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AN INERTIAL NON-MONOTONIC SELF-ADAPTIVE ITERATIVE ALGORITHM FOR SOLVING EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we introduce a modification of the extragradient algorithm with a non-monotonic stepsize rule to solve equilibrium problems. This modification is based on the inertial subgradient technique. Under mild conditions, such as, the Lipschitz continuity and the monotonicity of a bifunction (including the pseudomonotonicity), the strong convergence of the proposed algorithm is established in a real Hilbert space. The proposed algorithm uses a non-monotonic stepsize rule based on the local bifunction information rather than its Lipschitz-type constants or other line search methods. We present various numerical examples, which illustrate the strong convergence of the algorithm.
Keywords. Equilibrium problem; Inertial method; Lipschitz-type conditions; Non-monotonic stepsize rule; Strong convergence.

1. Introduction

Let $\mathbb{C}$ be a nonempty, convex, and closed subset of a real Hilbert space $\mathbb{H}$. Let $f : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be a bifunction with $f(y,y) = 0$ for each $y \in \mathbb{C}$. Recall that the known equilibrium problem is defined in the following way: Find $\wp^* \in \mathbb{C}$ such that

$$f(\wp^*, y) \geq 0, \ \forall y \in \mathbb{C}. \tag{EP}$$

This problem draw much attention due to the facts that it has lots of real applications and includes a number of mathematical problems, such as fixed point problems, vector and scalar minimization problems, variational inequalities, complementarity problems, and saddle point problems as special cases; see, e.g., [1, 2, 3, 4, 5] and the references therein. To the best of our knowledge, the term "equilibrium problem" was introduced in 1992 by Muu and Oettli [1] and

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further effectively studied by Blum and Oettli [2]. Problem (EP) is also known as the Ky Fan inequality because of his initial contribution to the field [6].

Recently, many researchers introduced and studied various iterative methods for solving problem (EP), such as proximal point methods, extragradient-like methods, and hybrid projection methods; see, e.g., [7, 8, 9, 10, 11, 12] and the references therein. One of useful methods is the famous extragradient method introduced by Flam et al. [13] and Tran et al. [10]. The method reads as follows. Select an arbitrary starting point $u_0 \in \mathbb{H}$. For the current iterate u_n, we obtain the next iteration u_{n+1} by the following iterative process

$$\begin{cases} u_n \in \mathbb{C}, \\ y_n = \underset{y \in \mathbb{C}}{\arg\min}\{\zeta f(u_n, y) + \frac{1}{2}\|u_n - y\|^2\}, \\ u_{n+1} = \underset{y \in \mathbb{C}}{\arg\min}\{\zeta f(y_n, y) + \frac{1}{2}\|u_n - y\|^2\}, \end{cases}$$

where c_1 and c_1 are Lipschitz constants of f_1 and f_2, and $0 < \zeta < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$. We remark that the extragradient method was first introduced and studied by Korpelevich [14] for solving saddle point problems. It is also important to note that the above results in Flam et al. [13] and Tran et al. [10] used a constant stepsize that is dependent on the Lipschitz-type constants of the bifunction. In [10], a weak convergence result was established. The above methods are limited from the viewpoint of computation because the Lipschitz-type constants are usually unknown or not easy to calculate. Recently, Hieu et al. [11] introduced a new gradient-based method for solving pseudomonotone equilibrium problems with the aid of the new stepsize rule, however, their stepsize sequence is nonincreasing and the results in Hieu et al. [11] may depend on the choice of initial stepsize.

Recently, the inertial-type method were extensively studied to accelerate original algorithms; see, e.g., [15, 16, 17, 18, 19] and the references therein. It was initially derived from the oscillator equation with a damping and conservative force restoration. This second-order dynamical system is called a heavy friction ball, which was first studied by Polyak [20]. The main feature of inertial-type methods is that we can use the two previous iterations to obtain the next iteration. Numerical findings confirm that the inertial effect strengthens the efficiency of algorithms; see, e.g., [21, 22] and the references therein. In view of the above results, the following question arises naturally. Can one devise a strongly convergent inertial extragradient-like algorithm with non-monotone stepsize rules to solve problem (EP) that does not depends on the Lipschitz-type constants and contractive mappings?

Motivated by the works of Censor et al. [23] and Hieu et al. [11], we give a positive answer to above question with the aid of inertial subgradient techniques in the context of infinite-dimensional real Hilbert spaces. Our main contributions in this paper are listed below. We introduce an inertial subgradient extragradient method with a non-monotone stepsize rule to solve the equilibrium problem in a real Hilbert space. The bifunction in problem (EP) is pseudomonotone. We obtain a strong convergence result without the aid of contractive mappings. Numerical experiments are provided to show that our algorithm is efficient and performs better than the existing ones.

The rest of the paper is organized as follows. Section 2 includes basic definitions and lemmas. In Section 3, we introduce the new inertial subgradient extragradient algorithm with a non-monotonic stepsize rule. In Section 4, the last section, we give numerical results to illustrate the behaviour of our algorithm.

2. Preliminaries

Let $\mathbb{C}$ be a nonempty, convex, and closed subset of a real Hilbert space $\mathbb{H}$. Let $f : \mathbb{H}\times\mathbb{H} \to \mathbb{R}$ be a bifunction. For problem (EP), we recall the following definitions (see [2, 24]).

(C1) f is said to be pseudomonotone on $\mathbb{C}$ if $f(u_1,u_2) \geq 0 \Longrightarrow f(u_2,u_1) \leq 0$, $\forall u_1,u_2 \in \mathbb{C}$.

(C2) f is said to be Lipschitz-type continuous on $\mathbb{C}$ if there exist two constants $c_1, c_2 > 0$ such that $f(u_1,u_3) \leq f(u_1,u_2) + f(u_2,u_3) + c_1\|u_1-u_2\|^2 + c_2\|u_2-u_3\|^2$, $\forall u_1,u_2,u_3 \in \mathbb{C}$.

(C3) $\limsup\limits_{n\to\infty} f(u_n,y) \leq f(q^*,y)$, where $\{u_n\}$ is a sequence weakly converging to q^*.

(C4) $f(u,\cdot)$ is convex and subdifferentiable on $\mathbb{H}$ for each fixed $u \in \mathbb{H}$, and the solution set $EP(f,\ \mathbb{C})$ is nonempty.

A metric projection $P_{\mathbb{C}}(u)$ of $u \in \mathbb{H}$ onto a closed and convex subset $\mathbb{C}$ of a Hilbert space $\mathbb{H}$ is defined by $P_{\mathbb{C}}(u) = \arg\min\{\|y-u\| : y \in \mathbb{C}\}$. A normal cone of $\mathbb{C}$ at $u \in \mathbb{C}$ is defined by $N_{\mathbb{C}}(u) = \{z \in \mathbb{H} : \langle z, y-u\rangle \leq 0, \forall y \in \mathbb{C}\}$. Let $\mho : \mathbb{C} \to \mathbb{R}$ be a convex function. Then, the subdifferential of $\mho$ at $u \in \mathbb{C}$ is defined by $\partial\mho(u) = \{z \in \mathbb{H} : \mho(y) - \mho(u) \geq \langle z, y-u\rangle, \forall y \in \mathbb{C}\}$.

Lemma 2.1. [25] *Let $\mho : \mathbb{C} \to \mathbb{R}$ be a convex, subdifferentiable and lower semi-continuous function on $\mathbb{C}$. An element $u \in \mathbb{C}$ is a minimizer of a function $\mho$ if and only if $0 \in \partial\mho(u) + N_{\mathbb{C}}(u)$, where $\partial\mho(u)$ stands for the subdifferentiable of $\mho$ at $u \in \mathbb{C}$, and $N_{\mathbb{C}}(u)$ stands for the normal cone of $\mathbb{C}$ at u.*

Lemma 2.2. [26] *Let $\{\mho_n\}$ be a sequence of non-negative real numbers such that $\mho_{n+1} \leq (1-b_n)\mho_n + b_n\ell_n$, where $\{b_n\} \subset (0,1)$ and $\{\ell_n\} \subset \mathbb{R}$ are two sequences such that $\lim_{n\to\infty} b_n = 0$, $\sum_{n=1}^{\infty} b_n = \infty$, and $\limsup_{n\to\infty} \ell_n \leq 0$. Then, $\lim_{n\to\infty} \mho_n = 0$.*

Lemma 2.3. [27] *Assume that $\{\mho_n\}$ is a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\mho_{n_i} < \mho_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then, there is a non decreasing sequence $m_k \subset \mathbb{N}$ such that $m_k \to \infty$ as $k \to \infty$, and the following conditions are fulfilled by all (sufficiently large) numbers $k \in \mathbb{N}$: $\mho_{m_k} \leq \mho_{m_{k+1}}$ and $\mho_k \leq \mho_{m_{k+1}}$. In fact, $m_k = \max\{j \leq k : \mho_j \leq \mho_{j+1}\}$.*

3. Main Results

In this section, we present a subgradient extragradient-like algorithm that combines both the non-monotonic stepsize rule and the inertial term and give strong convergence results in this section. Our algorithm is given as follows.

