J. Nonlinear Var. Anal. 6 (2022), No. 2, pp. 83-99
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.6.2022.2.06

CONSTRAINT QUALIFICATIONS IN NONSMOOTH OPTIMIZATION: CLASSIFICATION AND INTER-RELATIONS

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Abstract. The aim of this paper is to systematically study various known and some new constraint qualifications for a nonsmooth optimization problem constrained by inequality constraints where the functions involved are locally Lipschitz continuous. We classify the constraint qualifications into four levels by using the inclusion relations among the cones of interior constrained directions, feasible directions, attainable directions, tangent directions, and locally constrained directions. Numerous inter-relationships between the constraint qualifications are summarized schematically. We further discuss the nature of various cones of the feasible set by assuming the constraint functions to be semilocally convex, and establish the equivalence among some of the constraint qualifications.

Keywords. Constraint qualifications; Locally Lipschitz; Clarke subdifferential; Karush–Kuhn–Tucker optimality conditions; Semilocal Convexity.

1. Introduction

Constraint qualification is a condition imposed on the constraints of a nonlinear programming problem at a point, which guarantees that necessary optimality conditions, namely Karush-Kuhn-Tucker (KKT) conditions, hold at that point. Geometrically, a constraint qualification is an assumption, which ensures the similarity of the feasible set at a feasible point with its linear approximation in a neighborhood of that point. In fact, a constraint qualification guarantees that, for the local approximation of the feasible set at a feasible point, the analytic representation (the set of locally constrained directions) and the geometric representation (the set of tangent directions) are related. Based on this idea and the inclusion relations of the cones of tangent directions, attainable directions, feasible directions, and interior constrained directions, Wang *et al.* [1] categorized the constraint qualifications for smooth optimization problems into four levels by their relative strengths. Moreover, the authors reviewed many constraint qualifications available in the literature, identified the levels they belong to, and studied the inter-relationships among them. One may refer to the papers [2–9] for the classical definition of various constraint qualifications.

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Received December 24, 2021; Accepted February 28, 2022.

As many nonlinear nonsmooth problems exist both in theory and computation, the generalization of constraint qualifications and KKT conditions have been studied extensively in nonsmooth case. The KKT conditions were generalized by replacing the usual gradient by certain generalized gradients as in Rockafellar [10], Clarke [11], Michel and Penot [12], Ioffe [13], and Mordukhovich [14]. A detailed description about various generalized derivatives, namely Gâteaux, Dini, Dini-Hadamard, Clarke, and Michel-Penot derivatives, can be found in [15, Chapter 2]. The constraint qualifications in terms of such generalized gradients have been explored in the literature; see, e.g., [16–30] for more details. In fact, Giorgi *et al.* [16] introduced the numerous notions of cones of directions for nonsmooth optimization problems and investigated their inclusion relations. Moreover, they proved the KKT optimality conditions in terms of cone-directional derivatives and cone-subdifferentials.

For the specific case of locally Lipschitz objective and constraint functions, constraint qualifications in terms of Clarke generalized gradients have been studied mostly in the context of semi-infinite programming problems. Kanzi and Nobakhtian [20] extended the notion of Abadie and Zangwill constraint qualifications for a semi-infinite programming problem with a feasible set defined by inequality and equality constraints and a set constraint. Kanzi and Nobakhtian [21] considered Abadie, Guignard, and Zangwill constraint qualifications with inequality constraints only. Kanzi and Karimi [19] gave notions of Arrow-Hurwicz-Uzawa and Slater constraint qualification for the problems with inequality constraints. Further, Kanzi [22] gave analogues of the Guignard, Kuhn-Tucker, and Cottle constraint qualifications for a semi-infinite programming problem with the feasible set defined by inequality constraints. Later on, Kanzi [23, 24] considered the extended versions of the most of these constraint qualifications and the Mangasarian-Fromovitz constraint qualification for semi-infinite problems in terms of Michel-Penot subdifferentials.

In this paper, we classify and study the inter-relations of various constraint qualifications in terms of Clarke generalized gradients for a nonsmooth optimization problem with inequality constraints. Motivated by the classification of the constraint qualifications for the differentiable case proposed by Wang *et al.* in [1] and the inclusion relation of various cones for nonsmooth problems given in [16], we divide almost all the existing constraint qualifications into four levels. In this process, some new constraint qualifications are also proposed for the nonsmooth case. We also identify two constraint qualifications which were not considered in the differentiable case.

