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# A NOTE ON THE CARLESON MEASURE FOR DIRICHLET TYPE SPACES ON THE UNIT BALL OF $\mathbb{C}^n$

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**Abstract.** In this paper, we characterize the $q$-Carleson measure for Dirichlet type spaces $\mathscr{D}^p_\alpha$ on the unit ball when $0 < p < q < \infty$ and $\alpha > -1$ by using Carleson blocks and Carleson tubes.

**Keywords.** Carleson measure; Dirichlet type spaces; Lebesgue spaces.

## 1. Introduction

Let $\mathbb{B}$ be a open unit ball of $\mathbb{C}^n$, and let $\mathbb{S}$ be the boundary of $\mathbb{B}$. If $n = 1$, then $\mathbb{B}$ is the open unit disk in complex plane $\mathbb{C}$ and is always denoted by $\mathbb{D}$ in this paper. Let $H(\mathbb{B})$ denote the space of all holomorphic functions on $\mathbb{B}$. For any two points $z = (z_1, z_2, \cdots, z_n)$ and $w = (w_1, w_2, \cdots, w_n)$ in $\mathbb{C}^n$, we define $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ and

$$|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

Let $d\sigma$ and $dV$ be the normalized surface and volume measures on $\mathbb{S}$ and $\mathbb{B}$, respectively. For $-1 < \alpha < \infty$ and $0 < p < \infty$, the weighted Bergman space $A^p_\alpha(\mathbb{B})$( or $A^p_\alpha$) consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{A^p_\alpha} = \int_{\mathbb{B}} |f(z)|^p dV_\alpha(z) = \int_{\mathbb{B}} |f(z)|^p (1 - |z|^2)^\alpha dV(z) < \infty. \tag{1.1}$$

Obviously, we have

$$\|f\|_{A^p_\alpha} = 2n \int_0^1 r^{2n-1} M_p^p(r, f)(1 - r^2)^\alpha dr,$$

where

$$M_p(r, f) = \left(\int_{\mathbb{S}} |f(r\xi)|^p d\sigma(\xi)\right)^{\frac{1}{p}}.$$

It is well known that $f \in A^p_\alpha$ if and only if $\Re f \in A^p_{\alpha+p}$; see, e.g., [1, 2]. Here, $\Re f$ is the radial derivative of $f \in H(\mathbb{B})$, that is,

$$\Re f(z) = \sum_{k=1}^n z_k \frac{\partial f}{\partial z_k}(z).$$

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Let $0 < p < \infty$ and $\alpha > -1$. The Dirichlet type space $\mathscr{D}_\alpha^p$ on the unit ball is the space consisting of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{\mathscr{D}_\alpha^p}^p = \int_{\mathbb{B}} |\Re f(z)|^p (1-|z|^2)^\alpha dV(z) < \infty.$$

When $p \geq 1$, $\|\cdot\|_{\mathscr{D}_\alpha^p}$ is a semi-norm on $\mathscr{D}_\alpha^p$ and the norm on $\mathscr{D}_\alpha^p$ is often defined as $\|f\|_{\mathscr{D}_{\alpha,*}^p} = |f(0)| + \|f\|_{\mathscr{D}_\alpha^p}$. When $\alpha > p-1$, we see that rhe Dirichlet type space $\mathscr{D}_\alpha^p$ is just the weighted Bergman space $A_{\alpha-p}^p$. For more results about $\mathscr{D}_\alpha^p$ on the unit ball, we refer to [3, 4] and the references therein.

For $a \in \mathbb{B}$ and $r > 0$, let $D(a,r)$ denote the Bergman metric ball at $a$. Thus

$$D(a,r) = \left\{ z \in \mathbb{B} : \beta(a,z) = \frac{1}{2} \log \frac{1+|\varphi_a(z)|}{1-|\varphi_a(z)|} < r \right\},$$

where $\varphi_a$ is the involutive automorphism of $\mathbb{B}$ that interchanges 0 and $a$. $\beta(\cdot,\cdot)$ is called the Bergman metric on $\mathbb{B}$. If $\{a_j\}_{j=1}^\infty \subset \mathbb{B}$ satisfying

$$\inf_{i \neq j} \beta(a_i, a_j) \geq s > 0,$$

we say that $\{a_j\}_{j=1}^\infty \subset \mathbb{D}$ is $s$-separated or separated for simply. If $\{a_j\}_{j=1}^\infty \subset \mathbb{B}$ satisfies $\mathbb{B} = \cup_{j=1}^\infty D(a_j, r)$, we say that $\{a_j\}$ is $r$-covering.