Algorithm 3.1. ***Step 0:*** *Initially select $\alpha \geq 0$, $\zeta_1 > 0$, $\mu \in (0,1)$ and $u_0, u_1 \in \mathbb{C}$. Moreover, select a non-negative real sequence $\{\varphi_n\}$ such that $\sum_{n=1}^{\infty} \varphi_n < +\infty$ and choose $\{\beta_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = +\infty$.*

Step 1: *Compute* $\chi_n = u_n + \alpha_n(u_n - u_{n-1}) - \beta_n\big[u_n + \alpha_n(u_n - u_{n-1})\big]$ *and choose* α_n *such that*

$$0 \le \alpha_n \le \hat{\alpha}_n \quad \text{and} \quad \hat{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\varepsilon_n}{\|u_n - u_{n-1}\|}\right\} & \text{if} \quad u_n \neq u_{n-1}, \\ \alpha & \text{otherwise}, \end{cases} \tag{3.1}$$

where $\varepsilon_n = \circ(\beta_n)$ *a positive sequence such that* $\lim_{n\to\infty} \frac{\varepsilon_n}{\beta_n} = 0$.

Step 2: *Compute* $y_n = \underset{y\in\mathbb{C}}{\arg\min}\{\zeta_n f(\chi_n, y) + \frac{1}{2}\|\chi_n - y\|^2\}$. *If* $\chi_n = y_n$, *then STOP and* y_n *is a solution. Otherwise, go to* ***Step 3***.

Step 3: *Firstly choose* $\omega_n \in \partial_2 f(\chi_n, y_n)$ *satisfying* $\chi_n - \zeta_n\omega_n - y_n \in N_{\mathbb{C}}(y_n)$ *and construct a half-space*

$$\mathbb{H}_n = \{w \in \mathbb{H} : \langle \chi_n - \zeta_n\omega_n - y_n, w - y_n\rangle \le 0\}.$$

Compute $u_{n+1} = \underset{y\in\mathbb{H}_n}{\arg\min}\{\zeta_n f(y_n, y) + \frac{1}{2}\|\chi_n - y\|^2\}$.

Step 4: *Revise the stepsize* ζ_{n+1} *as follows:*

$$\zeta_{n+1} = \begin{cases} \min\left\{\zeta_n + \varphi_n, \ \frac{\mu\|\chi_n - y_n\|^2 + \mu\|u_{n+1} - y_n\|^2}{2[f(\chi_n, u_{n+1}) - f(\chi_n, y_n) - f(y_n, u_{n+1})]}\right\} \\ \qquad\qquad \text{if} \quad f(\chi_n, u_{n+1}) - f(\chi_n, y_n) - f(y_n, u_{n+1}) > 0, \\ \zeta_n + \varphi_n \qquad\qquad\qquad\qquad \text{otherwise}. \end{cases}$$

Set $n := n + 1$ *and go to* ***Step 1***.

Lemma 3.1. *Sequence* $\{\zeta_n\}$ *converges to* ζ *and* $\min\left\{\frac{\mu}{\max\{2c_1, 2c_2\}}, \zeta_1\right\} \le \zeta \le \zeta_1 + P$, *where* $P = \sum_{n=1}^{+\infty} \varphi_n$.

Proof. Due to the Lipschitz-type continuity, there exist constants $c_1 > 0$ and $c_2 > 0$. Assume that $f(\chi_n, u_{n+1}) - f(\chi_n, y_n) - f(y_n, u_{n+1}) > 0$ such that

$$\frac{\mu(\|\chi_n - y_n\|^2 + \|u_{n+1} - y_n\|^2)}{2[f(\chi_n, u_{n+1}) - f(\chi_n, y_n) - f(y_n, u_{n+1})]} \ge \frac{\mu(\|\chi_n - y_n\|^2 + \|u_{n+1} - y_n\|^2)}{2[c_1\|\chi_n - y_n\|^2 + c_2\|u_{n+1} - y_n\|^2]} \ge \frac{\mu}{2\max\{c_1, c_2\}}.$$

It follows that

$$\min\left\{\frac{\mu}{\max\{2c_1, 2c_2\}}, \zeta_1\right\} \le \zeta_n \le \zeta_1 + P.$$

Let $[\zeta_{n+1} - \zeta_n]^+ = \max\{0, \zeta_{n+1} - \zeta_n\}$ and $[\zeta_{n+1} - \zeta_n]^- = \max\{0, -(\zeta_{n+1} - \zeta_n)\}$. From the definition of $\{\zeta_n\}$, we have

$$\sum_{n=1}^{+\infty} (\zeta_{n+1} - \zeta_n)^+ = \sum_{n=1}^{+\infty} \max\{0, \zeta_{n+1} - \zeta_n\} \le P < +\infty.$$

That is, $\sum_{n=1}^{+\infty} (\zeta_{n+1} - \zeta_n)^+$ is convergent.

Next, we need to prove the convergence of $\sum_{n=1}^{+\infty} (\zeta_{n+1} - \zeta_n)^-$. Let $\sum_{n=1}^{+\infty} (\zeta_{n+1} - \zeta_n)^- = +\infty$. Hence, $\zeta_{n+1} - \zeta_n = (\zeta_{n+1} - \zeta_n)^+ - (\zeta_{n+1} - \zeta_n)^-$. This implies

$$\zeta_{k+1} - \zeta_1 = \sum_{n=0}^{k} (\zeta_{n+1} - \zeta_n) = \sum_{n=0}^{k} (\zeta_{n+1} - \zeta_n)^+ - \sum_{n=0}^{k} (\zeta_{n+1} - \zeta_n)^-. \tag{3.2}$$

By letting $k \to +\infty$ in (3.2), we have $\zeta_k \to -\infty$ as $k \to \infty$. This is a contradiction. Letting $k \to +\infty$ in (3.2), we obtain $\lim_{n\to\infty} \zeta_n = \zeta$. This completes the proof. □

Lemma 3.2. *Assume that conditions* (C1)–(C4) *are satisfied. Then, the sequence* $\{u_n\}$ *generated by Algorithm 3.1 is bounded.*

Proof. From (3.1), we have $\alpha_n\|u_n - u_{n-1}\| \leq \varepsilon_n$, $\forall n \in \mathbb{N}$. Due to $\lim_{n\to\infty}\left(\frac{\varepsilon_n}{\beta_n}\right) = 0$, we obtain

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| \leq \lim_{n\to\infty} \frac{\varepsilon_n}{\beta_n} = 0.$$

By using Lemma 2.1, we have

$$0 \in \partial_2\Big\{\zeta_n f(y_n,\cdot) + \frac{1}{2}\|\chi_n - \cdot\|^2\Big\}(u_{n+1}) + N_{\mathbb{H}_n}(u_{n+1}).$$

For $\omega \in \partial f(y_n, u_{n+1})$, there exists a vector $\overline{\omega} \in N_{\mathbb{H}_n}(u_{n+1})$ such that $\zeta_n\omega + u_{n+1} - \chi_n + \overline{\omega} = 0$. It follows that

$$\langle \chi_n - u_{n+1}, y - u_{n+1}\rangle = \zeta_n\langle \omega, y - u_{n+1}\rangle + \langle \overline{\omega}, y - u_{n+1}\rangle,\ \forall y \in \mathbb{H}_n.$$

Since $\overline{\omega} \in N_{\mathbb{H}_n}(u_{n+1})$ implies that $\langle \overline{\omega}, y - u_{n+1}\rangle \leq 0$, for all $y \in \mathbb{H}_n$, we have

$$\langle \chi_n - u_{n+1}, y - u_{n+1}\rangle \leq \zeta_n\langle \omega, y - u_{n+1}\rangle,\ \forall y \in \mathbb{H}_n. \tag{3.3}$$

