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FIXED POINTS OF GENERALIZED MULTI-VALUED CONTRACTIVE MAPPINGS IN METRIC TYPE SPACES

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Abstract. In the setting of metric type spaces, we establish some new results on the existence of fixed points for generalized multi-valued contractive mappings with respect to the w_b-distance. In support of our results, some examples are also presented.

Keywords. Fixed point; Metric type space; Multi-valued contractive mapping; w-distance.

1. INTRODUCTION

The well-known Banach contraction principle [1], which asserts that ”each single-valued contraction self-mapping on a complete metric space has a unique fixed point” plays a significant role in nonlinear functional analysis and optimization; see, e.g., [2, 3, 4, 5, 6] and the references therein. Recently, many fruitful generalizations of this classical fixed point result were established in various spaces. Using the Hausdorff-Pompieu metric, Nadler [7] established a multi-valued version of the Banach contraction principle, which is now one of the most useful result in metric fixed point theory. Due to this, Nadler's fixed point theorem has been generalized and investigated in various directions; see, e.g., [8, 9, 10, 11, 12] and the references therein. It is worth to mention that, for most cases, the existence part of the results can be obtained without using the Hausdorff-Pompieu metric; see, e.g., [13, 14, 15] and the references therein.

The concept of metric spaces was extended either reducing or modifying the metric axioms. One of the useful extensions is known as the metric type or b-metric spaces, which was introduced and studied by Bakhtin [16]. However, the authors [17, 18] initiated the study of the fixed points of self-mappings in a b-metric space and proved an analogue of the Banach contraction principle. Since then, various results on fixed points of single-valued and multi-valued operators have been established in b-metric spaces; see, e.g., [19, 20, 21] and the references therein.

In [22], Kada et al. introduced and studied the concept of the w-distance on a metric spaces. They improved several results by replacing the involved metric by a generalized distance. In [23], Suzuki and Takahashi introduced the notions of single-valued and multi-valued weakly

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contractive mappings with respect to the w-distance and proved some fixed point results for such mappings, which improved some classical fixed point results. Further work concerning the w-distance can be found in [24, 25, 26]. In [27], Hussain et al. defined the w-distance on metric type spaces, called the wt-distance (here, we call it w_b-distance) and studied fixed points and common fixed points of the single-valued mappings with respect to w_b-distances. For some recent results in this direction, we refer to [27, 28, 29] and the references therein.

In this paper, we present some general fixed point results for generalized multi-valued mappings on metric type spaces with respect to w_b-distances and improve/generalize a number of known fixed point results presented in Mizoguchi-Takahashi [10], Feng-Liu [14], Klim-Wardowski [15], Susuki-Takahashi [23], Hussain et al. [27], Demma et al. [28], Ćirić [30], Latif-Abdou [31], Latif-Albar [32], Latif et al. [33] and several others.

2. Preliminaries

Let (X,d) be a metric space. Let 2^X denote the collection of nonempty subsets of X, $Cl(X)$ denote a collection of nonempty closed subsets of X, $CB(X)$ denote a collection of nonempty closed bounded subsets of X, and $P(X)$ denote a collection of nonempty proximinal subsets of X. For any $L,M \in CB(X)$, define $H(L,M) = \max\{\sup_{x\in L} d(x,M), \sup_{y\in M} d(y,L)\}$, where $d(x,M) = \inf_{y\in M} d(x,y)$, and H is known as Hausdorff-Pompieu metric on $CB(X)$.

An element $x \in X$ is called a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in T(x)$. We denote the set of fixed points of T by $Fix(T) = \{x \in X : x \in T(x)\}$. A sequence $\{x_n\}$ in X is called an orbit of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all $n \geq 1$. A mapping $f : X \to \mathbb{R}$ is said to be lower semicontinuous if, for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$, $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

Using the Hausdorff-Pompieu metric, Nadler [7] established the following multi-valued version of the Banach contraction principle.

Theorem 2.1. [7] *Let (X,d) be a complete metric space, and let $T : X \to CB(X)$ be a map such that, for a fixed constant $h \in (0,1)$ and for each $x,y \in X$, $H(T(x),T(y)) \leq h\, d(x,y)$. Then $Fix(T) \neq \emptyset$.*

In [10], Mizoguchi and Takahashi proved the following result, which is a real generalization [11] of the Nadler's fixed point result.

Theorem 2.2. [10] *Let (X,d) be a complete metric space and let $T : X \to CB(X)$. Assume that there exists a function $k : [0,\infty) \to [0,1)$ such that, for every $t \in [0,\infty)$, $\limsup_{r\to t^+} k(r) < 1$, and, for all $x,y \in X$, $H(T(x),T(y)) \leq k(d(x,y))d(x,y)$. Then $Fix(T) \neq \emptyset$.*

In [14], Feng and Liu generalized Theorem 2.1 without using the Hausdorff-Pompieu metric.

Theorem 2.3. [14] *Let (X,d) be a complete metric space and let $T : X \to Cl(X)$ be a multi-valued mapping. Suppose that a real-valued function g on X, $g(x) = d(x,T(x))$ is lower semi-continuous. Then $Fix(T) \neq \emptyset$ provided that there exist constants $c,h \in (0,1)$, $h < c$ such that, for any $x \in X$, there is $y \in T(x)$ satisfying $cd(x,y) \leq g(x)$ and $g(y) \leq hd(x,y)$.*

In [15], Klim and Wardowski further generalized Theorem 2.3 in the following way.

Theorem 2.4. [15] *Let (X,d) be a complete metric space and let $T : X \to Cl(X)$ be a multi-valued mapping such that a real-valued function g on X defined by $g(x) = d(x,T(x))$ is lower* *semi-continuous. Then* $Fix(T) \neq \emptyset$ *provided that there exists* $c \in (0,1)$ *such that, for any* $x \in X$, *there is* $y \in T(x)$ *satisfying* $cd(x,y) \leq g(x)$ *and* $g(y) \leq k(d(x,y))d(x,y)$, *where k is a function from* $[0,\infty)$ *to* $[0,c)$ *with* $\limsup_{r\to t^+} k(r) < c$, *for every* $t \in [0,\infty)$.

In [30], Ćirić further unified and generalized the above mentioned theorems. In particular, the following result [30, Theorem 5] generalized Theorem 2.2.

