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# LINEAR FRACTIONAL OPTIMIZATION PROBLEMS ON JORDAN EUCLIDEAN ALGEBRAS

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**Abstract.** We consider a linear fractional optimization problem (LFOP) defined on an Euclidean Jordan algebras. We obtain an optimality theorem for the LFOP, which holds without any constraint qualification. Moreover, we formulate the non-fractional dual problem of the LFOP and then prove the duality theorems (weak duality theorem and strong duality theorem), which hold without any constraint qualification. Furthermore, we characterize the solution set of the LFOP by using the optimality conditions. We also discuss methodologies for the LFOP.

**Keywords.** Euclidean Jordan algebra; Linear fractional optimization problem; Optimality conditions; Strong duality theorem; Weak duality theorem.

## 1. INTRODUCTION AND PRELIMINARIES

First, we give definitions and preliminary results of Euclidean Jordan algebras.

**Definition 1.1.** [1] A finite-dimensional real vector space $V$ is called an algebra if a bilinear mapping $(x,y) \to x \circ y$ from $V \times V$ to $V$ is defined. An algebra $V$ is called a Jordan algebra if the following hold:

(i) for all $x,y \in V$, $x \circ y = y \circ x$

(ii) for all $x,y \in V$, $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$, where $x^2 = x \circ x$

A Jordan algebra $V$ is said to be Euclidean if

(iii) $x^2 + y^2 = 0$ implies $x = y = 0$, equivalently, there exists an inner product $(\cdot|\cdot)$ on $V$ such that $(x \circ y|z) = (y|x \circ z)$.

If there exists a (necessarily unique) element $e$ such that $x \circ e = e \circ x = x$ for all $x \in V$, we say that the Euclidean Jordan algebra $V$ has an identify It is known that all Euclidean Jordan algebras have the identity element $e$.

Let $V$ be an Euclidean Jordan algebra. Then since $V$ is finite-dimensional, given $x \in V$, there exists the minimal positive integer $k$ such that $e, x^1, \cdots, x^k$ are linearly dependent. Denote this integer by $m(x)$. We define the rank of $V$ as:

$$\mathrm{rank}(V) = r = \max\{m(x) \mid x \in V\}.$$

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An element $x \in V$ is said to be invertible if there exist $y \in \mathbb{R}[x]$ such that $x \circ y = e$, where $\mathbb{R}[x]$ is the algebra over $\mathbb{R}$ of polynomials in one variable with coefficients in $\mathbb{R}$. The inverse of $x$ is defined by $x^{-1}$. An element $v \in V$ is called idempotent if $v^2 = v$.

Let $Q = \{x^2 | \, x \in V\}$. Then $Q$ is a closed cone. Let $\Omega = \mathrm{int}Q$, where $\mathrm{int}\Omega$ is the interior of $Q$. Then $\overline{\Omega} = Q$, where $\overline{\Omega}$ is the closure of $\Omega$. The cone $\Omega$ is called the symmetric cone. The following conditions hold:

(i) for any $x, y \in \Omega$, there exists an invertible linear transformation $L : V \to V$ such that $L(\overline{\Omega}) = \overline{\Omega}$ and $L(x) = y$;

(ii) $\overline{\Omega}^* = \overline{\Omega}$, where $\overline{\Omega}^* = \{y \in V \mid (x|y) \geqq 0 \text{ for all } x \in \overline{\Omega}\}$.

In fact, $\Omega = \{x^2 \mid x \in V \text{ is invertible}\}$.

A Jordan algebra $V$ is said to be simple if it does not contain any non-trivial ideal.

**Definition 1.2.** [1] Let $V$ be an Euclidean Jordan algebra, and let $c_1, \cdots, c_k \in V$. Then $\{c_1, \cdots, c_k\}$ is said to be a Jordan frame if $c_i,\ i = 1, \cdots, k$ are non-zero and can not be written as sum of other two idempotents and the following properties hold:

$$c_i^2 = c_i$$
$$c_i \circ c_j = 0 \text{ if } i \neq j$$
$$\sum_{i=1}^{k} c_i = e.$$

**Theorem 1.1.** *[1] Let $V$ be an Euclidean Jordan algebra with rank $r$. For every $x \in V$, there exist a Jordan frame $\{c_1(x), \cdots, c_r(x)\}$ and real numbers $\lambda_1 x), \cdots \lambda_r(x)$ such that $x = \lambda_1(x)c_1(x) + \cdots + \lambda_r(x)c_r(x)$. The numbers $\lambda_i(x),\ i = 1, \cdots, r$, (with their multipliities) are uniquely determined by $x$. The numbers $\lambda_i(x),\ i = 1, \cdots, r$, are called the eigenvalues of $x$ and the trace of $x$ is defined by $\sum_{i=1}^{r} \lambda_i(x)$ and denoted by $tr(x)$, that is, $tr(x) = \sum_{i=1}^{r} \lambda_i(x)$.*

From the Proposition III.1.5 and Proposition III.4.1 in [1], $tr(x \circ y)$ is an inner product on a simple Euclidean Jordan algebra and, for the simple Euclidean algebra $V$, $\overline{\Omega}^* = \{y \in V \mid \mathrm{tr}(y \circ z) \geqq 0 \ \forall z \in \overline{\Omega}\}$. Thus a simple Euclidean Jordan algebra $V$ is a finite-dimensional Hilbert space with inner product $tr(\cdot \circ \cdot)$.

From the Proposition III, 4.4 in [1], an Eucliean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras.

Let us consider the examples of simple Euclidean Jordan algebras. The $n$-dimensional Euclidean space $\mathbb{R}^n$, which has the Jordan product $x \circ y = (x^T y, x_1 y_2 + y_1 x_2, \cdots, x_1 y_n + y_1 x_n)$ and the symmetric cone $\{x \in \mathbb{R}^n \mid x_1 > \sqrt{x_2^2 + \cdots + x_n^2}\}$, is a simple Euclidean Jordan algebra with rank 2. In this Euclidean Jordan algebra $\mathbb{R}^n$, $\mathrm{tr}(x \circ y) = 2x^T y$ for any $x, y \in \mathbb{R}^n$.

The space $S^n$ of real symmetric $n \times n$ matrices, which has the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$ and the symmetric cone $\{X \in S^n \mid X \text{ is positive definite}\}$, is a simple Euclidean Jordan algebra with rank $n$. In this Euclidean Jordan algebra $S^n$, $tr(X \circ Y) = Tr(XY)$ for any $X, Y \in S^n$.

The space of complex Hermitian $n \times n$ matrices, which has the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$, is a simple Euclidean Jordan algebra. The space of Hermitian $n \times n$ matrices with quaternion entries, which has the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$, is a simple Euclidean Jordan algebra.

The space of Hermitian $3 \times 3$ matrices with octonion entries, which has the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$, is a simple Euclidean Jordan algebra.

Let us give the definition of conjugate function for a proper, lower semicontinuous, and convex function. Let $h : V \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous, and convex function. The conjugate function $h^*$ of the function $h$ is defined by

$$h^*(v) = \sup\{tr(v \circ x) - h(x) \mid x \in \mathrm{dom}h\}$$

for any $v \in V$, where $\mathrm{dom}h := \{x \in V \mid h(x) < +\infty\}$. The epigraph of a function $g : V \to \mathbb{R} \cup \{+\infty\}$ is defined as

$$\mathrm{epi}g := \{(x, \alpha) \in V \times \mathbb{R} \mid g(x) \leqq \alpha\}.$$

Let $h_1, h_2 : V \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. If one of the functions $h_1$ and $h_2$ is continuous, then

$$\mathrm{epi}(h_1 + h_2)^* = \mathrm{epi}h_1^* + \mathrm{epi}h_2^*.$$

Jeyakumar *et al.* [2] proved the sequential Lagrange multiplier optimality conditions for convex optimization problems, which hold without any constraint qualification and were expressed by sequences. Such optimality conditions have been studied for many kinds of convex optimization problems. In particular, Kim *et al.* [3] studied the sequential Lagrange multiplier optimality conditions for a semidefinite linear fractional optimization problem, which holds without any constraint qualification. They [4] investigated the sequential Lagrange multiplier optimality conditions for a second-order cone linear fractional optimization problem, which holds without any constraint qualification.

The dual problem for a differentiable optimization problem was formulated by Wolfe [5]. He proved the weak duality theorem and the strong duality theorem which holds between the primal problem and its dual problem ( [6]). Many authors have studied such duality theorems for many kinds of optimization problems. In particular, using variable transformations, Craven [7] and Craven *et al.* [8] obtained the equivalent nonfractional linear optimization problem from a linear fractional optimization problem. He formulated the dual problem from the equivalent problem, and then gave duality theorems for the primal problem and its dual problem. So the dual problem is a nonfractional dual one for the linear fractional optimization problem which the duality theorems hold. By using the approach of Craven [7] and Craven *et al.* [8], Kim *et al.* formulated the dual problem for a semidefinite linear fractional optimization problem, and then proved directly duality theorems which holds under constraint qualification [9] and hold without any constraint qualification [3]. They [4] formulated the dual problem for a second-order cone linear fractional optimization problem, which is expressed by sequences, and then obtained the weak duality theorem and the strong duality theorem which hold between the primal problem and its dual problem. The strong duality theorem holds without any constraint qualification.