Moreover, $\omega \in \partial f(y_n, u_{n+1})$. Hence,

$$f(y_n, y) - f(y_n, u_{n+1}) \geq \langle \omega, y - u_{n+1}\rangle,\ \forall y \in \mathbb{H}. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\zeta_n f(y_n, y) - \zeta_n f(y_n, u_{n+1}) \geq \langle \chi_n - u_{n+1}, y - u_{n+1}\rangle,\ \forall y \in \mathbb{H}_n. \tag{3.5}$$

Hence,

$$\zeta_n\langle \omega_n, u_{n+1} - y_n\rangle \geq \langle \chi_n - y_n, u_{n+1} - y_n\rangle. \tag{3.6}$$

By using $\omega_n \in \partial f(\chi_n, y_n)$, we obtain

$$f(\chi_n, y) - f(\chi_n, y_n) \geq \langle \omega_n, y - y_n\rangle,\ \forall y \in \mathbb{H}.$$

By letting $y = u_{n+1}$, we have

$$f(\chi_n, u_{n+1}) - f(\chi_n, y_n) \geq \langle \omega_n, u_{n+1} - y_n\rangle,\ \forall y \in \mathbb{H}. \tag{3.7}$$

Combining (3.6) and (3.7), we arrive at

$$\zeta_n\big\{f(\chi_n, u_{n+1}) - f(\chi_n, y_n)\big\} \geq \langle \chi_n - y_n, u_{n+1} - y_n\rangle. \tag{3.8}$$

By substituting $y = \wp^*$ into (3.5), we have

$$\zeta_n f(y_n, \wp^*) - \zeta_n f(y_n, u_{n+1}) \geq \langle \chi_n - u_{n+1}, \wp^* - u_{n+1}\rangle. \tag{3.9}$$

Observe that $\wp^* \in EP(f, \mathbb{C})$ implies $f(\wp^*, y_n) \geq 0$. From the pseudomonotonicity of bifunction f, we obtain $f(y_n, \wp^*) \leq 0$. Thanks to (3.9), we arrive at

$$\langle \chi_n - u_{n+1}, u_{n+1} - \wp^*\rangle \geq \zeta_n f(y_n, u_{n+1}). \tag{3.10}$$

From definition of ζ_{n+1}, we obtain

$$f(\chi_n, u_{n+1}) - f(\chi_n, y_n) - f(y_n, u_{n+1}) \leq \frac{\mu\|\chi_n - y_n\|^2 + \mu\|u_{n+1} - y_n\|^2}{2\zeta_{n+1}} \tag{3.11}$$

(3.10) and (3.11) yield that

$$\begin{aligned}\langle \chi_n - u_{n+1}, u_{n+1} - \wp^* \rangle \geq\ & \zeta_n \{ f(\chi_n, u_{n+1}) - f(\chi_n, y_n) \} \\ & - \frac{\mu \zeta_n}{2\zeta_{n+1}} \|\chi_n - y_n\|^2 - \frac{\mu \zeta_n}{2\zeta_{n+1}} \|u_{n+1} - y_n\|^2. \end{aligned} \tag{3.12}$$

Combining (3.8) and (3.12), we have

$$\begin{aligned}\langle \chi_n - u_{n+1}, u_{n+1} - \wp^* \rangle \geq\ & \langle \chi_n - y_n, u_{n+1} - y_n \rangle \\ & - \frac{\mu \zeta_n}{2\zeta_{n+1}} \|\chi_n - y_n\|^2 - \frac{\mu \zeta_n}{2\zeta_{n+1}} \|u_{n+1} - y_n\|^2. \end{aligned} \tag{3.13}$$

Observe that

$$-2\langle \chi_n - u_{n+1}, u_{n+1} - \wp^* \rangle = -\|\chi_n - \wp^*\|^2 + \|u_{n+1} - \chi_n\|^2 + \|u_{n+1} - \wp^*\|^2.$$

$$2\langle y_n - \chi_n, y_n - u_{n+1} \rangle = \|\chi_n - y_n\|^2 + \|u_{n+1} - y_n\|^2 - \|\chi_n - u_{n+1}\|^2. \tag{3.14}$$

Combining (3.13) and (3.14), we have

$$\|u_{n+1} - \wp^*\|^2 \leq \|\chi_n - \wp^*\|^2 - \Big(1 - \frac{\mu \zeta_n}{\zeta_{n+1}}\Big) \|\chi_n - y_n\|^2 - \Big(1 - \frac{\mu \zeta_n}{\zeta_{n+1}}\Big) \|u_{n+1} - y_n\|^2. \tag{3.15}$$

It follows from the definition of $\{\chi_n\}$ that

$$\begin{aligned}\big\|\chi_n - \wp^*\big\| &= \big\|(1-\beta_n)(u_n - \wp^*) + (1-\beta_n)\alpha_n(u_n - u_{n-1}) - \beta_n \wp^*\big\| \\ &\leq (1-\beta_n)\big\|u_n - \wp^*\big\| + (1-\beta_n)\alpha_n \big\|u_n - u_{n-1}\big\| + \beta_n \big\|\wp^*\big\| \\ &\leq (1-\beta_n)\|u_n - \wp^*\| + \beta_n K_1, \end{aligned} \tag{3.16}$$

where

$$(1-\beta_n)\frac{\alpha_n}{\beta_n}\big\|u_n - u_{n-1}\big\| + \big\|\wp^*\big\| \leq K_1.$$

It is given that $\zeta_n \to \zeta$, so there exists a fixed number $\Im \in (0, 1-\mu)$ such that

$$\lim_{n\to\infty} \Big(1 - \frac{\mu \zeta_n}{\zeta_{n+1}}\Big) = 1 - \mu > \Im > 0.$$

Thus, there exists a fixed finite number $N_1 \in \mathbb{N}$ such that

$$\Big(1 - \frac{\mu \zeta_n}{\zeta_{n+1}}\Big) > \Im > 0, \ \forall n \geq N_1.$$

From (3.15), we rewrite

$$\|u_{n+1} - \wp^*\|^2 \leq \|\chi_n - \wp^*\|^2, \ \forall n \geq N_1. \tag{3.17}$$

Combining (3.16) and (3.17), we obtain

$$\begin{aligned}\|u_{n+1} - \wp^*\| &\leq (1-\beta_n)\|u_n - \wp^*\| + \beta_n K_1 \\ &\leq \max\big\{\|u_n - \wp^*\|, K_1\big\} \\ &\ \ \vdots \\ &\leq \max\big\{\|u_{N_1} - \wp^*\|, K_1\big\}. \end{aligned}$$

This infers that $\{u_n\}$ is a bounded sequence. □

Theorem 3.1. *Let $\{u_n\}$ be te sequence generated by Algorithm 3.1, and let* (C1)–(C4) *be satisfied. Then, $\{u_n\}$ converges strongly to some $\wp^*$. Moreover, $P_{EP(f,\, \mathbb{C})}(0) = \wp^*$.*

Proof. From (3.16), we have

$$\begin{aligned}\left\|\chi_n - \wp^*\right\|^2 &\le (1-\beta_n)^2\|u_n - \wp^*\|^2 + \beta_n^2 K_1^2 + 2K_1\beta_n(1-\beta_n)\|u_n - \wp^*\| \\ &\le \|u_n - \wp^*\|^2 + \beta_n\left[\beta_n K_1^2 + 2K_1(1-\beta_n)\|u_n - \wp^*\|\right] \\ &\le \|u_n - \wp^*\|^2 + \beta_n K_2,\end{aligned} \tag{3.18}$$

for some $K_2 > 0$. Combining (3.15) and (3.18), we obtain

$$\begin{aligned}\|u_{n+1} - \wp^*\|^2 \le \|u_n - \wp^*\|^2 + \beta_n K_2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|\chi_n - y_n\|^2 \\ - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|u_{n+1} - y_n\|^2.\end{aligned} \tag{3.19}$$

Due to the Lipschitz-continuity and pseudomonotonicity of f, we have that $EP(f,\, \mathbb{C})$ is a closed and convex set (see [10]). Since $\wp^* = P_{EP(f,\, \mathbb{C})}(0)$, we have $\langle 0 - \wp^*, y - \wp^*\rangle \le 0$, $\forall y \in EP(f,\, \mathbb{C})$. The rest of the proof is divided into the following two cases.