Theorem 2.5. [30] *Let* (X,d) *be a complete metric space and let* $T : X \to Cl(X)$ *be a multi-valued mapping such that a real-valued function g on X defined by* $g(x) = d(x,T(x))$ *is lower semi-continuous. Then* $Fix(T) \neq \emptyset$ *provided that, for any* $x \in X$, *there is* $y \in T(x)$ *satisfying* $d(x,y) \leq (2-k(d(x,y)))\,g(x)$ *and* $g(y) \leq k(d(x,y))d(x,y)$, *where k is a function from* $[0,\infty)$ *to* $[0,1)$ *with* $\limsup_{r\to t^+} k(r) < 1$, *for every* $t \in [0,\infty)$.

In [22], Kada et al. introduced the concept of the w-distance on a metric spaces as follows.

Let (X,d) be a metric space. A function $p : X \times X \to [0,\infty)$ is called a w-distance on X if it satisfies the following conditions. For each $x,y,z \in X$,

(i) $p(x,z) \leq p(x,y) + p(y,z)$;
(ii) a function $p(x,\cdot) : X \to [0,\infty)$ is lower semi-continuous, i.e., if a sequence $\{y_n\}$ in X with $y_n \to y \in X$, then $p(x,y) \leq \liminf_{n\to\infty} p(x,y_n)$;
(iii) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \varepsilon$.

Clearly, metric d is a w-distance on X. Let $(Y,\|.\|)$ be a normed space. Then the functions $p_1, p_2 : Y \times Y \to [0,\infty)$ defined by $p_1(x,y) = \|y\|$ and $p_2(x,y) = \|x\| + \|y\|$ for all $x,y \in Y$ are w-distances [22]. Furthermore, the examples and properties of the w-distance, we refer to [22, 34]. Using the concept of the w-distances, Kada et al. [22] improved some classical results in metric fixed point theory. In [23], Susuki and Takahashi established fixed point results for single and multi-valued contractive type mappings with respect to the w-distance. They improved the Banach contraction principle and Nadler's fixed point result (Theorem 2.1). On the other hand, a useful extension of a metric space is known as the metric type or b-metric space, which was introduced in [17, 18] as follows.

Let X be a nonempty set, $b \geq 1$ and $D : X \times X \to [0,\infty)$ be a function satisfying the following restrictions, for all $x,y,z \in X$,

(i) $D(x,y) = 0$ if and only if $x = y$;
(ii) $D(x,y) = D(y,x)$;
(iii) $D(x,y) \leq b\,[D(x,z) + D(z,y)]$.

Then D is called a b-metric on X, and (X,D) is called a b-metric space (also known as a metric type space [19]). In the sequel, we also call it a metric type space.

Remark 2.1. Clearly, every metric space is a metric type space. But, the converse may not be true [17]. Thus, the family of metric type spaces is effectively larger than the one of metric spaces.

Example 2.1. [16] Consider the set $X = [0,1]$ endowed with the function $D : X \times X \to [0,\infty)$, which is defined by $D(x,y) = |x-y|^2$, for any $x,y \in X$. Then (X,D) is a metric type space with $b = 2$, but it is not a metric space.

Example 2.2. [27] Let $D:\mathbb{R}\times\mathbb{R}\to[0,\infty)$ be defined by $D(x,y)=|x-y|^2+\left|\frac{1}{x}-\frac{1}{y}\right|^2$ for any $x,y\in\mathbb{R}$. Then $(\mathbb{R},D)$ is a metric type space with $b=2$.

Example 2.3. [35] For $0<p<1$, the space $l_p=\left\{(x_n)_{n\in\mathbb{N}}\subset\mathbb{R},\text{where}\sum_{n=1}^{\infty}x_n<\infty\right\}$, which together with the mapping $D:l_p\times l_p\to\mathbb{R}$ defined by

$$D(x,y)=\left(\sum_{n=1}^{\infty}|x_n-y_n|^p\right)^{1/p},$$

is a metric type space with $b=2^{1/p}>1$.

For further examples of metric type spaces, we refer to [17, 20, 35]. The notions of convergent sequences, Cauchy sequences, and complete spaces in the setting of metric type spaces can be defined similarly as in metric spaces; see, e.g., [19]. A subset K of the metric type space (X,D) is said to be open if and only if, for any $u\in K$, there exists $\varepsilon>0$ such that the open ball $B_o(u,\varepsilon)\subset K$. The family of all open subsets of X, denoted by τ, defines a topology on (X,D). Further, any nonempty subset K of X is closed if and only if, for any sequence $\{x_n\}$ in K which converges to x, $x\in K$, for detail see [19].

Lemma 2.1. [18] *Let A be a closed subset of a metric type space (X,D) and $x\in X$. Then $D(x,A)=0\Leftrightarrow x\in\overline{A}=A$, where $D(x,A)=\inf\{D(x,y):y\in A\}$, and the closure of the set A is denoted by $\overline{A}$.*

Motivated by the work of Kada et al. [22], Hussain et al. [27] introduced the *wt*-distance (here we say it w_b-distance) in the setting of metric type space as follows.

Let (X,D) be a metric type space with constant $b\geq 1$. Then a function $p_b:X\times X\to[0,\infty)$ is called a w_b-distance on X if, for any $x,y,z\in X$, the following conditions are satisfied:

(i) $p_b(x,z)\leq b\,[p_b(x,y)+p_b(y,z)]$;

(ii) $p_b(x,\cdot):X\to[0,\infty)$ is b-lower semi-continuous (i.e., if, for any sequence $\{y_n\}$ in X, $y_n\to y\in X$, then $p_b(x,y)\leq\liminf_{n\to\infty} b\,p_b(x,y_n)$);

(iii) for any $\varepsilon>0$, there exists $\delta>0$ such that $p_b(z,x)\leq\delta$ and $p_b(z,y)\leq\delta$ imply $D(x,y)\leq\varepsilon$.

Note that, for $b=1$, each w_b-distance is a w-distance. Also, each b-metric D is a w_b-distance on X.

Example 2.4. [27] Let $X=\mathbb{R}$ and $D(x,y)=(x-y)^2$. Then

(i) The function $p_b:X\times X\to[0,\infty)$ defined by $p_b(x,y)=|x|^2+|y|^2$ for every $x,y\in X$ is a w_b-distance on X.

(ii) The function $p_b:X\times X\to[0,\infty)$ defined by $p_b(x,y)=|y|^2$ for every $x,y\in X$ is a w_b-distance on X.

