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A CERTAIN CLASS OF θ_L-TYPE NON-LINEAR OPERATORS AND SOME RELATED FIXED POINT RESULTS

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Abstract. This paper presents a systematic investigation of an extension of the developments concerning the θ-contractions, which were proposed in 2014 by Jleli and Samet. This paper generalizes the notion of the θ-contractions to the case of non-linear θ_L-contraction mappings, and prove multi-valued fixed point results in $\mathfrak{b}$-metric-like spaces. The paper also includes a tangible example, which displays the motivation for such investigations as those presented here. This paper is completed by giving an application of the proposed non-linear θ_L-contractions to the Liouville-Caputo fractional derivatives and fractional differential equations.
Keywords. Fixed points; $\mathfrak{b}$-metric-like spaces; θ_L-type non-linear operators; Liouville-Caputo fractional derivative; Liouville-Caputo fractional differential equations.

1. INTRODUCTION

The Banach fixed point (or contraction) theorem opened a door to a new area of pure mathematics and functional analysis. It states the conditions sufficient for the existence and uniqueness of a fixed point, and the theorem also provides an iterative system by which we can find approximations to the fixed point and the error bounds. Iterative systems are used in several branches of applied mathematics, and the criteria of convergence proof and the error estimates are very often produced by an application of the Banach fixed point theorem. The idea of fixed point theory was explored and furthered by a good number of researchers (see, e.g., [1, 2, 3, 4]). Banach's concept was prolonged by either extending metric spaces in many different ways or

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by modifying the structure of the contraction itself. In 2012, Wardowski [5] developed the concept of a contraction principle mapping, which is known as the F-contraction, and proved some fixed point theorems, which provide generalizations of the Banach contraction principle. Also, while extending the idea of the F-contractive principle mappings to the case of non-linear F-contractions and proving fixed point theorems via the dynamic iterative system, Jleli and Samet [6] presented a study of a new class of contractions, which they called the θ-contractions. This has provided new aspects of the Banach contraction principle in a unified approach.

2. Preliminaries

Recently, Jleli and Samet [6] introduced the following concept:

Let $(\Gamma,\eth)$ be a complete metric space and $I:\Gamma\to\Gamma$. Then there exist $\theta\in\Xi$ and $\mathfrak{a}\in(0,1)$ such that

$$\lambda_1,\lambda_2\in\Gamma,\ \eth(I\lambda_1,I\lambda_2)>0 \implies \theta\big(\eth(I\lambda_1,I\lambda_2)\big)\leqq\theta^{\mathfrak{a}}\big(\eth(\lambda_1,\lambda_2)\big),$$

where

$$\theta^{\mathfrak{a}}\big(\eth(\lambda_1,\lambda_2)\big)=\Big(\theta\big(\eth(\lambda_1,\lambda_2)\big)\Big)^{\mathfrak{a}},$$

and Ξ is the set of mappings $\theta:(0,\infty)\to(1,\infty)$ satisfying each of the conditions (θ_{i}), (θ_{ii}), and (θ_{iii}):

(θ_{i}) θ is non-decreasing and right-continuous;
(θ_{ii}) for each $\{s_\tau\}$ in $(0,\infty)$, it is asserted that

$$\lim_{\tau\to\infty}\theta(s_\tau)=1 \iff \lim_{\tau\to\infty}(s_\tau)=0;$$

(θ_{iii}) there exist $r\in(0,1)$ and $\lambda\in(0,\infty]$ such that

$$\lim_{s\to 0+}\frac{\theta(s)-1}{s^r}=\lambda.$$

Then I has at least one fixed point.

Example 2.1. The functions $\theta:(0,\infty)\to(1,\infty)$ defined, in terms of the exponential function e^x, by

$$\theta_1(x)=\mathrm{e}^x,\ \ \theta_2(x)=\mathrm{e}^{\sqrt{x}},\ \ \theta_3(x)=\mathrm{e}^{x\mathrm{e}^x},\ \ \theta_4(x)=\mathrm{e}^{\sqrt{x\mathrm{e}^x}}\quad\text{and}\quad\theta_5(x)=1+\sqrt{x}$$

are all in Ξ.

Nadler [7] used the idea of the Pompeiu-Hausdorff metric, and discussed the contraction theorem for many-valued functions rather than single-valued functions. We recall here the definition of the Hausdorff-Pompeiu metric and some related results.

Let $(\Gamma,\eth)$ be a $\mathfrak{b}$-metric-like space. For $\lambda\in\Gamma$ and $P_1\subseteq\Gamma$, let

$$\eth_{\mathfrak{b}}(\lambda,P_1)=\inf\{\eth(\lambda,y):y\in P_2\}.$$

Define a mapping $H_{\mathfrak{b}}:\mathrm{CB}(\Gamma)\times\mathrm{CB}(\Gamma)\to[0,\infty)$ by

$$H_{\mathfrak{b}}(P_1,P_2)=\max\left\{\sup_{\lambda\in P_1}\eth_{\mathfrak{b}}(\lambda,P_2),\sup_{y\in P_2}\eth_{\mathfrak{b}}(y,P_1)\right\}$$

for all $P_1, P_2 \in \mathrm{CB}(\Gamma)$. Then $H_{\mathfrak{b}}$ is known as the Hausdorff-Pompeiu $\mathfrak{b}$-metric-like induced by $\eth$-metric-like on $\mathrm{CB}(\Gamma)$, where $\mathrm{CB}(\Gamma)$ is the class of all non-empty, closed, and bounded subsets of Γ. A point $\lambda \in \Gamma$ is said to be a fixed point of $I : \Gamma \to \mathrm{CL}(\Gamma)$ such that $\lambda \in I\lambda$, where $\mathrm{CL}(\Gamma)$ is the class of all non-empty and closed subsets of Γ.

Some other related recent developments are being recalled here as Theorem 2.1, Lemma 2.1, and Lemma 2.2.

Theorem 2.1. (see [7]) *Let* $(\Gamma, \eth)$ *be a complete metric space, and let* $I : \Gamma \to \mathrm{CB}(\Gamma)$. *Then* $(\Gamma, \eth)$ *is a Nadler contraction if there is* $\gamma \in [0,1)$ *such that*

$$H_{\mathfrak{b}}(I\lambda_1, I\lambda_2) \leqq \gamma \eth(\lambda_1, \lambda_2) \qquad (\forall\, \lambda_1, \lambda_2 \in \Gamma).$$

Furthermore, I *possesses at least one fixed point.*

Lemma 2.1. (see [7]) *Let* $(\Gamma, \eth)$ *be a metric space,* $P_2 \in \mathrm{CB}(\Gamma)$*, and* $\lambda \in \Gamma$. *Then, for all* $\varepsilon > 0$, *there exists* $y \in P_2$ *such that*

$$\eth(\lambda, y) \leqq \eth(\lambda, P_2) + \varepsilon.$$

Lemma 2.2. (see [3]) *Let* $(\Gamma, \eth)$ *be a metric space, and let* $P_1, P_2 \in \mathrm{CB}(\Gamma)$ *with* $H_{\mathfrak{b}}(P_1, P_2) > 0$. *Then, for all* $h > 1$ *and* $\lambda \in P_1$, *there exists* $y = y(\lambda) \in P_2$ *such that*

$$\eth(\lambda, y) < hH_{\mathfrak{b}}(P_1, P_2).$$

Matthews [8] put forward the concept of a partial metric, which was augmented by Alghamdi *et al.* [9] to $\mathfrak{b}$-metric-like spaces. While comparing it with the partial metric, they proved that every partial metric is an $\mathfrak{b}$-metric-like, but that the converse is not true, confirming that a $\mathfrak{b}$-metric-like space is more general than the partial metric space. They also derived some interesting results on fixed points in this newly-defined approach. Their idea was further exploited by various authors in many ways (see, e.g., [10, 11]).

Definition 2.1. (see [9]) A $\mathfrak{b}$-metric-like space on $\Gamma \neq \phi$ is a function $\mathfrak{b} : \Gamma \times \Gamma \to \mathbb{R}^+ \cup \{0\}$ such that, for each $\lambda_1, \lambda_2, \lambda_3 \in \Gamma$ with $s \geqq 1$, the following conditions hold true:
(b_{i}) the condition:

$$\eth(\lambda_1, \lambda_2) = 0$$

implies that $\lambda_1 = \lambda_2$;
(b_{ii}) the following condition is satisfied:

$$\eth(\lambda_1, \lambda_2) = \eth(\lambda_2, \lambda_1);$$

(b_{iii}) the following condition is satisfied:

$$\eth(\lambda_1, \lambda_3) \leqq s[\eth(\lambda_1, \lambda_2) + \eth(\lambda_2, \lambda_3)].$$

The pair $(\Gamma, \eth)$ is called a $\mathfrak{b}$-metric-like space.

Example 2.2. Let $\Gamma = \{0,1,2\}$, and let $\eth : \Gamma \times \Gamma \to \mathbb{R}^+ \cup \{0\}$ be given by

$$\begin{pmatrix} \eth(0,0) & \eth(1,1) & \eth(2,2) & \eth(0,1) & \eth(0,2) & \eth(1,2) \\ 0 & 2 & 2 & 4 & 2 & 1 \end{pmatrix}$$

with

$$\eth(\lambda_1, \lambda_2) = \eth(\lambda_2, \lambda_1)$$

for every $\lambda_1, \lambda_2 \in \Gamma$. Then $(\Gamma, \eth)$ is a $\mathfrak{b}$-metric-like space with $s = 2$. Owing to Example 2.2 above, it is neither a $\mathfrak{b}$-metric nor a metric-like space since

$$\eth(0,1) = 4 > 3 = \eth(0,2) + \eth(2,1).$$

Remark 2.1. Every partial metric is a $\mathfrak{b}$-metric-like space, but the converse is not true in general (see [9]).