The rest of the paper is organised as follows. Section 2 consists of some definitions and results from convex analysis and nonsmooth analysis, which will be needed in the sequel. Based on the set inclusion relation among various cones of directions given in Section 2, we categorize the constraint qualifications into four levels in Section 3 and study their inter-relations which are summarized in Figure 2. Some counter examples are also given to validate the strict implications, wherever exist, in above relations. Section 4 provides modified relations of various constraint qualifications considered in Section 3 by imposing additional assumption of semilocal convexity at a point on the active constraints. For this case, we prove that the tangent cone is the closure of the cone of feasible directions, which is employed to establish the equivalence of some of the constraint qualifications. We conclude the paper with some remarks in Section 5.

2. Preliminaries

For a nonempty set $S \subseteq \mathbb{R}^n$, the closure, interior, convex hull, and conical hull of S are denoted by $\text{cl}(S)$, $\text{int}(S)$, $\text{conv}(S)$, and $\text{cone}(S)$, respectively. The negative polar and strict negative polar cones of S are defined by

$$S^* := \{d \in \mathbb{R}^n : \langle x, d\rangle \leq 0, \text{ for all } x \in S\},$$
$$S^- := \{d \in \mathbb{R}^n : \langle x, d\rangle < 0, \text{ for all } x \in S\},$$

respectively. Observe that if $S^- \neq \emptyset$, then $\text{cl}(S^-) = S^*$.

We denote open ball of radius r and center at $x \in \mathbb{R}^n$ by $B_r(x)$.

Lemma 2.1. [31, Proposition 4.2.7] *Let S_1 and S_2 be any nonempty subsets of $\mathbb{R}^n$.*

(i) *S_1^* is a closed convex cone and $S_1^{**} = \text{cl}(\text{cone}(S_1))$.*
(ii) *If $S_1 \subseteq S_2$, then $S_2^* \subseteq S_1^*$.*

We now recall various notions of cones at a point in a set. These cones are local approximation of the set at that point. One may refer to [16] for more details about these cones.

Definition 2.1. Let S be a nonempty subset of $\mathbb{R}^n$ and $\bar{x} \in S$.

(i) [9, p. 37] The cone of feasible directions at $\bar{x}$, denoted by $D(S,\bar{x})$, is defined as
$$D(S,\bar{x}) := \left\{d \in \mathbb{R}^n : \text{there exists } \bar{t} > 0 \text{ with } \bar{x} + td \in S, \text{ for all } t \in [0,\bar{t}\,]\right\}.$$
(ii) [3, p. 177] The cone of attainable directions at $\bar{x}$, denoted by $A(S,\bar{x})$, is defined as
$$A(S,\bar{x}) := \Big\{d \in \mathbb{R}^n : \text{there exist a continuous path } \gamma : \mathbb{R} \to \mathbb{R}^n,\ \bar{t} > 0 \text{ such that}$$
$$\gamma(0) = \bar{x},\ \gamma(t) \in S \text{ for all } t \in [0,\bar{t}\,] \text{ and } d = \lim_{t\to 0_+} \frac{\gamma(t) - \gamma(0)}{t}\Big\}.$$
(iii) [2, p. 32] The contingent cone at $\bar{x}$, denoted by $T(S,\bar{x})$, is defined as
$$T(S,\bar{x}) := \Big\{d \in \mathbb{R}^n : \text{there exist sequences } (x_k)_{k\in\mathbb{N}} \subseteq S \text{ with } x_k \to \bar{x},\ t_k \downarrow 0$$
$$\text{such that } \frac{x_k - \bar{x}}{t_k} \to d\Big\}.$$

Clearly, the cones $A(S,\bar{x})$ and $T(S,\bar{x})$ can be equivalently defined as

$$A(S,\bar{x}) := \big\{d \in \mathbb{R}^n : \text{for every } t_k \downarrow 0, \text{ there exists a sequence } (d_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^n \text{ with}$$
$$d_k \to d \text{ such that } \bar{x} + t_k d_k \in S\big\},$$
$$T(S,\bar{x}) := \big\{d \in \mathbb{R}^n : \text{there exist sequences } (d_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^n, \text{ with } d_k \to d \text{ and } t_k \downarrow 0$$
$$\text{such that } \bar{x} + t_k d_k \in S\big\}.$$

The cones $A(S,\bar{x})$ and $T(S,\bar{x})$ are closed cones, but $D(S,\bar{x})$ is not necessarily closed. If S is convex, then $D(S,\bar{x})$, $A(S,\bar{x})$ and $T(S,\bar{x})$ are all convex cones. One may refer to [32] for more details.