Lemma 2.2. [27] *Let (X,D) be a metric type space with constant $b\geq 1$, and let p_b be a w_b-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to zero, and let $x,y,z\in X$. Then the following assertions hold:*

(i) *if $p_b(x_n,y)\leq\alpha_n$ and $p_b(x_n,z)\leq\beta_n$ for any $n\in\mathbb{N}$, then $y=z$. In particular, if $p_b(x,y)=0$ and $p_b(x,z)=0$, then $y=z$;*

(ii) *if $p_b(x_n,y_n)\leq\alpha_n$ and $p_b(x_n,z)\leq\beta_n$ for any $n\in\mathbb{N}$, then $D(y_n,z)\to 0$;*

(iii) *if* $p_b(x_n, x_m) \leq \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence;*
(iv) *if* $p_b(y, x_n) \leq \alpha_n$ *for any* $n \in \mathbb{N}$, *then* $\{x_n\}$ *is a Cauchy sequence.*

Lemma 2.3. [33] *Let* A *be a closed subset of a metric type space* (X, D)*, and let* p_b *be a* w_b*-distance on* X*. Suppose that there exists* $u \in X$ *such that* $p_b(u, u) = 0$*. Then* $p_b(u, A) = 0 \Leftrightarrow u \in A$*, where* $p_b(u, A) = \inf\{p_b(u, v) : v \in A\}$.

The objective of this paper is to present some more general results on the existence of fixed points for multi-valued mappings with respect to the w_b-distance. We unify and generalize a number of known fixed point results in the metric fixed point theory, including the corresponding fixed results presented in this section.

3. Main Results

First, we prove a fixed point result, which is a generalization of Ćirić [30, Theorem 5], in the setting of metric type spaces.

Theorem 3.1. *Let* (X, D) *be a complete metric type space with a* w_b*-distance* p_b*. Let* $T : X \to Cl(X)$ *be a multi-valued mapping such that a real-valued function* g *on* X *defined by* $g(x) = p_b(x, T(x))$ *is b-lower semi-continuous. Assume that the following conditions hold:*

(i) *there is a function* φ *from* $[0, +\infty)$ *to* $[0, 1)$ *with* $\limsup_{r \to t^+} \varphi(r) < 1$*, for every* $t \in [0, +\infty)$*;*
(ii) *for any* $x \in X$*, there is* $y \in T(x)$ *satisfying*

$$p_b(x, y) \leq (2 - \varphi(p_b(x, y)))\, g(x)$$

and

$$g(y) \leq \varphi(p_b(x, y))\, p_b(x, y).$$

Then, there exists $u_0 \in X$ *such that* $g(u_0) = 0$*. Further, if* $p_b(u_0, u_0) = 0$*, then* $u_0 \in T(u_0)$*.*

Proof. Let $x_0 \in X$ be an arbitrary but fixed element of X. From (ii), we see that there exists $x_1 \in T(x_0)$ such that

$$p_b(x_0, x_1) \leq (2 - \varphi(p_b(x_0, x_1)))\, p_b(x_0, T(x_0))$$

and

$$p_b(x_1, T(x_1)) \leq \varphi(p_b(x_0, x_1))\, p_b(x_0, x_1).$$

Thus,

$$p_b(x_1, T(x_1)) \leq \varphi(p_b(x_0, x_1))\,(2 - \varphi(p_b(x_0, x_1)))\, p_b(x_0, T(x_0)). \tag{3.1}$$

Define a function $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \varphi(t)(2 - \varphi(t)) = 1 - (1 - \varphi(t))^2. \tag{3.2}$$

Since, for each $t \in [0, +\infty)$, $\varphi(t) < 1$, and $\limsup_{r \to t^+} \varphi(r) < 1$, we obtain $\psi(t) < 1$ and

$$\limsup_{r \to t^+} \psi(r) < 1, \quad \text{for all } t \in [0, +\infty). \tag{3.3}$$

From (3.1) and (3.2), we have

$$p_b(x_1, T(x_1)) \leq \psi(p_b(x_0, x_1))\, p_b(x_0, T(x_0)).$$

Similarly, for $x_1 \in X$, there exists $x_2 \in T(x_1)$ such that

$$p_b(x_1,x_2) \leq (2-\varphi(p_b(x_1,x_2)))\, p_b(x_1,T(x_1))$$

and

$$p_b(x_2,T(x_2)) \leq \psi(p_b(x_1,x_2))\, p_b(x_1,T(x_1)).$$

Continuing this process, we obtain an orbit $\{x_n\}$ of T at $x_0 \in X$ such that $x_{n+1} \in T(x_n)$ satisfies

$$p_b(x_n,x_{n+1}) \leq (2-\varphi(p_b(x_n,x_{n+1})))\, p_b(x_n,T(x_n)) \tag{3.4}$$

and

$$p_b(x_{n+1},T(x_{n+1})) \leq \psi(p_b(x_n,x_{n+1}))\, p_b(x_n,T(x_n)), \tag{3.5}$$

for each $n \geq 0$. Since $\psi(t) < 1$ for each $t \in [0,+\infty)$, we have, for all $n \geq 0$

$$p_b(x_{n+1},T(x_{n+1})) < p_b(x_n,T(x_n)).$$

Thus, the sequence of non-negative real numbers $\{p_b(x_n,T(x_n))\}$ is strictly decreasing and bounded below, thus convergent. Therefore, there is some $\alpha \geq 0$ such that

$$\lim_{n\to\infty} p_b(x_n,T(x_n)) = \alpha. \tag{3.6}$$

Note that, for each n, $p_b(x_n,T(x_n)) \leq p_b(x_n,x_{n+1})$, and since $\varphi(t) < 1$ for each $t \in [0,+\infty)$, we conclude from (3.4) that, for each n,

$$p_b(x_n,x_{n+1}) < 2p_b(x_n,T(x_n)). \tag{3.7}$$

Thus, the sequence $\{p_b(x_n,x_{n+1})\}$ is bounded. Therefore, there is some $\beta \geq 0$ such that

$$\liminf_{n\to\infty} p_b(x_n,x_{n+1}) = \beta. \tag{3.8}$$

Note that $\alpha \leq \beta$. First, we show that $\alpha = \beta$. Suppose that $\alpha = 0$. Then, $\lim_{n\to\infty} p_b(x_n,x_{n+1}) = 0$, and thus $\beta = 0$. Now, for $\alpha > 0$, we suppose that $\alpha \neq \beta$. Then $\beta - \alpha > 0$, and it follows from (3.6) and (3.8) that there is a positive integer n_0 such that

$$p_b(x_n,T(x_n)) < \alpha + \frac{\beta-\alpha}{4}, \quad \forall n \geq n_0, \tag{3.9}$$

and

$$\beta - \frac{\beta-\alpha}{4} < p_b(x_n,x_{n+1}), \quad \forall n \geq n_0. \tag{3.10}$$