Let $\xi \in \mathbb{S}$ and $0 < r < 1$. The Carleson tube $S_\xi^*(r)$ is defined by

$$S_\xi^*(r) = \{z \in \mathbb{B} : |1 - \langle z, \xi \rangle| < r^2\}.$$

Let $a \in \mathbb{B} \backslash \{0\}$. We define

$$S_a = \left\{ z \in \mathbb{B} : 1-|a| < |z| < 1, |1 - \langle \frac{z}{|z|}, \frac{a}{|a|} \rangle| < 1-|a| \right\}.$$

The set $S_a$ is called a Carleson block and introduced in [5]. From [5], we see that the Carleson block plays an important role when studying some holomorphic function spaces on the unit ball.

Suppose $0 < q < \infty$, $\mu$ is a positive Borel measure, and $X$ is a (quasi-) Bananch space. If the identity operator $Id : X \to L_\mu^q$ is bounded, we say that $\mu$ is a $q$-Carleson measure for $X$. By the closed graph theorem, $Id : X \to L_\mu^q$ is bounded if and only if $X \subset L_\mu^q$. As we know, $q$-Carleson measures for Bergman spaces $A_\alpha^p$ were characterized neatly and completely many years ago; see, e.g., [1, 6–8] and the references therein. Among others, we state the following theorem, which is a special case of [1, Theorem 50] and [5, Theorem 1] and will be useful in the proof of our main results in this paper.

**Theorem 1.1.** *Suppose $\mu$ is a positive Borel measure on $\mathbb{B}$, $0 < p \leq q < \infty$ and $\alpha > -1$. Then the following statements are equivalent.*

*(i) $A_\alpha^p \subset L_\mu^q$.*

*(ii) For each (or some) $s > 0$,*

$$\sup_{z \in \mathbb{B}} \int_{\mathbb{B}} \frac{(1-|z|^2)^s}{|1 - \langle z, w \rangle|^{s + \frac{q}{p}(n+1+\alpha)}} d\mu(w) < \infty.$$

*(iii) There is a constant $C > 0$ such that $\mu(S_\xi^*(r)) \leq C r^{\frac{2q}{p}(n+1+\alpha)}$ for all $r > 0$ and $\xi \in \mathbb{S}$.*

*(iv) There is a constant $C > 0$ such that $\mu(S_a) \leq C(1-|a|)^{\frac{q}{p}(n+1+\alpha)}$ for all $a \in \mathbb{B}$.*

(v) *For each (or some) $r > 0$, there exists $C > 0$ such that $\mu(D(a,r)) \leq C(1-|a|^2)^{\frac{q}{p}(n+1+\alpha)}$ for all $a \in \mathbb{B}$.*

However, it is very difficult to give a complete characterization of the $q$-Carleson measure for $\mathscr{D}^p_\alpha$. Some cases are still open. We refer to [3,4,9,10] for several characterizations of $q$-Carleson measure for $\mathscr{D}^p_\alpha$ when $1 < p \leq q < \infty$. But most of these characterizations are very complicated in the case of the unit ball. Fortunately, when $0 < p < q < \infty$, Girela and Peláez [10] described the $q$-Carelson measure for $\mathscr{D}^p_\alpha$ on the unit disk neatly and completely by using Carleson square. Motivated by [10], we characterize the $q$-Carelson measure for $\mathscr{D}^p_\alpha$ when $0 < p < q < \infty$ on the unit ball by using Carleson blocks and Carleson tubes. We state the main result in this paper as follows.