Optimization problems often have multiple solutions. Mangasarian [10] presented simple and elegant characterizations of the solution set for a convex optimization problem over a convex set when one solution is known. These characterizations were extended to various classes of optimization problems. In particular, Jeyakumar *et al.* [11] gave Lagrange multiplier characterizations of the solution set of a convex optimization problem involving convex inequality constraints in terms of Lagrange multipliers of a known solution.

In this paper, considering simple Euclidean Jordan algebra as an underlined space, we are intend to unify our results for semidefinite liner fractional problems and second-order cone linear fractional problems in [3,4,9,12]; see [13–15] for more results on the linear optimization problems defined on Euclidean Jordan algebras.

Throughout this paper, we assume that $V$ is a simple Euclidean Jordan algebra with rank $r$.

In this paper, we consider the following linear fractional optimization problem:

$$\text{(LFOP)} \qquad \text{Minimize} \quad \frac{\text{tr}(c\circ x)+\alpha}{\text{tr}(d\circ x)+\beta}$$
$$\text{subject to} \quad \text{tr}(a_i\circ x)=b_i,\ i=1,\ldots,m$$
$$x\in\overline{\Omega},$$

where $c,d\in V$, $\alpha,\beta$ are given real numbers, $a_i, i=1,\cdots,m$ and $b_i, i=1,\cdot,m$ are give real numbers, and $\overline{\Omega}=\{x^2 \mid x\in V\}$. Let $\triangle=\{x\in\overline{\Omega} \mid \text{tr}(a_i\circ x)=b_i,\ i=1,\cdots,m\}$ and $D_F=\{x\in V \mid \text{tr}(d\circ x)+\beta>0\}$. Throughout this paper, we assume that $\triangle\subset D_F$.

In this paper, we obtain an optimality theorem for the LFOP, which holds without any constraint qualification. We formulate the non-fractional dual problem of the LFOP, and then prove the duality theorems (weak duality theorem and strong duality theorem), which hold without any constraint qualification. Moreover, we establish the duality theorems (weak duality theorem, strong duality theorem, and converse duality theorem), which are considered under a constraint qualification. We characterize the solution sets of the LFOP by using the optimality conditions. Furthermore, We discuss methodologies for the LFOP.

## 2. Optimality Theorems

We now can prove the following proposition.

**Proposition 2.1.** *Let $q(x)=\frac{tr(c\circ x)+\alpha}{tr(d\circ x)+\beta}, x\in V$ where $c,d\in V$, $\alpha,\beta$ are given real numbers. Let $D_q=\{x\in V \mid tr(d\circ x)+\beta\neq 0\}(\neq\emptyset)$. Then*

*(i) for any $x\in D_q$, $q$ is differentiable at $x$ and*

$$\nabla q(x)=\frac{[tr(d\circ x)+\beta]c-[tr(c\circ x)+\alpha]d}{[tr(d\circ x)+\beta]^2}$$

*(ii) for any $x,y\in D_q$,*

$$q(y)-q(x)=\frac{tr(d\circ x)+\beta}{tr(d\circ y)+\beta}\ tr(\nabla q(x)\circ(y-x)).$$

Now we obtain the following sequential Lagrange multiplier optimality theorem for the LFOP, which holds without any constraint qualification: following the proof of the Theorem 2.1 in [3], we can prove the theorem. But for the completeness, we give its proof.

**Theorem 2.1.** *Let $\bar{x}\in\triangle$. Then $\bar{x}$ is an optimal solution of (LFOP) if and only if there exist $\lambda_i^l\in\mathbb{R}$, $i=1,\cdots,m$ and $v^l\in\overline{\Omega}$ such that*

$$c-q(\bar{x})d+\lim_{l\to\infty}\left[\sum_{i=1}^{m}\lambda_i^l a_i-v^l\right]=0$$
$$\textit{and}\ \lim_{l\to\infty} tr(v^l\circ\bar{x})=0$$

*where $q(\bar{x})=\frac{tr(c\circ\bar{x})+\alpha}{tr(d\circ\bar{x})+\beta}$.*

*Proof.* ($\Rightarrow$) Let $\bar{x}$ be an optimal solution of the LFOP. Let $q(x)=\frac{\mathrm{tr}(c\circ x)+\alpha}{\mathrm{tr}(d\circ x)+\beta}$. By Proposition 2.1, $q(x)-q(\bar{x})=\frac{\mathrm{tr}(d\circ\bar{x})+\beta}{\mathrm{tr}(d\circ x)+\beta}\,\mathrm{tr}(\nabla q(\bar{x})\circ(x-\bar{x}))$. It follows that

$$\frac{\mathrm{tr}(d\circ\bar{x})+\beta}{\mathrm{tr}(d\circ x)+\beta}\,\mathrm{tr}(\nabla q(\bar{x})\circ(x-\bar{x}))\geqq 0$$

for any $x\in\triangle$. Since $\frac{\mathrm{tr}(d\circ\bar{x})+\beta}{\mathrm{tr}(d\circ x)+\beta}>0$, $\mathrm{tr}(\nabla q(\bar{x})\circ(x-\bar{x}))\geqq 0$, for any $x\in\triangle$, one has $\mathrm{tr}(\nabla q(\bar{x})\circ x)-\mathrm{tr}(\nabla q(\bar{x})\circ\bar{x})\geqq 0\ \forall x\in\triangle$. Let $g(x)=\mathrm{tr}(\nabla q(\bar{x})\circ x)-\mathrm{tr}(\nabla q(\bar{x})\circ\bar{x})$. Then $g(x)+\delta_\triangle(x)\geqq g(\bar{x})+\delta_\triangle(\bar{x})=0$, where $\delta_\triangle$ is the indicator function defined by $\delta_\triangle(x)=0$ if $x\in\triangle$ and $\delta_\triangle(x)=\infty$ if $x\notin\triangle$. Thus $(g+\delta_\triangle)^*(0)\leqq 0$ and so $(0,0)\in\mathrm{epi}(g+\delta_\triangle)^*$. Since $g$ is continuous, we have $(0,0)\in\mathrm{epi}g^*+\mathrm{epi}\delta_\triangle^*$. We can check that

$$\mathrm{epi}g^*=\{(\nabla q(\bar{x}),\mathrm{tr}(\nabla q(\bar{x})\circ\bar{x}))\}+\{0\}\times\mathbb{R}^+,$$
$$\mathrm{epi}\delta_\triangle^*=\mathrm{cl}\{\bigcup_{\lambda_i\in\mathbb{R}}\sum_{i=1}^m\lambda_i(a_i,b_i)+(-\overline{\Omega})\times\mathbb{R}^+\},$$

where cl$A$ is the closure of a set $A$. So we have

$$(0,0)\in\{\nabla q(\bar{x}),\mathrm{tr}(\nabla q(\bar{x})\circ\bar{x})\}+\{0\}\times\mathbb{R}^++\mathrm{cl}\{\bigcup_{\lambda_i\in\mathbb{R}}\sum_{i=1}^n\lambda_i(a_i,b_i)+(-\overline{\Omega})\times\mathbb{R}^+\}.$$

Thus there exist $\widetilde{\lambda}_i^l\in\mathbb{R},\ i=1,\cdots,m,\ \widetilde{v}^l\in\overline{\Omega},\ \widetilde{r}\in\mathbb{R}^+$ and $\widetilde{r}^l\in\mathbb{R}^+$ such that

$$\nabla q(\bar{x})+\lim_{l\to\infty}\Big[\sum_{i=1}^m\widetilde{\lambda}_i^l a_i-\widetilde{v}^l\Big]=0$$
$$\text{and}\ \ \mathrm{tr}(\nabla q(\bar{x})\circ\bar{x})+\widetilde{r}+\lim_{l\to\infty}(\sum_{i=1}^m\lambda_i^l b_i+\widetilde{r}^l)=0.$$

Hence we obtain

$$\nabla q(\bar{x})+\lim_{l\to\infty}\Big[\sum_{i=1}^m\widetilde{\lambda}_i^l a_i-\widetilde{v}^l\Big]=0$$
$$\text{and}\ \ \lim_{l\to\infty}\mathrm{tr}(\widetilde{v}^l\circ\bar{x})=0.$$

Since $\nabla q(\bar{x})=\frac{[\mathrm{tr}(d\circ\bar{x})+\beta]c-[\mathrm{tr}(c\circ\bar{x})+\alpha]d}{(\mathrm{tr}(d\circ\bar{x})+\beta)^2}$, letting $\lambda_i=(\mathrm{tr}(d\circ\bar{x})+\beta)\widetilde{\lambda}_i^l$ and $v^l=\mathrm{tr}(d\circ\bar{x})+\beta)\widetilde{v}^l$, we obtain, $v^l\in\overline{\Omega}$, and

$$c-q(\bar{x})d+\lim_{l\to\infty}\Big[\sum_{i=1}^m\lambda_i^l a_i-v^l\Big]=0$$
$$\text{and}\ \ \lim_{l\to\infty}\mathrm{tr}(v^l\circ\bar{x})=0.$$

The converse can be easily proved. □

From the proof of Theorem 2.1, we can obtain the following Lagrange optimality theorem for the LFOP. This theorem means that the closedness of the set $\bigcup_{\lambda_i\in\mathbb{R}}\{\sum_{i=1}^m\lambda_i(a_i,b_i)\}+(-\overline{\Omega})\times\mathbb{R}^+$ can be used as a constraint qualification for (LFOP).