Using (3.4), (3.9) and (3.10), we have

$$\begin{aligned}\beta - \frac{\beta-\alpha}{4} &< p_b(x_n,x_{n+1}) \\ &\leq (2-\varphi(p_b(x_n,x_{n+1})))\, p_b(x_n,T(x_n)) \\ &< (2-\varphi(p_b(x_n,x_{n+1})))\left[\alpha + \frac{\beta-\alpha}{4}\right].\end{aligned}$$

Thus, for all $n \geq n_0$, we have

$$(2-\varphi(p_b(x_n,x_{n+1}))) > \frac{3\beta+\alpha}{3\alpha+\beta},$$

that is,

$$1 + (1-\varphi(p_b(x_n,x_{n+1}))) > 1 + \frac{2(\beta-\alpha)}{3\alpha+\beta}.$$

It follows that

$$-(1-\varphi(p_b(x_n,x_{n+1})))^2 < -\left[\frac{2(\beta-\alpha)}{3\alpha+\beta}\right]^2.$$

Thus, for all $n \geq n_0$, we have

$$\begin{aligned}\psi(p_b(x_n,x_{n+1})) &= 1-(1-\varphi(p_b(x_n,x_{n+1})))^2 \\ &< 1-\left[\frac{2(\beta-\alpha)}{3\alpha+\beta}\right]^2.\end{aligned} \tag{3.11}$$

Take $\lambda = 1-\left[\frac{2(\beta-\alpha)}{3\alpha+\beta}\right]^2$. From (3.5) and (3.11), one has

$$p_b(x_{n+1},T(x_{n+1})) < \lambda\, p_b(x_n,T(x_n)), \quad \forall n \geq n_0. \tag{3.12}$$

Since $\beta > \alpha$, one has $\lambda < 1$. From (3.12), one has, for any $l \geq 1$,

$$p_b(x_{n_0+l},T(x_{n_0+l})) < \lambda^l p_b(x_{n_0},T(x_{n_0})). \tag{3.13}$$

As $\alpha > 0$ and $\lambda < 1$, there is a positive integer l_0 such that $\lambda^{l_0} p_b(x_{n_0},T(x_{n_0})) < \alpha$. Note that $\alpha \leq p_b(x_n,T(x_n))$ for each $n \geq 0$. Thus, it follows from (3.13) that

$$\alpha \leq p_b(x_{n_0+l_0},T(x_{n_0+l_0})) < \lambda^{l_0} p_b(x_{n_0},T(x_{n_0})) < \alpha,$$

which is a contradiction. Hence $\alpha = \beta$. Now we show that $\alpha = 0$. Suppose $\alpha > 0$. Since

$$\beta = \alpha \leq p_b(x_n,T(x_n)) \leq p_b(x_n,x_{n+1}),$$

then from (3.8) we can find $\liminf_{n\to\infty} p_b(x_n,x_{n+1}) = \beta^+$, where β^+ means $\beta > 0$. So, there exists a subsequence $\{p_b(x_{n_l},x_{n_l+1})\}$ of $\{p_b(x_n,x_{n+1})\}$ such that $\lim_{l\to\infty} p_b(x_{n_l},x_{n_l+1}) = \beta^+$. Now from (3.3) we have

$$\limsup_{p_b(x_{n_l},x_{n_l+1})\to\beta^+} \psi(p_b(x_{n_l},x_{n_l+1})) < 1, \tag{3.14}$$

and from (3.5), we have

$$p_b(x_{n_l+1},T(x_{n_l+1})) \leq \psi(p_b(x_{n_l},x_{n_l+1}))p_b(x_{n_l},T(x_{n_l})).$$

Letting $l \to \infty$ and using (3.6), we arrive at

$$\begin{aligned}\alpha &= \limsup_{l\to\infty} p_b(x_{n_l+1},T(x_{n_l+1})) \\ &\leq \left(\limsup_{l\to\infty}\left(\psi(p_b(x_{n_l},x_{n_l+1}))\right)\right)\left(\limsup_{l\to\infty}\left(p_b(x_{n_l},T(x_{n_l}))\right)\right) \\ &= \left(\limsup_{p_b(x_{n_l},x_{n_l+1})\to\beta+} \psi(p_b(x_{n_l},x_{n_l+1}))\right)\alpha.\end{aligned}$$

Since $\alpha > 0$, then it follows from the last inequality that

$$1 \leq \limsup_{p_b(x_{n_l},x_{n_l+1})\to\beta^+} \psi(p_b(x_{n_l},x_{n_l+1})),$$

which contradicts (3.14). Hence, $\alpha = 0$. In view of (3.6) and (3.7), we obtain

$$\lim_{n\to\infty} p_b(x_n,T(x_n)) = 0, \tag{3.15}$$

and thus $\lim_{n\to\infty} p_b(x_n,x_{n+1}) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Let

$$\delta = \limsup_{p_b(x_{n_l},x_{n_l+1})\to 0} \psi(p_b(x_{n_l},x_{n_l+1})).$$

From (3.3), one has $\delta < 1$. We choose $k \in (0,b^{-1})$ such that $\delta < k < 1$. Then there exists some $n_0 \in \mathbb{N}$ such that $\psi(p_b(x_n,x_{n+1})) < k$, $\forall n \geq n_0$. From (3.5), we have

$$p_b(x_{n+1},T(x_{n+1})) \leq k p_b(x_n,T(x_n)), \quad \forall n \geq n_0.$$

By induction, we obtain

$$p_b(x_{n+1},T(x_{n+1})) \leq k^{n+1-n_0} p_b(x_{n_0},T(x_{n_0})), \quad \forall n \geq n_0. \tag{3.16}$$

A combination of (3.7) and (3.16) yields

$$p_b(x_n,x_{n+1}) \leq 2k^{n-n_0} p_b(x_{n_0},T(x_{n_0})), \quad \forall n \geq n_0. \tag{3.17}$$