**Theorem 1.2.** *Suppose that $0 < p < q < \infty$, $\alpha > -1$, and $\mu$ is a positive Borel measure on $\mathbb{B}$. Then the following statements hold.*

(i) *When $p < \alpha+n+1$, $\mathscr{D}^p_\alpha \subset L^q_\mu$ if and only if*

$$\sup_{a\in\mathbb{B}} \frac{\mu(S_a)}{(1-|a|)^{\frac{q}{p}(\alpha+n+1-p)}} < \infty. \tag{1.2}$$

(ii) *When $p = \alpha+n+1$, if $1 < p < 2+\frac{1}{n-1}$, $\mathscr{D}^p_\alpha \subset L^q_\mu$ if and only if*

$$\mu(S_a) \lesssim \left(\log\frac{1}{1-|a|}\right)^{q\left(\frac{1}{p}-1\right)}; \tag{1.3}$$

*if $p \geq 2+\frac{1}{n-1}$, (1.3) implies $\mathscr{D}^p_\alpha \subset L^q_\mu$.*

(iii) *When $p > \alpha+n+1$, $\mathscr{D}^p_\alpha \subset L^q_\mu$ if and only if $\mu$ is finite.*

In this paper, constants, which are denoted by $C$, are positive and may differ from one occurrence to the next. We say that $A \lesssim B$ if there exists a constant $C$ such that $A \leq CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

## 2. Preliminaries

In this section, we state some definitions and some well-known results for the proof of main result in this paper. The following Forelli-Rudin's estimate is well known.

**Lemma 2.1.** *[2, Theorem 1.12] If $c > 0$ and $t > -1$, then*

$$\int_{\mathbb{B}} \frac{(1-|w|^2)^t}{|1-\langle z,w\rangle|^{n+1+t+c}} dV(w) \approx \frac{1}{(1-|z|^2)^c}.$$

As we know, $|1-\langle\cdot,\cdot\rangle|^{\frac{1}{2}}$ is the non-isotropic metric on the unit ball. For any $\xi \in \mathbb{S}$ and $0 < r < \sqrt{2}$, the non-isotropic ball $Q(\xi,r)$ is defined by

$$Q(\xi,r) = \{\eta \in \mathbb{S} : |1-\langle\xi,\eta\rangle| \leq r^2\}.$$

The following lemma can be obtained by some straightforward calculations; see, e.g., [11, Lemma 11].

**Lemma 2.2.** *For any $0 < r < 1$, there exist $\xi_{r,1}, \xi_{r,2}, \cdots, \xi_{r,N_r}$ in $\mathbb{S}$ such that*

(i) *$Q(\xi_{r,i},r) \cap Q(\xi_{r,j},r) = \emptyset$, if $1 \leq i < j \leq N_r$;*

*(ii)* $\mathbb{S} = \cup_{i=1}^{N_r} Q(\xi_{r,i}, 2r)$;
*(iii)* $N_r \approx r^{-2n}$.

*Then, we can choose* $Q_{r,j}(j = 1, 2, \cdots, N_r)$ *properly such that*

*(iv)* $Q(\xi_{r,j}, r) \subset Q_{r,j} \subset Q(\xi_{r,j}, 2r)$ *for all* $1 \le j \le N_r$;
*(v)* $\mathbb{S} = \cup_{j=1}^{N_r} Q_{r,j}$;
*(vi)* $Q_{r,i} \cap Q_{r,j} = \emptyset$ *when* $1 \le i < j \le N_r$.

Using Lemma 2.2, we can define a Bergman tree as follows. For $k = 1, 2, \cdots$, let

$$N_k = N_{\frac{1}{\sqrt{2^k}}},\ \xi_{k,j} = \xi_{\frac{1}{\sqrt{2^k}}, j},\ \text{and}\ Q_{k,j} = Q_{\frac{1}{\sqrt{2^k}}, j}.$$

Define

$$R_{k,j} = \left\{ z \in \mathbb{B} : 1 - \frac{1}{2^k} \le |z| < 1 - \frac{1}{2^{k+1}}, \frac{z}{|z|} \in Q_{k,j} \right\},$$

$Q_{0,1} = \mathbb{S}$, and $R_{0,1} = \frac{1}{2}\mathbb{B}$. Let

$$\Upsilon = \{R_{k,j} : k = 0, 1, 2, \cdots, j = 1, 2, \cdots, N_k\}.$$

Then, $\mathbb{B} = \cup_{k=0}^{\infty} \cup_{j=1}^{N_k} R_{k,j}$ and $N_k \approx 2^{nk}$.

Next, we define a tree structure on the collection $\Upsilon$ by declaring that $R_{k+1,j'}$ is a child of $R_{k,j}$, which will be written as $R_{k,j} \prec R_{k+1,j'}$ if $\xi_{k+1,j'} \in Q_{k,j}$ and $k \ge 1$. In the case $k = 0$, we declare that every $R_{1,j}$ is a child of the "root" $R_{0,1}$ and write as $R_{0,1} \prec R_{1,j}$.