Example 2.3. Let $\Gamma = \{0,1,2\}$, and suppose that

$$\eth(\lambda_1, \lambda_2) = \begin{cases} 2, & \lambda_1 = \lambda_2 = 0, \\ \dfrac{1}{2}, & \text{otherwise.} \end{cases}$$

Then $(\Gamma, \eth)$ is a $\mathfrak{b}$-metric-like space with $s = 2$.

Definition 2.2. Let $(\Gamma, \eth)$ be a $\mathfrak{b}$-metric-like space. Then
(i) a sequence $\{\lambda_\tau\}$ of Γ converges to $\lambda \in \Gamma$ if and only if

$$\lim_{\tau \to \infty} \eth(\lambda, \lambda_\tau) = \eth(\lambda, \lambda);$$

(ii) a sequence $\{\lambda_\tau\}$ of $\mathfrak{b}$-metric-like space $(\Gamma, \eth)$ is said to be a Cauchy-sequence if and only if

$$\lim_{\tau, m \to \infty} \eth(\lambda_\tau, \lambda_m)$$

exists (and is finite);
(iii) a $\mathfrak{b}$-metric-like space $(\Gamma, \eth)$ is said to be complete if and only if every Cauchy-sequence $\{\lambda_\tau\} \in \Gamma$ converges with respect to $\tau(\eth)$ to $\lambda \in \Gamma$ in such a way that

$$\lim_{\tau, m \to \infty} \eth(\lambda_\tau, \lambda_m) = \eth(\lambda, \lambda) = \lim_{\tau \to \infty} \eth(\lambda, \lambda_\tau).$$

Definition 2.3. Let $(\Gamma, \eth)$ be a $\mathfrak{b}$-metric-like space. Then each $\mathfrak{b}$-metric-like space on Γ generates a T_0 topology $\tau(\eth)$ which is the family of open $\eth$-balls, given by

$$\{B_\eth(\lambda_1, r) : \lambda_1 \in \Gamma \quad \text{and} \quad r > 0\},$$

where

$$B_\eth(\lambda_1, r) = \{\lambda_2 \in \Gamma : |\eth(\lambda_1, \lambda_2) - \eth(\lambda_1, \lambda_1)| < r\}$$

for all $\lambda_1 \in \Gamma$ and $r > 0$.

Definition 2.4. The criterion of a convergent sequence is not to be necessarily unique in a $\mathfrak{b}$-metric-like space. Indeed, if we let $\lambda_\tau = 2$ for all $\tau = 1, 2, 3, \cdots$, then

$$\lim_{\tau \to \infty} \eth(\lambda_\tau, 2) = \eth(2,2)$$

and

$$\lim_{\tau \to \infty} \eth(\lambda_\tau, 1) = \eth(1,1).$$

Definition 2.5. Let $(\Gamma, \eth)$ be a $\mathfrak{b}$-metric-like space.
(i) The function $I : \Gamma \to \Gamma$ is said to be continuous at $t \in \Gamma$ if, for a given $\varepsilon > 0$, there exists $\delta^l > 0$ such that

$$I\left(B_\eth\left(t, \delta^l\right)\right) \subseteq B_\eth(It, \varepsilon).$$

Indeed, we call $\eth$ to be continuous on Γ if I is continuous for all $t \in \Gamma$;
(ii) the function $I : \Gamma \to \Gamma$ is said to be sequentially continuous on $v \in \Gamma$ if, for all sequences $\{u_j\}$ in Γ, $I(u_j) \to I(v)$, that is, if

$$\lim_{j\to\infty} \eth\left(Iu_j, Iv\right) = \eth\left(Iv, Iv\right).$$

The function I is said to be sequentially continuous on Γ if it is sequentially continuous on each $v \in \Gamma$;
(iii) the continuous mapping $I : G \to G$ such that

$$\lim_{\tau\to\infty} \eth\left(\lambda_\tau, \lambda\right) = \eth\left(\lambda, \lambda\right) \longrightarrow \lim_{\tau\to\infty} \eth(I\lambda_\tau, I\lambda) = \eth\left(I\lambda, I\lambda\right).$$

We now recall the following result on a sequence in the $\mathfrak{b}$-metric-like space.

Lemma 2.3. (see [9]) *Let* $\{\lambda_j\}_{j=0}^{\tau}$ *be a sequence in the* $\mathfrak{b}$*-metric-like space* $(\Gamma, \eth)$ *with constants* $s \geqq 1$. *Then*

$$\eth\left(\lambda_\tau, \lambda_0\right) \leqq s\eth\left(\lambda_0, \lambda_1\right) + s^2\eth\left(\lambda_1, \lambda_2\right) + \cdots + s^{\tau-1}\eth\left(\lambda_{\tau-1}, \lambda_\tau\right).$$

The main purpose of this paper is to introduce and investigate a new concept of the θ_L-contraction in $\mathfrak{b}$-metric-like spaces. We relax the restrictions on the function $\theta : (0,\infty) \to (1,\infty)$, which was considered by Jleli and Samet [6], by eliminating the condition (θ_{iii}). We suppose instead that there exist $r \in (0,1)$ and $\lambda \in (0,\infty)$ such that

$$\lim_{s\to 0+} \frac{\theta(s) - 1}{s^r} = \lambda.$$

We then prove some multi-valued Suzuki type fixed point results in a $\mathfrak{b}$-metric-like space. This paper also includes an interesting example which displays the motivation for such investigations as those presented here. Our investigation is completed by giving an application of the proposed non-linear θ_L-contractions to the Liouville-Caputo fractional derivatives and fractional differential equations.

3. Main Results

First, in order to give our general definition, we denote by Ξ_L the family of functions $\theta_L : (0,\infty) \to (1,\infty)$ such that the following statements are true:
(θ_{i}) θ_L is non-decreasing and θ_L is right-continuous;
(θ_{ii}) for each $\{\lambda_\tau\} \subset (0,\infty)$, $\lim_{\tau\to\infty} \theta_L(\lambda_\tau) = 1 \Longleftrightarrow \lim_{\tau\to\infty}(\lambda_\tau) = 0$.

Lemma 3.1. *Let* $(\Gamma, \eth, s)$ *be a* $\mathfrak{b}$*-metric like space. For a mapping* $I : \Delta \subseteq \Gamma \to K(\Gamma)$, *where* $K(\Gamma)$ *is a compact subset of* Γ, *the following assertions are equivalent*:
(a) *the mapping* I *is continuous;*
(b) *since* $K(\Gamma)$ *is a compact subset of* Γ, *for every convergence subsequence* $\lambda_\tau \to \lambda^*$,

$$I\lambda_\tau \to I\lambda^*, \quad \forall \lambda \in \Delta.$$

Proof. (a) $\Longrightarrow$ (b): Given an $\varepsilon > 0$, there is $\delta > 0$ such that

$$|\eth(\lambda, \lambda^*) - \eth(\lambda^*, \lambda^*)| < \delta \quad \text{and} \quad |\eth(\lambda, \lambda^*) - \eth(\lambda, \lambda)| < \delta,$$

which imply that

$$|\eth(I\lambda, I\lambda^*) - \eth(I\lambda^*, I\lambda^*)| < \varepsilon \quad \text{and} \quad |\eth(I\lambda, I\lambda^*) - \eth(I\lambda, I\lambda)| < \varepsilon.$$

Now, owing to the fact that the sequence $\lambda_{i(\tau)} \to \lambda^*$ so that $I\lambda_{i(\tau)} \to I\lambda$, we have

$$\left|\eth\left(\lambda_{i(\tau)},\lambda^*\right)-\eth\left(\lambda^*,\lambda^*\right)\right|<\delta \quad \text{and} \quad \left|\eth\left(\lambda_{i(\tau)},\lambda^*\right)-\eth\left(\lambda_{i(\tau)},\lambda_{i(\tau)}\right)\right|<\delta,$$

which yield

$$\left|\eth\left(I\lambda_{i(\tau)},I\lambda^*\right)-\eth\left(I\lambda^*,I\lambda^*\right)\right|<\varepsilon \quad \text{and} \quad \left|\eth\left(I\lambda_{i(\tau)},I\lambda^*\right)-\eth\left(I\lambda_{i(\tau)},I\lambda_{i(\tau)}\right)\right|<\varepsilon.$$

Consequently, $I\lambda_{i(\tau)} \to I\lambda^*$. We have thus proved that (a) $\Longrightarrow$ (b).

(b) $\Longrightarrow$ (a): Assuming, on the contrary, that I is not continuous, there is $\varepsilon > 0$ such that $\delta > 0$,

$$|\eth(\lambda,\lambda^*)-\eth(\lambda^*,\lambda^*)|<\delta \quad \text{and} \quad |\eth(\lambda,\lambda^*)-\eth(\lambda,\lambda)|<\delta,$$

which imply that

$$|\eth(I\lambda,I\lambda^*)-\eth(I\lambda^*,I\lambda^*)| \geqq \varepsilon \quad \text{and} \quad |\eth(I\lambda,I\lambda^*)-\eth(I\lambda,I\lambda)| \geqq \varepsilon.$$

Setting

$$\delta = \frac{1}{\lambda_{i(\tau)}} \qquad (\forall\, i \in \mathbb{N} := \{1,2,3,\cdots\}),$$

we find that

$$\left|\eth\left(\lambda_{i(\tau)},\lambda^*\right)-\eth\left(\lambda^*,\lambda^*\right)\right| < \frac{1}{\lambda_{i(\tau)}}$$

and

$$\left|\eth\left(\lambda_{i(\tau)},\lambda^*\right)-\eth\left(\lambda_{i(\tau)},\lambda_{i(\tau)}\right)\right| < \frac{1}{\lambda_{i(\tau)}},$$

which yield

$$\left|\eth\left(I\lambda_{i(\tau)},I\lambda^*\right)-\eth\left(I\lambda^*,I\lambda^*\right)\right| \geqq \varepsilon$$

and

$$\left|\eth\left(I\lambda_{i(\tau)},I\lambda^*\right)-\eth\left(I\lambda_{i(\tau)},I\lambda_{i(\tau)}\right)\right| \geqq \varepsilon.$$