Before proceeding further, we recall from [11] that a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is upper semicontinuous at $\bar{x} \in \mathbb{R}^n$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $F(x) \subseteq F(\bar{x}) + B_\varepsilon(0)$, for all $x \in B_\delta(\bar{x})$.

We next consider some notions and results of nonsmooth analysis from [11].

Definition 2.2. [11, pp. 25–27] The generalized Clarke directional derivative of a locally Lipschitz function $f:\mathbb{R}^n \to \mathbb{R}$ at $\bar{x} \in \mathbb{R}^n$ in the direction of $d \in \mathbb{R}^n$ is defined as

$$f^\circ(\bar{x},d) := \limsup_{\substack{y\to\bar{x}\\ t\downarrow 0}} \frac{f(y+td)-f(y)}{t}.$$

The Clarke subdifferential of f at $\bar{x}$ is defined as

$$\partial_c f(\bar{x}) := \{\xi \in \mathbb{R}^n : \langle \xi, d\rangle \leq f^\circ(\bar{x},d), \text{ for all } d \in \mathbb{R}^n\}.$$

The set $\partial_c f(\bar{x})$ is a nonempty, compact, and convex subset of $\mathbb{R}^n$. By Proposition 2.1.1 (a) and Proposition 2.1.5 (d) of [11], the set-valued map $\partial_c f$ is upper semicontinuous at every $x \in \mathbb{R}^n$, and $d \mapsto f^\circ(\bar{x},d)$ is a positively homogeneous subadditive function on $\mathbb{R}^n$.

We now state the Lebourg mean-value theorem in terms of Clarke subdifferential.

Theorem 2.1. [11, Theorem 2.3.7] *Let $x,u \in \mathbb{R}^n$ and $f:\mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function on an open set containing $[x,u]$. Then there exists $z \in (x,u)$ such that*

$$f(u)-f(x) \in \langle \partial_c f(z), u-x\rangle,$$

where $\langle \partial_c f(z), u-x\rangle = \big\{\langle \xi, u-x\rangle : \xi \in \partial_c f(z)\big\}$.

Following are the notions of pseudoconcavity and pseudolinearity defined in terms of the Clarke directional derivative. One may refer to [15, Chapter 3] for a unified study of generalized nonsmooth convexity in terms of various generalized derivatives.

Definition 2.3. [33, Definition 7] A locally Lipschitz function $f:\mathbb{R}^n \to \mathbb{R}$ is said to be locally pseudoconvex at $\bar{x} \in \mathbb{R}^n$ if there exists a neighbourhood N of $\bar{x}$ such that

$$f^\circ(\bar{x}, x-\bar{x}) \geq 0,\ x \in N \Longrightarrow\ f(x) \geq f(\bar{x}).$$

The function f is said to be locally pseudoconcave at $\bar{x}$ if $-f$ is locally pseudoconvex at $\bar{x}$. Moreover, f is said to be locally pseudolinear at $\bar{x}$ if it is both locally pseudoconvex and locally pseudoconcave at $\bar{x}$.

Consider the following nonsmooth optimization problem

$$\begin{aligned} \text{Minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1,2,\ldots,m, \\ & x \in \mathbb{R}^n, \end{aligned} \tag{P}$$

where f and $g_i,\ i=1,2,\ldots,m$, are real valued locally Lipschitz functions defined on $\mathbb{R}^n$. Let $I = \{1,2,\ldots,m\}$ and S denote the set of feasible points of (P), that is,

$$S := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i \in I\}.$$

For any $\bar{x} \in S$, the set of active indices at $\bar{x}$, denoted by $I(\bar{x})$, is defined as

$$I(\bar{x}) := \{i \in I : g_i(\bar{x}) = 0\}.$$

For $\bar{x} \in S$, we consider the set

$$B(\bar{x}) := \bigcup_{i \in I(\bar{x})} \partial_c g_i(\bar{x}).$$

Clearly, $B(\bar{x})$ is a compact subset of $\mathbb{R}^n$ being a finite union of compact sets. One can easily see that $B(\bar{x})^* = (\text{cone}(B(\bar{x})))^* = (\text{cl}(\text{cone}(B(\bar{x}))))^*$.