Case 1. Assume that there exists a fixed number $N_2 \in \mathbb{N}$ $(N_2 \ge N_1)$ such that $\|u_{n+1} - \wp^*\| \le \|u_n - \wp^*\|$, $\forall n \ge N_2$. The above relation implies that $\lim_{n\to\infty}\|u_n - \wp^*\|$ exists. Let $\lim_{n\to\infty}\|u_n - \wp^*\| = l$, for some $l \ge 0$. From (3.19) we can write

$$\left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|\chi_n - y_n\|^2 + \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|u_{n+1} - y_n\|^2 \le \|u_n - \wp^*\|^2 + \beta_n K_2 - \|u_{n+1} - \wp^*\|^2. \tag{3.20}$$

Due to existence of the limit of $\|u_n - \wp^*\|$, and the fact that $\beta_n \to 0$, we conclude that

$$\|\chi_n - y_n\| \to 0 \quad \text{and} \quad \|u_{n+1} - y_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.21}$$

Hence,

$$\lim_{n\to\infty}\|\chi_n - u_{n+1}\| \le \lim_{n\to\infty}\|\chi_n - y_n\| + \lim_{n\to\infty}\|y_n - u_{n+1}\| = 0. \tag{3.22}$$

Next, we estimate

$$\begin{aligned}\|\chi_n - u_n\| &= \|u_n + \alpha_n(u_n - u_{n-1}) - \beta_n\left[u_n + \alpha_n(u_n - u_{n-1})\right] - u_n\| \\ &\le \beta_n\frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| + \beta_n\|u_n\| + \beta_n^2\frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| \longrightarrow 0,\end{aligned} \tag{3.23}$$

which implies that

$$\lim_{n\to\infty}\|u_n - u_{n+1}\| \le \lim_{n\to\infty}\|u_n - \chi_n\| + \lim_{n\to\infty}\|\chi_n - u_{n+1}\| = 0. \tag{3.24}$$

Thus, $\{\chi_n\}$ and $\{y_n\}$ are bounded sequences. The reflexivity of $\mathbb{H}$ and the boundedness of $\{u_n\}$ guarantee that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\} \rightharpoonup \hat{u} \in \mathbb{H}$ as $k \to \infty$. Next, we prove that $\hat{u} \in EP(f,\, \mathbb{C})$. From (3.5), we have

$$\zeta_{n_k} f(y_{n_k}, y) \ge \zeta_{n_k} f(y_{n_k}, u_{n_{k+1}}) + \langle \chi_{n_k} - u_{n_{k+1}}, y - u_{n_{k+1}}\rangle,\ \forall y \in \mathbb{H}_n.$$

It follows from (3.11) that

$$\zeta_{n_k} f(y_{n_k}, u_{n_k+1}) \geq \zeta_{n_k}\{f(\chi_{n_k}, u_{n_k+1}) - f(\chi_{n_k}, y_{n_k})\} - \frac{\mu\zeta_{n_k}}{2\zeta_{n_k+1}}\|\chi_{n_k} - y_{n_k}\|^2 - \frac{\mu\zeta_{n_k}}{2\zeta_{n_k+1}}\|u_{n_k+1} - y_{n_k}\|^2,$$

which implies that

$$\zeta_{n_k} f(y_{n_k}, y) \geq \zeta_{n_k}\{f(\chi_{n_k}, u_{n_k+1}) - f(\chi_{n_k}, y_{n_k})\} + \langle \chi_{n_k} - u_{n_{k+1}}, y - u_{n_{k+1}}\rangle - \frac{\mu\zeta_{n_k}}{2\zeta_{n_k+1}}\|\chi_{n_k} - y_{n_k}\|^2 - \frac{\mu\zeta_{n_k}}{2\zeta_{n_k+1}}\|u_{n_k+1} - y_{n_k}\|^2.$$

where y is an arbitrary element of set $\mathbb{H}_n$. From (3.21), (3.22), (3.23), (3.24), and the boundedness of $\{u_n\}$, we conclude that the right-hand side of the above expression goes to zero. In view of $\zeta_{n_k} > 0$, and $y_{n_k} \rightharpoonup \hat{u}$, we have $0 \leq \limsup_{k\to\infty} f(y_{n_k}, y) \leq f(\hat{u}, y)$, $\forall y \in \mathbb{H}_n$. Hence, $\hat{u} \in EP(f, \mathbb{C})$. Note that

$$\limsup_{n\to\infty}\langle \wp^*, \wp^* - u_n\rangle = \lim_{k\to\infty}\langle \wp^*, \wp^* - u_{n_k}\rangle = \langle \wp^*, \wp^* - \hat{u}\rangle \leq 0. \tag{3.25}$$

From the fact that $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$ and (3.25), we deduce that

$$\limsup_{n\to\infty}\langle \wp^*, \wp^* - u_{n+1}\rangle \leq \limsup_{n\to\infty}\langle \wp^*, \wp^* - u_n\rangle + \limsup_{n\to\infty}\langle \wp^*, u_n - u_{n+1}\rangle \leq 0. \tag{3.26}$$

Observe that

$$\begin{aligned}
&\|\chi_n - \wp^*\|^2 \\
&= \|u_n + \alpha_n(u_n - u_{n-1}) - \beta_n u_n - \alpha_n\beta_n(u_n - u_{n-1}) - \wp^*\|^2 \\
&= \|(1-\beta_n)(u_n - \wp^*) + (1-\beta_n)\alpha_n(u_n - u_{n-1}) - \beta_n\wp^*\|^2 \\
&\leq \|(1-\beta_n)(u_n - \wp^*) + (1-\beta_n)\alpha_n(u_n - u_{n-1})\|^2 + 2\beta_n\langle -\wp^*, \chi_n - \wp^*\rangle \\
&= (1-\beta_n)^2\|u_n - \wp^*\|^2 + (1-\beta_n)^2\alpha_n^2\|u_n - u_{n-1}\|^2 \\
&\quad + 2\alpha_n(1-\beta_n)^2\|u_n - \wp^*\|\|u_n - u_{n-1}\| + 2\beta_n\langle -\wp^*, \chi_n - u_{n+1}\rangle + 2\beta_n\langle -\wp^*, u_{n+1} - \wp^*\rangle \\
&\leq (1-\beta_n)\|u_n - \wp^*\|^2 + \alpha_n^2\|u_n - u_{n-1}\|^2 + 2\alpha_n(1-\beta_n)\|u_n - \wp^*\|\|u_n - u_{n-1}\| \\
&\quad + 2\beta_n\|\wp^*\|\|\chi_n - u_{n+1}\| + 2\beta_n\langle -\wp^*, u_{n+1} - \wp^*\rangle \\
&= (1-\beta_n)\|u_n - \wp^*\|^2 + \beta_n\Big[\alpha_n\|u_n - u_{n-1}\|\frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| \\
&\quad + 2(1-\beta_n)\|u_n - \wp^*\|\frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| + 2\|\wp^*\|\|\chi_n - u_{n+1}\| + 2\langle \wp^*, \wp^* - u_{n+1}\rangle\Big],
\end{aligned}$$

which together with (3.17) yields that

$$\begin{aligned}
&\|u_{n+1} - \wp^*\|^2 \\
&\leq (1-\beta_n)\|u_n - \wp^*\|^2 + \beta_n\Big[\alpha_n\|u_n - u_{n-1}\|\frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| \\
&\quad + 2(1-\beta_n)\|u_n - \wp^*\|\frac{\alpha_n}{\beta_n}\|u_n - u_{n-1}\| + 2\|\wp^*\|\|\chi_n - u_{n+1}\| + 2\langle \wp^*, \wp^* - u_{n+1}\rangle\Big].
\end{aligned} \tag{3.27}$$

Using (3.22), (3.26), (3.27), and Lemma 2.2, we conclude that $\left\|u_n - \wp^*\right\| \to 0$ as $n \to \infty$.