Since p_b is the w_b-distance, for any $n,m \in \mathbb{N}, m > n$, we have

$$\begin{aligned} p_b(x_n,x_m) &\leq b[p_b(x_n,x_{n+1}) + p_b(x_{n+1},x_m)] \\ &\leq b p_b(x_n,x_{n+1}) + b(b[p_b(x_{n+1},x_{n+2}) + p_b(x_{n+2},x_m)]) \\ &= b p_b(x_n,x_{n+1}) + b^2 p_b(x_{n+1},x_{n+2}) + b^2 p_b(x_{n+2},x_m) \\ &\leq b p_b(x_n,x_{n+1}) + b^2 p_b(x_{n+1},x_{n+2}) \\ &\quad + b^2(b[p_b(x_{n+2},x_{n+3}) + p_b(x_{n+3},x_m)]) \\ &= b p_b(x_n,x_{n+1}) + b^2 p_b(x_{n+1},x_{n+2}) \\ &\quad + b^3(p_b(x_{n+2},x_{n+3}) + p_b(x_{n+3},x_m)) \\ &\ \ \vdots \\ &\leq b p_b(x_n,x_{n+1}) + b^2 p_b(x_{n+1},x_{n+2}) + \ldots \\ &\quad + b^{m-n-1}(p_b(x_{m-2},x_{m-1}) + p_b(x_{m-1},x_m)). \end{aligned} \tag{3.18}$$

Using (3.17) and (3.18) we obtain

$$\begin{aligned} p_b(x_n,x_m) &\leq 2bk^{n-n_0} p_b(x_{n_0},T(x_{n_0})) + 2b^2 k^{n-n_0+1} p_b(x_{n_0},T(x_{n_0})) + \ldots \\ &\quad + 2b^{m-n-1} k^{m-n_0-2} p_b(x_{n_0},T(x_{n_0})) + 2b^{m-n-1} k^{m-n_0-1} p_b(x_{n_0},T(x_{n_0})) \\ &= 2bk^{n-n_0}\left(1 + bk + (bk)^2 + \ldots + (bk)^{m-n-2} + b^{m-n-2}k^{m-n-1}\right) p_b(x_{n_0},T(x_{n_0})). \end{aligned}$$

As $bk < 1$, we obtain, for all $m,n \in \mathbb{N}$ with $m > n \geq n_0$,

$$p_b(x_n,x_m) \leq \frac{2bk^{n-n_0}}{1-bk} p_b(x_{n_0},T(x_{n_0})).$$

Since $\frac{2bk^{n-n_0}}{1-bk} \to 0$ as $n \to +\infty$, we conclude from Lemma 2.2 (iii) that $\{x_n\}$ is a Cauchy sequence in X. Due to the completeness of X, there exists some $u_0 \in X$ such that $\{x_n\}$ converges to u_0. Now, by using the b-lower semi-continuity of function g and (3.15), we obtain

$$0 \leq g(u_0) \leq \liminf_{n\to\infty} bg(x_n) = 0,$$

and hence, $g(u_0) = p_b(u_0,T(u_0)) = 0$. If $p_b(u_0,u_0) = 0$, then it follows from Lemma 2.3 that $u_0 \in T(u_0)$ as $T(u_0)$ is closed. □

Remark 3.1. (i) Theorem 3.1 generalizes Theorem 2.5 ([30, Theorem 5]) and the fixed point results [10, Theorem 5]. Indeed, if $b=1$ and $p_b=d$ in Theorem 3.1, then we obtain Theorem 2.5 immediately.

(ii) Theorem 3.1 includes the fixed point result [31, Theorem 2.1] as a special case when $b=1$.

Replacing the b-lower semi-continuity of the real-valued function g of Theorem 3.1 with another suitable condition, we obtain the following fixed point result, which generalizes [31, Theorem 2.2].

Theorem 3.2. *Suppose that all the hypotheses of Theorem 3.1 hold except the b-lower semi-continuity of g. Assume that* $\inf\{p_b(x,u)+p_b(x,T(x)) : x\in X\}>0$, *for every* $u\in X$ *with* $u\notin T(u)$. *Then* $Fix(T)\neq\emptyset$.

Proof. Following the proof of Theorem 3.1, there exists an orbit $\{x_n\}$ of T, which is a Cauchy sequence in X. Due to the completeness of X, there exists $u_0\in X$ such that $\{x_n\}$ converges to u_0. Since $p_b(x_n,\cdot)$ is b-lower semi-continuous and $\lim_{m\to\infty}x_m=u_0\in X$, it follows from the proof of Theorem 3.1 that, for all $n\geq n_0$,

$$p_b(x_n,u_0)\leq \liminf_{m\to\infty} b\,p_b(x_n,x_m)\leq \frac{2b^2k^{n-n_0}}{1-bk}p_b(x_{n_0},T(x_{n_0})).$$

On the other hand, we have

$$p_b(x_n,T(x_n))\leq p_b(x_n,x_{n+1})\leq 2k^{n-n_0}p_b(x_{n_0},T(x_{n_0})).$$

Letting $u_0\notin T(u_0)$, we have

$$\begin{aligned}
0 &< \inf\{p_b(x,u_0)+p_b(x,T(x)) : x\in X\}\\
&\leq \inf\{p_b(x_n,u_0)+p_b(x_n,T(x_n)) : n\geq n_0\}\\
&\leq \inf\left\{\frac{2b^2k^{n-n_o}}{1-bk}p_b(x_{n_0},T(x_{n_0}))+2k^{n-n_0}p_b(x_{n_0},T(x_{n_0})) : n\geq n_0\right\}\\
&= \frac{2k^{-n_0}(b^2-bk+1)}{1-bk}p_b(x_{n_0},T(x_{n_0}))\inf\{k^n : n\geq n_0\}=0,
\end{aligned}$$

which is impossible. Hence, $u_0\in Fix(T)$. □

Now, we present a general result on the existence of fixed points for multi-valued mappings, which improve/generalize a number of known fixed point results.

Theorem 3.3. *Let* (X,D) *be a complete metric type space with a* w_b*-distance* p_b. *Let* $T : X\to Cl(X)$ *be a multi-valued mapping such that the real-valued function g on X defined by* $g(x)=p_b(x,T(x))$ *is b-lower semi-continuous. Assume that the following conditions hold:*

(i) *there are functions* $\varphi : [0,+\infty)\to(0,1)$ *and* $\mu : [0,+\infty)\to[c,1)$, *with* $c>0$. *Here* μ *is nondecreasing such that* $\varphi(t)<\mu(t)$ *and* $\limsup_{r\to t^+}\varphi(r)<\limsup_{r\to t^+}\mu(r)$, *for every* $t\in[0,+\infty)$;

(ii) *for any* $x\in X$, *there exists* $y\in T(x)$ *satisfying*

$$\mu\left(p_b(x,y)\right)p_b(x,y)\leq g(x)$$

and

$$g(y)\leq\varphi\left(p_b(x,y)\right)p_b(x,y).$$

Then there exists $u_0 \in X$ *such that* $g(u_0) = 0$*. Further, if* $p_b(u_0, u_0) = 0$*, then* $u_0 \in T(u_0)$*.*