By the definition of tree, if $k \ge 0$ and $s \ge 1$, we say that $R_{k,j_0} \prec R_{k+s,j_s}$ if there exist $R_{k+1,j_1}, \cdots, R_{k+s-1,j_{s-1}}$ such that

$$R_{k,j_0} \prec R_{k+1,j_1} \prec \cdots \prec R_{k+s-1,j_{s-1}} \prec R_{k+s,j_s}.$$

For convenience, let $R_{k,j} \prec R_{k,j}$. Obviously, $\prec$ is a partial order.

The following lemma describes the relationship between Bergman trees and Bergman metric balls, which can be found in, for example, [11].

**Lemma 2.3.** *The following statements hold.*

*(i) Let* $r > 0$ *be fixed. There exists* $N = N(r)$ *such that, for any* $z \in \mathbb{B}$, $D(z, r)$ *can be covered by a subsets of* $\{R_{k,j}\}$ *with no more than N elements.*

*(ii) For any given* $0 < s < r < 1$, *there exists* $M = M(s, r)$ *such that if* $\{a_j\}$ *is s-separated and r-covering, then each* $R_{k,j}$ *can be covered by a subset of* $\{D(a_j, r)\}$ *with no more than M elements.*

## 3. Proof of Theorem 1.2

*Proof. (i).* Suppose that $p < \alpha + n + 1$ and the identity operator $Id : \mathscr{D}_\alpha^p \to L_\mu^q$ is bounded. For $a \in \mathbb{B}$, set

$$f_a(z) = \frac{(1 - |a|^2)^s}{(1 - \langle z, a \rangle)^{\frac{n+1+\alpha}{p} - 1 + s}}$$

for some $s$ large enough. By Lemma 2.1,

$$\int_{\mathbb{B}} |\Re f_a(z)|^p (1 - |z|^2)^\alpha dV(z) \lesssim \int_{\mathbb{B}} \frac{(1 - |a|^2)^{ps} (1 - |z|^2)^\alpha}{|1 - \langle z, a \rangle|^{n+1+\alpha+ps}} dV(z) \approx 1. \tag{3.1}$$

Hence $f_a \in \mathscr{D}^p_\alpha$. For any $z \in S_a$, we see from [5] that $|1-\langle z,a\rangle| \approx 1-|a|$. Therefore, it follows from (3.1) that

$$\frac{\mu(S_a)}{(1-|a|)^{\frac{q(\alpha+n+1-p)}{p}}} \approx \int_{S_a} |f_a(z)|^q d\mu(z) \le \|f_a\|^q_{L^q_\mu} \lesssim \|f_a\|^q_{\mathscr{D}^p_\alpha} \lesssim 1,$$

which implies (1.2).

The proof of the sufficiency will be divided into four cases.

*Case (ia):* $0 < p < \alpha+1$.

Suppose that (1.2) holds. By Littlewood-Paley's formula, we have

$$|f(0)|^p + \|f\|^p_{\mathscr{D}^p_\alpha} \approx \int_{\mathbb{B}} |f(z)|^p (1-|z|^2)^{\alpha-p} dV(z) = \|f\|^p_{A^p_{\alpha-p}}.$$

Then, In view of Theorem 1.1, we obtain the desired result.

*Case (ib):* $p = \alpha+1$.

Suppose that (1.2) holds. We claim that

$$M_t(r,f) \lesssim M_s(\frac{r+1}{2},f)(1-r)^{\frac{n}{t}-\frac{n}{s}}, \tag{3.2}$$

for $0 < s < t < \infty$, $f \in H(\mathbb{B})$, and $r \in (\frac{1}{2},1)$.

In fact, for a constant $\delta > 0$ small enough, we have

$$M_t^t(r,f) \le M_s^s(r,f) \sup_{\xi\in\mathbb{S}} |f(r\xi)|^{t-s},$$

and

$$\begin{aligned}
\sup_{\xi\in\mathbb{S}} |f(r\xi)|^{t-s} &\lesssim \sup_{\xi\in\mathbb{S}} \left(\frac{1}{(1-r)^{n+1}} \int_{D(r\xi,\delta)} |f(z)|^s dV(z)\right)^{\frac{t-s}{s}} \\
&\le \left(\frac{1}{(1-r)^{n+1}} \int_{r-\frac{1-r}{2}\le|z|\le r+\frac{1-r}{2}} |f(z)|^s dV(z)\right)^{\frac{t-s}{s}} \\
&\lesssim M_s^{t-s}(\frac{1+r}{2},f)(1-r)^{\frac{n(s-t)}{s}},
\end{aligned}$$