Case 2: Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i} - \wp^*\| \leq \|u_{n_{i+1}} - \wp^*\|, \ \forall i \in \mathbb{N}.$$

From Lemma 2.3, we see that there exists a sequence $\{m_k\} \subset \mathbb{N}$ as $\{m_k\} \to \infty$ such that

$$\|u_{m_k} - \wp^*\| \leq \|u_{m_{k+1}} - \wp^*\| \quad \text{and} \quad \|u_k - \wp^*\| \leq \|u_{m_{k+1}} - \wp^*\|, \text{ for all } k \in \mathbb{N}. \tag{3.28}$$

Following Case 1, we see that (3.20) yields

$$\begin{aligned}&\left(1 - \frac{\mu \zeta_{m_k}}{\zeta_{m_k+1}}\right)\|\chi_{m_k} - y_{m_k}\|^2 + \left(1 - \frac{\mu \zeta_{m_k}}{\zeta_{m_k+1}}\right)\|u_{m_k+1} - y_{m_k}\|^2 \\ &\leq \|u_{m_k} - \wp^*\|^2 + \beta_{m_k}K_2 - \|u_{m_k+1} - \wp^*\|^2.\end{aligned}$$

Since $\beta_{m_k} \to 0$, we can deduce $\lim_{k\to\infty}\|\chi_{m_k} - y_{m_k}\| = \lim_{k\to\infty}\|u_{m_k+1} - y_{m_k}\| = 0$. Hence,

$$\lim_{k\to\infty}\|u_{m_{k+1}} - \chi_{m_k}\| \leq \lim_{k\to\infty}\|u_{m_{k+1}} - y_{m_k}\| + \lim_{k\to\infty}\|y_{m_k} - \chi_{m_k}\| = 0.$$

Next, we estimate

$$\begin{aligned}\|\chi_{m_k} - u_{m_k}\| &= \|u_{m_k} + \alpha_{m_k}(u_{m_k} - u_{m_k-1}) - \beta_{m_k}\left[u_{m_k} + \alpha_{m_k}(u_{m_k} - u_{m_k-1})\right] - u_{m_k}\| \\ &\leq \alpha_{m_k}\|u_{m_k} - u_{m_k-1}\| + \beta_{m_k}\|u_{m_k}\| + \alpha_{m_k}\beta_{m_k}\|u_{m_k} - u_{m_k-1}\| \\ &= \beta_{m_k}\frac{\alpha_{m_k}}{\beta_{m_k}}\|u_{m_k} - u_{m_k-1}\| + \beta_{m_k}\|u_{m_k}\| + \beta_{m_k}^2\frac{\alpha_{m_k}}{\beta_{m_k}}\|u_{m_k} - u_{m_k-1}\| \longrightarrow 0.\end{aligned}$$

Observe that $\lim_{k\to\infty}\|u_{m_k} - u_{m_k+1}\| \leq \lim_{k\to\infty}\|u_{m_k} - \chi_{m_k}\| + \lim_{k\to\infty}\|\chi_{m_k} - u_{m_k+1}\| = 0$. Following Case 1, we have

$$\limsup_{k\to\infty}\langle \wp^*, \wp^* - u_{m_k+1}\rangle \leq 0. \tag{3.29}$$

From (3.27) and (3.28), we have

$$\begin{aligned}&\left\|u_{m_k+1} - \wp^*\right\|^2 \\ &\leq (1 - \beta_{m_k})\left\|u_{m_k} - \wp^*\right\|^2 + \beta_{m_k}\Big[\alpha_{m_k}\left\|u_{m_k} - u_{m_k-1}\right\|\frac{\alpha_{m_k}}{\beta_{m_k}}\left\|u_{m_k} - u_{m_k-1}\right\| \\ &\quad + 2(1 - \beta_{m_k})\left\|u_{m_k} - \wp^*\right\|\frac{\alpha_{m_k}}{\beta_{m_k}}\left\|u_{m_k} - u_{m_k-1}\right\| + 2\left\|\wp^*\right\|\left\|\chi_{m_k} - u_{m_k+1}\right\| + 2\langle \wp^*, \wp^* - u_{m_k+1}\rangle\Big] \\ &\leq (1 - \beta_{m_k})\left\|u_{m_{k+1}} - \wp^*\right\|^2 + \beta_{m_k}\Big[\alpha_{m_k}\left\|u_{m_k} - u_{m_k-1}\right\|\frac{\alpha_{m_k}}{\beta_{m_k}}\left\|u_{m_k} - u_{m_k-1}\right\| \\ &\quad + 2(1 - \beta_{m_k})\left\|u_{m_k} - \wp^*\right\|\frac{\alpha_{m_k}}{\beta_{m_k}}\left\|u_{m_k} - u_{m_k-1}\right\| + 2\left\|\wp^*\right\|\left\|\chi_{m_k} - u_{m_k+1}\right\| + 2\langle \wp^*, \wp^* - u_{m_k+1}\rangle\Big].\end{aligned}$$

Thus,

$$\begin{aligned}&\left\|u_{m_k+1} - \wp^*\right\|^2 \\ &\leq \Big[\alpha_{m_k}\left\|u_{m_k} - u_{m_k-1}\right\|\frac{\alpha_{m_k}}{\beta_{m_k}}\left\|u_{m_k} - u_{m_k-1}\right\| \\ &\quad + 2(1 - \beta_{m_k})\left\|u_{m_k} - \wp^*\right\|\frac{\alpha_{m_k}}{\beta_{m_k}}\left\|u_{m_k} - u_{m_k-1}\right\| + 2\left\|\wp^*\right\|\left\|\chi_{m_k} - u_{m_k+1}\right\| + 2\langle \wp^*, \wp^* - u_{m_k+1}\rangle\Big].\end{aligned} \tag{3.30}$$

Since $\beta_{m_k} \to 0$ and $\left\|u_{m_k} - \wp^*\right\|$ is a bounded, (3.29) and (3.30) imply that $\|u_{m_k+1} - \wp^*\|^2 \to 0$ as $k \to \infty$. Further, we have $\lim_{n\to\infty} \|u_k - \wp^*\|^2 \le \lim_{n\to\infty} \|u_{m_k+1} - \wp^*\|^2 \le 0$. Hence, $u_n \to \wp^*$. This complete the proof. □

From Theorem 3.1, we obtain the following result to solve variational inequalities.

Corollary 3.1. *Assume that $\mathscr{L} : \mathbb{C} \to \mathbb{H}$ is a pseudomonotone, weakly continuous, and L-Lipschitz continuous operator and the solution set $VI(\mathscr{L},\mathbb{C})$ is nonempty. Let $\{u_n\}$ be a sequence generated in the following algorithm:*
Step 0: *Select $\alpha \ge 0$, $\zeta_1 > 0$, $\mu \in (0,1)$, and $u_0, u_1 \in \mathbb{C}$. Moreover, select a non-negative real sequence $\{\varphi_n\}$ such that $\sum_{n=1}^{\infty} \varphi_n < +\infty$, and a sequence $\{\beta_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = +\infty$.*
Step 1: *Compute $\chi_n = u_n + \alpha_n(u_n - u_{n-1}) - \beta_n\big[u_n + \alpha_n(u_n - u_{n-1})\big]$, where $\{\alpha_n\}$ is modified on each iteration as follows:*

$$0 \le \alpha_n \le \hat{\alpha}_n \quad \textit{and} \quad \hat{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\varepsilon_n}{\|u_n - u_{n-1}\|}\right\} & \textit{if} \quad u_n \ne u_{n-1}, \\ \alpha & \textit{otherwise}. \end{cases} \tag{3.31}$$

Step 2: *Compute*

$$\begin{cases} y_n = P_{\mathbb{C}}(\chi_n - \zeta_n \mathscr{L}(\chi_n)), \\ u_{n+1} = P_{\mathbb{H}_n}(\chi_n - \zeta_n \mathscr{L}(y_n)), \end{cases}$$

where $\mathbb{H}_n = \{z \in \mathbb{H} : \langle \chi_n - \zeta_n \mathscr{L}(\chi_n) - y_n, z - y_n\rangle \le 0\}$. The stepsize rule for next iteration is evaluated as follows:

$$\zeta_{n+1} = \begin{cases} \min\left\{\zeta_n + \varphi_n, \frac{\mu\|\chi_n - y_n\|^2 + \mu\|u_{n+1} - y_n\|^2}{2\langle \mathscr{L}(\chi_n) - \mathscr{L}(y_n), u_{n+1} - y_n\rangle}\right\} & \textit{if} \quad \big\langle \mathscr{L}(\chi_n) - \mathscr{L}(y_n), u_{n+1} - y_n\big\rangle > 0, \\ \zeta_n + \varphi_n & \textit{otherwise}. \end{cases}$$

Then, $\{u_n\}$ converge strongly to $\wp^* \in VI(\mathscr{L},\mathbb{C})$.

4. Numerical Experiments

In this section, we include a numerical illustration to demonstrate the validity of the proposed method. The MATLAB codes were run in MATLAB version 9.5 (R2018b) on the Intel(R) Core(TM)i5-6200 CPU PC @ 2.30GHz 2.40GHz, RAM 8.00 GB.

