \operatorname{epi}(f+\delta_A)^*$, which is equivalent to $v(P_{f-p}) \geq -r$, thanks to (4.6). Then, by the stable strong Lagrange duality, there exists $(\overline{\lambda}_{t_i})_{t_i\in T} \in S(D_{f-p})$ such that

$$v(P_{f-p}) = v(D_{f-p}) = \inf_{x\in C}\{f(x) - \langle p,x\rangle + \sum_{t_i\in T} \overline{\lambda}_{t_i}\big(w_{t_i}(x) - k_{t_i}^{-1}(0)\big)\}.$$

Combine this with (2.3) and (2.13), we see that

$$\begin{aligned}
-r &\leq \inf_{x\in X}\{f(x) + \delta_C(x) - \langle p,x\rangle + \sum_{t_i\in T} \overline{\lambda}_{t_i}\big(w_{t_i}(x) - k_{t_i}^{-1}(0)\big)\} \\
&= -\sup_{x\in X}\{\langle p - \sum_{t_i\in T} \overline{\lambda}_{t_i} w_{t_i}, x\rangle - \big(f + \delta_C - \sum_{t_i\in T} \overline{\lambda}_{t_i} k_{t_i}^{-1}(0)\big)(x)\} \\
&= -\big(f + \delta_C - \sum_{t_i\in T} \overline{\lambda}_{t_i} k_{t_i}^{-1}(0)\big)^*(p - \sum_{t_i\in T} \overline{\lambda}_{t_i} w_{t_i}) \\
&= -(f+\delta_C)^*(p - \sum_{t_i\in T} \overline{\lambda}_{t_i} w_{t_i}) - \sum_{t_i\in T} \overline{\lambda}_{t_i} k_{t_i}^{-1}(0).
\end{aligned}$$

This implies that $\big(p - \sum_{t_i\in T} \overline{\lambda}_{t_i} w_{t_i}, r - \sum_{t_i\in T} \overline{\lambda}_{t_i} k_{t_i}^{-1}(0)\big) \in \operatorname{epi}(f+\delta_C)^*$. Consequently,

$$\begin{aligned}
(p,r) &= \big(p - \sum_{t_i\in T} \overline{\lambda}_{t_i} w_{t_i}, r - \sum_{t_i\in T} \overline{\lambda}_{t_i} k_{t_i}^{-1}(0)\big) + \big(\sum_{t_i\in T} \overline{\lambda}_{t_i} w_{t_i}, \sum_{t_i\in T} \overline{\lambda}_{t_i} k_{t_i}^{-1}(0)\big) \\
&\in \operatorname{epi}(f+\delta_C)^* + \operatorname{cone}\operatorname{co}\bigcup_{t\in T}\{(w_t,\delta) \in X^* \times \mathbb{R} : k_t^{-1}(0) \leq \delta\}.
\end{aligned}$$

Therefore, (3.17) is proved. □

The following corollary is a direct consequence of Theorem 4.1 when $p = 0$.

Corollary 4.1. *Suppose that the family $\{\delta_C; g_i : i \in I\}$ has the conical* (Q-WEHP)$_f$. *Then the strong Lagrange duality between (P_f) and (D_f) holds.*

As in [7], we used $\Lambda(X)$ to denote the class of all proper convex functions on X. For a convex set Z, we define

$$\Lambda_Z(X) := \{h \in \Lambda(X) : \operatorname{dom} h \cap Z \neq \emptyset\}.$$

For $h \in \Lambda(X)$, let $\operatorname{cont} h$ denote the set of all continuity points of h, that is,

$$\operatorname{cont} h := \{x \in X : h \text{ is continuous at } x\}.$$

The following theorem gives a characterization for the conical (Q-WEHP)$_f$ with $f = 0$.

Theorem 4.2. *The following statements are equivalent:*

(i) *The family* $\{\delta_C; g_i : i \in I\}$ *has the conical* (Q-WEHP)$_0$*, that is,*

$$\text{epi}\delta_A^* = K. \tag{4.11}$$

(ii) *If* $h \in \Lambda_A(X)$ *is such that*

$$\text{epi}(h+\delta_A)^* = \text{epi}h^* + \text{epi}\delta_A^*, \tag{4.12}$$

then the strong Lagrange duality between (P_h) *and* (D_h) *holds.*

(iii) *If* $h \in \Lambda_A(X)$ *and* $\text{cont}h \cap A \neq \emptyset$*, then the strong Lagrange duality between* (P_h) *and* (D_h) *holds.*

(iv) *If* $p \in X^*$*, then the strong Lagrange duality between* (P_p) *and* (D_p) *holds.*

Proof. (i) $\Rightarrow$ (ii). Assume (i) holds. Let $h \in \Lambda_A(X)$ be such that (4.12) holds. Then, by (3.3) and (2.11),

$$\begin{aligned} \text{epi}(h+\delta_A)^* &= \text{epi}h^* + K \\ &\subseteq \text{epi}(h+\delta_C)^* + \text{cone co} \cup_{t\in T} \{(w_t, \delta) \in X^* \times \mathbb{R} : k_t^{-1}(0) \leq \delta\}. \end{aligned}$$

This implies that the family $\{\delta_C; g_i : i \in I\}$ satisfies the $(\text{Q-WEHP})_h$ by Remark 3.1(i). Thus, applying Corollary 4.1, we complete the proof of the implication (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Note that (4.12) is satisfied if $h \in \Lambda_A(X)$ is continuous at some point in A (see Lemma 2.2). Thus it is immediate that (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv). This is trivial.