Proof. Let $x_0 \in X$ be an arbitrary but fixed element of X. Then there exists $x_1 \in T(x_0)$ such that

$$\mu\left(p_b(x_0, x_1)\right) p_b(x_0, x_1) \leq p_b(x_0, T(x_0))$$

and

$$p_b(x_1, T(x_1)) \leq \varphi\left(p_b(x_0, x_1)\right) p_b(x_0, x_1).$$

It follows that

$$p_b(x_1, T(x_1)) \leq \frac{\varphi\left(p_b(x_0, x_1)\right)}{\mu\left(p_b(x_0, x_1)\right)} p_b(x_0, T(x_0)). \tag{3.19}$$

Define a function $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \frac{\varphi(t)}{\mu(t)}, \quad \forall t \in [0, +\infty).$$

Since, for each $t \in [0, +\infty)$, $\varphi(t) < \mu(t)$, we have $\psi(t) < 1$ and

$$\limsup_{r \to t^+} \psi(r) < 1, \quad \forall t \in [0, +\infty). \tag{3.20}$$

From (3.19), we have

$$p_b(x_1, T(x_1)) \leq \psi\left(p_b(x_0, x_1)\right) p_b(x_0, T(x_0)).$$

Similarly, for $x_1 \in X$, there exists $x_2 \in T(x_1)$ such that

$$\mu\left(p_b(x_1, x_2)\right) p_b(x_1, x_2) \leq p_b(x_1, T(x_1))$$

and

$$p_b(x_2, T(x_2)) \leq \varphi\left(p_b(x_1, x_2)\right) p_b(x_1, x_2).$$

Thus

$$p_b(x_2, T(x_2)) \leq \psi\left(p_b(x_1, x_2)\right) p_b(x_1, T(x_1)).$$

Continuing this process, we can obtain an orbit $\{x_n\}$ of T in X satisfying

$$\mu\left(p_b(x_n, x_{n+1})\right) p_b(x_n, x_{n+1}) \leq p_b(x_n, T(x_n)), \tag{3.21}$$

and

$$p_b(x_{n+1}, T(x_{n+1})) \leq \varphi\left(p_b(x_n, x_{n+1})\right) p_b(x_n, x_{n+1}). \tag{3.22}$$

Hence, we have

$$p_b(x_{n+1}, T(x_{n+1})) \leq \psi\left(p_b(x_n, x_{n+1})\right) p_b(x_n, T(x_n)), \tag{3.23}$$

for all $n \geq 0$. Since $\psi(t) < 1$ for each $t \in [0, +\infty)$, we have, for all $n \geq 0$

$$p_b(x_{n+1}, T(x_{n+1})) < p_b(x_n, T(x_n)).$$

Thus, $\{p_b(x_n, T(x_n))\}$ is strictly decreasing and bounded below, thus it is convergent. Therefore, there is some $\alpha \geq 0$ such that $\lim_{n\to\infty} p_b(x_n, T(x_n)) = \alpha$. Now, we need to show that $\{p_b(x_n, x_{n+1})\}$ is also strictly decreasing. Assume that $p_b(x_n, x_{n+1}) \leq p_b(x_{n+1}, x_{n+2})$. Since $\mu(t)$ is nondecreasing, we have

$$\mu\left(p_b(x_n, x_{n+1})\right) \leq \mu\left(p_b(x_{n+1}, x_{n+2})\right). \tag{3.24}$$

Using (3.21), (3.22) and (3.24) with $n = n+1$, we arrive at

$$\begin{aligned} p_b(x_{n+1},x_{n+2}) &\leq \frac{\varphi\left(p_b(x_n,x_{n+1})\right)}{\mu\left(p_b(x_{n+1},x_{n+2})\right)} p_b(x_n,x_{n+1}) \\ &\leq \frac{\varphi\left(p_b(x_n,x_{n+1})\right)}{\mu\left(p_b(x_n,x_{n+1})\right)} p_b(x_n,x_{n+1}) \\ &= \psi\left(p_b(x_n,x_{n+1})\right) p_b(x_n,x_{n+1}) \\ &< p_b(x_n,x_{n+1}), \end{aligned}$$

which is a contradiction. Hence, $\{p_b(x_n,x_{n+1})\}$ is strictly decreasing. Therefore, there exists some $\beta \geq 0$ such that $\lim_{n\to\infty} p_b(x_n,x_{n+1}) = \beta^+$. In view of (3.23), we obtain

$$\begin{aligned} \alpha &\leq \left(\limsup_{n\to\infty} \psi(p_b(x_n,x_{n+1}))\right)\alpha \\ &= \left(\limsup_{p_b(x_n,x_{n+1})\to\beta^+} \psi(p_b(x_n,x_{n+1}))\right)\alpha. \end{aligned}$$

By using (3.20), we conclude that $\alpha = 0$. In view of $\mu \geq c$, we conclude from (3.21) that

$$p_b(x_n,x_{n+1}) \leq \frac{1}{c} p_b(x_n,T(x_n)), \quad \forall n \geq n_0. \tag{3.25}$$

Since $\{p_b(x_n,T(x_n))\} \to 0$, we have $\{p_b(x_n,x_{n+1})\} \to 0$. Now, we let

$$\delta = \limsup_{n\to\infty} \psi(p_b(x_n,x_{n+1})).$$

From (3.20), we have $\delta < 1$. Choose $k \in (0,b^{-1})$ such that $\delta < k < 1$. Then there exists some $n_0 \in \mathbb{N}$ such that

$$\psi(p_b(x_n,x_{n+1})) < k, \quad \forall n \geq n_0.$$

It follows from (3.23),

$$p_b(x_{n+1},T(x_{n+1})) \leq k p_b(x_n,T(x_n)), \quad \forall n \geq n_0.$$

By induction, we have

$$p_b(x_{n+1},T(x_{n+1})) \leq k^{n+1-n_0} p_b(x_{n_0},T(x_{n_0})), \quad \forall n \geq n_0. \tag{3.26}$$