Consequently, $\lambda_{i(\tau)} \to \lambda^*$ while $I\lambda_{i(\tau)}$ does not converge to $I\lambda^*$, which is a conradiction. We have thus proved by contradiction that (b) $\Longrightarrow$ (a). □

Definition 3.1. Let $(\Gamma,\eth,s)$ be a $\mathfrak{b}$-metric-like space. A mapping $I:\Gamma \to \mathrm{CB}(\Gamma)$ is called multi-valued θ_L-contraction if there exists $\theta \in \Xi_L$ such that

$$\frac{1}{2s}\,\eth(\lambda_1,I\lambda_1) < \eth(\lambda_1,\lambda_2) \implies \theta\big(H_{\mathfrak{b}}(I\lambda_1,I\lambda_2)\big) \leqq \theta\big(\Omega(\lambda_1,\lambda_2)\big), \tag{3.1}$$

where

$$\Omega(\lambda_1,\lambda_2) = [\theta^a\eth(\lambda_1,\lambda_2)]\left[\theta^{\mathfrak{b}}\big(\eth(\lambda_1,I\lambda_1)\big)\right][\theta^c\eth(\lambda_2,I\lambda_2)]\,\frac{[\theta^d\eth(\lambda_2,I\lambda_1)]}{2s}\,\frac{[\theta^e\eth(\lambda_1,I\lambda_2)]}{2s}$$

for all $\lambda_1,\lambda_2 \in \Delta$, a, b, c, d, $e \in \mathbb{R}^+$, $H_{\mathfrak{b}}(I\lambda_1,I\lambda_2) > 0$, and $0 \leqq a+b+c+d+e < 1$.

Remark 3.1. Let $(\Gamma,\eth,s)$ be a $\mathfrak{b}$-metric-like space, and let $I:\Lambda \to \mathrm{CB}(\Gamma)$ be a multi-valued mapping satisfying the following condition:

$$\ln\theta\big(H_{\mathfrak{b}}(I\lambda_1,I\lambda_2)\big) \leqq \gamma\ln\theta\big(\eth(\lambda_1,\lambda_2)\big) < \ln\theta(\eth(\lambda_1,\lambda_2)).$$

Thus, in view of (θ_{i}), we have

$$H_{\mathfrak{b}}(I\lambda_1,I\lambda_2) < \eth(\lambda_1,\lambda_2) \qquad (\forall\, \lambda_1,\lambda_2 \in \Lambda;\ I\lambda_1 \neq I\lambda_2).$$

Theorem 3.1. *Let $(\Gamma,\eth,s)$ be a complete $\mathfrak{b}$-metric-like space, and let $I:\Lambda\to CB(\Gamma)$ be a multi-valued θ_L-contraction. Then I has a fixed point.*

Proof. Let $\lambda_0\in\Lambda$ and define the class of iterative sequences $\{\lambda_\tau\}$ by $\lambda_{\tau+1}=I\lambda_\tau$. In case that there is $\lambda_{\tau_0}\in\mathbb{N}$ such that $\lambda_{\tau_0}=\lambda_{\tau_{0+1}}$ for some $\tau_0\in\mathbb{N}$, our proof of Theorem 3.1 proceeds as follows. If we let $\lambda_\tau\neq\lambda_{\tau+1}$ for every $\tau\in\mathbb{N}$, then

$$\frac{1}{2s}\,\eth(\lambda_\tau,I\lambda_\tau)<\eth(\lambda_\tau,\lambda_{\tau+1})\qquad(\forall\,\tau\in\mathbb{N}).\tag{3.2}$$

From (3.1), we can write

$$\begin{aligned}\theta(\eth(\lambda_{\tau+1},I\lambda_{\tau+1}))&\leqq\theta(H_{\mathfrak{b}}(I\lambda_\tau,I\lambda_{\tau+1}))\\&\leqq[\theta^a(\eth(\lambda_\tau,\lambda_{\tau+1}))]\left[\theta^b(\eth(\lambda_\tau,I\lambda_\tau))\right][\theta^c(\eth(\lambda_{\tau+1},I\lambda_{\tau+1}))]\\&\quad\cdot\frac{\left[\theta^d(\eth(\lambda_\tau,I\lambda_{\tau+1}))\right]}{2s}\,\frac{[\theta^e(\eth(\lambda_{\tau+1},I\lambda_\tau))]}{2s}\qquad(\forall\,\tau\in\mathbb{N}).\end{aligned}\tag{3.3}$$

Next, we examine the following inequality:

$$\eth(\lambda_{\tau+1},I\lambda_{\tau+1})<\eth(\lambda_\tau,I\lambda_\tau).\tag{3.4}$$

Assume, based on the contrary, that there is $\tau_0\in\mathbb{N}$ such that

$$\eth(\lambda_{\tau_0+1},I\lambda_{\tau_0+1})\geqq\eth(\lambda_{\tau_0},I\lambda_{\tau_0}).$$

Then, in view of (3.3), we have

$$\begin{aligned}\theta(\eth(\lambda_{\tau_0+1},I\lambda_{\tau_0+1}))&\leqq\theta(H_{\mathfrak{b}}(I\lambda_{\tau_0},I\lambda_{\tau_0+1}))\\&\leqq[\theta^a(\eth(\lambda_{\tau_0},\lambda_{\tau_0+1}))]\left[\theta^b(\eth(\lambda_{\tau_0},I\lambda_{\tau_0}))\right][\theta^c(\eth(\lambda_{\tau_0+1},I\lambda_{\tau_0+1}))]\\&\quad\cdot\frac{\left[\theta^d(\eth(\lambda_{\tau_0},I\lambda_{\tau_0+1}))\right]}{2s}\,\frac{[\theta^e(\eth(\lambda_{\tau_0+1},I\lambda_{\tau_0}))]}{2s}\\&\leqq[\theta^a(\eth(\lambda_{\tau_0},\lambda_{\tau_0+1}))]\left[\theta^b(\eth(\lambda_{\tau_0},I\lambda_{\tau_0}))\right][\theta^c(\eth(\lambda_{\tau_0+1},I\lambda_{\tau_0+1}))]\\&\quad\cdot\frac{\left[\theta^d(s\eth(\lambda_{\tau_0},\lambda_{\tau_0+1})+s\eth(\lambda_{\tau_0+1},I\lambda_{\tau_0+1}))\right]}{2s}\,\frac{[\theta^e(2s\eth(\lambda_{\tau_0},I\lambda_{\tau_0}))]}{2s}\\&\leqq[\theta^a(\eth(\lambda_{\tau_0},\lambda_{\tau_0+1}))]\left[\theta^b(\eth(\lambda_{\tau_0},I\lambda_{\tau_0}))\right][\theta^c(\eth(\lambda_{\tau_0+1},I\lambda_{\tau_0+1}))]\\&\quad\cdot\left[\theta^d(\eth(\lambda_{\tau_0+1},I\lambda_{\tau_0+1}))\right][\theta^e(\eth(\lambda_{\tau_0},I\lambda_{\tau_0}))],\end{aligned}$$

which implies that

$$\theta(\eth(\lambda_{\tau_0+1},I\lambda_{\tau_0+1}))\leqq\theta^{\frac{a+b+e}{1-c-d}}(\eth(\lambda_{\tau_0},\lambda_{\tau_0+1}))<\theta(\eth(\lambda_{\tau_0},\lambda_{\tau_0+1})).$$

From this, together with the equation (3.2), we have

$$\eth(\lambda_{\tau_0+1},I\lambda_{\tau_0+1})<\eth(\lambda_{\tau_0},\lambda_{\tau_0+1}),$$

which is a contradiction. Thus, (3.4) holds true.

In view of the above observations, $\eth(\lambda_\tau,\lambda_{\tau+1})$ is a decreasing sequence with respect to real numbers and is bounded from below. Suppose that there is $\Lambda\geqq 0$ such that

$$\Lambda=\lim_{\tau\to\infty}\eth(\lambda_\tau,\lambda_{\tau+1})=\inf\{\eth(\lambda_\tau,\lambda_{\tau+1}):\tau\in\mathbb{N}\}.\tag{3.5}$$

We now show that $\Lambda = 0$. Assume, based on the contrary, that $\Lambda > 0$. Then, for every $\varepsilon > 0$, there is $\alpha \in \mathbb{N}$ such that

$$\theta\left[\eth\left(\lambda_{\alpha}, \lambda_{\alpha+1}\right)\right] \leqq \theta\left[\Lambda + \varepsilon\right]. \tag{3.6}$$

Also, by applying (3.2), we have

$$\frac{1}{2s}\,\eth\left(\lambda_{\alpha}, I\lambda_{\alpha}\right) < \eth\left(\lambda_{\alpha}, \lambda_{\alpha+1}\right). \tag{3.7}$$