**Theorem 2.2.** *Let $\bar{x} \in \triangle$. Suppose that $\bigcup_{\lambda_i \in \mathbb{R}} \{\sum_{i=1}^{m} \lambda_i (a_i, b_i)\} + (-\overline{\Omega}) \times \mathbb{R}^+$ is closed in $V \times \mathbb{R}$. Then $\bar{x}$ is an optimal solution of the LFOP if and only if there exist $\lambda_i \in \mathbb{R},\ i = 1, \cdots, m$ and $v \in \overline{\Omega}$ such that*

$$c - q(\bar{x})d + \sum_{i=1}^{m} \lambda_i a_i - v = 0$$
$$\text{and } \ tr(v \circ \bar{x}) = 0.$$

Now we give an example to illustrate how to solve semidefinite linear optimization problems by using Theorem 2.1. But we show that we can not apply Theorem 2.2 to this example.

**Example 2.1.** [3] Consider the following semidefinite linear fractional problem:

$$\text{(SLFP)} \qquad \text{Minimize} \qquad \frac{\mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1}{\mathrm{Tr}\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1}$$

$$\text{subject to} \qquad \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in S^2_+,$$
$$\mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} = 0,$$

where $S^2_+$ is the set of $2 \times 2$ symmetric and positve semidefinite matrices, which is the closure of the symmetric cone of the Euclidean Jordan algebra $S^2$. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$, and

$$\triangle = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in S^2_+ \ \middle| \ \mathrm{Tr}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} = 0 \right\}.$$

Let

$$q\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} = \frac{\mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1}{\mathrm{Tr}\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} + 1}.$$

Then $\triangle = \left\{ \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in S^2 \ \middle| \ x_1 \geqq 0, x_2 = x_3 = 0 \right\}$ and $q\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} = 1$ for any $\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in \triangle$. Furthermore, for any $X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \in \triangle$,

$$C - q(X)D + \lim_{n \to \infty}\left[ nA - \begin{pmatrix} \frac{1}{n} & \frac{1}{2} \\ \frac{1}{2} & n \end{pmatrix} \right] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\lim_{n \to \infty} \mathrm{Tr}\begin{pmatrix} \frac{1}{n} & \frac{1}{2} \\ \frac{1}{2} & n \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} = \lim_{n \to \infty} \frac{1}{n} x_1 = 0.$$

It follows from Theorem 2.1 that $\triangle$ is the one of all the optimal solutions of the SLFP.

But, there does not exist $X \in \triangle$ such that

$$C - q(X)D + \lambda A \in S^2_+, \ \lambda \in \mathbb{R}$$
$$\text{and } \mathrm{Tr}(C - q(X)D + \lambda A)X = 0.$$

This means that Theorem 2.2 can not be applied to this example.

Now we give an example, which is a second-order linear fractional optimization problem. This example is obtained by modifying a known example in the Internet (https://math.stackexchange.com/questions/712188/how-can-the-sum-of-two-closed-cones-be-not-closed).

**Example 2.2.** [4] Let $\overline{\Omega} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \geqq \sqrt{x_2^2 + x_3^2}, \ x_1 \geqq 0\}$, and let $a_1 = (1, 0, -1)$ and $b_1 = 0$. Consider the following second-order linear fractional optimization problem:

$$\text{(SLF)} \qquad \text{Minimize} \quad \frac{x_1 + x_3 + 1}{x_1 - x_2 + x_3 + 1}$$
$$\text{subject to} \quad (x_1, x_2, x_3) \in \bar{\Omega}$$
$$a_1 (x_1, x_2, x_3)^T = b_1.$$

Let $c = (1, 0, 1)$, $d = (1, -1, 1)$ and $\triangle = \{(x_1, x_2, x_3) \in \overline{\Omega} \mid a_1(x_1, x_2, x_3)^T = b_1\}$. Let $q(x_1, x_2, x_3) = \frac{x_1 + x_3 + 1}{x_1 - x_2 + x_3 + 1}$. Then $\triangle = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \geqq 0, \ x_1 = x_3, \ x_2 = 0\}$ and $q(x_1, x_2, x_3) = 1$ for any $(x_1, x_2, x_3) \in \triangle$. Furthermore, for any $(x_1, x_2, x_3) \in \triangle$,

$$\begin{aligned}
& c - q(x_1, x_2, x_3)d + \lim_{n\to\infty} \left[ n a_1 - \left( \sqrt{n^2 + \left(1 + \frac{1}{n}\right)^2}, 1 + \frac{1}{n}, -n \right) \right] \\
&= (0, 1, 0) + \lim_{n\to\infty} \left( n - \sqrt{n^2 + \left(1 + \frac{1}{n}\right)^2}, -1 - \frac{1}{n}, 0 \right) \\
&= (0, 0, 0)
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{n\to\infty} \left( \sqrt{n^2 + \left(1 + \frac{1}{n}\right)^2}, 1 + \frac{1}{n}, -n \right) (x_1, x_2, x_3)^T \\
&= \lim_{n\to\infty} \left( \sqrt{n^2 + \left(1 + \frac{1}{n}\right)^2} - n \right) x_1 \\
&= 0.
\end{aligned}$$

Thus, by Theorem 2.1, $\triangle$ is the one of all the optimal solutions of the SLF. But there does not exist $(x_1, x_2, x_3) \in \triangle$ such that

$$c - q(x_1, x_2, x_3)d + \lambda a_1 \in \bar{\Omega}, \ \lambda \in \mathbb{R}$$
$$\text{and } \ (c - q(x_1, x_2, x_3)d + \lambda a_1)(x_1, x_2, x_3)^T = 0.$$

This means that Theorem 2.2 can not applied to this example.

By [16, Proposition 2.1], we can obtain the following proposition. For the completeness, following the proof of [16, Proposition 2.1], we prove the following proposition.

**Proposition 2.2.** *Let $\bar{x} \in \triangle$, and assume that $a_1, \cdots, a_m$ are linearly independent in $V$. Suppose that there exists $\widetilde{x} \in \Omega$ such that $tr(a_i \circ \widetilde{x}) = b_i, \ i = 1, \cdots, m$. Then $\bigcup_{\lambda_i \in \mathbb{R}} \sum_{i=1}^m \lambda_i (a_i, b_i) + (-\overline{\Omega}) \times \mathbb{R}^+$ is closed in $\mathbb{R}^n \times \mathbb{R}$.*

*Proof.* Let $g_i(x) = \mathrm{tr}(a_i \circ x)$ and $h_i(x) = g_i(x) - b_i$, $x \in V$, $i = 1, \cdots, m$. Let $G(x) = \begin{pmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{pmatrix}$ and $H(x) = \begin{pmatrix} h_1(x) \\ \vdots \\ h_m(x) \end{pmatrix}$. Since $a_1, \cdots, a_m$ are linearly independent in $V$, one has hat $G : V \to \mathbb{R}^m$ is surjective. By open mapping theorem, one see that $H(\Omega)$ is open in $\mathbb{R}^m$. By assumption, one obtains $0 \in H(\Omega)$. Since $H(\Omega)$ is open, one concludes $0 \in \mathrm{int} H(\overline{\Omega})$. From the proof of Theorem 2.1, one sees that

$$\mathrm{epi}\delta^*_\triangle = \mathrm{cl}\,\{\bigcup_{\lambda_i \in \mathbb{R}} \sum_{i=1}^m \lambda_i(a_i, b_i) + (-\overline{\Omega}) \times \mathbb{R}^+\}.$$

Since $\bigcup\limits_{\lambda_i \in \mathbb{R}} \sum_{i=1}^m \lambda_i(a_i, b_i) + (-\overline{\Omega}) \times \mathbb{R}^+ = \bigcup\limits_{\lambda_i \in \mathbb{R}} \mathrm{epi}(\sum_{i=1}^m \lambda_i h_i)^* + \mathrm{epi}\delta^*_{\overline{\Omega}}$, one has

$$\mathrm{epi}\delta^*_\triangle = \mathrm{cl}(\bigcup_{\lambda_i \in \mathbb{R}} \mathrm{epi}(\sum_{i=1}^m \lambda_i h_i)^* + \mathrm{epi}\delta^*_{\overline{\Omega}}).$$