Using (3.25) and (3.26), we have

$$p_b(x_n,x_{n+1}) \leq \frac{1}{c} p_b(x_n,T(x_n)) \leq \frac{1}{c} k^{n-n_0} p_b(x_{n_0},T(x_{n_0})), \quad \forall n \geq n_0. \tag{3.27}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. From (3.27), for all $m > n \geq n_0$, we have

$$\begin{aligned}
p_b(x_n,x_m) &\leq b\,[p_b(x_n,x_{n+1})+p_b(x_{n+1},x_m)]\\
&\leq bp_b(x_n,x_{n+1})+b\,(b\,[p_b(x_{n+1},x_{n+2})+p_b(x_{n+2},x_m)])\\
&= bp_b(x_n,x_{n+1})+b^2p_b(x_{n+1},x_{n+2})+b^2p_b(x_{n+2},x_m)\\
&\leq bp_b(x_n,x_{n+1})+b^2p_b(x_{n+1},x_{n+2})+b^2\,(b\,[p_b(x_{n+2},x_{n+3})+p_b(x_{n+3},x_m)])\\
&= bp_b(x_n,x_{n+1})+b^2p_b(x_{n+1},x_{n+2})+b^3\,(p_b(x_{n+2},x_{n+3})+p_b(x_{n+3},x_m))\\
&\vdots\\
&\leq bp_b(x_n,x_{n+1})+b^2p_b(x_{n+1},x_{n+2})+\ldots\\
&\quad +b^{m-n-1}\,(p_b(x_{m-2},x_{m-1})+p_b(x_{m-1},x_m))\\
&\leq \frac{1}{c}bk^{n-n_0}p_b(x_{n_0},T(x_{n_0}))+\frac{1}{c}b^2k^{n-n_0+1}p_b(x_{n_0},T(x_{n_0}))+\ldots\\
&\quad +\frac{1}{c}b^{m-n-1}k^{m-n_0-2}p_b(x_{n_0},T(x_{n_0}))+\frac{1}{c}b^{m-n-1}k^{m-n_0-1}p_b(x_{n_0},T(x_{n_0}))\\
&= \frac{1}{c}bk^{n-n_0}\left(1+bk+(bk)^2+\ldots+(bk)^{m-n-2}+b^{m-n-2}k^{m-n-1}\right)p_b(x_{n_0},T(x_{n_0}))\\
&\leq \frac{1}{c}bk^{n-n_0}(1+bk+(bk)^2+\ldots)p_b(x_{n_0},T(x_{n_0})).
\end{aligned}$$

Since $bk<1$, for $m,n\in\mathbb{N}$ with $m>n\geq n_0$, we obtain

$$p_b(x_n,x_m)\leq\frac{1}{c}\left(\frac{bk^{n-n_0}}{1-bk}\right)p_b(x_{n_0},T(x_{n_0})).$$

Since $\frac{1}{c}\left(\frac{bk^{n-n_0}}{1-bk}\right)\to 0$ as $n\to+\infty$, we conclude from Lemma 2.2 (iii) that $\{x_n\}$ is a Cauchy sequence. Thanks to the completeness of X, there exists some $u_0\in X$ such that $\{x_n\}$ converges to u_0. Since g is b-lower semi-continuous, we have

$$0\leq g(u_0)\leq\liminf_{n\to\infty}bg(x_n)=0,$$

and thus, $g(u_0)=p_b(u_0,T(u_0))=0$. Now, if $p_b(u_0,u_0)=0$, then it follows from Lemma 2.3 that $u_0\in T(u_0)$ as $T(u_0)$ is closed. □

Remark 3.2. (i) If we consider $b=1$ and the w_b-distance function p_b equals to the distance d in Theorem 3.3, then it reduces to [30, Theorem 6] and thus it also generalizes Theorem 2.2, [10, Theorem 5].

(ii) Theorem 3.3 includes Theorem 2.4, [15, Theorem 2.1], as a special case. Indeed, if we consider a constant mapping $\mu(t)=a$ in Theorem 3.3, for all $t\in[0,\infty)$ and $a\in(0,1)$, $b=1$, and the w_b-distance function $p_b=d$, then we obtain [15, Theorem 2.1].

(iii) If we take $b=1$ in Theorem 3.3, then we have a fixed point result [31, Theorem 2.3].

The conclusion of Theorem 3.3 is still valid if we replace the b-lower semi-continuity of real-valued function g with another suitable assumption. Consequently, the following result includes [31, Theorem 2.4] as a special case.

Theorem 3.4. *Suppose that all the hypotheses of Theorem 3.3 hold except the b-lower semi-continuity of the function g. Assume that* $\inf\{p_b(x,u)+p_b(x,T(x)):x\in X\}>0$, *for every* $u\in X$ *with* $u\notin T(u)$. *Then* $Fix(T)\neq\emptyset$.

Finally, we present a fixed point result, which is a generalization of the results [15, Theorem 2.2], [30, Theorem 7], and [31, Theorem 2.5].

Theorem 3.5. *Let (X,D) be a complete metric type space with a w_b-distance p_b. Let $T: X \to P(X)$ be a multi-valued mapping such that a real-valued function g on X defined by $g(x) = p_b(x,T(x))$ is b-lower semi-continuous. Assume that the following conditions hold:*

(i) there is a function $\varphi: [0,+\infty) \to [0,1)$ such that, for each $t \in [0,+\infty)$, $\limsup_{r\to t^+} \varphi(r) < 1$;

(ii) for any $x \in X$, there is $y \in T(x)$ satisfying $p_b(x,y) = g(x)$ and $g(y) \leq \varphi(p_b(x,y))\, p_b(x,y)$.

Then there exists $u_0 \in X$ such that $g(u_0) = 0$. Further, if $p_b(u_0,u_0) = 0$, then $u_0 \in T(u_0)$.

Proof. Let $x_0 \in X$ be an arbitrary but fixed element of X. From (ii), we can choose $x_1 \in T(x_0)$ such that $p_b(x_0,x_1) = p_b(x_0,T(x_0))$, $p_b(x_1,T(x_1)) \leq \varphi(p_b(x_0,x_1))\, p_b(x_0,x_1)$, and $\varphi(p_b(x_0,x_1)) < 1$. Continuing this process, we obtain a sequence $\{x_n\}$ such that $x_{n+1} \in T(x_n)$ and $p_b(x_n,x_{n+1}) = p_b(x_n,T(x_n))$, $p_b(x_{n+1},T(x_{n+1})) \leq \varphi(p_b(x_n,x_{n+1}))\, p_b(x_n,x_{n+1})$, and $\varphi(p_b(x_n,x_{n+1})) < 1$. Following the method in the proof of Lemma 2.1 [33], one can easily show that $\{x_n\}$ is a Cauchy sequence in X. Thanks to the completeness of X, there exists $u_0 \in X$ such that $\{x_n\}$ converges to u_0. Since g is b-lower semi-continuous, we have $0 \leq g(u_0) \leq \liminf_{n\to\infty} bg(x_n) = 0$, and thus, $g(u_0) = p_b(u_0,T(u_0)) = 0$. Since $p_b(u_0,u_0) = 0$, and $T(u_0)$ is closed, it follows from Lemma 2.3 that $u_0 \in T(u_0)$. □

Remark 3.3. (i) Theorem 3.5 generalizes the fixed point result due to Ćirić[30, Theorem 7] and thus also generalizes [15, Theorem 2.2]. Indeed, if $b = 1$ and $p_b = d$ in Theorem 3.5, then it reduces to [30, Theorem 7].