Since I is a multi-valued θ_L-contraction, we obtain

$$\begin{aligned}
\theta\left(\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha+1}\right)\right) &\leqq \theta\left(H_{\mathfrak{b}}\left(I\lambda_{\alpha}, I\lambda_{\alpha+1}\right)\right)\\
&\leqq \left[\theta^{a}\left(\eth\left(\lambda_{\alpha}, \lambda_{\alpha+1}\right)\right)\right]\left[\theta^{b}\left(\eth\left(\lambda_{\alpha}, I\lambda_{\alpha+1}\right)\right)\right]\left[\theta^{c}\left(\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha+1}\right)\right)\right]\\
&\quad\cdot \frac{\left[\theta^{d}\left(\eth\left(\lambda_{\alpha}, I\lambda_{\alpha+1}\right)\right)\right]}{2s}\,\frac{\left[\theta^{e}\left(\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha}\right)\right)\right]}{2s}\\
&\leqq \left[\theta^{a}\left(\eth\left(\lambda_{\alpha}, \lambda_{\alpha+1}\right)\right)\right]\left[\theta^{b}\left(\eth\left(\lambda_{\alpha}, I\lambda_{\alpha}\right)\right)\right]\left[\theta^{c}\left(\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha+1}\right)\right)\right]\\
&\quad\cdot \frac{\left[\theta^{d}\left(s\eth\left(\lambda_{\alpha}, \lambda_{\alpha+1}\right) + \eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha+1}\right)\right)\right]}{2s}\,\frac{\left[\theta^{e}\left(2s\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha}\right)\right)\right]}{2s}\\
&\leqq \left[\theta^{a}\left(\eth\left(\lambda_{\alpha}, \lambda_{\alpha+1}\right)\right)\right]\left[\theta^{b}\left(\eth\left(\lambda_{\alpha}, I\lambda_{\alpha}\right)\right)\right]\left[\theta^{c}\left(\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha+1}\right)\right)\right]\\
&\quad\cdot \left[\theta^{d}\left(\eth\left(\lambda_{\alpha}, \lambda_{\alpha+1}\right)\right)\right]\left[\theta^{e}\left(\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha}\right)\right)\right],
\end{aligned}$$

which implies that

$$\theta\left(\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha+1}\right)\right) \leqq \theta^{\frac{a+b+e+d}{1-c}}\left(\eth\left(\lambda_{\alpha}, \lambda_{\alpha+1}\right)\right).$$

Since

$$\frac{1}{2s}\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha+1}\right) < \eth\left(\lambda_{\alpha+1}, \lambda_{\alpha+2}\right),$$

by appealing to (3.1), we can write

$$\begin{aligned}
\theta\left(\eth\left(\lambda_{\alpha+2}, I\lambda_{\alpha+2}\right)\right) &\leqq \theta\left(H_{\mathfrak{b}}\left(I\lambda_{\alpha+1}, I\lambda_{\alpha+2}\right)\right)\\
&\leqq \left[\theta^{a}\left(\eth\left(\lambda_{\alpha+1}, \lambda_{\alpha+2}\right)\right)\right]\left[\theta^{b}\left(\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha+1}\right)\right)\right]\left[\theta^{c}\left(\eth\left(\lambda_{\alpha+2}, I\lambda_{\alpha+2}\right)\right)\right]\\
&\quad\cdot \frac{\left[\theta^{d}\left(\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha+2}\right)\right)\right]}{2s}\,\frac{\left[\theta^{e}\left(\eth\left(\lambda_{\alpha+2}, I\lambda_{\alpha+1}\right)\right)\right]}{2s}\\
&\leqq \left[\theta^{a}\left(\eth\left(\lambda_{\alpha+1}, \lambda_{\alpha+2}\right)\right)\right]\left[\theta^{b}\left(\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha+1}\right)\right)\right]\left[\theta^{c}\left(\eth\left(\lambda_{\alpha+2}, I\lambda_{\alpha+2}\right)\right)\right]\\
&\quad\cdot \frac{\left[\theta^{d}\left(\eth\left(\lambda_{\alpha+1}, \lambda_{\alpha+2}\right) + \eth\left(\lambda_{\alpha+2}, I\lambda_{\alpha+2}\right)\right)\right]}{2s}\,\frac{\left[\theta^{e}\left(2s\eth\left(\lambda_{\alpha+2}, I\lambda_{\alpha+1}\right)\right)\right]}{2s}\\
&\leqq \left[\theta^{a}\left(\eth\left(\lambda_{\alpha+1}, \lambda_{\alpha+2}\right)\right)\right]\left[\theta^{b}\left(\eth\left(\lambda_{\alpha+1}, I\lambda_{\alpha+1}\right)\right)\right]\left[\theta^{c}\left(\eth\left(\lambda_{\alpha+2}, I\lambda_{\alpha+2}\right)\right)\right]\\
&\quad\cdot \left[\theta^{d}\left(\eth\left(\lambda_{\alpha+1}, \lambda_{\alpha+2}\right)\right)\right]\left[\theta^{e}\left(\eth\left(\lambda_{\alpha+2}, I\lambda_{\alpha+1}\right)\right)\right],
\end{aligned}$$

which implies that

$$\theta\left(\eth\left(\lambda_{\alpha+2}, I\lambda_{\alpha+2}\right)\right) \leqq \theta^{\frac{a+b+e+d}{1-c}}\left(\eth\left(\lambda_{\alpha+1}, \lambda_{\alpha+2}\right)\right).$$

Continuing the above iterative process along with (3.6), we have

$$\begin{aligned}\theta\left(\eth\left(\lambda_{\alpha+\tau}, I\lambda_{\alpha+(\tau+1)}\right)\right) &\leqq \theta\left(H\left(I\lambda_{\alpha+\tau}, I\lambda_{\alpha+(\tau+1)}\right)\right) \\ &\leqq \theta^{\frac{a+b+e+d}{1-c}}\left(\eth\left(\lambda_{\alpha+(\tau-1)}, \lambda_{\alpha+\tau}\right)\right) \\ &\leqq \theta^{\left[\frac{a+b+e+d}{1-c}\right]^2}\left(\eth\left(\lambda_{\alpha+(\tau-2)}, \lambda_{\alpha+(\tau-1)}\right)\right) \\ &\leqq \cdots \leqq \theta^{\left(\frac{a+b+e+d}{1-c}\right)^\tau}\left(\eth\left(\lambda_\alpha, I\lambda_\alpha\right)\right) \\ &\leqq \theta^{\left(\frac{a+b+e+d}{1-c}\right)^\tau}\left[\Lambda+\varepsilon\right]. \end{aligned} \tag{3.8}$$

Upon letting $\tau \to \infty$, we obtain

$$\lim_{\tau\to\infty} \theta\left(\eth\left(\lambda_{\alpha+\tau}, I\lambda_{\alpha+(\tau+1)}\right)\right) = 1.$$

Also, in view of (θ_{ii}), we have $\lim_{\tau\to\infty} \eth\left(\lambda_{\alpha+\tau}, I\lambda_{\alpha+(\tau+1)}\right) = 0$. So, there is $\mathbb{N}_1 \in \mathbb{N}$ such that

$$\eth\left(\lambda_{\alpha+\tau}, I\lambda_{\alpha+(\tau+1)}\right) < \Lambda \qquad (\forall\, \tau > \mathbb{N}_1),$$

which is a contradiction with respect to Λ. Therefore, we have

$$\lim_{\tau\to\infty} \eth\left(\lambda_\tau, \lambda_{\tau+1}\right) = 0. \tag{3.9}$$

Next, we show that

$$\lim_{\tau,m\to\infty} \eth\left(\lambda_\tau, \lambda_m\right) = 0. \tag{3.10}$$

On the contrary, let us assume that, for a given $\varepsilon > 0$, $\zeta(\tau)$ and $\xi(\tau)$ are sequences in $\mathbb{N}$ such that

$$\eth(\lambda_{\zeta(\tau)}, \lambda_{\xi(\tau)}) \geqq \varepsilon\,,\ \eth(\lambda_{\zeta(\tau)-1}, \lambda_{\xi(\tau)}) < \varepsilon,\ \zeta(\tau) > \xi(\tau) > \tau \qquad (\forall\, \tau \in \mathbb{N}). \tag{3.11}$$

So, we have

$$\begin{aligned}\eth(\lambda_{\zeta(\tau)}, \lambda_{\xi(\tau)}) &\leqq s\eth(\lambda_{\zeta(\tau)}, \lambda_{\zeta(\tau)-1}) + s\eth(\lambda_{\zeta(\tau)-1}, \lambda_{\xi(\tau)}) \\ &< s\eth\left(\lambda_{\zeta(\tau)}, \lambda_{\zeta(\tau)-1}\right) + s\varepsilon. \end{aligned} \tag{3.12}$$

In light of (3.9), there is $\mathbb{N}_2 \in \mathbb{N}$ such that

$$\eth\left(\lambda_{\zeta(\tau)}, \lambda_{\zeta(\tau)-1}\right) < \varepsilon, \eth\left(\lambda_{\zeta(\tau)}, \lambda_{\zeta(\tau)+1}\right) < \varepsilon \quad \text{and} \quad \eth(\lambda_{\xi(\tau)}, \lambda_{\xi(\tau)+1}) < \varepsilon \qquad (\forall\, \tau > \mathbb{N}_2), \tag{3.13}$$

which together with (3.12) yields

$$\eth\left(\lambda_{\zeta(\tau)}, \lambda_{\xi(\tau)}\right) < 2s\varepsilon \qquad (\forall\, \tau > \mathbb{N}_2). \tag{3.14}$$

By (3.11) and (3.13), we obtain

$$\frac{1}{2s}\,\eth\left(\lambda_{\zeta(\tau)}, \lambda_{\zeta(\tau)+1}\right) < \frac{\varepsilon}{2s} < \eth\left(\lambda_{\zeta(\tau)}, \lambda_{\xi(\tau)}\right) \qquad (\forall\, \tau > \mathbb{N}_2). \tag{3.15}$$

Applying the triangle inequality, we find that

$$\varepsilon \leqq \eth\left(\lambda_{\zeta(\tau)}, \lambda_{\zeta(\tau)}\right) \leqq s\eth\left(\lambda_{\zeta(\tau)}, \lambda_{\zeta(\tau)+1}\right) + s^2\eth\left(\lambda_{\zeta(\tau)+1}, \lambda_{\xi(\tau)+1}\right) + s^2\eth\left(\lambda_{\xi(\tau)+1}, \lambda_{\xi(\tau)}\right). \tag{3.16}$$

Now, if we proceed to the limit as $\tau \to \infty$ in (3.16), and make use of (3.9), then we have

$$\frac{\varepsilon}{s^2} \leqq \lim_{\tau\to\infty} \inf \eth(\lambda_{\zeta(\tau)+1}, \lambda_{\xi(\tau)+1}).$$