(iv) $\Rightarrow$ (i). The proof is complete by letting $f = 0$ in Theorem 4.1. □

5. Total Lagrange Duality

Recall that the problem (P_{f-p}) and the corresponding dual problem (D_{f-p}) are defined by (4.1) and (4.2), respectively. For each $p \in X^*$, we use $S(P_{f-p})$ and $S(D_{f-p})$ to denote the optimal solution sets of (P_{f-p}) and (D_{f-p}), respectively. We write $S(P_f)$ for $S(P_{f-0})$, the optimal solution set of the problem (P_f). It is easy to see that

$$x_0 \in S(P_{f-p}) \Longleftrightarrow p \in \partial(f+\delta_A)(x_0). \tag{5.1}$$

This section is devoted to the study of characterizing the total lagrange dualities, which are defined as follows.

Definition 5.1. Between the problems (P_f) and (D_f), we say that

(i) the total Lagrange duality holds if the strong Lagrange duality holds provided that $S(P_f) \neq \emptyset$;

(ii) the stable total Lagrange duality holds if, for each $p \in X^*$, the strong Lagrange duality holds between (P_{f-p}) and (D_{f-p}) provided that $S(P_{f-p}) \neq \emptyset$.

The following theorem provides a sufficient and necessary condition to ensure the stable total Lagrange duality between (P_f) and (D_f) holds.

Theorem 5.1. *The family* $\{\delta_C; g_i : i \in I\}$ *has* $(\text{Q-WBCQ})_f$ *if and only if the stable total Lagrange duality between* (P_f) *and* (D_f) *holds.*

Proof. Let $p \in X^*$ and $x_0 \in S(P_{f-p})$. By (5.1), we have $p \in \partial(f+\delta_A)(x_0)$, and hence, by the assumption of (Q-WBCQ)$_f$,

$$p \in \partial(f+\delta_C)(x_0) + \text{cone co} \bigcup_{t \in T(x_0)} \{w_t\}.$$

Noting that $\partial(w_{t_i}(\cdot) - k_{t_i}^{-1}(0))(x_0) = \{w_{t_i}\}$ for each $t_i \in T$. There exists $(\overline{\lambda}_{t_i})_{t_i \in T} \in \mathbb{R}_+^{(T)}$ such that

$$p \in \partial(f+\delta_C)(x_0) + \sum_{t_i \in T} \overline{\lambda}_{t_i} \partial\big(w_{t_i} - k_{t_i}^{-1}(0)\big)(x_0) \tag{5.2}$$

and

$$\overline{\lambda}_{t_i}\big(w_{t_i}(x_0) - k_{t_i}^{-1}(0)\big) = 0 \text{ for each } t_i \in T. \tag{5.3}$$

Thus,

$$p \in \partial\big(f + \delta_C + \sum_{t_i \in T} \overline{\lambda}_{t_i}(w_{t_i} - k_{t_i}^{-1}(0)\big)(x_0).$$

Combining this with (2.5) and (5.3), we see that, for each $x \in C$,

$$\langle p, x - x_0 \rangle \le f(x) + \sum_{t_i \in T} \overline{\lambda}_{t_i}\big(w_{t_i}(x) - k_{t_i}^{-1}(0)\big) - f(x_0).$$

Therefore,

$$v(P_{f-p}) \le \inf_{x \in C}\{f(x) - \langle p, x\rangle + \sum_{t_i \in T} \overline{\lambda}_{t_i}\big(w_{t_i}(x) - k_{t_i}^{-1}(0)\big)\} \le v(D_{f-p}).$$

This together with the stable weak Lagrange duality implies that $v(D_{f-p}) = v(P_{f-p})$ and $\overline{\lambda} \in S(D_{f-p})$.