Let $\Lambda = \bigcup\limits_{\lambda_i \in \mathbb{R}} \mathrm{epi}(\sum_{i=1}^m \lambda_i h_i)^* + epi\delta^*_{\overline{\Omega}}$. $\mathrm{epi}\delta^*_\triangle$ is closed. To see that the set $\Omega$ is closed, it is sufficient to prove that $\Lambda = \mathrm{epi}\delta^*_\triangle$. Let $(v, \alpha) \in \Lambda$. Then there exist $\lambda_i \in \mathbb{R}$, $i = 1, \cdots, m$, $(v_1, \alpha_1) \in \mathrm{epi}(\sum_{i=1}^m \lambda_i h_i)^*$ and $(v_2, \alpha_2) \in \mathrm{epi}\delta^*_{\overline{\Omega}}$ such that $v = v_1 + v_2$ and $\alpha = \alpha_1 + \alpha_2$. Since $(\sum_{i=1}^m \lambda_i h_i)^*(v_1) \leqq \alpha_1$ and $\delta^*_{\overline{\Omega}}(v_2) \leqq \alpha_2$, one has

$$\mathrm{tr}(v_1 \circ x) - \sum_{i=1}^m \lambda_i h_i(x) \leqq \alpha_1 \text{ for any } x \in V \text{ and}$$

$$\mathrm{tr}(v_2 \circ x) \leqq \alpha_2 \text{ for any } x \in \overline{\Omega}.$$

Thus $\mathrm{tr}(v_1 \circ x) + tr(v_2 \circ x) \leqq \alpha_1 + \alpha_2$, that is,

$$\mathrm{tr}((v_1 + v_2) \circ x) \leqq \alpha_1 + \alpha_2 \text{ for any } x \in \triangle.$$

Thus $(v, \alpha) \in \mathrm{epi}\delta^*_\triangle$. Hence $\triangle \subset \mathrm{epi}\delta^*_\triangle$. Let $(v, r) \in \mathrm{epi}\delta^*_\triangle$. Let

$$T = \{(p, y) \in \mathbb{R} \times \mathbb{R}^m \mid \text{there exist } x \in \overline{\Omega} \text{ such that } \mathrm{tr}(v \circ x) \geqq p,\ H(x) = y\}.$$

Then $T$ is a nonempty convex set. Since $\delta^*_\triangle(v) = \sup\limits_{x \in \triangle} tr(v \circ x) < +\infty$, for any $\varepsilon > 0$, there exists $\widetilde{x} \in \triangle$ such that $\delta^*_\triangle(v) - \varepsilon < v(\widetilde{x})$ and so $(\delta^*_\triangle(v) - \varepsilon, 0) \in T$. But $(\delta^*_\triangle(v) + \varepsilon, 0) \notin T$ for any $\varepsilon > 0$. Thus $(\delta^*_\triangle(v), 0)$ is a boundary point of $T$. Since $\mathrm{tr}(v \circ \cdot)$ is continuous, one has that $(\mathrm{tr}(v \circ \cdot))^{-1}(-r, r)$ is an open neighborhood of 0 for any fixed $r > 0$. By the assumption, there exists $\widehat{x} \in \Omega$ such that $H(\widehat{x}) = 0$. Let $B = (\mathrm{tr}(v \circ \widehat{x}) - r - \mu, \mathrm{tr}(v \circ \widehat{x}) - r) \times H(\widehat{x} + \mathrm{tr}(v \circ \cdot)^{-1}(-r, r) \cap \Omega)$ for fixed $\mu > 0$. Since $H$ is an open mapping, we see that $B$ is open for any $x \in (\widehat{x} + \mathrm{tr}(v \circ \cdot)^{-1}(-r, r)) \cap \Omega$

$$\mathrm{tr}(v \circ \widehat{x}) - r < \mathrm{tr}(v \circ x) < \mathrm{tr}(v \circ \widehat{x}) + r.$$

When $(\alpha, w) \in B$, there exist $u \in (\widehat{x} + \mathrm{tr}(v \circ \cdot)^{-1}(-r, r)) \cap \Omega$ such that $w = H(u)$ and $tr(v \circ \widehat{x}) < \alpha < \mathrm{tr}(v \circ \widehat{x}) - r < \mathrm{tr}(v \circ u)$, and so $(\alpha, w) \in T$. Thus $B \subset T$ and $\mathrm{int}T \neq \emptyset$. Since $(\delta^*_\triangle(v), 0)$

is a boundary point of $T$, one has $(\delta^*_\triangle(v),0) \notin \mathrm{int}T$. By the separation theorem, there exists $(\theta,\lambda) \in \mathbb{R}\times\mathbb{R}^m$ such that $(\theta,\lambda) \neq (0,0)$

$$\theta\delta^*_\triangle(v) \leqq \theta p + \lambda^T y \text{ for any } (p,y) \in T.$$

If $\theta = 0$, then $0 \leqq \lambda^T y$ for any $y \in \{y \in \mathbb{R}^m \mid y = H(x),\ x \in \overline{\Omega}\}$. Since $0 \in \mathrm{int}H(\overline{\Omega})$, one has $\lambda = 0$. This is a contradiction. Hence $\theta \neq 0$. Since $(\delta^*_\triangle(v) - \varepsilon, 0) \in T$ for any $\varepsilon > 0$, $\theta \leqq 0$. Thus we may assume that $\theta = -1$. Hence $\delta^*_\triangle(v) \geqq p - \lambda^T y$ for any $(p,y) \in T$. So, $\delta^*_\triangle(v) \geqq \mathrm{tr}(v\circ x) - \lambda^T H(x)$ for any $x \in \overline{\Omega}$. Thus $\delta^*_\triangle(v) \geqq \mathrm{tr}(v\circ x) - \lambda^T H(x) - \delta_{\overline{\Omega}}(x)$ for any $x \in V$. Hence

$$\delta^*_\triangle(v) \geqq (\delta_{\overline{\Omega}} + \sum_{i=1}^m \lambda_i h_i)^*(v).$$

Since $(v,r) \in \mathrm{epi}\delta^*_\triangle$, then $r \geqq \delta^*_\triangle(v) \geqq (\delta_{\overline{\Omega}} + \sum_{i=1}^m \lambda_i h_i)^*(v)$. Thus $(v,r) \in \mathrm{epi}(\sum_{i=1}^m \lambda_i h_i + \delta_{\overline{\Omega}})^*$. Since $\sum_{i=1}^m \lambda_i h_i$ is continuous, we have

$$\mathrm{epi}(\sum_{i=1}^m \lambda_i h_i + \delta_{\overline{\Omega}})^* = \mathrm{epi}(\sum_{i=1}^m \lambda_i h_i)^* + \mathrm{epi}\delta^*_{\overline{\Omega}} = \Lambda.$$

Hence $(v,r) \in \Lambda$ and $\mathrm{epi}\delta^*_\triangle \subset \Lambda$. Consequently, $\Lambda = \mathrm{epi}\delta^*_\triangle$, so $\Lambda$ is closed. □

## 3. Sequential Duality Theorems

By using the sequential Lagrange multiplier optimality theorem (Theorem 2.1), we formulate the dual problem for the LFOP, which is expressed by sequences, as follows;

$$\begin{array}{lll} \text{(LFOD)} & \text{Maximize} & r \\ & \text{subject to} & c - rd + \lim_{l\to\infty}\left[\sum_{i=1}^m y_i^l a_i - v^l\right] = 0 \\ & & -r\beta + \alpha \geqq \limsup_{l\to\infty} \sum_{i=1}^m y_i^l b_i \\ & & y_i^l \in \mathbb{R}, i = 1,\cdots,m,\ v^l \in \overline{\Omega}, l = 1,2,\cdots. \end{array}$$

Following the proof of [3, Theorem 3.1 ], we can obtain the following weak duality theorem.

**Theorem 3.1.** *(Weak Duality) Let $x$ be feasible for the LFOP, and let $(r,\{y_i^l\},\{v^l\})$ be feasible for the LFOD. Then $\frac{tr(c\circ x)+\alpha}{tr(d\circ x)+\beta} \geqq r$.*

*Proof.* Let $x$ be feasible for the LFOP, and let $(r,\{y_i^l\},\{v^l\})$ be feasible for the LFOD. Suppose to the contrary that

$$\frac{\mathrm{tr}(c\circ x)+\alpha}{\mathrm{tr}(d\circ x)+\beta} < r.$$

Since $\mathrm{tr}(d\circ x)+\beta > 0$, one has $\mathrm{tr}(c\circ x)+\alpha - r[\mathrm{tr}(d\circ x)+\beta] < 0$. So, $\alpha - r\beta + \mathrm{tr}((c-rd)\circ x) < 0$, which implies $\alpha - r\beta + \mathrm{tr}\Big[-\lim_{l\to\infty}(\sum_{i=1}^m y_i^l a_i - v^l)\circ x\Big] < 0$. Thus $\alpha - r\beta - \lim_{l\to\infty}\Big[\mathrm{tr}(\sum_{i=1}^m y_i^l a_i \circ$