The following notions of cones are from [16, p. 450], where the authors also provided a more general form of these cones in [16, Definition 4.8.1].

Definition 2.4. Let $\bar{x} \in S$.

(i) The cone of locally constrained directions at $\bar{x}$, denoted by $G(S,\bar{x})$, is defined as

$$G(S,\bar{x}) := \Big\{ d \in \mathbb{R}^n : \langle \xi, d \rangle \leq 0, \text{ for all } \xi \in \bigcup_{i \in I(\bar{x})} \partial_c g_i(\bar{x}) \Big\}.$$

In other words, $G(S,\bar{x})$ is the negative polar cone of $B(\bar{x})$.

(ii) The cone of interior constrained directions at $\bar{x}$, denoted by $G^-(S,\bar{x})$, is defined as the strict negative polar cone of $B(\bar{x})$, that is,

$$G^-(S,\bar{x}) := \Big\{ d \in \mathbb{R}^n : \langle \xi, d \rangle < 0, \text{ for all } \xi \in \bigcup_{i \in I(\bar{x})} \partial_c g_i(\bar{x}) \Big\}.$$

It is obvious from the above definition that $0 \notin G^-(S,\bar{x})$ and $G^-(S,\bar{x}) = \operatorname{int}(G(S,\bar{x}))$. Moreover, $\operatorname{cl}(G^-(S,\bar{x})) = G(S,\bar{x})$ provided $G^-(S,\bar{x}) \neq \emptyset$.

The following theorem relates various cones defined so far. The proof follows from Theorem 4.8.15 of [16] and the inclusion diagram of cones given in [16, p. 243].

Theorem 2.2. *For $\bar{x} \in S$, $G^-(S,\bar{x}) \subseteq D(S,\bar{x}) \subseteq A(S,\bar{x}) \subseteq T(S,\bar{x})$.*

We now have a consequence of Theorem 2.2 and Lemma 2.1, which is required in the sequel to classify various constraint qualifications into different levels.

Corollary 2.1. *For $\bar{x} \in S$, $T(S,\bar{x})^* \subseteq A(S,\bar{x})^* \subseteq D(S,\bar{x})^* \subseteq G^-(S,\bar{x})^*$.*

The following Karush-Kuhn-Tucker optimality condition holds under a constraint qualification and an additional closedness assumption of the cone generated by collection of subdifferentials of the active constraints at an optimal point.

Theorem 2.3. [23, Theorem 4.3] *Let $\bar{x}$ be an optimal solution of (P), $T(S,\bar{x})^* \subseteq G(S,\bar{x})^*$, and* $\operatorname{cone}(B(\bar{x}))$ *be closed. If the function $d \mapsto f^\circ(\bar{x},d)$ is concave or $G(S,\bar{x}) \subseteq T(S,\bar{x})$, then there exist scalars $\lambda_i \geq 0$, $i \in I(\bar{x})$, such that*

$$0 \in \partial_c f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial_c g_i(\bar{x}).$$

Remark 2.1. For Clarke regular function (see Remark 3.2 in [34]), the above inclusion condition can be equivalently written as

$$0 \in \partial_c \left(f + \sum_{i \in I(\bar{x})} \lambda_i g_i \right)(\bar{x}).$$

Remark 2.2. A sufficient condition for the $\operatorname{cone}(B(\bar{x}))$ to be closed is $0 \notin \operatorname{conv}(B(\bar{x}))$ as given in [31, Proposition 1.4.7]. Moreover, as illustrated in [16, pp. 442–443], the closedness of $\operatorname{cone}(B(\bar{x}))$ follows from the condition that $G^-(S,\bar{x}) \neq \emptyset$. However, in the smooth case, we note that $\operatorname{cone}(B(\bar{x}))$ is a closed set being a finitely generated cone.