Problem 4.1. Let $\mathbb{C} \subset \mathbb{R}^5$ be defined as $\mathbb{C} := \{u \in \mathbb{R}^5 : -5 \le u_i \le 5\}$. Let $f : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ be defined as $f(u,y) = \langle Pu + Qy + d, y - u\rangle$, $\forall u,y \in \mathbb{C}$, which is the Nash-Cournot equilibrium model [10]. To see the numerical efficiency of the different methods, we consider P and Q as follows:

$$P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix},$$

where $d = (1,-2,-1,2,-1)^T$ and Lipschitz-type constants are $c_1 = c_2 = \frac{1}{2}\|P - Q\|$.

Problem 4.2. Let $\mathbb{C} \subset \mathbb{R}^N$ be defined as $\mathbb{C} = \{u \in \mathbb{R}^N : Au \leq b\}$, where A is an $100 \times N$ matrix. Let $f : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ be defined as $f(u,y) = \langle \mathscr{L}(u), y-u \rangle$, $\forall u,y \in \mathbb{C}$, where $\mathscr{L} : \mathbb{R}^N \to \mathbb{R}^N$ is an operator defined by $\mathscr{L}(u) = Pu + r$, where $r \in \mathbb{R}^N$ and $P = QQ^T + R + S$, where Q is an $N \times N$ matrix, R is an $N \times N$ skew-symmetric matrix, and S is an $N \times N$ positive definite diagonal matrix. It is easy to see that f is monotone and the Lipschitz constants are $2c_1 = 2c_2 = \|P\|$ (see [28, 29] for details).

Problem 4.3. Let $\mathbb{C} \subset \mathbb{R}^2$ be defined as $\mathbb{C} = \{u \in \mathbb{R}^2 : (u_1-2)^2 + (u_2-2)^2 \leq 1\}$. Let $\mathscr{L} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\mathscr{L}(u) = \begin{pmatrix} 0.5u_1u_2 - 2u_2 - 10^7 \\ -4u_1 - 0.1u_2^2 - 10^7 \end{pmatrix}.$$

We can evaluate that $\mathscr{L}$ is Lipschitz continuous with $L = 5$, and pseudomonotone. Assume that $f(u,y) = \langle \mathscr{L}(u), y-u \rangle$, and $c_1 = c_2 = \frac{5}{2}$.

Problem 4.4. Let $\mathbb{H} = L^2([0,1])$ be a Hilbert space with inner product $\langle u,y \rangle = \int_0^1 u(t)y(t)dt$, for all $u,y \in \mathbb{H}$ and induced norm $\|u\| = \sqrt{\int_0^1 |u(t)|^2 dt}$. Let $f : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ be defined by $f(u,y) = \langle \mathscr{L}(u), y-u \rangle$, $\forall u,y \in \mathbb{C}$, where $\mathbb{C} := \{u \in L^2([0,1]) : \|u\| \leq 1\}$. Moreover, assume that $\mathscr{L} : \mathbb{C} \to \mathbb{H}$ is defined by

$$\mathscr{L}(u)(t) = \int_0^1 \left[u(t) - H(t,s)f(u(s))\right] ds + g(t),$$

where

$$H(t,s) = \frac{2tse^{(t+s)}}{e\sqrt{e^2-1}}, \quad f(u) = \cos(u) \quad \text{and} \quad g(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

From [30], we see that f is monotone (hence pseudomonotone) with Lipschitz constants $2c_1 = 2c_1 = 2$, and the solution set is nonempty.

Now, we use Problem 4.1 to compare the numerical efficiency of Algorithm 3.1, Algorithm 3.2 in [31], Algorithm 4.1 in [11], and Algorithm 3 in [12] by taking the different initials of $u_1 = y_0$ and the fixed values of $u_0 = y_{-1} = (0,0,0,0,0)^T$. We can see how these initials affect the convergence of the iterative sequence. Figures 1-4 show a number of results obtained by enabling a tolerance 10^{-5}. Information about the control parameters are taken as follows: (1) Algorithm 3.2 in [31] (Alg3.2): $\lambda = \frac{1}{\max\{4c_1, 4c_2\}}, \alpha_n = \frac{1}{20(n+2)}, D_n = \|u_n - y_n\|^2$. (2) Algorithm 4.1 in [11] (Alg4.1): $\lambda_0 = 0.25, \mu = 0.33, \alpha_n = \frac{1}{(n+1)^{0.5}}, D_n = \max\{\|u_{n+1} - y_n\|^2, \|u_n - y_n\|^2\}$. (3) Algorithm 3 in [12] (Alg3): $\lambda = \frac{1}{\max\{4c_1, 4c_2\}}, \theta = 0.50, \varepsilon_n = \frac{1}{(n+1)^2}, \gamma_n = \frac{1}{20(n+2)}, \beta_n = \frac{7}{10}(1-\gamma_n), D_n = \|\chi_n - y_n\|^2$. (4) Algorithm 3.1 (Alg1): $\zeta_1 = 0.25, \mu = 0.33, \alpha = 0.5, \varepsilon_n = \frac{1}{(n+1)^2}, \beta_n = \frac{1}{20(n+2)}, D_n = \|\chi_n - y_n\|^2, \varphi_n = \frac{100}{(n+1)^2}$.

Next, we use Problem 4.1 to compare the numerical efficiency of Algorithm 3.1, Algorithm 3.2 in [31], Algorithm 4.1 in [11], and Algorithm 3 in [12] by letting different tolerance values, and $u_1 = y_0 = (1,1,1,1,1)^T$ and $u_0 = y_{-1} = (0,0,0,0,0)^T$. Figures 5 and 6 and the following table show a number results obtained by letting different tolerance values. Information about the control parameters are same as the in the experiment above.

Next, we use Problem 4.2 to compare the numerical effectiveness of Algorithm 3.1, Algorithm 3.2 in [31], Algorithm 4.1 in [11], and Algorithm 3 in [12] by letting different dimension

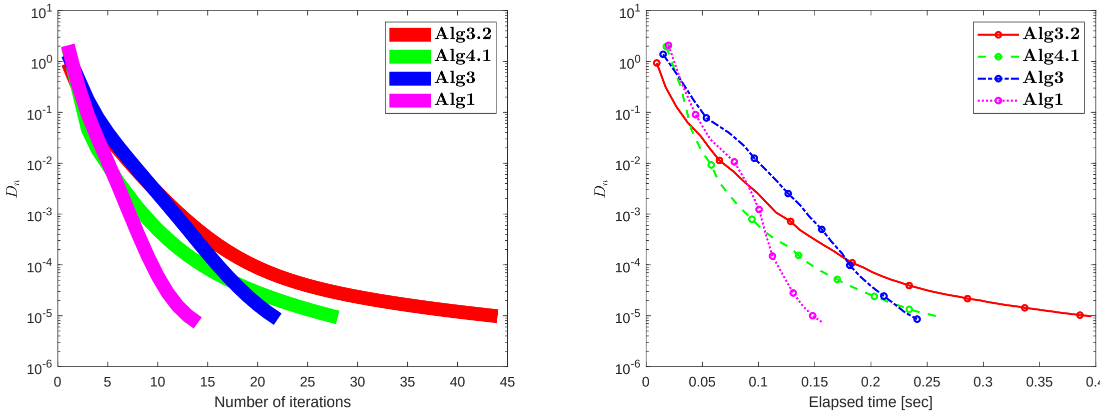


FIGURE 1. $u_1 = (1,0,1,0,1)^T$ and the number of iterations are 44, 28, 22, 14 and elapsed time are 0.3961, 0.2625, 0.2410, 0.1573, respectively.

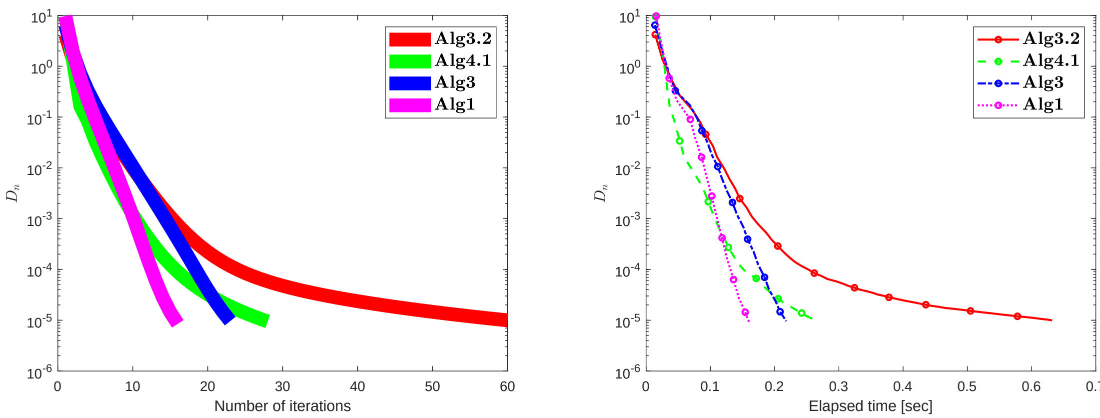


FIGURE 2. $u_1 = (2,1,-2,1,2)^T$ and the number of iterations are 60, 28, 23, 16 and the elapsed time are 0.6321, 0.2676, 0.2180, 0.1613, respectively.