Also, there is $\mathbb{N}_3 \in \mathbb{N}$ such that $\eth\left(\lambda_{\zeta(\tau)+1}, \lambda_{\xi(\tau)+1}\right) > 0$, $\forall \tau > \mathbb{N}_3$. Further, from (3.1), we can write

$$\begin{aligned}
&\theta\left(\eth(\lambda_{\zeta(\tau)+1}, \lambda_{\xi(\tau)+1})\right) \\
&\leqq \theta\left(H_{\mathfrak{b}}\left(I\lambda_{\zeta(\tau)}, I\lambda_{\xi(\tau)}\right)\right) \\
&\leqq \left[\theta^a\left(\eth\left(\lambda_{\zeta(\tau)}, \lambda_{\xi(\tau)}\right)\right)\right]\left[\theta^b\left(\eth\left(\lambda_{\zeta(\tau)}, I\lambda_{\zeta(\tau)}\right)\right)\right]\left[\theta^c\left(\eth\left(\lambda_{\xi(\tau)}, I\lambda_{\xi(\tau)}\right)\right)\right] \\
&\quad \cdot \frac{\left[\theta^d\left(\eth\left(\lambda_{\zeta(\tau)}, I\lambda_{\xi(\tau)}\right)\right)\right]}{2s} \frac{\left[\theta^e\left(\eth\left(\lambda_{\xi(\tau)}, I\lambda_{\zeta(\tau)}\right)\right)\right]}{2s} \\
&\leqq \left[\theta^a\left(\eth\left(\lambda_{\zeta(\tau)}, \lambda_{\xi(\tau)}\right)\right)\right]\left[\theta^b\left(\eth\left(\lambda_{\zeta(\tau)}, I\lambda_{\zeta(\tau)}\right)\right)\right]\left[\theta^c\left(\eth\left(\lambda_{\xi(\tau)}, I\lambda_{\xi(\tau)}\right)\right)\right] \\
&\quad \cdot \frac{\left[\theta^d\left(\eth\left(\lambda_{\zeta(\tau)}, \lambda_{\xi(\tau)}\right) + \eth\left(\lambda_{\xi(\tau)}, I\lambda_{\xi(\tau)}\right)\right)\right]}{2} \\
&\quad \cdot \frac{\left[\theta^e\left(\eth\left(\lambda_{\xi(\tau)}, \lambda_{\zeta(\tau)}\right) + \eth\left(\lambda_{\zeta(\tau)}, I\lambda_{\zeta(\tau)}\right)\right)\right]}{2}
\end{aligned} \tag{3.17}$$

for $\tau > \max\{\mathbb{N}_2, \mathbb{N}_3\}$. In view of (3.13) and (3.14), inequality (3.17) yields

$$\begin{aligned}
\theta\left(\eth(\lambda_{\zeta(\tau)+1}, \lambda_{\xi(\tau)+1})\right) \leqq \left[\theta^a\left(\eth\left(2s\varepsilon\right)\right)\right]&\left[\theta^b\left(\eth\left(\lambda_{\zeta(\tau)}, I\lambda_{\zeta(\tau)}\right)\right)\right]\left[\theta^c\left(\eth\left(\lambda_{\xi(\tau)}, I\lambda_{\xi(\tau)}\right)\right)\right] \\
&\cdot \frac{\left[\theta^d\left(2s\varepsilon\right) + \varepsilon\right]}{2} \frac{\left[\theta^e\left(2s\varepsilon\right) + \varepsilon\right]}{2}
\end{aligned} \tag{3.18}$$

for $\tau > \max\{\mathbb{N}_2, \mathbb{N}_3\}$. Taking the limit as $\tau \to \infty$ in (3.18), we have

$$\lim_{\tau \to \infty} \theta\left(\eth(\lambda_{\zeta(\tau)+1}, \lambda_{\xi(\tau)+1})\right) = 1,$$

which, by vertue of (θ_{ii}), implies that $\lim_{\tau \to \infty} \eth(\lambda_{\zeta(\tau)+1}, \lambda_{\xi(\tau)+1}) = 0$.

Next, (3.16) implies that $\lim_{\tau \to \infty} \eth(\lambda_{\zeta(\tau)}, \lambda_{\xi(\tau)}) = 0$, which contradicts (3.11). Hence (3.10) holds true. Therefore, $\{\lambda_\tau\}$ is a Cauchy sequence in Δ. Since Δ is complete $\mathfrak{b}$-metric-like, there is a point $\lambda^* \in \Delta$ such that

$$\eth(\lambda^*, \lambda^*) = \lim_{\tau \to \infty}\{\eth(\lambda_\tau, \lambda^*)\} = \lim_{\tau, m \to \infty}\{\eth(\lambda_\tau, \lambda_m)\} = 0. \tag{3.19}$$

We show further that

$$\frac{1}{2s}\eth(\lambda_\tau, I\lambda_\tau) < \eth(\lambda_\tau, \lambda^*) \text{ or } \frac{1}{2s}\eth(\lambda_{\tau+1}, I\lambda_{\tau+1}) < \eth(\lambda_{\tau+1}, \lambda^*). \tag{3.20}$$

We assume, on the contrary, that there is $m_0 \in \mathbb{N}$ such that

$$\frac{1}{2s}\eth(\lambda_{m_0}, I\lambda_{m_0}) \geqq \eth(\lambda_{m_0}, \lambda^*) \text{ or } \frac{1}{2s}\eth(\lambda_{m_0+1}, I\lambda_{m_0+1}) \geqq \eth(\lambda_{m_0+1}, \lambda^*). \tag{3.21}$$

Then, from (3.4) and (3.21), we have

$$\begin{aligned}
\eth\left(\lambda_{m_0}, \lambda_{m_0+1}\right) &\leqq s\eth\left(\lambda_{m_0}, \lambda^*\right) + s\eth\left(\lambda^*, \lambda_{m_0+1}\right) \\
&\leqq \frac{1}{2}\eth\left(\lambda_{m_0}, I\lambda_{m_0}\right) + \frac{1}{2}\eth\left(\lambda_{m_0+1}, I\lambda_{m_o+1}\right) \\
&< \frac{1}{2}\eth(\lambda_{m_0}, \lambda_{m_0+1}) + \frac{1}{2}\eth(\lambda_{m_0}, \lambda_{m_0+1}) \\
&= \eth\left(\lambda_{m_0}, \lambda_{m_0+1}\right),
\end{aligned}$$

which is a contradiction. Hence (3.20) holds true. Thus we can write

$$\begin{aligned}\theta\left(\eth(\lambda_{\tau+1},I\lambda^*\right) &\leqq \theta\big(H_{\mathfrak{b}}\left(I\lambda_\tau,I\lambda^*\right)\big)\\ &\leqq \left[\theta^a\big(\eth\left(\lambda_\tau,\lambda^*\right)\big)\right]\left[\theta^b\big(\eth\left(\lambda_\tau,I\lambda_\tau\right)\big)\right]\left[\theta^c\big(\eth\left(\lambda^*,I\lambda^*\right)\big)\right]\\ &\quad\cdot\frac{\left[\theta^d\big(\eth\left(\lambda_\tau,I\lambda^*\right)\big)\right]}{2s}\frac{\left[\theta^e\big(\eth\left(\lambda^*,I\lambda_\tau\right)\big)\right]}{2s}\end{aligned}\tag{3.22}$$

or

$$\begin{aligned}\theta\left(\eth(\lambda_{\tau+2},I\lambda^*\right) &\leqq \theta\left(H_{\mathfrak{b}}\left(I\lambda_{\tau+1},I\lambda^*\right)\right)\\ &\leqq \left[\theta^a\left(\eth\left(\lambda_{\tau+1},\lambda^*\right)\right)\right]\left[\theta^b\left(\eth\left(\lambda_{\tau+1},I\lambda_{\tau+1}\right)\right)\right]\left[\theta^c\left(\eth\left(\lambda^*,I\lambda^*\right)\right)\right]\\ &\quad\cdot\frac{\left[\theta^d\left(\eth\left(\lambda_{\tau+1},I\lambda^*\right)\right)\right]}{2s}\frac{\left[\theta^e\left(\eth\left(\lambda^*,I\lambda_{\tau+1}\right)\right)\right]}{2s}.\end{aligned}\tag{3.23}$$

Let us now discuss the following two cases.

Case 1. Assume that (3.22) holds true. By (3.22), we have

$$\begin{aligned}\theta\left(\eth(\lambda_{\tau+1},I\lambda^*\right) &\leqq \theta\left(H_{\mathfrak{b}}\left(I\lambda_\tau,I\lambda^*\right)\right)\\ &\leqq \left[\theta^a\left(\eth\left(\lambda_\tau,\lambda^*\right)\right)\right]\left[\theta^b\left(\eth\left(\lambda_\tau,I\lambda_\tau\right)\right)\right]\left[\theta^c\left(\eth\left(\lambda^*,I\lambda^*\right)\right)\right]\\ &\quad\cdot\frac{\left[\theta^d\left(\eth\left(\lambda_\tau,\lambda^*\right)+\eth\left(\lambda^*,I\lambda^*\right)\right)\right]}{2}\\ &\quad\cdot\frac{\left[\theta^e\left(\eth\left(\lambda^*,\lambda_\tau\right)+\eth\left(\lambda_\tau,I\lambda_\tau\right)\right)\right]}{2}.\end{aligned}\tag{3.24}$$

By (3.9) and (3.19), we see that there is $\mathbb{N}_4\in\mathbb{N}$ such that, for some $\varepsilon_0>0$,

$$\eth\left(\lambda_\tau,\lambda^*\right)<\varepsilon_0 \quad\text{and}\quad \eth\left(\lambda_\tau,I\lambda_\tau\right)<\varepsilon_0(\forall\,\tau>\mathbb{N}_4).\tag{3.25}$$

From (3.24) and (3.25), we have

$$\begin{aligned}\theta\left(\eth(\lambda_{\tau+1},I\lambda^*\right) &\leqq \theta\left(H_{\mathfrak{b}}\left(I\lambda_\tau,I\lambda^*\right)\right)\\ &\leqq \left[\theta^a\left(\eth\left(\lambda_\tau,\lambda^*\right)\right)\right]\left[\theta^b\left(\eth\left(\lambda_\tau,I\lambda_\tau\right)\right)\right]\left[\theta^c\left(\eth\left(\lambda^*,I\lambda^*\right)\right)\right]\\ &\quad\cdot\frac{\left[\theta^d\left(\varepsilon_0+\eth\left(\lambda^*,I\lambda^*\right)\right)\right]}{2}\left[\theta^e\left(\varepsilon_0\right)\right]\end{aligned}\tag{3.26}$$

for $\tau>\mathbb{N}_4$. Taking the limit as $\tau\to\infty$ in (3.25), we find that

$$\lim_{\tau\to\infty}\theta\left(\eth\left(\lambda_{\tau+1},I\lambda^*\right)\right)=1.$$

By means of (θ_{ii}), we have

$$\lim_{\tau\to\infty}\eth\left(\lambda_{\tau+1},I\lambda^*\right)=0.\tag{3.27}$$

On the other hand, we see that

$$\eth\left(\lambda^*,I\lambda^*\right)\leqq s\eth\left(\lambda^*,\lambda_{\tau+1}\right)+s\eth\left(\lambda_{\tau+1},I\lambda^*\right).$$

Taking the limit as $\tau\to\infty$, and by (3.19) and (3.27), we obtain $\eth\left(\lambda^*,I\lambda^*\right)=0$. Thus, clearly, λ^* is a fixed point of I.