which implies (3.2). Choose $t$ and $x$ such that

$$nt = (n+x+1)p, \ \ p < t < q, \ \ x > -1.$$

For any $f \in \mathscr{D}^p_{p-1}$, we have

$$M_p(r,\Re f) = o(\frac{1}{1-r}), \ r \to 1.$$

By (3.2), we have

$$\|\Re f\|^t_{A^t_{x+t}} \lesssim \int_0^1 M_p^p(\frac{r+1}{2},\Re f)(1-r)^{(p-t)(1+\frac{n}{p})+x+t} dr \lesssim \|f\|^p_{\mathscr{D}^p_{p-1}}.$$

Thus, $\mathscr{D}^p_{p-1} \subset A^t_x$. Since

$$\sup_{a\in\mathbb{B}} \frac{\mu(S_a)}{(1-|a|)^{\frac{(x+n+1)q}{t}}} = \sup_{a\in\mathbb{B}} \frac{\mu(S_a)}{(1-|a|)^{\frac{nq}{p}}} < \infty,$$

Theorem 1.1 implies $A^t_x \subset L^q_\mu$. So, $\mathscr{D}^p_{p-1} \subset L^q_\mu$.

*Case (ic):* $\alpha+1<p<\alpha+n+1$ *and* $p\leq 1$.

Suppose that (1.2) holds. By the proof of [2, Theorem 2.16], for any $g\in H(\mathbb{B})$ with $g(0)=0$, one has

$$|g(z)|^p \lesssim \int_{\mathbb{B}} \frac{|\Re g(w)|^p}{|1-\langle z,w\rangle|^{(n+\beta)p}}(1-|w|^2)^{\gamma}dV(w),$$

when $\beta>0$ is large enough, where $\gamma=(n+1+\beta)p-(n+1)>\alpha+p$. Minkowski's inequality implies

$$\begin{aligned}\|g\|_{L^q_\mu}^p &\lesssim \left(\int_{\mathbb{B}}\left(\int_{\mathbb{B}} \frac{|\Re g(w)|^p}{|1-\langle z,w\rangle|^{(n+\beta)p}}(1-|w|^2)^{\gamma}dV(w)\right)^{\frac{q}{p}}d\mu(z)\right)^{\frac{p}{q}}\\ &\leq \int_{\mathbb{B}}|\Re g(w)|^p(1-|w|^2)^{\gamma}\left(\int_{\mathbb{B}}\frac{1}{|1-\langle z,w\rangle|^{(n+\beta)q}}d\mu(z)\right)^{\frac{p}{q}}dV(w).\end{aligned}$$

By (1.2) and [1, Theorem 45], we have

$$\int_{\mathbb{B}}\frac{1}{|1-\langle z,w\rangle|^{(n+\beta)q}}d\mu(z)\lesssim \frac{1}{(1-|w|^2)^{(n+\beta)q-\frac{q}{p}(\alpha+n+1-p)}}.$$

So,

$$\|g\|_{L^q_\mu}^p \lesssim \int_{\mathbb{B}}|\Re g(w)|^p(1-|w|^2)^{\alpha}dV(w).$$

That is, $\mathscr{D}^p_\alpha\subset L^q_\mu$.

*Case (id):* $\alpha+1<p<\alpha+n+1$ *and* $p>1$.

Suppose that (1.2) holds. Let $\rho(R_{k,j})=2^{-k(\alpha+n+1-p)}$. We claim that, for any given $R_{k,j}$, there exist at most $N$ (independent of $R_{k,j}$) points $a_1,a_2,\cdots,a_N$ in $\mathbb{B}$ such that

$$|a_1|=\cdots=|a_N|=1-\frac{1}{2^k},\quad \bigcup_{R\in\Upsilon,R_{k,j}\prec R} R\subset \cup_{k=1}^N S(a_k). \tag{3.3}$$