Conversely, suppose that the stable total Lagrange duality holds between (P_f) and (D_f). Let $x_0 \in \text{dom} f \cap A$. In view of Remark 3.1, we only need to show

$$\partial(f+\delta_A)(x_0) \subseteq \partial(f+\delta_C)(x_0) + \text{cone co} \bigcup_{t \in T(x_0)} \{w_t\}. \tag{5.4}$$

To do this, Suppose that $p \in \partial(f+\delta_A)(x_0)$. Then, by (5.1), $x_0 \in S(P_{f-p})$. Thus, by the stable total Lagrange duality holds between (P_f) and (D_f), there exists $\overline{\lambda} \in S(D_{f-p})$ such that

$$\begin{aligned} f(x_0) - \langle p, x_0\rangle &= \inf_{x \in C}\{f(x) - \langle p, x\rangle + \sum_{t_i \in T} \overline{\lambda}_{t_i}\big(w_{t_i}(x) - k_{t_i}^{-1}(0)\big)\} \\ &\le f(x_0) - \langle p, x_0\rangle + \sum_{t_i \in T} \overline{\lambda}_{t_i}\big(w_{t_i}(x_0) - k_{t_i}^{-1}(0)\big) \\ &\le f(x_0) - \langle p, x_0\rangle, \end{aligned} \tag{5.5}$$

where the last inequality holds as $w_{t_i}(x_0) - k_{t_i}^{-1}(0) \le 0$ for each $t_i \in T$. Hence,

$$\sum_{t_i \in T} \overline{\lambda}_{t_i}\big(w_{t_i}(x_0) - k_{t_i}^{-1}(0)\big) = 0$$

and for each $x \in C$,

$$\langle p, x - x_0\rangle \le \big(f + \sum_{t_i \in T} \overline{\lambda}_{t_i}(w_{t_i} - k_{t_i}^{-1}(0))\big)(x) - \big(f + \sum_{t_i \in T} \overline{\lambda}_{t_i}(w_{t_i} - k_{t_i}^{-1}(0))\big)(x_0).$$

This implies that $t_i \in T(x_0)$ and

$$p \in \partial\big(f + \delta_C + \sum_{t_i \in T} \overline{\lambda}_{t_i}(w_{t_i} - k_{t_i}^{-1}(0))\big)(x_0).$$

Consequently,

$$\begin{aligned} p &\in \partial\big(f + \delta_C + \sum_{t_i \in T(x_0)} \overline{\lambda}_{t_i}(w_{t_i} - k_{t_i}^{-1}(0))\big)(x_0) = \partial(f + \delta_C)(x_0) + \sum_{t_i \in T(x_0)} \overline{\lambda}_{t_i} w_{t_i} \\ &\subseteq \partial(f + \delta_C)(x_0) + \text{cone co} \bigcup_{t \in T(x_0)} \{w_{t_i}\}. \end{aligned}$$

Therefore, the $(\text{Q-WBCQ})_f$ holds. □

The following corollary is a direct consequence of Theorem 5.1 and generalizes [18, Theorem 3.7].

Corollary 5.1. *The family* $\{\delta_C; g_i : i \in I\}$ *has* $(\text{Q-WBCQ})_f$*. Then the total Lagrange duality between* (P_f) *and* (D_f) *holds.*

Specializing with $f = 0$, we have the following theorem.

Theorem 5.2. *The following statements are equivalent:*

(i) *The family* $\{\delta_C; g_i : i \in I\}$ *satisfies* $(\text{Q-WBCQ})_0$*, that is*

$$N_A(x) = N_C(x) + \text{cone co} \bigcup_{t \in T(x)} \{w_t\}. \tag{5.6}$$

(ii) *If* $h \in \Lambda_A(X)$ *is such that*

$$\partial(h + \delta_A)(x) = \partial h(x) + N_A(x) \tag{5.7}$$

and $S(P_h) \neq \emptyset$*, then the total Lagrange duality between* (P_h) *and* (D_h) *holds.*

(iii) *If* $h \in \Lambda_A(X)$*,* $S(P_h) \neq \emptyset$ *and* $\text{cont}h \cap A \neq \emptyset$*, then the total Lagrange duality between* (P_h) *and* (D_h) *holds.*

(iv) *If* $h \in X^*$ *and* $S(P_h) \neq \emptyset$*, then the total Lagrange duality between* (P_h) *and* (D_h) *holds.*

Proof. (i) ⇒ (ii). Let $h \in \Lambda_A(X)$ be such that (5.7) holds. Then, it follows from (5.6) and (2.12) that

$$\partial(h + \delta_A)(x) \subseteq \partial(h + \delta_C)(x) + \text{cone co} \bigcup_{t \in T(x)} \{w_t\}.$$

This shows that the family $\{\delta_C; g_i : i \in I\}$ has the $(\text{Q-WBCQ})_h$ (see Remark 3.1). Applying Corollary 5.1, we complete the proof of the implication (i) ⇒ (ii).

(ii) ⇒ (iii). This is clear that (5.7) holds for all $h \in \Lambda_A(x)$ with $\text{cont}h \cap A \neq \emptyset$.

(iii) ⇒ (iv). This is trivial.

(iv) ⇒ (i). It is a straightforward consequence of Theorem 5.1 (applied $0, h$ in place of $f, -p$). □

Acknowledgments

This research was supported in part by the National Natural Science Foundation of China (Grant 11861033), Hunan Provincial Natural Science Foundation of China (Grant 2020JJ4494).

Yeong-Cheng Liou was partially supported by MOST project in Taiwan with grand no. MOST 110-2410-H-037-001 and NKUST and KMU joint R&D Project (NKUST- KMU 110KK002).

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