(ii) If $b = 1$ in Theorem 3.5, then we obtain [31, Theorem 2.5].

4. Examples

The following example shows that Theorem 3.1 is a real generalization of Ćirić [30, Theorem 5] and Latif and Abdou [31, Theorem 2.1].

Example 4.1. Let $X = [0,1]$. Define $D(x,y) = (x-y)^2$ for all $x,y \in X$. Then X is a metric type space with $b = 2$. Define a w_b-distance function on X by $p_b(x,y) = y^2$, for all $x,y \in X$. Let $T: X \to Cl(X)$ be defined by

$$T(x) = \begin{cases} \left\{\frac{1}{2}x^2\right\}, & x \in \left[0,\frac{15}{32}\right) \cup \left(\frac{15}{32},1\right], \\ \left\{\frac{17}{96},\frac{1}{4}\right\}, & x = \frac{15}{32}, \end{cases}$$

and define a function $\varphi: [0,\infty) \to [0,1)$ by

$$\varphi(t) = \begin{cases} \frac{3}{4}t, & t \in \left[0,\frac{1}{2}\right), \\ \frac{3}{8}, & t \in \left[\frac{1}{2},\infty\right). \end{cases}$$

Then,

$$g(x) = p_b(x,T(x)) = \begin{cases} \frac{1}{4}x^4, & x \in \left[0,\frac{15}{32}\right) \cup \left(\frac{15}{32},1\right], \\ \left(\frac{17}{96}\right)^2, & x = \frac{15}{32}, \end{cases}$$

and g is b-lower semi-continuous. For each $x \in [0, \frac{15}{32}) \cup (\frac{15}{32}, 1]$, there exists $y \in T(x) = \left\{\frac{1}{2}x^2\right\}$ such that $p_b(x,y) = p_b(x,T(x)) = \frac{1}{4}x^4$. Thus, for each $x \in [0,1], x \neq \frac{15}{32}$, we have

$$p_b(x,y) = \frac{1}{4}x^4 \leq (2-\varphi(p_b(x,y)))\frac{1}{4}x^4 = (2-\varphi(p_b(x,y)))\, p_b(x,T(x))$$

and

$$\begin{aligned} p_b(y,T(y)) &= p_b\left(\frac{1}{2}x^2, \frac{1}{2}\left(\frac{1}{2}x^2\right)^2\right) = \left(\frac{1}{8}x^4\right)^2 = \frac{1}{16}x^4 p_b(x,y) \\ &\leq \frac{3}{16}x^4 p_b(x,y) = \varphi(p_b(x,y))\, p_b(x,y). \end{aligned}$$

Letting $x = \frac{15}{32}$, we have $T(x) = \{\frac{17}{96}, \frac{1}{4}\}$. Clearly, there exists $y = \frac{17}{96} \in T(x)$ such that

$$p_b(x,y) = \left(\frac{17}{96}\right)^2 \leq \left(2 - \frac{3}{4}\left(\frac{17}{96}\right)^2\right)\left(\frac{17}{96}\right)^2 = (2-\varphi(p_b(x,y)))\, p_b(x,T(x))$$

and

$$\begin{aligned} p_b(y,T(y)) &= p_b\left(\frac{17}{96}, \frac{1}{2}\left(\frac{17}{96}\right)^2\right) = \frac{1}{4}\left(\frac{17}{96}\right)^4 \\ &= \frac{1}{4}\left(\frac{17}{96}\right)^2 p_b(x,y) \\ &\leq \frac{3}{4}\left(\frac{17}{96}\right)^2 p_b(x,y) = \varphi(p_b(x,y))p_b(x,y). \end{aligned}$$

Thus, for each $x \in [0,1]$, all the conditions of Theorem 3.1 are satisfied. Hence, it follows that $Fix(T) \neq \emptyset$. Note that $Fix(T) = \{0\}$. Clearly, the w_b-distance p_b is not a metric d, so T does not satisfy the hypotheses of Theorem 2.5 [30, Theorem 5]. In addition, the w_b-distance p_b is not a w-distance p, so T does not satisfy the hypotheses of [31, Theorem 2.1].

Finally, we present an example, which shows that Theorem 3.5 is a real generalization of Klim-Wardowski [15, Theorem 2.2], Ćirić [30, Theorem 7], and Latif and Abdou [31, Theorem 2.5].

Example 4.2. For $b = 2$, consider the metric type space $X = [0,\infty)$ with $D(x,y) = (x-y)^2$ for all $x,y \in X$. Define a w_b-distance function on X by $p_b(x,y) = x^2 + y^2$, for all $x,y \in X$. Now, for any real number $a > 1$, define $T : X \to Cl(X)$ by

$$T(x) = \left\{\frac{x}{a}\right\} \cup [(1+2x), \infty), \quad \forall x \in [0,\infty),$$

and define a constant function $\varphi : [0,\infty) \to [0,1)$ by $\varphi(t) = \frac{1}{a^2}$, for all $t \in [0,\infty)$. Note that $\varphi(t) < 1$ for all $t \in [0,\infty)$. For each $x \in X$, we have

$$g(x) = p_b(x,T(x)) = x^2 + \left(\frac{x}{a}\right)^2 = \left(\frac{a^2+1}{a^2}\right)x^2.$$

Further, for each $x \in X$, there exists $y = \frac{x}{a} \in T(x)$ such that

$$p_b(x,y) = p_b\left(x, \frac{x}{a}\right) = p_b(x,T(x)),$$

and

$$p_b(y,T(y)) = p_b(\frac{x}{a},\frac{x}{a^2}) = \frac{x^2}{a^2}+\frac{x^2}{a^4} = \frac{1}{a^2}\left(\frac{a^2+1}{a^2}\right)x^2 = \varphi(p_b(x,y))p_b(x,y).$$

Therefore, all the assumptions of Theorem 3.5 are satisfied. We conclude $Fix(T)=\{0\}$. Note that $T(x)$ is not a compact for all $x\in X$, and the w_b-distance p_b is not a metric d (even not a w-distance p on X). Consequently [15, Theorem 2.2], [30, Theorem 7], and [31, Theorem 2.5] are not applicable.

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