As a fact, if $k,s\geq 0$ and $j_0=j$ for every $z\in R_{k+s,j_s}$ and $R_{k,j_0}\prec R_{k+s,j_s}$, then

$$\begin{aligned}|1-\langle \frac{z}{|z|},\xi_{k,j_0}\rangle|^{\frac12} &\leq |1-\langle \frac{z}{|z|},\xi_{k+s,j_s}\rangle|^{\frac12}+\sum_{i=1}^{s}|1-\langle \xi_{k+i,j_i},\xi_{k+i-1,j_{i-1}}\rangle|^{\frac12}\\ &\leq \frac{1}{\sqrt{2^{k+s}}}+\sum_{i=1}^{s}\frac{1}{\sqrt{2^{k+i-1}}}\leq \frac{\sqrt{2}}{\sqrt{2}-1}\frac{1}{\sqrt{2^k}}.\end{aligned}\tag{3.4}$$

So,

$$z\in\left\{\eta\in\mathbb{B}:1-\frac{1}{2^k}\leq|\eta|<1,\left|1-\left\langle\frac{\eta}{|\eta|},\xi_{k,j}\right\rangle\right|<\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right)^2\frac{1}{2^k}\right\}.$$

By [5, Proposition 1], we have that (3.3) holds. Therefore, if $k \geq 1$, then it follows from (1.2) that

$$\begin{aligned}
\sum_{R_{k,j} \prec R} \mu(R) \leq \sum_{i=1}^{N} \mu(S_{a_i}) &\lesssim 2^{-\frac{q}{p}(\alpha+n+1-p)k} \\
&\lesssim \left( \int_0^{1-\frac{1}{2^k}} (1-t)^{-\frac{\alpha+n+1-p}{p-1}-1} dt \right)^{-\frac{q(p-1)}{p}} \\
&= \left( \sum_{i=0}^{k-1} \int_{1-\frac{1}{2^i}}^{1-\frac{1}{2^{i+1}}} (1-t)^{-\frac{\alpha+n+1-p}{p-1}-1} dt \right)^{-\frac{q(p-1)}{p}} \\
&\lesssim \left( \sum_{R \prec R_{k,j}} \rho(R)^{-\frac{1}{p-1}} \right)^{-\frac{q(p-1)}{p}}. \qquad (3.5)
\end{aligned}$$

When $k = 0$, we see that (3.5) holds obviously.

For any $f \in \mathscr{D}_\alpha^p$ with $f(0) = 0$, let $z_{k,j}, w_{k,j} \in \overline{R_{k,j}}$ such that

$$|f(z_{k,j})| = \sup_{z \in R_{k,j}} |f(z)|, \ |\nabla f(w_{k,j})| = \sup_{z \in R_{k,j}} |\nabla f(z)|.$$

Here, $\nabla f(z) = (\frac{\partial f}{\partial z_1}(z), \cdots, \frac{\partial f}{\partial z_n}(z))$ is the gradient of $f$. For any $R_{k,j} (k \geq 1)$, let $R_{k-1,j'}$ be the father of it, that is, $R_{k,j}$ is a child of $R_{k-1,j'}$. Define

$$\mathscr{I} \phi_{k,j} = \sum_{R_{s,t} \prec R_{k,j}} \phi_{s,t},$$

in which, $\phi_{0,1} = |f(z_{0,1})|$ and

$$\phi_{k,j} = |f(z_{k,j})| - |f(z_{k-1,j'})|, \ \text{ when } \ k = 1, 2, \cdots.$$

Then, by (3.5) and Theorem 4 in [3], we have

$$\begin{aligned}
\left( \sum_{R_{k,j} \in \Upsilon} \mu(R_{k,j}) |f(z_{k,j})|^q \right)^{\frac{1}{q}} &= \left( \sum_{R_{k,j} \in \Upsilon} \mu(R_{k,j}) |\mathscr{I} \phi_{k,j}|^q \right)^{\frac{1}{q}} \\
&\lesssim \left( \sum_{R_{k,j} \in \Upsilon} \rho(R_{k,j}) |\phi_{k,j}|^p \right)^{\frac{1}{p}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|f\|_{L_\mu^q} &\leq \left( \sum_{R_{k,j} \in \Upsilon} \mu(R_{k,j}) |f(z_{k,j})|^q \right)^{\frac{1}{q}} \\
&\lesssim \left( \sum_{R_{k,j} \in \Upsilon} \rho(R_{k,j}) |f(z_{k,j}) - f(z_{k-1,j'})|^p \right)^{\frac{1}{p}}. \qquad (3.6)
\end{aligned}$$