3. Classification of Constraint Qualifications

In this section, we categorize constraint qualifications into four categories on the basis of inclusion relations given in Corollary 2.1. We also investigate the inter-relations among various constraint qualifications and justify the strict relations through various examples of constraint functions in nonsmooth setting. For $\bar{x} \in S$, the levels are categorized in terms of the following four inclusions,

(i) $T(S,\bar{x})^* \subseteq G(S,\bar{x})^*$;
(ii) $A(S,\bar{x})^* \subseteq G(S,\bar{x})^*$;
(iii) $D(S,\bar{x})^* \subseteq G(S,\bar{x})^*$;
(iv) $G^-(S,\bar{x})^* \subseteq G(S,\bar{x})^*$.

We note that the relative strength of the constraint qualifications in the different levels increases as we move from one level to the next level. We now investigate the constraint qualifications in each level separately.

Level 1: Constraint qualifications in this level ensure the relation $T(S,\bar{x})^* \subseteq G(S,\bar{x})^*$. Hence, constraint qualifications given by the relations between $G(S,\bar{x})$ and $T(S,\bar{x})$ and their closed or convex hulls belong to this level. As $T(S,\bar{x})$ is a closed cone, the possibilities in this case are $G(S,\bar{x}) \subseteq T(S,\bar{x})$, $G(S,\bar{x}) \subseteq \text{conv}(T(S,\bar{x}))$, $G(S,\bar{x}) \subseteq \text{cl}(\text{conv}(T(S,\bar{x})))$, and $T(S,\bar{x})^* \subseteq G(S,\bar{x})^*$. By Lemma 2.1,

$$G(S,\bar{x}) \subseteq \text{cl}(\text{conv}(T(S,\bar{x}))) \iff T(S,\bar{x})^* \subseteq G(S,\bar{x})^*.$$

Thus, there are three different constraint qualifications in this class. Two of the constraint qualifications in this class are identified in literature, namely Abadie constraint qualification (ACQ) [21, Definition 3.1(a)] and Guignard constraint qualification (GCQ) [21, Definition 3.1(c)], which are given by conditions $G(S,\bar{x}) \subseteq T(S,\bar{x})$ and $G(S,\bar{x}) \subseteq \text{cl}(\text{conv}(T(S,\bar{x})))$, respectively.

For the differentiable case, Wang *et al.* [1] identified only these two constraint qualifications in this level as they assumed that $\text{conv}(T(S,\bar{x})) = \text{cl}(\text{conv}(T(S,\bar{x})))$. However, Fenchel [35] pointed out that, for a cone K in $\mathbb{R}^n$, $\text{cl}(\text{conv}(K))) = \text{conv}(\text{cl}(K))$ provided $n \leq 2$ or K is polyhedral. Figure 1 illustrates the cone considered in [35], where $K = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : (x_1 - |x_3|)^2 + x_2^2 \leq x_3^2\}$. It can be seen that $\text{conv}(K) = \{(d_1,d_2,d_3) \in \mathbb{R}^3 : d_1 > 0, \text{ or } d_1 = d_2 = 0\}$ and $\text{cl}(\text{conv}(K)) = \{(d_1,d_2,d_3) \in \mathbb{R}^3 : d_1 \geq 0\}$.

We now define the third constraint qualification in this level.

Definition 3.1. Let $\bar{x} \in S$. Then the constraint qualification CQ1 holds at $\bar{x}$ if

$$G(S,\bar{x}) \subseteq \text{conv}(T(S,\bar{x})).$$

Kanzi and Nobakhtian in [21, Definition 3.1(b)] defined a constraint qualification, namely basic constraint qualification (BCQ), which holds at $\bar{x}$ if $T(S,\bar{x})^* \subseteq \text{cone}(B(\bar{x}))$. As $\text{cone}(B(\bar{x}))$ is assumed to be closed, we conclude from Lemma 2.1 that BCQ and GCQ are equivalent.

From the above definitions, we have the following implication relations.

$$\text{ACQ} \implies \text{CQ1} \implies \text{GCQ}.$$

However, the converse of above implications may fail to hold as illustrated in the following examples.