$x) - \mathrm{tr}(v^l \circ x)\Big] < 0$. Hence

$$\begin{aligned}\alpha - r\beta &< \lim_{l\to\infty}\Big[\mathrm{tr}(\sum_{i=1}^m y_i^l a_i \circ x) - \mathrm{tr}(v^l \circ x)\Big]\\ &\leqq \limsup_{l\to\infty}\mathrm{tr}(\sum_{i=1}^m y_i^l a_i \circ x) + \limsup_{l\to\infty}(-\mathrm{tr}(v^l \circ x))\\ &\leqq \limsup_{l\to\infty}\sum_{i=1}^m y_i^l \mathrm{tr}(a_i \circ x)\\ &= \limsup_{l\to\infty}\sum_{i=1}^m y_i^l b_i.\end{aligned}$$

Thus, $\alpha - r\beta < \limsup_{l\to\infty}\sum_{i=1}^m y_i^l b_i$. This is a contradiction. Consequently, $\frac{\mathrm{tr}(c\circ x)+\alpha}{\mathrm{tr}(d\circ x)+\beta} \geqq r$. □

**Theorem 3.2.** *(Strong Duality) Let $\bar{x}$ be an optimal solution of the LFOP. Let $\bar{r} = \frac{tr(c\circ\bar{x})+\alpha}{tr(d\circ\bar{x})+\beta}$. Then there exist $y_i^l \in \mathbb{R}$, $v^l \in \overline{\Omega}$ such that $(\bar{r}, \{y_i^l\}, \{v^l\})$ is feasible for the LFOD. In addition, $(\bar{r}, \{y_i^l\}, \{v^l\})$ is an optimal solution.*

*Proof.* Let $\bar{x}$ be an optimal solution of the LFOP. By Theorem 2.2, there exist $y_i^l \in \mathbb{R}$, $v^l \in \overline{\Omega}$ such that

$$c - \frac{\mathrm{tr}(c\circ\bar{x})+\alpha}{\mathrm{tr}(d\circ\bar{x})+\beta}\, d + \lim_{l\to\infty}(\sum_{i=1}^m y_i^l a_i - v^l) = 0$$
$$\text{and } \lim_{l\to\infty}\mathrm{tr}(v^l \circ \bar{x}) = 0.$$

Let $\bar{r} = \frac{\mathrm{tr}(c\circ\bar{x})+\alpha}{\mathrm{tr}(d\circ\bar{x})+\beta}$. Then $\mathrm{tr}(c - \bar{r}d)\circ\bar{x} + \lim_{l\to\infty}\sum_{i=1}^m y_i^l\ \mathrm{tr}(a_i\circ\bar{x}) = 0$, and so, $\mathrm{tr}(c\circ\bar{x}) - \bar{r}\ \mathrm{tr}(d\circ\bar{x}) + \lim_{l\to\infty}\sum_{i=1}^m y_i^l b_i = 0$. Since $\bar{r} = \frac{\mathrm{tr}(c\circ\bar{x})+\alpha}{\mathrm{tr}(d\circ\bar{x})+\beta}$, one has $\mathrm{tr}(c\circ\bar{x}) - \bar{r}\ \mathrm{tr}(d\circ\bar{x}) = -\alpha + \bar{r}\beta$. Hence, $-\alpha + \bar{r}\beta + \lim_{l\to\infty}\sum_{i=1}^m y_i^l b_i = 0$. Thus $(\bar{x}, \{y_i^l\}, \{v^l\})$ is feasible for the LFOD. Hence, it follows from Theorem 3.1 (the weak duality theorem) that $(\bar{r}, \{y_i^l\}, \{v^l\})$ is an optimal solution to the LFOD. □

## 4. Duality Theorems under Constraint Qualifications

Using the Lagrange multiplier optimality theorem (Theorem 2.2), which holds under a constraint qualification, we formulate the dual problem for the LFOP as follows:

$$\begin{aligned}&\text{(LFODC)} \quad \text{Maximize} \quad r\\ &\qquad\qquad \text{subject to} \quad c - rd + \sum_{i=1}^m y_i a_i \in \overline{\Omega}\\ &\qquad\qquad\qquad\qquad -\beta r + \alpha \geqq \sum_{i=1}^m y_i b_i.\end{aligned}$$

Let $F = \{(r,y) \in \mathbb{R}\times V \mid c - rd + \sum_{i=1}^m y_i a_i \in \overline{\Omega},\ -\beta r + \alpha \geqq \sum_{i=1}^m y_i b_i\}$. We can prove the following theorems (weak duality theorem and strong duality theorem).

**Theorem 4.1.** *(Weak Duality) Let $x$ be feasible for the LFOP, and let $(r,y)$ be feasible for the LFODC. Then* $\frac{tr(c\circ x)+\alpha}{tr(d\circ x)+\beta} \geqq r$.

**Theorem 4.2.** *(Strong Duality) Let $\bar{x}$ be an optimal solution to the LFOP. Suppose that* $\bigcup_{\lambda_i\in\mathbb{R}} \sum_{i=1}^{m}\{\lambda_i(a_i,b_i)\} + (-\overline{\Omega}) \times \mathbb{R}^+$ *is closed in* $V \times \mathbb{R}_+$. *Let* $\bar{r} = \frac{tr(c\circ\bar{x})+\alpha}{tr(d\circ\bar{x})+\beta}$. *Then there exist* $\bar{y} \in \mathbb{R}^m$ *such that* $(\bar{r},\bar{y})$ *is an optimal solution to the LFODC.*

Now we prove the converse duality theorem.

**Theorem 4.3.** *(Converse Duality) Assume that for any* $x \in \triangle$, $tr(d\circ x)+\beta > 0$ *and that* $\bigcup_{q\in\overline{\Omega}} \big\{\big(tr(q\circ d), -tr(q\circ a_1), \cdots, -tr(q\circ a_m), tr(q\circ c)\big)\big\} + \bigcup_{\lambda\geqq 0}\{\lambda(\beta, b_1,\cdots,b_m,\alpha)\} + \{(0,0)\}\times\mathbb{R}_+$ *is closed in* $\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}$. *If* $(\bar{r},\bar{y}) \in F$ *is an optimal solution to the LFODC, then there exist* $\bar{q} \in \overline{\Omega}$ *and* $\bar{\lambda} \geqq 0$ *such that*

$$\begin{aligned} &tr(\bar{q}\circ d)+\bar{\lambda}\beta+1=0\\ &tr(\bar{q}\circ a_i)-\bar{\lambda}b_i=0,\ i=1,\cdots,m,\\ &tr(\bar{q}\circ c)+\bar{\lambda}\alpha+\bar{r}\leqq 0.\end{aligned}$$

*In addition, if $\triangle$ is bounded, then $\frac{\bar{q}}{\bar{\lambda}}$ is an optimal solution to the LFOP and* $\bar{r} = \dfrac{tr\left(c\circ\frac{\bar{q}}{\bar{\lambda}}\right)+\alpha}{tr\left(d\circ\frac{\bar{q}}{\bar{\lambda}}\right)+\beta}$.

*Proof.* Let $f(r,y) = -r,\ g(r,y) = -c + rd - \sum_{i=1}^{m} y_i a_i$ and $h(r,y) = \sum_{i=1}^{m} y_i b_i + \beta r - \alpha$. Then the LFODC becomes:

$$\begin{array}{ll} \text{Minimize}_{r,y} & f(r,y) \\ \text{subject to} & \begin{pmatrix} g(r,y) \\ h(r,y) \end{pmatrix} \in (-\overline{\Omega})\times(-\mathbb{R}_+). \end{array}$$

Let $G = \{(r,y)\in\mathbb{R}\times\mathbb{R}^m \mid g(r,y)\in -\overline{\Omega}\}$ and $H = \{(r,y)\mid h(r,y)\in -\mathbb{R}_+\}$. Let $(\bar{r},\bar{y})\in F$ be an optimal solution to the LFODC. Then $f(r,y)+\delta_{G\cap H}(r,y) \geqq f(\bar{r},\bar{y})+\delta_{G\cap H}(\bar{r},\bar{y})$ and

$$(f+\delta_{G\cap H})^*(0,0) \leqq -f(\bar{r},\bar{y}) - \delta_{G\cap H}(\bar{r},\bar{y}).$$

Thus,

$$\begin{aligned} (0,0,-f(\bar{r},\bar{y})) &\in \text{epi}(f+\delta_{G\cap H})^* \\ &= \text{epi} f^* + \text{epi}\delta^*_{G\cap H} \\ &= \text{epi} f^* + \text{cl}(\text{epi}\delta^*_G + \text{epi}\delta^*_H). \end{aligned} \tag{4.1}$$

We can easily check that $\text{epi} f^* = \{(1,0)\}\times\mathbb{R}_+$. Since $G = \{(r,y)\mid \forall q\in\overline{\Omega},\ \text{tr}(q\circ g(r,y))\leqq 0\}$, one has

$$\delta_G(r,y) = \sup_{q\in\overline{\Omega}} tr(q\circ g(r,y)).$$

It follows that

$$\begin{aligned} \text{epi}\delta^*_G &= \text{epi}(\sup_{q\in\overline{\Omega}} \text{tr}(q\circ g(\cdot,\cdot)))^* \\ &= \text{cl}\bigcup_{q\in\overline{\Omega}} \text{epi}(\text{tr}(q\circ g(\cdot,\cdot))^*. \end{aligned}$$