Let $I_{k,j}$ be the shortest curve of those contained in $R_{k,j}\cup R_{k-1,j'}$ and connecting $z_{k,j}$ and $z_{k-1,j'}$. When $k\geq 2$, we have

$$\begin{aligned}|f(z_{k,j})-f(z_{k-1,j'})| &\lesssim (|\nabla f(w_{k,j})|+|\nabla f(w_{k-1,j'})|)(\mathrm{diam}(R_{k,j})+\mathrm{diam}(R_{k-1,j'}))\\ &\lesssim \frac{1}{2^k}(|\nabla f(w_{k,j})|+|\nabla f(w_{k-1,j'})|).\end{aligned}$$

Here, $\mathrm{diam}(R_{k,j})$ denotes the diameter of $R_{k,j}$. Then $\rho(R_{k,j})\approx\rho(R_{k-1,j'})$ implies

$$\sum_{k\geq 2,R_{k,j}\in\Upsilon}\rho(R_{k,j})|f(z_{k,j})-f(z_{k-1,j'})|^p\lesssim\sum_{R_{k,j}\in\Upsilon}\frac{\rho(R_{k,j})}{2^{kp}}|\nabla f(w_{k,j})|^p. \tag{3.7}$$

By Lemma 2.3 and the subharmonicity of $\frac{\partial f}{\partial z_j}(j=1,2,\cdots,n)$ for any given separated and $r$-covering sequence $\{b_j\}_{j=1}^{\infty}$ in $\mathbb{B}$, we have by [1, Theorem 13] that

$$\begin{aligned}\sum_{k\geq 2,R_{k,j}\in\Upsilon}\frac{\rho(R_{k,j})}{2^{kp}}|\nabla f(w_{k,j})|^p &= \sum_{k\geq 2,R_{k,j}\in\Upsilon}2^{-k(\alpha+n+1)}|\nabla f(w_{k,j})|^p\\ &\lesssim\sum_{j=1}^{\infty}\int_{D(b_j,r)}|\nabla f(z)|^p(1-|z|^2)^{\alpha}dV(z)\\ &\approx\|f\|^p_{\mathscr{D}^p_\alpha}.\end{aligned} \tag{3.8}$$

By (3.7) and (3.8), we have

$$\sum_{k\geq 2,R_{k,j}\in\Upsilon}\rho(R_{k,j})|f(z_{k,j})-f(z_{k-1,j'})|^p\lesssim\|f\|^p_{\mathscr{D}^p_\alpha}.$$

By (3.6) and the obvious fact that

$$\sum_{k<2,R_{k,j}\in\Upsilon}\rho(R_{k,j})|f(z_{k,j})-f(z_{k-1,j'})|^p<\infty,$$

we have $f\in L^q_\mu$, i.e., $\mathscr{D}^p_\alpha\subset L^q_\mu$.

*(ii).* From [4, Theorem 3.1], it is enough to prove that (1.3) holds if and only if, for all $k=1,2,\cdots,j=1,2,\cdots,N_k$,

$$\left(\sum_{R_{k,j}\prec R}\mu(R)\right)^{\frac{1}{q}}\left(\sum_{R\prec R_{k,j}}1\right)^{1-\frac{1}{p}}\lesssim 1. \tag{3.9}$$

Let (1.3) hold. By (3.3), we have

$$\sum_{R_{k,j}\prec R}\mu(R)\leq\sum_{i=1}^{N}\mu(S_{a_i})\lesssim\left(\log\frac{1}{1-(1-\frac{1}{2^k})}\right)^{q(\frac{1}{p}-1)}\approx k^{q(\frac{1}{p}-1)}.$$

From the fact $\sum_{R\prec R_{k,j}}1=k+1$, we have that (3.9) holds.