Case 2. Let us assume that (3.23) holds true. Then, by (3.23), we have

$$\begin{aligned}\theta\big(\eth\left(\lambda_{\tau+2}, I\lambda^*\right)\big) &\leqq \theta\left(H_{\mathfrak{b}}\left(I\lambda_{\tau+1}, I\lambda^*\right)\right)\\ &\leqq \left[\theta^a\left(\eth\left(\lambda_{\tau+1},\lambda^*\right)\right)\right]\Big[\theta^b\left(\eth\left(\lambda_{\tau+1}, I\lambda_{\tau+1}\right)\right)\Big]\left[\theta^c\left(\eth\left(\lambda^*, I\lambda^*\right)\right)\right]\\ &\quad\cdot\frac{\left[\theta^d\left(\eth\left(\lambda_{\tau+1},\lambda^*\right)+\eth\left(\lambda^*, I\lambda^*\right)\right)\right]}{2}\\ &\quad\cdot\frac{\left[\theta^e\left(\eth\left(\lambda^*,\lambda_{\tau+1}\right)+\eth\left(\lambda_{\tau+1}, I\lambda_{\tau+1}\right)\right)\right]}{2}.\end{aligned}\tag{3.28}$$

By (3.9) and (3.19), we see that there is $\mathbb{N}_5 \in \mathbb{N}$ such that, for some $\varepsilon_1 > 0$,

$$\eth\left(\lambda_{\tau+1},\lambda^*\right) < \varepsilon_1 \quad \text{and} \quad \eth\left(\lambda_{\tau+1}, I\lambda_{\tau+1}\right) < \varepsilon_1 \qquad (\forall\, \tau > \mathbb{N}_5).\tag{3.29}$$

Also, from (3.28) and (3.29), we have

$$\begin{aligned}\theta\left(\eth(\lambda_{\tau+2}, I\lambda^*\right) &\leqq \theta\left(H_{\mathfrak{b}}\left(I\lambda_{\tau+1}, I\lambda^*\right)\right)\\ &\leqq \left[\theta^a\left(\eth\left(\lambda_{\tau+1},\lambda^*\right)\right)\right]\Big[\theta^b\left(\eth\left(\lambda_{\tau+1}, I\lambda_{\tau+1}\right)\right)\Big]\left[\theta^c\left(\eth\left(\lambda^*, I\lambda^*\right)\right)\right]\\ &\quad\cdot\frac{\left[\theta^d\left(\varepsilon_1+\eth\left(\lambda^*, I\lambda^*\right)\right)\right]}{2}\left[\theta^e\left(\varepsilon_1\right)\right]\end{aligned}\tag{3.30}$$

for $\tau > \mathbb{N}_5$. Thus, taking the limit as $\tau \to \infty$ in (3.30), we find that

$$\lim_{\tau\to\infty}\theta\left(\eth\left(\lambda_{\tau+2}, I\lambda^*\right)\right) = 1.$$

Also, in light of (θ_{ii}), we have

$$\lim_{\tau\to\infty}\eth\left(\lambda_{\tau+2}, I\lambda^*\right) = 0.\tag{3.31}$$

On the other hand, we see that

$$\eth\left(\lambda^*, I\lambda^*\right) \leqq s\eth\left(\lambda^*,\lambda_{\tau+2}\right) + s\eth\left(\lambda_{\tau+2}, I\lambda^*\right).$$

If we now proceed to the limit as $\tau \to \infty$ and apply (3.19) and (3.31), then we obtain

$$\eth\left(\lambda^*, I\lambda^*\right) = 0.$$

Thus, λ^* is a fixed point of I. This completes our proof of Theorem 3.1. □

Corollary 3.1. *Let* $(\Gamma, \eth, s)$ *be a* $\mathfrak{b}$*-metric-like space. Suppose that the mapping* $I : \Delta \to \mathrm{CB}(\Gamma)$ *is multi-valued* θ_L*-contraction, that is, there exists* $\theta \in \Xi_L$ *such that*

$$\frac{1}{2s}\eth\left(\lambda_1, I\lambda_1\right) < \eth\left(\lambda_1,\lambda_2\right) \Longrightarrow \theta\left(H_{\mathfrak{b}}\left(I\lambda_1, I\lambda_2\right)\right) \leqq \theta\big(\Omega\left(\lambda_1,\lambda_2\right)\big),\tag{3.32}$$

where

$$\Omega\left(\lambda_1,\lambda_2\right) = \left[\theta^a\eth\left(\lambda_1,\lambda_2\right)\right]\Big[\theta^b\big(\eth\left(\lambda_1, I\lambda_1\right)\big)\Big]\left[\theta^c\eth\left(\lambda_2, I\lambda_2\right)\right]$$

for all $\lambda_1, \lambda_2 \in \Delta$*,* a*,* b*,* $c \in \mathbb{R}^+$*,* $H_{\mathfrak{b}}\left(I\lambda_1, I\lambda_2\right) > 0$*, and* $0 \leqq a+b+c < 1$*. Then* I *has a fixed point.*

Corollary 3.2. *Let* $(\Gamma, \eth, s)$ *be a* $\mathfrak{b}$*-metric-like space. Suppose that the mapping* $I : \Delta \to \mathrm{CB}(\Gamma)$ *is multi-valued* θ_L*-contraction, that is, there exists* $\theta \in \Xi_L$ *such that*

$$\frac{1}{2s}\eth\left(\lambda_1, I\lambda_1\right) < \eth\left(\lambda_1,\lambda_2\right) \Longrightarrow \theta\big(H_{\mathfrak{b}}\left(I\lambda_1, I\lambda_2\right)\big) \leqq \theta^a\big(\eth\left(\lambda_1,\lambda_2\right)\big)\tag{3.33}$$

for all $\lambda_1, \lambda_2 \in \Delta$*,* $a \in \mathbb{R}^+$*,* $H_{\mathfrak{b}}\left(I\lambda_1, I\lambda_2\right) > 0$*, and* $0 \leqq a < 1$*. Then* I *has a fixed point.*

Here, if we consider $\theta(r) = \mathrm{e}^{\sqrt{r}}\alpha$ in Theorem 3.1, then we obtain the following results.

Corollary 3.3. *Let $(\Gamma, \eth, s)$ be a $\mathfrak{b}$-metric-like space. Suppose that the mapping $I : \Delta \to \mathrm{CB}(\Gamma)$ is multi-valued θ_L-contraction, that is, there exists $\theta \in \Xi_L$ such that*

$$\frac{1}{2s}\eth(\lambda_1, I\lambda_1) < \eth(\lambda_1, \lambda_2)$$

implies that

$$\begin{aligned}\sqrt{H_{\mathfrak{b}}(I\lambda_1, I\lambda_2)} \leqq a\sqrt{\eth(\lambda_1,\lambda_2)} + b\sqrt{\eth(\lambda_1, I\lambda_1)} + c\sqrt{\eth(\lambda_2, I\lambda_2)} \\ + d\sqrt{\frac{\eth(\lambda_2, I\lambda_1)}{2s}} + e\sqrt{\frac{\eth(\lambda_1, I\lambda_2)}{2s}}\end{aligned} \tag{3.34}$$

for all $\lambda_1, \lambda_2 \in \Delta$, a, b, c, $d, e \in \mathbb{R}^+$, $H_{\mathfrak{b}}(I\lambda_1, I\lambda_2) > 0$, and $0 \leqq a+b+c+d+e < 1$. Then I has a fixed point.

Corollary 3.4. *Let $(\Gamma, \eth, s)$ be a $\mathfrak{b}$-metric-like space. Suppose that the mapping $I : \Delta \to \mathrm{CB}(\Gamma)$ is multi-valued θ_L-contraction, that is, there exists $\theta \in \Xi_L$ such that*

$$\theta(H_{\mathfrak{b}}(I\lambda_1, I\lambda_2)) \leqq \theta\big(\Omega(\lambda_1, \lambda_2)\big), \tag{3.35}$$

where

$$\begin{aligned}\Omega(\lambda_1,\lambda_2) = [\theta^a\eth(\lambda_1,\lambda_2)]\left[\theta^b(\eth(\lambda_1, I\lambda_1))\right][\theta^c\eth(\lambda_2, I\lambda_2)] \\ \cdot\frac{\left[\theta^d\eth(\lambda_2, I\lambda_1)\right]}{2s}\,\frac{[\theta^e\eth(\lambda_1, I\lambda_2)]}{2s}\end{aligned}$$

for all $\lambda_1, \lambda_2 \in \Delta$, a, $\mathfrak{b}$, c, $d, e \in \mathbb{R}^+$, $H_{\mathfrak{b}}(I\lambda_1, I\lambda_2) > 0$, and $0 \leqq a+b+c+d+e < 1$. If $\eth(I\lambda_1, I\lambda_2) \leqq \eth(\lambda_1, \lambda_1)$, then I has a fixed point.