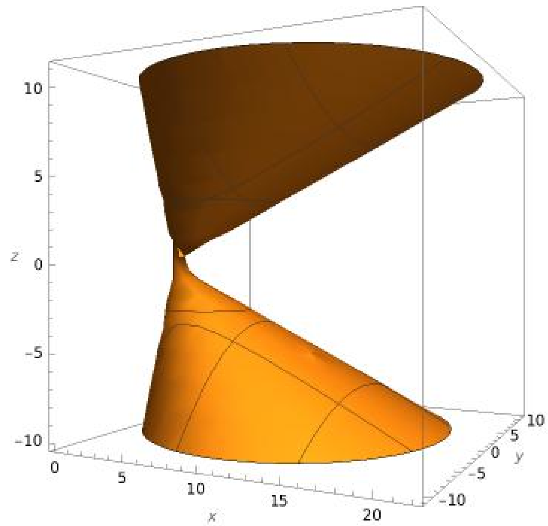

FIGURE 1. $K = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : (x_1 - |x_3|)^2 + x_2^2 \leq x_3^2\}$

Example 3.1. Let $g_i : \mathbb{R}^3 \to \mathbb{R}, i = 1,2$ be functions defined as

$$g_1(x_1,x_2,x_3) = (x_1 - |x_3|)^2 + x_2^2 - x_3^2,$$
$$g_2(x_1,x_2,x_3) = -x_1.$$

Clearly, the feasible set is $S = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : (x_1 - |x_3|)^2 + x_2^2 - x_3^2 \leq 0\}$. At the feasible point $\bar{x} = (0,0,0)$, we have $I(\bar{x}) = \{1,2\}$ and $T(S,\bar{x}) = S$. Since $\nabla g_1(\bar{x}) = \{(0,0,0)\}$ and $\nabla g_2(\bar{x}) = \{(-1,0,0)\}$, we observe that $G(S,\bar{x}) = \text{cl}(\text{conv}(T(S,\bar{x})))$, but $G(S,\bar{x}) \not\subseteq \text{conv}(T(S,\bar{x}))$. Thus, GCQ holds at $\bar{x}$ but CQ1 fails to hold at $\bar{x}$.

Example 3.2. Let $g_i : \mathbb{R}^2 \to \mathbb{R}, i = 1,2$ be functions defined as

$$g_1(x_1,x_2) = x_1|x_2|$$
$$g_2(x_1,x_2) = -x_1|x_2|.$$

Then the feasible set is $S = \{(x_1,x_2) \in \mathbb{R}^2 : x_1x_2 = 0\}$. At the feasible point $\bar{x} = (0,0)$, we have $I(\bar{x}) = \{1,2\}$ and $\partial_c g_i(\bar{x}) = \{(0,0)\}$ for $i \in I(\bar{x})$. Also, $T(S,\bar{x}) = S$ and $G(S,\bar{x}) = \mathbb{R}^2 = \text{conv}(T(S,\bar{x}))$. Hence CQ1 holds at $\bar{x}$ but ACQ fails to hold at $\bar{x}$.

Level 2: Constraint qualifications involving relations between $G(S,\bar{x})$ and $A(S,\bar{x})$ and their closed or convex hulls which ensure the relation $A(S,\bar{x})^* \subseteq G(S,\bar{x})^*$, correspond to this level. As $A(S,\bar{x})$ is also a closed set, we observe as in Level 1, the only different possibilities in this class are $G(S,\bar{x}) \subseteq A(S,\bar{x})$, $G(S,\bar{x}) \subseteq \text{conv}(A(S,\bar{x}))$, and $G(S,\bar{x}) \subseteq \text{cl}(\text{conv}(A(S,\bar{x})))$. In literature, the condition $G(S,\bar{x}) \subseteq A(S,\bar{x})$ is known as Kuhn-Tucker constraint qualification (KTCQ) [22, Definition 3.1(b)] and $G(S,\bar{x}) \subseteq \text{cl}(\text{conv}(A(S,\bar{x})))$ is a generalization of Arrow-Hurwicz-Uzawa constraint qualification (AHUCQ) [3, p. 178] to problem (P).

We now define the third constraint qualification belonging to this level.