Conversely, suppose that (3.9) holds. For any fixed $|a|\geq\frac{1}{2}$, there is only one $R_{k,j}$ such that $a\in R_{k,j}$. It follows that

$$1-\frac{1}{2^k}\leq|a|<1-\frac{1}{2^{k+1}}\ \text{and}\ \log\frac{1}{1-|a|}\approx k.$$

For any $z \in S_a$, there exist $s \geq k$ and $t > 0$ such that $z \in R_{s,t}$. Let $R_{k,j'} \prec R_{s,t}$. By (3.4), we have

$$\begin{aligned}|1-\langle \xi_{k,j},\xi_{k,j'}\rangle|^{\frac{1}{2}} &\leq |1-\langle \xi_{k,j},\frac{a}{|a|}\rangle|^{\frac{1}{2}}+|1-\langle \frac{a}{|a|},\frac{z}{|z|}\rangle|^{\frac{1}{2}}+|1-\langle \frac{z}{|z|},\xi_{k,j'}\rangle|^{\frac{1}{2}}\\ &\leq \frac{1}{\sqrt{2^k}}+(1-|a|)^{\frac{1}{2}}+\frac{\sqrt{2}}{\sqrt{2}-1}\frac{1}{\sqrt{2^k}} \leq \frac{C}{\sqrt{2^k}}.\end{aligned}$$

Fix $C$ for a moment. Let $Q_{k,j,C} = \{\xi \in \mathbb{S} : |1-\langle \xi,\xi_{k,j}|^{\frac{1}{2}} < \frac{C}{\sqrt{2^k}}\}$ and $Q_{k,j,C} \subset \cup_{i=1}^{M} Q_{k,j_i}$ such that $Q_{k,j,C} \cap Q_{k,j_i} \neq \varnothing$ for $i = 1,2,\cdots,M$. Then

$$\cup_{i=1}^{M} Q_{k,j_i} \subset \left\{\xi \in \mathbb{S} : |1-\langle \xi,\xi_{k,j}\rangle|^{\frac{1}{2}} < \frac{C+4}{\sqrt{2^k}}\right\} = Q_{k,j,C+4}.$$

It follows from [2, Lemma 4.6] that

$$\sigma(Q_{k,j_i}) \approx \sigma(Q_{k,j,C+4}) \approx 2^{-nk}.$$

Thus, $M$ is independent of $R_{k,j}$. Using (3.9), we have

$$\mu(S_a) \leq \sum_{i=1}^{M}\left(\sum_{R_{k,j_i}\prec R}\mu(R)\right) \lesssim (k+1)^{-q(1-\frac{1}{p})} \approx \left(\log\frac{1}{1-|a|}\right)^{-q(1-\frac{1}{p})}.$$

Therefore, (1.3) holds.

*(iii).* Suppose $p > \alpha+n+1$. By [2, Theore 2.1], for all $|z| > \frac{1}{2}$ and $\eta = \frac{z}{|z|}$,

$$\left|f(z)-f(\frac{1}{2}\eta)\right| = \left|\int_{\frac{1}{2}}^{|z|}\Re f(t\eta)\frac{dt}{t}\right| \lesssim \int_{\frac{1}{2}}^{|z|}\frac{\|f\|_{D_\alpha^p}}{(1-t)^{(n+1+\alpha)/p}}dt \lesssim \|f\|_{\mathscr{D}_\alpha^p}.$$

Then, $\mathscr{D}_\alpha^p \subset H^\infty$. So, $\mu$ is finite if and only if $\mathscr{D}_\alpha^p \subset L_\mu^q$ in this case. The proof is complete. $\square$

For the benefit of readers, by using Propositions 1 and 2 in [5], we give the following characterization by using Carleson tubes.

**Theorem 1.2′.** *Suppose $0 < p < q < \infty$, $\alpha > -1$ and $\mu$ is a positive Borel measure on $\mathbb{B}$. Then the following statements holds.*

*(i) When $p < \alpha+n+1$, $\mathscr{D}_\alpha^p \subset L_\mu^q$ if and only if*

$$\sup_{\xi\in\mathbb{S},0<r<1}\frac{\mu(S_\xi^*(r))}{r^{\frac{2q}{p}(\alpha+n+1-p)}} < \infty.$$

*(ii) When $p = \alpha+n+1$, if $1 < p < 2+\frac{1}{n-1}$, $\mathscr{D}_\alpha^p \subset L_\mu^q$ if and only if*

$$\mu(S_\xi^*(r)) \lesssim \left(\log\frac{1}{r}\right)^{q\left(\frac{1}{p}-1\right)},\ \xi \in \mathbb{S}, r \in (0,1); \tag{3.10}$$

*if $p \geq 2+\frac{1}{n-1}$, (3.10) implies $\mathscr{D}_\alpha^p \subset L_\mu^q$.*

*(iii) When $p > \alpha+n+1$, $\mathscr{D}_\alpha^p \subset L_\mu^q$ if and only if $\mu$ is finite.*

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