Since

$$(\mathrm{tr}(q \circ g(\cdot,\cdot))^*(s,v) = \begin{cases} \mathrm{tr}(q\circ c) & \text{if } (s,v) = \Big(\mathrm{tr}(q\circ d), -\mathrm{tr}(q\circ a_1), \cdots, -\mathrm{tr}(q\circ a_m)\Big) \\ +\infty & \text{otherwise,} \end{cases}$$

one has $\mathrm{epi}\delta_G^* = \mathrm{cl}\Big[\bigcup_{q\in\overline{\Omega}} \{(\mathrm{tr}(q\circ d), -\mathrm{tr}(q\circ a_1), \cdots, -\mathrm{tr}(q\circ a_m), \mathrm{tr}(q\circ c)\} + \{(0,0)\}\times\mathbb{R}_+\Big]$. Since $H = \{(r,y) \mid \forall\lambda \geqq 0,\ \lambda h(r,y) \leqq 0\}$, one has $\delta_H(r,y) = \sup_{\lambda\geqq 0} \lambda h(r,y)$. So,

$$\mathrm{epi}\delta_H^* = \mathrm{cl}\bigcup_{\lambda\geqq 0} \mathrm{epi}(\lambda h(\cdot,\cdot))^* = \{\lambda(\beta, b_1, \ldots, b_m, \alpha)\} + \{(0,0)\}\times\mathbb{R}_+.$$

From (4.1), one has

$$\begin{aligned}(0,0,-\bar{r}) \in\ & \{(1,0)\}\times\mathbb{R}_+ \\ & +\mathrm{cl}\Big(\bigcup_{q\in\overline{\Omega}} \{(\mathrm{tr}(q\circ d), -\mathrm{tr}(q\circ a_1), \cdots, -\mathrm{tr}(q\circ a_m), \mathrm{tr}(q\circ c)\} \\ & + \bigcup_{\lambda\geqq 0} \{\lambda(\beta, b_1, \cdots, b_m, \alpha)\} + \{(0,0)\}\times\mathbb{R}_+\Big).\end{aligned}$$

Thus, there exist $\bar{q}\in\overline{\Omega}$, $\bar{\lambda}\geqq 0$ and $\bar{r}_+\geqq 0$ such that $(0,0,-\bar{r}) = (1,0,\bar{r}_+) + \Big(\mathrm{tr}(\bar{q}\circ d), -\mathrm{tr}(\bar{q}\circ a_1), \cdots, -\mathrm{tr}(\bar{q}\circ a_m), \mathrm{tr}(\bar{q}\circ c)\Big) + \bar{\lambda}(\beta, b_1, \cdots, b_m, \alpha)$, that is,

$$\begin{aligned} 0 &= 1 + \mathrm{tr}(\bar{q}\circ d) + \bar{\lambda}\beta \\ 0 &= -\mathrm{tr}(\bar{q}\circ a_i) + \lambda b_i,\ \ i = 1, \cdots, m \\ -\bar{r} &= \bar{r}_+ + \mathrm{tr}(\bar{q}\circ c) + \bar{\lambda}\alpha. \end{aligned}$$

Hence, there exist $\bar{q}\in\overline{\Omega}$ and $\bar{\lambda}\geqq 0$ such that

$$\mathrm{tr}(\bar{q}\circ d) + \bar{\lambda}\beta + 1 = 0 \tag{4.2}$$

$$\mathrm{tr}(\bar{q}\circ a_i) - \bar{\lambda} b_i = 0,\ \ i = 1, \cdots, m \tag{4.3}$$

$$\mathrm{tr}(\bar{q}\circ c) + \bar{\lambda}\alpha + \bar{r} \leqq 0. \tag{4.4}$$

Now we assume that $\triangle$ is bounded. Suppose that $\bar{\lambda} = 0$. It follows from (4.2) that $\mathrm{tr}(d\circ\bar{q}) + 1 = 0$ and hence $\bar{q}\neq 0$. For any $x\in\triangle$ and any $\lambda\geqq 0$, $x + \lambda\bar{q}\in\overline{\Omega}$ and $\mathrm{tr}(a_i\circ(x+\lambda\bar{q})) = \mathrm{tr}(a_i\circ x) + \lambda\mathrm{tr}(a_i\circ\bar{q}) = \mathrm{tr}(a_i\circ x) = b_i$, $i = 1, \cdots, m$. So, $x + \lambda\bar{q}\in\triangle$, which contradicts the boundedness of $\triangle$. Thus $\bar{\lambda} > 0$. From (4.3), one has $\frac{1}{\bar{\lambda}}\bar{q}\in\triangle$. From (4.2), one has

$$\bar{r}\,\mathrm{tr}(\bar{q}\circ d) + \bar{\lambda}\bar{r}\beta + \bar{r} = 0. \tag{4.5}$$

In view of (4.4) and (4.5), we have

$$\bar{r}\mathrm{tr}(\bar{q}\circ d) + \bar{\lambda}\bar{r}\beta - \mathrm{tr}(\bar{q}\circ c) - \bar{\lambda}\alpha \geqq 0. \tag{4.6}$$

Since $(\bar{r},\bar{y}) \in F$, we have

$$\mathrm{tr}(\bar{q}\circ c) - \bar{r}\ \mathrm{tr}(\bar{q}\circ d) + \sum_{i=1}^{m} \bar{y}_i \mathrm{tr}(\bar{q}\circ a_i) \geqq 0$$

$$\text{and}\ \ \bar{\lambda}(-\bar{r}\beta + \alpha - \sum_{i=1}^{m} \bar{y}_i b_i) \geqq 0.$$

So, $\bar{r}\ \mathrm{tr}(q\circ d) + \bar{\lambda}\bar{r}\beta - \mathrm{tr}(\bar{q}\circ c) - \bar{\lambda}\alpha \leqq 0$. From (4.6), we conclude

$$\bar{r}\ \mathrm{tr}(\bar{q}\circ d) + \bar{\lambda}\bar{r}\beta - \mathrm{tr}(\bar{q}\circ c) - \bar{\lambda}\alpha = 0.$$

Thus $\bar{r}\ \mathrm{tr}\left(\frac{1}{\bar{\lambda}}\bar{q}\circ d\right) + \bar{r}\beta - \mathrm{tr}\left(\frac{1}{\bar{\lambda}}\bar{q}\circ c\right) - \alpha = 0$ and so $\bar{r} = \dfrac{\mathrm{tr}\left(\frac{1}{\bar{\lambda}}\bar{q}\circ c\right)+\alpha}{\mathrm{tr}\left(\frac{1}{\bar{\lambda}}\bar{q}\circ d\right)+\beta}$. Since $\frac{1}{\bar{\lambda}}\bar{q} \in \triangle$, it follows from the weak duality theorem that $\frac{1}{\bar{\lambda}}\bar{q}$ is an optimal solution to the LFOP. This completes the proof. □

Since the Slater condition for the LFODC implies that the set

$$\bigcup_{q\in\overline{\Omega}} \{(\ \mathrm{tr}(q\circ d), -\mathrm{tr}(q\circ a_1), \cdots, -\mathrm{tr}(q\circ a_m), \mathrm{tr}(q\circ c))\}$$
$$+ \bigcup_{\lambda\geqq 0} \{\lambda(\beta, b_1, \cdots, b_m, \alpha)\} + \{(0,0)\}\times\mathbb{R}_+$$

is closed in $\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}$, we can obtain the following theorem from the above theorem by following the proof of [16, Proposition 5.1]. For the completeness, we still give its proof.

**Theorem 4.4.** *Assume that, for any $x \in \triangle$, $tr(d\circ x)+\beta > 0$, and that $\triangle$ is bounded. Further, assume that there exist $\widetilde{y} \in \mathbb{R}^m$ and $\widetilde{r} \in \mathbb{R}$ such that $c + \sum_{i=1}^m \widetilde{y}_i a_i - \widetilde{r} d \in \Omega$ and $\beta\widetilde{r} + \sum_{i=1}^m \widetilde{y}_i a_i < \alpha$. If $(\bar{r},\bar{y})$ is an optimal solution to the LFODC, then there exists $\bar{x} \in \triangle$ such that $\bar{x}$ is an optimal solution to the LFOP and $\bar{r} = \frac{tr(c\circ\bar{x})+\alpha}{tr(d\circ\bar{x})+\beta}$.*

*Proof.* Let

$$K = \bigcup_{q\in\overline{\Omega}} \{(tr(q\circ d), -\mathrm{tr}(q\circ a_1), \cdots, -\mathrm{tr}(q\circ a_m), \mathrm{tr}(q\circ c))\}$$
$$+ \bigcup_{\lambda\geqq 0} \{\lambda(\beta, b_1, \cdots, b_m, \alpha)\} + \{(0,0)\}\times\mathbb{R}_+.$$