Tolerance	0.01		0.001		0.0001		0.00001	
Algo.	**iter.**	**time**	**iter.**	**time**	**iter.**	**time**	**iter.**	**time**
Algorithm 3.2 in [31]	7	0.07259	12	0.12327	21	0.19863	55	0.50759
Algorithm 4.1 in [11]	5	0.06205	10	0.10908	18	0.17806	30	0.27618
Algorithm 3 in [12]	7	0.08036	12	0.11527	16	0.16133	22	0.21190
Algorithm 3.1	5	0.05530	8	0.09403	10	0.11141	14	0.14207

N, and $u_1 = y_0 = (1,1,1,1,1)^T$ and $u_0 = y_{-1} = (0,0,0,0,0)^T$. Figure 7 show a number of results obtained by letting a tolerance 10^{-4}. Information about the control parameters is considered as follows: (1) Algorithm 3.2 in [31] (Alg3.2): $\lambda = \frac{1}{\max\left\{3c_1, 3c_2\right\}}, \alpha_n = \frac{1}{10(n+2)}, D_n = \|u_n - y_n\|^2$. (2) Algorithm 4.1 in [11] (Alg4.1): $\lambda_0 = 0.50, \mu = 0.40, \alpha_n = \frac{1}{(n+1)^{0.5}}, D_n = \max\left\{\|u_{n+1} - y_n\|^2, \|u_n - y_n\|^2\right\}$. (3) Algorithm 3 in [12] (Alg3): $\lambda = \frac{1}{\max\left\{3c_1, 3c_2\right\}}, \theta = 0.50, \varepsilon_n = \frac{1}{(n+1)^2}, \gamma_n =$

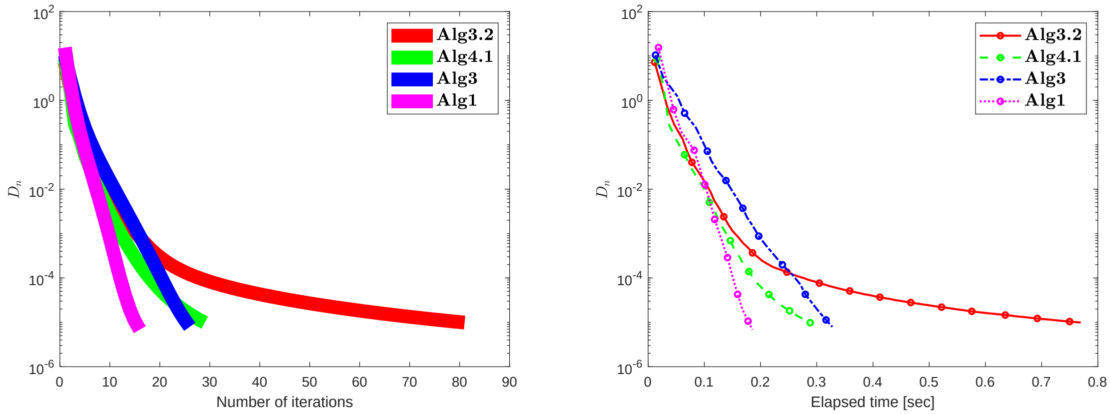


FIGURE 3. $u_1 = (1, 3, -1, 2, 3)^T$ and the number of iterations are 81, 29, 26, 16 and the elapsed time are 0.7697, 0.2884, 0.3278, 0.1857, respectively.

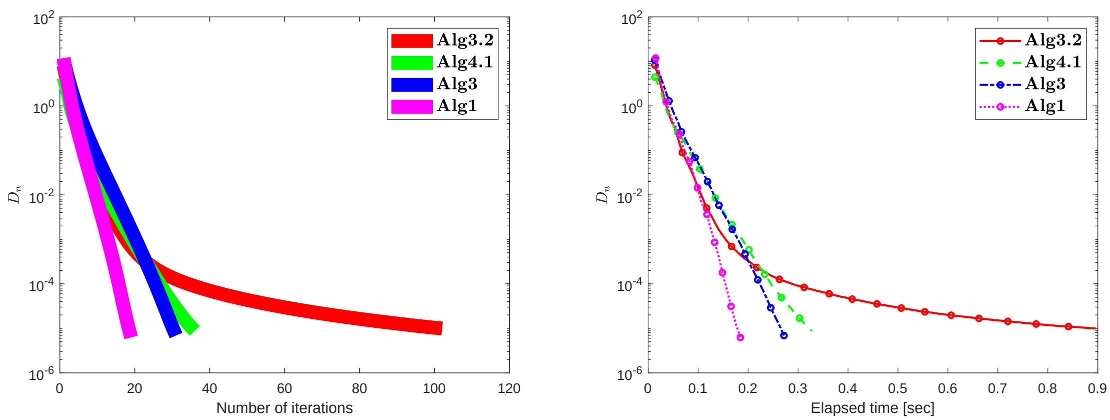
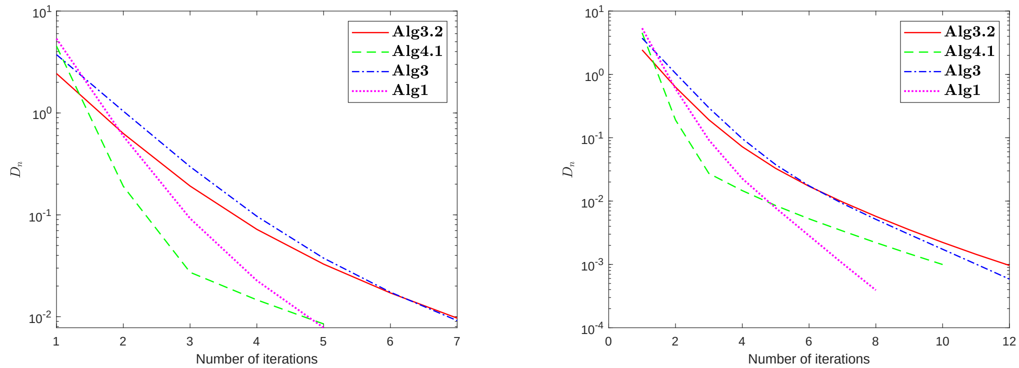


FIGURE 4. $u_1 = (2, -1, 3, -4, 5)^T$ and the number of iterations are 102, 36, 31, 19 and the elapsed time are 0.8959, 0.3275, 0.2718, 0.1845, respectively.

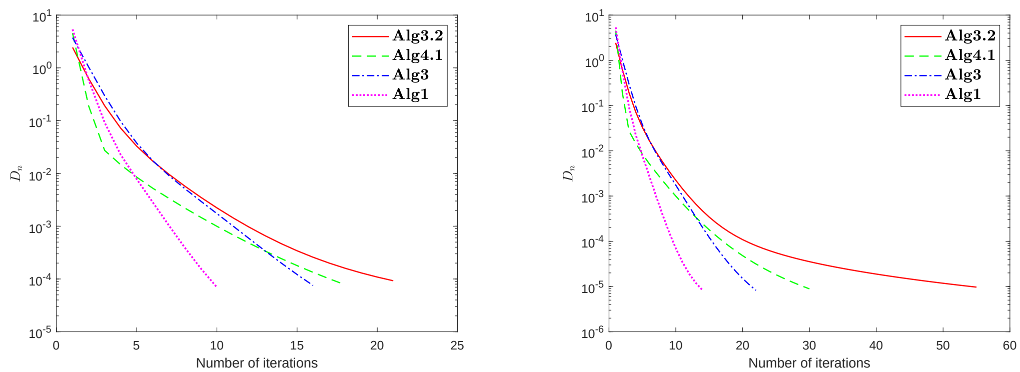
$\frac{1}{10(n+2)}, \beta_n = \frac{5}{10}(1-\gamma_n), D_n = \|\chi_n - y_n\|^2$. (4) Algorithm 3.1 (Alg1): $\zeta_1 = 0.50, \mu = 0.40, \alpha = 0.50, \varepsilon_n = \frac{1}{(n+1)^2}, \beta_n = \frac{1}{10(n+2)}, D_n = \|\chi_n - y_n\|^2, \varphi_n = \frac{100}{(n+1)^2}$.