Corollary 3.5. *Let $(\Gamma, \eth, s)$ be a $\mathfrak{b}$-metric-like space. Suppose that the mapping $I : \Delta \to \mathrm{CB}(\Gamma)$ is multi-valued θ_L-contraction, that is, there exists $\theta \in \Xi_L$ such that*

$$\theta(H_{\mathfrak{b}}(I\lambda_1, I\lambda_2)) \leqq [\theta^a\eth(\lambda_1,\lambda_2)]\left[\theta^b(\eth(\lambda_1, I\lambda_1))\right][\theta^c\eth(\lambda_2, I\lambda_2)] \tag{3.36}$$

for all $\lambda_1, \lambda_2 \in \Delta$, a, b, $c \in \mathbb{R}^+$, $H_{\mathfrak{b}}(I\lambda_1, I\lambda_2) > 0$, and $0 \leqq a+b+c < 1$. If $\eth(I\lambda_1, I\lambda_2) \leqq \eth(\lambda_1, \lambda_1)$, then I has a fixed point.

Corollary 3.6. *Let $(\Gamma, \eth, s)$ be a $\mathfrak{b}$-metric-like space. Suppose that the mapping $I : \Delta \to \mathrm{CB}(\Gamma)$ is multi-valued θ_L-contraction, that is, there exists $\theta \in \Xi_L$ such that*

$$\begin{aligned}\sqrt{H_{\mathfrak{b}}(I\lambda_1, I\lambda_2)} \leqq a\sqrt{\eth(\lambda_1,\lambda_2)} + b\sqrt{\eth(\lambda_1, I\lambda_1)} + c\sqrt{\eth(\lambda_2, I\lambda_2)} \\ + d\sqrt{\frac{\eth(\lambda_2, I\lambda_1)}{2s}} + e\sqrt{\frac{\eth(\lambda_1, I\lambda_2)}{2s}}\end{aligned} \tag{3.37}$$

for all $\lambda_1, \lambda_2 \in \Delta$, a, b, c, $d, e \in \mathbb{R}^+$, $H_{\mathfrak{b}}(I\lambda_1, I\lambda_2) > 0$, and $0 \leqq a+b+c+d+e < 1$ with $\theta(r) = \mathrm{e}^{\sqrt{r}}\alpha$ as in Theorem 3.1. If $\eth(I\lambda_1, I\lambda_2) \leqq \eth(\lambda_1, \lambda_1)$, then I has a fixed point.

Corollary 3.7. *Let* $(\Gamma, \eth, s)$ *be a* $\mathfrak{b}$*-metric-like space. Suppose that the mapping* $I : \Delta \to \mathrm{CB}(\Gamma)$ *is multi-valued* θ_L*-contraction, that is, there exists* $\theta \in \Xi_L$ *such that the following implication holds true*:

$$\frac{1}{2s} \min\{\eth(\lambda_1, I\lambda_1), \eth(\lambda_2, I\lambda_2)\} < \eth(\lambda_1, \lambda_2)$$
$$\implies \theta(H_{\mathfrak{b}}(I\lambda_1, I\lambda_2)) \leqq \theta^a\big(U_i(\lambda_1, \lambda_2)\big), \tag{3.38}$$

where

$$U_1(\lambda_1, \lambda_2) = \eth(\lambda_1, \lambda_2)$$

and

$$U_2(\lambda_1, \lambda_2) = [\theta^a \eth(\lambda_1, \lambda_2)] \left[\theta^b(\eth(\lambda_1, I\lambda_1))\right] [\theta^c \eth(\lambda_2, I\lambda_2)] \frac{[\theta^d \eth(\lambda_2, I\lambda_1)]}{2s} \frac{[\theta^e \eth(\lambda_1, I\lambda_2)]}{2s}$$

for all $\lambda_1, \lambda_2 \in \Delta$, $a, b, c, d, e \in \mathbb{R}^+$, $i = 1, 2$, $H_{\mathfrak{b}}(I\lambda_1, I\lambda_2) > 0$, *and* $0 \leqq a + b + c + d + e < 1$. *Then I has a fixed point.*

Example 3.1. Let $\Gamma = [-10, \infty)$ be an $\mathfrak{b}$-metric-like $\eth$ defined by

$$\eth(\lambda_1, \lambda_2) = (\max\{\lambda_1, \lambda_2\})^2 \qquad (\forall\, \lambda_1, \lambda_2 \in \Gamma).$$

Also let $I : \Lambda \to \mathrm{CB}(\Gamma)$ be defined, in terms of the exponential function e^x, as follows:

$$I(\lambda) = \begin{cases} \left[0, \dfrac{\lambda}{2\mathrm{e}^2}\right] & (\lambda \in [0, 4]) \\ \{0, \lambda\} & \big(\lambda \in [-10, 0) \cup (4, \infty)\big). \end{cases}$$

Clearly, we have

$$\frac{1}{2s} \eth(\lambda_1, I\lambda_1) < \eth(\lambda_1, \lambda_2) \iff \lambda_1, \lambda_2 \in [0, 4].$$

Firstly, we claim that I satisfies the inequality (3.1) with

$$a = \frac{1}{\mathrm{e}} \qquad \text{and} \qquad \theta(r) = \mathrm{e}^{\sqrt{r \mathrm{e}^r}}.$$

For $\lambda_1, \lambda_2 \in [0, 4]$, we thus find that

$$\theta[H(I\lambda_1, I\lambda_2)] = \theta\left(\frac{(\max\{\lambda_1, \lambda_2\})^2}{2\mathrm{e}^2}\right)$$
$$\leqq \mathrm{e}^{\frac{1}{\mathrm{e}}\sqrt{\frac{(\max\{\lambda_1, \lambda_2\})^2}{2} \mathrm{e}^{\frac{(\max\{\lambda_1, \lambda_2\})^2}{2}}}} = \big[\theta\big(\eth(\lambda_1, \lambda_2)\big)\big]^a.$$

Hence the requirement of Corollary 3.2 are fulfilled, and 0 is a fixed point of I. For $\lambda_1 = 0$ and $\lambda_2 = 5$, we find that

$$\theta[H(I\lambda_1, I\lambda_2)] = \theta[H(I0, I5)] = \theta(25) > [\theta(25)]^a = \big[\theta\big(\eth(\lambda_1, \lambda_2)\big)\big]^a$$

for all $\theta \in \Xi_L$ and $a \in [0, 1)$. Thus, clearly, Corollary 3.2 cannot be satisfied.

4. An Application

In this section, we deal with some new aspects of the Liouville-Caputo fractional differential equations in the context of the $\mathfrak{b}$-metric-like spaces. Several earlier developments on fixed point theory and its applications involving fractional calculus can be found in, for example, [12, 13, 14], and also in the references cited in each of these works.

We first define the Liouville-Caputo fractional differential equation based on the Liouville-Caputo fractional derivative operator ${}^{\mathrm{LC}}\check{D}_\kappa$ of order κ, which is defined by (see, for details, [15]; see also the recent developments in [16, 17, 18, 19, 20]).

$$ {}^{\mathrm{LC}}\check{D}_\kappa\big(\Phi(g)\big) = \frac{1}{\Gamma(n-\kappa)} \int_0^g (g-t)^{n-\kappa-1}\ \Phi^{(n)}(t)\,dt, \tag{4.1} $$

where $n \in \mathbb{N}$, $n-1 < \kappa < n$ $(n = [\kappa]+1)$, $\Phi \in C^n([0,\infty])$ and Γ denotes the familiar (Euler's) Gamma function.

Let the $\mathfrak{b}$-metric-like space $\delta_\varsigma : C(I) \times C(I) \to \mathbb{R}^+$ be given by

$$ \delta_\varsigma(\varepsilon_{s-1}, \varepsilon_s)\Big|_{s=2} = \delta_\varsigma(\varepsilon_1, \varepsilon_2) = \sup_{k \in I} \left(|\varepsilon_1(k)| + |\varepsilon_2(k)|\right)^2. \tag{4.2} $$

We also consider the following family of fractional differential equations and its integral boundary-valued problem:

$$ {}^{\mathrm{LC}}\check{D}_\kappa\big(\Psi(g)\big) = L\big(g, \Psi(g)\big), \tag{4.3} $$

where $g \in (0,1)$, $\kappa \in (1,2]$, and

$$ \begin{cases} \Psi,(0) = 0, \\ \Psi(1) = \displaystyle\int_0^\vartheta \Psi(g)\,dg, \qquad \big(\vartheta \in (0,1)\big), \end{cases} \tag{4.4} $$

where $I = [0,1]$, $\Psi \in C(I,\mathbb{R})$, and $L : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $P : \Delta \to \Delta$ be defined by

$$ \begin{aligned} Pv(r) = \frac{1}{\Gamma(\kappa)} \int_0^g (g-t)^{\kappa-1} L(t, v(t))\,dt - \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^1 (1-t)^{\kappa-1} L(t, v(t))\,dt \\ + \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^\vartheta \left(\int_0^{g_1} (g_1 - t_1)^{\kappa-1} L(t_1, v(t_1))\,dt_1 \right) dt \end{aligned} \tag{4.5} $$

for $v \in \Delta$ and $g \in [0,1]$. We now state our proposed application as Theorem 4.1 below.