From Theorem 4.3, it is enough to prove that $K$ is closed. Let $g(r,y) = -c + rd - \sum_{i=1}^m y_i a_i$ and $h(r,y) = \sum_{i=1}^m y_i b_i + \beta r - \alpha$. Then by assumption, there exist $\widetilde{r} \in \mathbb{R}$ and $\widetilde{y} \in \mathbb{R}^m$ such that $g(\widetilde{r},\widetilde{y}) \in -\Omega$ and $h(\widetilde{r},\widetilde{y}) < 0$. From the proof of Theorem 4.3, one has

$$K = \bigcup_{(q,\lambda)\in\overline{\Omega}\times\mathbb{R}_+} \mathrm{epi}\Big(tr(q\circ g(\cdot,\cdot)) + \lambda h(\cdot,\cdot)\Big)^*. \tag{4.7}$$

Let $\{(s_n^*, y_n^*, \alpha_n^*)\}$ be a sequence in $K$ such that $(s_n^*, y_n^*, \alpha_n^*) \to (s^*, y^*, \alpha^*) \in \mathbb{R}\times\mathbb{R}^m\times\mathbb{R}$. Then from (4.7), there exist $q_n \in \overline{\Omega}$ and $\mu_n \in \mathbb{R}_+$ such that

$$\mathrm{tr}(q_n \circ g(s_n^*, y_n^*)) + \mu_n h(s_n^*, y_n^*) \leqq \alpha_n^*. \tag{4.8}$$

Since $\mathrm{int}(\overline{\Omega}\times\mathbb{R}_+) \neq \emptyset$, there exist a compact subset $B$ of $\overline{\Omega}\times\mathbb{R}_+$ such that, for any $(q,r) \in B$, $(q,r) \neq (0,0)$ and $\overline{\Omega}\times\mathbb{R}_+ = \{\lambda(q,r) \mid \lambda \geqq 0\ \text{ and }\ (q,r) \in B\}$. Thus there exist $\mu_n \geqq 0$ and

$(b_n,t_n) \in B$ such that $(q_n,\lambda_n) = \mu_n(b_n,t_n)$. Since $B$ is compact, we may assume that $(b_n,t_n) \to (b,t) \in B$.

(i) We consider the case that $\mu_n \to \mu > 0$. Then we may assume that $\mu_n \neq 0$ for all $n$. Since $(q_n,\lambda_n) = \mu_n(b_n,t_n)$, it follows from (4.8) that

$$\Big(\mathrm{tr}(b_n \circ g) + t_n h\Big)^* \Big(\frac{1}{\mu_n}s_n^*, \frac{1}{\mu_n}y_n^*\Big) \leqq \frac{1}{\mu_n}\alpha_n^*$$

for all $n$. For any $(r,y) \in \mathbb{R}\times\mathbb{R}^m$, we have

$$\begin{aligned}\frac{1}{\mu_n}\alpha_n^* &\geqq \Big(\mathrm{tr}(b_n \circ g) + t_n h\Big)^* \Big(\frac{1}{\mu_n}s_n^*, \frac{1}{\mu_n}y_n^*\Big)\\ &\geqq \frac{1}{\mu_n}s_n^* r + \mathrm{tr}\Big(\frac{1}{\mu_n}y_n^* \circ y\Big) - \mathrm{tr}(b_n \circ g(r,y)) - t_n h(r,y).\end{aligned}$$

So, letting $n \to \infty$, we have, for any $(r,y) \in \mathbb{R}\times\mathbb{R}^m$,

$$\frac{1}{\mu}\alpha^* \geqq \frac{1}{\mu}s^* r + \mathrm{tr}\Big(\frac{1}{\mu}y^{*T}y\Big) - \mathrm{tr}(b\circ g(r,y)) - t\ h(r,y).$$

Hence $(\mathrm{tr}(b\circ g)+th)^*(s^*,y^*) \leqq \alpha^*$ and so $(s^*,y^*,\alpha^*) \in \mathrm{epi}(\mathrm{tr}(b\circ g)+t\ h)^* \subset K$. Thus, in this case, $K$ is closed.

(ii) We consider the case that $\mu_n \to 0$. Then $(q_n,\lambda_n) \to 0$, so, for any $(r,y) \in \mathbb{R}\times\mathbb{R}^m$, $\mathrm{tr}(q_n \circ g(r,y)) + (\lambda_n h)(r,y) \to 0$. From (4.8), one has $s_n^* r + \mathrm{tr}(y_n^* \circ y) - \mathrm{tr}(q_n \circ g(r,y)) - \lambda_n h(r,y) \leqq \alpha_n^*$ for any $(r,y) \in \mathbb{R}\times\mathbb{R}^m$. Letting $n \to \infty$, we have

$$s^* r + \mathrm{tr}(y^* \circ y) \leqq \alpha^*$$

for all $(r,y) \in \mathbb{R}\times\mathbb{R}^m$. Thus $(s^*,y^*,\alpha^*) \in \mathrm{epi}(\mathrm{tr}(0\cdot g)+0h)^*$. Since $(0,0) \in \bar{\Omega}\times\mathbb{R}_+$, then $(s^*,y^*,\alpha^*) \in K$. Thus, in this case, $K$ is closed.

(iii) We consider the case that $\mu_n \to +\infty$. Since the sequence $\{(s_n^*,y_n^*,\alpha_n^*)\}$ is bounded, one has

$$\frac{1}{\mu_n}s_n^* \to 0, \quad \frac{1}{\mu_n}y_n^* \to 0 \ \text{ and } \ \frac{1}{\mu_n}\alpha_n^* \to 0.$$

From (4.8), for any $(r,y) \in \mathbb{R}\times\mathbb{R}^m$, one has

$$\frac{\alpha_n}{\mu_n} \geqq \frac{1}{\mu_n}s_n^* r + \mathrm{tr}\Big(\frac{1}{\mu_n}y_n^* \circ y\Big) - \mathrm{tr}(b_n \circ g(r,y)) - t_n h(r,y).$$

Letting $n \to \infty$, we have, for any $(r,y) \in \mathbb{R}\times\mathbb{R}^m$,

$$\mathrm{tr}(b\circ g(r,y)) + t\ h(r,y) \geqq 0. \tag{4.9}$$

From the assumption, there exists $(\tilde{r},\tilde{y}) \in \mathbb{R}\times\mathbb{R}^m$ such that

$$g(\tilde{r},\tilde{y}) \in -\Omega \ \text{ and } \ h(\tilde{r},\tilde{y}) < 0.$$

Since $b \in \bar{\Omega}$ and $t \in \mathbb{R}_+$ and $(b,t) \neq (0,0)$, one has

$$\mathrm{tr}(b\circ g(\tilde{r},\tilde{y})) + t\ h(\tilde{r},\tilde{y}) < 0.$$

This contradicts (4.9). Thus this case does not occur. □

## 5. Characterizations of Solution Sets

Let $\bar{S}$ be the set of solutions of the LFOP. Let $\bar{x} \in \bar{S}$. Then by Theorem 2.1, there exist a sequence $\{y_i^l\}$ in $\mathbb{R}$, $i = 1, \cdots, m$ and a sequence $\{v^l\}$ in $\overline{\Omega}$ such that

$$c - q(\bar{x})d + \lim_{l\to\infty}\Big[\sum_{i=1}^m y_i^l a_i - v^l\Big] = 0$$
$$\text{and } \lim_{l\to\infty} \mathrm{tr}(v^l \circ \bar{x}) = 0,$$

where $q(\bar{x}) = \frac{\mathrm{tr}(c\circ\bar{x})+\alpha}{\mathrm{tr}(d\circ\bar{x})+\beta}$.

By using the above sequences $\{y_i^l\}$ and $\{v^l\}$, we can characterize the solution set $\bar{S}$ as follows:

**Theorem 5.1.** *The set $\bar{S}$ of solutions of the LFOP is as follows:*

$$\bar{S} = \{\widetilde{x} \in F \mid c - q(\widetilde{x})d + \lim_{l\to\infty}\Big[\sum_{i=1}^m y_i^l a_i - v^l\Big] = 0,\ \lim_{l\to\infty} tr(v^l \circ \widetilde{x}) = 0\},$$

*where $q(\widetilde{x}) = \frac{tr(c\circ\widetilde{x})+\alpha}{tr(d\circ\widetilde{x})+\beta}$.*

*Proof.* Let $\widetilde{x} \in \bar{S}$ be any fixed. Then $q(\bar{x}) = q(\widetilde{x})$, so $\mathrm{tr}(c\circ\bar{x}) - q(\bar{x})\mathrm{tr}(d\circ\bar{x}) = \mathrm{tr}(c\circ\widetilde{x}) - q(\widetilde{x})\mathrm{tr}(d\circ\widetilde{x})$. Since $c - q(\bar{x})d + \lim_{l\to\infty}\Big[\sum_{i=1}^m y_i^l a_i - v^l\Big] = 0$, one has

$$\mathrm{tr}(c\circ\bar{x}) - q(\bar{x})\mathrm{tr}(d\circ\bar{x}) + \lim_{l\to\infty}\Big[\sum_{i=1}^m y_i^l \mathrm{tr}(a_i\circ\bar{x}) - \mathrm{tr}(v^l\circ\bar{x})\Big] = 0$$

and

$$\mathrm{tr}(c\circ\widetilde{x}) - q(\widetilde{x})\mathrm{tr}(d\circ\widetilde{x}) + \lim_{l\to\infty}\Big[\sum_{i=1}^m y_i^l \mathrm{tr}(a_i\circ\widetilde{x}) - \mathrm{tr}(v^l\circ\widetilde{x})\Big] = 0.$$