Next, we work on Problem 4.3 to see the numerical performance of Algorithm 3.1, Algorithm 3.2 in [31], Algorithm 4.1 in [11], and Algorithm 3 in [12] by choosing the different starting points $u_1 = y_0$ and the fixed values of $u_0 = y_{-1} = (0,0)^T$. Figure 8 show a number of results obtain by letting a tolerance 10^{-4}. Information about the control parameters is taken as follows: (1) Algorithm 3.2 in [31] (Alg3.2): $\lambda = \frac{1}{\max\left\{3c_1, 3c_2\right\}}, \alpha_n = \frac{1}{5(n+2)}, D_n = \|u_n - y_n\|^2$. (2) Algorithm 4.1 in [11] (Alg4.1): $\lambda_0 = 0.40, \mu = 0.33, \alpha_n = \frac{1}{(n+1)^{0.5}}, D_n = \max\left\{\|u_{n+1} - y_n\|^2, \|u_n - y_n\|^2\right\}$. (3) Algorithm 3 in [12] (Alg3): $\lambda = \frac{1}{\max\left\{3c_1, 3c_2\right\}}, \theta = 0.50, \varepsilon_n = \frac{1}{(n+1)^2}, \gamma_n = \frac{1}{5(n+2)}, \beta_n = \frac{4}{10}(1-\gamma_n), D_n = \|\chi_n - y_n\|^2$. (4) Algorithm 3.1 (Alg1): $\zeta_1 = 0.40, \mu = 0.33, \alpha = 0.50, \varepsilon_n = \frac{1}{(n+1)^2}, \beta_n = \frac{1}{5(n+2)}, D_n = \|\chi_n - y_n\|^2, \varphi_n = \frac{100}{(n+1)^2}$.



(A) tolerance = 10^{-2}. (B) tolerance = 10^{-3}.

FIGURE 5



(A) tolerance = 10^{-4}. (B) tolerance = 10^{-5}.

FIGURE 6

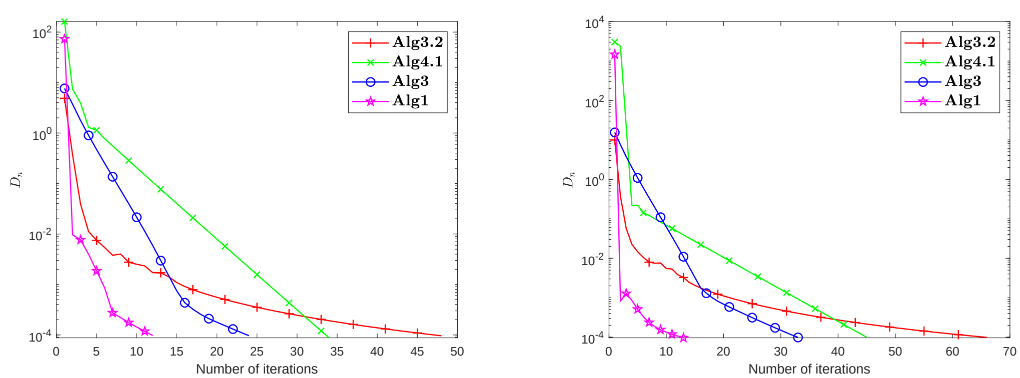


FIGURE 7. $N = 5$, $N = 10$, and the number of iterations are 48, 34, 24, 12 and 66, 45, 33, 13 respectively.

Finally, we work on Problem 4.4 to see the numerical effectiveness of Algorithm 3.1, Algorithm 3.2 in [31], Algorithm 4.1 in [11], and Algorithm 3 in [12] by choosing the different starting points of $u_1 = y_0$, and the fixed values of $u_0 = y_{-1} = t$. Figure 9 show a number of results

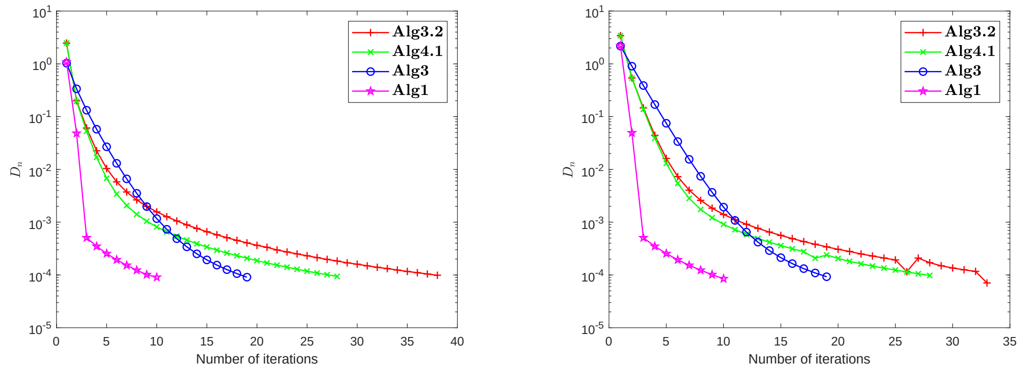


FIGURE 8. Numerical comparison when $u_1 = (1.5, 1.7)^T$, $u_1 = (1.0, 2.0)^T$ and the number of iterations are 38, 28, 19, 10 and 33, 28, 19, 10, respectively.

achieved by setting up a tolerance 10^{-4}. Information about the control parameters is considered as follows: (1) Algorithm 3.2 in [31] (Alg3.2): $\lambda = \frac{1}{\max\left\{5c_1, 5c_2\right\}}, \alpha_n = \frac{1}{5(n+2)}, D_n = \|u_n - y_n\|^2$. (2) Algorithm 4.1 in [11] (Alg4.1): $\lambda_0 = 0.40, \mu = 0.33, \alpha_n = \frac{1}{(n+1)^{0.5}}, D_n = \max\left\{\|u_{n+1} - y_n\|^2, \|u_n - y_n\|^2\right\}$. (3) Algorithm 3 in [12] (Alg3): $\lambda = \frac{1}{\max\left\{5c_1, 5c_2\right\}}, \theta = 0.50, \varepsilon_n = \frac{1}{(n+1)^2}, \gamma_n = \frac{1}{5(n+2)}, \beta_n = \frac{4}{10}(1 - \gamma_n), D_n = \|\chi_n - y_n\|^2$. (4) Algorithm 3.1 (Alg1): $\zeta_1 = 0.40, \mu = 0.33, \alpha = 0.50, \varepsilon_n = \frac{1}{(n+1)^2}, \beta_n = \frac{1}{5(n+2)}, D_n = \|\chi_n - y_n\|^2, \varphi_n = \frac{100}{(n+1)^2}$.

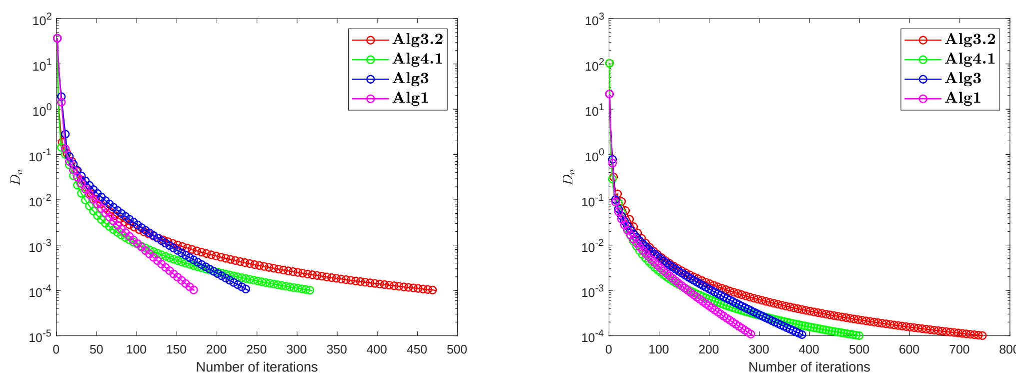


FIGURE 9. Numerical efficiency comparison when $u_1 = (t^3 - 2t + 4)e^t$, $u_1 = (3t^2 - 1)\cos(t)$ and the number of iterations are 474, 319, 240, 172 and 747, 500, 390, 288, respectively.

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