Theorem 4.1. *Let $L : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function and non-decreasing on the second variable. Suppose also that there exists $\theta \in \Xi_L$ such that, for $\varepsilon_1, \varepsilon_2 \in \Delta$, $g \in [0,1]$ and $\alpha \in [0,1)$, $\frac{1}{2s}\,\eth(\varepsilon_1, I\varepsilon_1 \cap \Lambda) < \eth(\varepsilon_1, \varepsilon_2)$ implies that*

$$ \big(|P\varepsilon_1(r)| + |P\varepsilon_2(r)|\big)^2 \leqq \Omega \left(\left[1 + \sqrt{\max_{g \in I} \{\eth(\varepsilon_1, \varepsilon_2)(r)\}} \right]^\alpha - 1 \right)^2, \tag{4.6} $$

where

$$ \Omega = \frac{(2\kappa - 1)\ \Gamma(\kappa+1)}{2(5\kappa+2)}. $$

Then the equations (4.3) and (4.4) have precisely one solution, that is, $\varepsilon^* \in \Delta$.

Proof. For each $g \in I$ and owing to the operator P, we can write

$$
\begin{aligned}
&\left(|P\varepsilon_1(r)| + |P\varepsilon_2(r)|\right)^2 \\
&= \Bigg| \Bigg(\frac{1}{\Gamma(\kappa)} \int_0^g (g-t)^{\kappa-1} L\big(t, \varepsilon_1(t)\big) dt \\
&\qquad - \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^1 (1-t)^{\kappa-1} L\big(t, \varepsilon_{i-1}(t)\big) dt \\
&\qquad + \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^\vartheta \left(\int_0^{g_1} (g_1-t_1)^{\kappa-1} L\big(t_1, \varepsilon_1(t_1)\big) dt_1 \right) dt \Bigg) \\
&\qquad - \Bigg(\bigg[\frac{1}{\Gamma(\kappa)} \int_0^g (g-t)^{\kappa-1} L\big(t, \varepsilon_2(t)\big) dt \\
&\qquad - \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^1 (1-t)^{\kappa-1} L\big(t, \varepsilon_2(t)\big) dt \\
&\qquad + \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^\vartheta \left(\int_0^{g_1} (g_1-t_1)^{\kappa-1} L\big(t_1, \varepsilon_2(t_1)\big) dt_1 \right) dt \bigg] \Bigg) \Bigg| \\
&\leqq \frac{1}{\Gamma(\kappa)} \int_0^g (g-t)^{\kappa-1} \left| L\big(t, \varepsilon_1(t)\big) - L(t, \varepsilon_2(t)) \right| dt \\
&\qquad + \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^1 (1-t)^{\kappa-1} \left| L\big(t, \varepsilon_1(t)\big) - L\big(t, \varepsilon_2(t)\big) \right| dt \\
&\qquad + \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^\vartheta \left| \int_0^{g_1} (g_1-t_1)^{\kappa-1} \left[L\big(t_1, \varepsilon_1(t_1)\big) - L\big(t, \varepsilon_2(t)\big) \right] dt_1 \right| dt.
\end{aligned}
$$

Next, we observe that

$$
\begin{aligned}
&\left(|P\varepsilon_1(r)| + |P\varepsilon_2(r)|\right)^2 \\
&\leqq \frac{1}{\Gamma(\kappa)} \int_0^g (g-t)^{\kappa-1} \Omega \left(\left[1 + \sqrt{\max_{g \in I} \eth(\varepsilon_1, \varepsilon_2)(r)} \right]^\alpha - 1 \right)^2 dt \\
&\qquad + \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^1 (1-t)^{\kappa-1} \Omega \left(\left[1 + \sqrt{\max_{g \in I} \eth(\varepsilon_1, \varepsilon_2)(r)} \right]^\alpha - 1 \right)^2 dt \\
&\qquad + \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^\vartheta \int_0^{g_1} (g_1-t_1)^{\kappa-1} \Omega \left(\left[1 + \sqrt{\max_{g \in I} \eth(\varepsilon_1, \varepsilon_2)(r)} \right]^\alpha - 1 \right)^2 dt_1\, dt \\
&\leqq \frac{\Omega}{\Gamma(\kappa)} \left(\left[1 + \sqrt{\max_{g \in I} \eth(\varepsilon_1, \varepsilon_2)(r)} \right]^\alpha - 1 \right)^2 \left\{ \begin{array}{c} \int_0^g (g-t)^{\kappa-1} dt \\ + \frac{2g}{(2-\vartheta^2)} \int_0^1 (1-t)^{\kappa-1} dt \\ + \frac{2g}{2-\vartheta^2} \int_0^\vartheta \int_0^{g_1} (g_1-t_1)^{\kappa-1}\; dt_1\, dt, \end{array} \right\},
\end{aligned}
$$

which yields

$$\begin{aligned}\left(\left|P\varepsilon_1(r)\right|+P\varepsilon_2(r)\right|\right)^2 &\leqq \frac{\Omega}{\Gamma(\kappa)}\left(\left[1+\sqrt{\max_{g\in I}\eth(\varepsilon_1,\varepsilon_2)(r)}\right]^{\alpha}-1\right)^2\\
&\quad\cdot\left(\frac{g^{\kappa}}{\kappa}+\frac{2g}{2-\vartheta^2}\,\frac{1}{\kappa}+\frac{2g}{2-\vartheta^2}\,\frac{\vartheta^{\kappa+1}}{\kappa(\kappa+1)}\right)\\
&\leqq \Omega\left(\left[1+\sqrt{\max_{g\in I}\eth(\varepsilon_1,\varepsilon_2)(r)}\right]^{\alpha}-1\right)^2\\
&\quad\cdot\sup_{g\in(0,1)}\left\{g^{\kappa}+\frac{2g}{(2-\vartheta^2)}+\frac{2g}{(2-\vartheta^2)}\,\frac{\vartheta^{\kappa+1}}{(\kappa+1)}\right\}\\
&=\frac{2\kappa-1}{2(5\kappa+2)}\left(\left[1+\sqrt{\max_{g\in I}\eth(\varepsilon_1,\varepsilon_2)(r)}\right]^{\alpha}-1\right)^2\\
&\quad\cdot\sup_{g\in(0,1)}\left\{g^{\kappa}+\frac{2g}{(2-\vartheta^2)}+\frac{2g}{(2-\vartheta^2)}\,\frac{\vartheta^{\kappa+1}}{(\kappa+1)}\right\}\\
&=\frac{2\kappa-1}{2(5\kappa+2)}\left(\left[1+\sqrt{\max_{g\in I}U(\varepsilon_1,\varepsilon_2)(r)}\right]^{\alpha}-1\right)^2.\end{aligned}$$

This last observation implies that

$$\left(\left|P\varepsilon_1(r)\right|+\left|P\varepsilon_2(r)\right|\right)^2 \leqq \left(\left[1+\sqrt{\max_{g\in I}\eth(\varepsilon_1,\varepsilon_2)(r)}\right]^{\alpha}-1\right)^2. \tag{4.7}$$

Therefore, we have

$$\begin{aligned}\left(\left|P\varepsilon_1(r)\right|+\left|P\varepsilon_2(r)\right|\right)^2 &= \sup_{k\in I}\left(\left|P\varepsilon_1(r)\right|+\left|P\varepsilon_2(r)\right|\right)^2\\
&\leqq \left(\left[1+\sqrt{\max_{g\in I}\eth(\varepsilon_1,\varepsilon_2)(r)}\right]^{\alpha}-1\right)^2.\end{aligned} \tag{4.8}$$

Now, by applying (4.8), we find that

$$1+\sqrt{\eth(\varepsilon_1,\varepsilon_2)} \leqq \left[1+\sqrt{\eth(\varepsilon_1,\varepsilon_2)}\right]^{\alpha}.$$

Thus, by the contractive condition (3.33) upon setting

$$\theta(\varepsilon)=1+\sqrt{\varepsilon} \qquad (\varepsilon\in\mathbb{R}),$$

we obtain

$$\theta\left[H_d(I\varepsilon_1,I\varepsilon_2)\right] \leqq \left[\theta\left(\eth(\varepsilon_1,\varepsilon_2)\right)\right]^{a} \qquad (\forall\, \varepsilon_1,\varepsilon_2\in\Delta),$$

where $a\in[0,1)$. Thus, all of the required hypotheses of Corollary (3.2) are satisfied, and we conclude that the problem P involving the equations (4.3) and (4.4) has at least one solution. □

Example 4.1. Based upon the Liouville-Caputo fractional derivative operator ${}^{\mathrm{LC}}\check{D}_\kappa$ of order κ, let us consider the following integral boundary-value problem:

$$ {}^{\mathrm{LC}}\check{D}_{\frac{3}{2}}\big(\Psi(g)\big) = \frac{1}{(g+3)^2}\,\frac{|\Psi(g)|}{1+|\Psi(g)|} \tag{4.9} $$

and, analogous to the equations (4.3) and (4.4),

$$ \begin{cases} \Psi(0)=0, \\ \Psi(1)=\int_0^{\frac{3}{4}} \Psi(g)\,dg \qquad \left(\vartheta=\frac{3}{4}\in(0,1)\right), \end{cases} \tag{4.10} $$

where we have set $\kappa=\frac{3}{2}$, $\vartheta=\frac{3}{4}$, and

$$ L\big(g,\Psi(g)\big)=\frac{1}{(g+3)^2}\frac{|\Psi(g)|}{1+|\Psi(g)|}. $$

The above setting is an example of equations (4.3) and (4.4). Hence, the pair of equations (4.9) and (4.10) has at least one solution.

5. Concluding Remarks and Observations

In this paper, we introduced and systematically studied an extension of the developments concerning the θ-contractions which were proposed, in the year of 2014, by Jleli and Samet [6]. We fruitfully generalized the notion of the θ-contractions to the case of non-linear θ_L-contraction mappings and proved several multi-valued fixed point results in $\mathfrak{b}$-metric-like spaces. This paper also includes an interesting example, which displays the motivation for such investigations as those presented here. An application of the proposed non-linear θ_L-contractions to the Liouville-Caputo fractional derivatives and fractional differential equations was given. With a view to motivating the interested readers, we included some related recent developments (see, for example, [21] and [22]) on fixed points, contractive and multi-valued mappings, and their applications to integral equations (see also the related works [23, 24, 25, 26, 27, 28]).

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