Hence $\lim_{l\to\infty}\Big[\sum_{i=1}^m y_i^l \mathrm{tr}(a_i\circ\bar{x}) - \mathrm{tr}(v^l\circ\bar{x})\Big] = \lim_{l\to\infty}\Big[\sum_{i=1}^m y_i^l \mathrm{tr}(a_i\circ\widetilde{x}) - \mathrm{tr}(v^l\circ\widetilde{x})\Big]$. Since $\lim_{l\to\infty}\mathrm{tr}(v^l\circ\bar{x}) = 0$, one has

$$\lim_{l\to\infty}\sum_{i=1}^m y_i^l \mathrm{tr}(a_i\circ\bar{x}) = \lim_{l\to\infty}\Big[\sum_{i=1}^m y_i^l \mathrm{tr}(a_i\circ\widetilde{x}) - \mathrm{tr}(v^l\circ\widetilde{x})\Big].$$

Since $\bar{x} \in \bar{S}$ and $\widetilde{x} \in \bar{S}$, one has

$$\lim_{l\to\infty}\sum_{i=1}^m y_i^l b_i = \lim_{l\to\infty}\Big[\sum_{i=1}^m y_i^l b_i - \mathrm{tr}(v^l\circ\widetilde{x})\Big].$$

Thus $\lim_{l\to\infty}\mathrm{tr}(v^l\circ\widetilde{x}) = 0$. Hence

$$\bar{S} \subset \{\widetilde{x} \in F \mid c - q(\widetilde{x})d + \lim_{l\to\infty}\Big[\sum_{i=1}^m y_i^l a_i - v^l\Big] = 0,\ \lim_{l\to\infty}\mathrm{tr}(v^l\circ\widetilde{x}) = 0\}.$$

The converse is true by Theorem 2.1. Consequently,

$$\bar{S} = \{\widetilde{x} \in F \mid c - q(\widetilde{x})d + \lim_{l\to\infty}\Big[\sum_{i=1}^m y_i^l a_i - v^l\Big] = 0,\ \lim_{l\to\infty}\mathrm{tr}(v^l\circ\widetilde{x}) = 0\}.$$

□

If $d=0$, $\alpha=0$, and $\beta=1$, then the LFOP becomes the following conic linear optimization problem (CLOP):

$$\text{(CLOP)} \quad \begin{array}{ll} \text{Minimize} & \text{tr}(c\circ x) \\ \text{subject to} & x\in\overline{\Omega},\ \text{tr}(a_i\circ x)=b_i, i=1,\cdots,m. \end{array}$$

Let $\bar{\bar{S}}$ be the set of solutions of the CLOP and let $\bar{x}\in\bar{\bar{S}}$. Then, by Theorem 2.1, there exist a sequence $\{y_i^l\}$ in $\mathbb{R}$, $i=1,\cdots,m$ and a sequence $\{v^l\}$ in $\overline{\Omega}$ such that

$$c+\lim_{l\to\infty}\Big[\sum_{i=1}^m y_i^l a_i - v^l\Big]=0$$
$$\text{and } \lim_{l\to\infty}\text{tr}(v^l\circ\bar{x})=0.$$

By using the above sequences $\{y_i^l\}$ and $\{v^l\}$, we have from Theorem 4.1 the following theorem:

**Theorem 5.2.** *The set $\bar{\bar{S}}$ of soluttions of (CLOP) is as follows:*

$$\bar{\bar{S}}=\{\widetilde{x}\in F \mid \lim_{l\to\infty} tr(v^l\circ\widetilde{x})=0\}.$$

Suppose that $\bigcup_{\lambda_i\in\mathbb{R}}\sum_{i=1}^m\lambda_i(a_i,b_i)+(-\overline{\Omega})\times\mathbb{R}^+$ is closed in $\mathbb{R}^n\times\mathbb{R}$. Let $\bar{S}$ be the set of solutions of the LFOP, and let $\bar{x}\in\bar{S}$. Then by Theorem 2.2, there exist $y_i\in\mathbb{R}, i=1,\cdots,m$ and $v\in\overline{\Omega}$ such that

$$\sum_{i=1}^m y_i a_i-\frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta}d+c-v=0 \text{ and}$$
$$\text{tr}(v\circ\bar{x})=0.$$

By using the above $y_i$ and $v$, we can characterize the solution set $\bar{S}$ as follows:

**Theorem 5.3.** *We have the solution set $\bar{S}$:*

$$\bar{S}=\{\widetilde{x}\in F \mid c-q(\widetilde{x})d+\sum_{i=1}^m y_i a_i-v=0,\ \ tr(v\circ\widetilde{x})=0\},$$

*where $q(\widetilde{x})=\frac{tr(c\circ\widetilde{x})+\alpha}{tr(d\circ\widetilde{x})+\beta}$.*

## 6. Discussion on Methodologies for (LFOP)

Let $V$ be a simple Euclidean Jordan algebra. Recall our linear fractional optimization problem (LFOP):

$$\text{(LFOP)} \quad \begin{array}{ll} \text{Minimize} & \dfrac{\text{tr}(c\circ x)+\alpha}{\text{tr}(d\circ x)+\beta} \\ \text{subject to} & \text{tr}(a_i\circ x)=b_i,\ i=1,\ldots,m \\ & x\in\overline{\Omega}, \end{array}$$

where $c,d\in V$, $\alpha,\beta$ are given real numbers, $a_i, i=1,\cdots,m$ and $b_i, i=1,\cdot,m$ are give real numbers, and $\overline{\Omega}=\{x^2\mid x\in V\}$. Let $\triangle=\{x\in\overline{\Omega}\mid \text{tr}(a_i\circ x)=b_i,\ i=1,\cdots,m\}$.

Consider the following linear optimization problem (LFOPL):

$$\begin{array}{lll}(\text{LFOPL}) & \text{Maximize} & \operatorname{tr}(c\circ y)+\alpha t \\ & \text{subject to} & \operatorname{tr}(d\circ y)+\beta t=1 \\ & & \operatorname{tr}(a_i\circ y)-tb_i=0,\ i=1,\cdots,m.\end{array}$$

Let $\Gamma=\{(y,t)\in\overline{\Omega}\times\mathbb{R}_+|tr(c\circ y)+\alpha t=1,\ tr(a_i\circ y)-tb_i=0,\ i=1,\cdots,m\}$.

Following Charnes and Cooper [17], Craven and Mond [8], and Craven [7], we can prove the following theorem.

**Theorem 6.1.** (Transformation Techniques) *Assume that no point $(y,0)$ with $y\geqq 0$ is feasible for the LFOPL, and for any $x\in\triangle$, $tr(d\circ x)+\beta>0$. Then the followings are true:*

*(i) Define $q(x)=(\frac{1}{tr(d\circ x)+\beta}x,\frac{1}{tr(d\circ x)+\beta})$ for any $x\in\triangle$. Then $q(\triangle)=\Gamma$, and $q$ is one to one and onto between $\triangle$ and $\Gamma$ and $q^{-1}(y,t)=\frac{1}{t}y$ for any $(y,t)\in\Gamma$.*

*(ii) If $x^*$ is an optimal solution to the LFOP, then $(\frac{1}{tr(d\circ x^*)+\beta}x^*,\frac{1}{tr(d\circ x^*)+\beta})$ is an optimal solution to the LFOPL.*

*(iii) If $(y^*,t^*)$ is an optimal solution to the LFOPL, then $\frac{1}{t^*}y^*$ is an optimal solution to the LFOP.*

**Remark 6.1.** Theorem 6.1 means that we can use the LFOPL to solve the LFOP. Define, for any $x,y\in V$ and any $s,t\in\mathbb{R}$, $(x,s)\circ(y,t)=x\circ y+st$ and $((x,s)|(y,t))=(x|y)+st$. Since $V$ is a simple Euclidean Jordan algebra, $V\times\mathbb{R}$ is an Euclidean Jordan algebra with product $\circ$ and inner product $(\cdot|\cdot)$. So, by using primal-dual interior methods (path-following methods), for example, Baes [13], Choi and Lee [15], and Faybusovich [14] and the assumptions in Theorem 6.1, we can obtain algorithms for solving the LFOP.

## 7. The Conclusion

In this paper, by using the nonfractional technique, we obtained the optimality theorems, duality theorems and the characterizations of solution sets for linear fractional optimization problems, which hold without any constraint qualifications and under constraint qualifications. By using the transformation technique, we discussed primal-dual interior methods for linear fractional optimization problems. In the near future, we would like to study the existence of optimal solution, genericity properties, and sensitivity and stability results for linear fractional optimization problems.

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