Definition 3.2. The constraint qualification CQ2 is said to be satisfied at $\bar{x} \in S$ if

$$G(S,\bar{x}) \subseteq \text{conv}(A(S,\bar{x})).$$

It follows immediately from the above definitions that

$$\text{KTCQ} \implies \text{CQ2} \implies \text{AHUCQ}.$$

However, the reverse implications need not be true in general, for instance, in Example 3.1, AHUCQ holds at $\bar{x}$ but CQ2 fails to hold at $\bar{x}$ as $A(S,\bar{x}) = T(S,\bar{x})$. Similarly, from Example 3.2 we infer that CQ2 is strictly weaker than KTCQ.

Level 3: Constraint qualifications defined by the inclusion relations of $G(S,\bar{x})$ and $D(S,\bar{x})$ and their closed or convex cones which fulfill the condition $D(S,\bar{x})^* \subseteq G(S,\bar{x})^*$ are included in this level. Following the similar argument as in Level 1 and using the fact that $D(S,\bar{x})$ is not necessarily closed, we have four different possibilities in this level. They are $G(S,\bar{x}) \subseteq D(S,\bar{x})$, $G(S,\bar{x}) \subseteq \text{cl}(D(S,\bar{x}))$, $G(S,\bar{x}) \subseteq \text{conv}(D(S,\bar{x}))$, and $G(S,\bar{x}) \subseteq \text{cl}(\text{conv}(D(S,\bar{x})))$. In literature, in the nonsmooth setting only one constraint qualification has been considered in this class, namely Zangwill constraint qualification (ZCQ) [21, Definition 3.1(d)] which is given by the condition $G(S,\bar{x}) \subseteq \text{cl}((D(S,\bar{x})))$.

Based on other possibilities, we next define remaining constraint qualifications belonging to this level.

Definition 3.3. Let $\bar{x} \in S$.

(i) The constraint qualification CQ3 holds at $\bar{x}$ if $G(S,\bar{x}) \subseteq D(S,\bar{x})$.
(ii) The constraint qualification CQ4 holds at $\bar{x}$ if $G(S,\bar{x}) \subseteq \text{conv}(D(S,\bar{x}))$.
(iii) The constraint qualification CQ5 holds at $\bar{x}$ if $G(S,\bar{x}) \subseteq \text{cl}(\text{conv}(D(S,\bar{x})))$.

Remark 3.1. From the above definitions, it is clear that

(i) CQ3 $\Longrightarrow$ ZCQ $\Longrightarrow$ CQ5;
(ii) CQ3 $\Longrightarrow$ CQ4 $\Longrightarrow$ CQ5.

We now provide counter examples to above implications. Observe that, in Example 3.2, CQ5 and CQ4 hold at $\bar{x}$, but both ZCQ and CQ3 fail to hold at $\bar{x}$ as $D(S,\bar{x}) = T(S,\bar{x})$.

The following example illustrates that both ZCQ and CQ5 hold at a feasible point while neither CQ3 nor CQ4 hold at that point.

Example 3.3. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function defined as

$$g(x_1,x_2) = \max\{x_1^2, x_1\} - x_2.$$

At the feasible point $\bar{x} = (0,0)$, we observe that $D(S,\bar{x}) = \{(d_1,d_2) : d_1 - d_2 \leq 0,\ d_2 > 0\} \cup \{(0,0)\}$ and $g^\circ(\bar{x},d) = \max\{d_1 - d_2, -d_2\}$. Thus, $G(S,\bar{x}) = \text{cl}\,(D(S,\bar{x}))$. Hence, ZCQ holds at $\bar{x}$, but CQ3 fails to hold at $\bar{x}$. Moreover, as $D(S,\bar{x})$ is a convex set, CQ5 holds at $\bar{x}$ while CQ4 does not hold at $\bar{x}$.

Level 4: We design constraint qualifications belonging to this level, as in other levels, in terms of relations between $G(S,\bar{x})$ and $G^-(S,\bar{x})$ and their closed or convex hulls which ensure the relation $G^-(S,\bar{x})^* \subseteq G(S,\bar{x})^*$. This relation obviously holds when $G^-(S,\bar{x}) \neq \emptyset$. In this situation, we know that $\text{cl}(G^-(S,\bar{x})) = G(S,\bar{x})$. Thus, it is easy to see that all the possibilities hold only when $G^-(S,\bar{x}) \neq \emptyset$. Hence, this condition is the only constraint qualification in this class and known as Cottle's constraint qualification (CCQ) [22, Definition 3.1(c)].

We consider other constraint qualifications also in this level which guarantee the condition $G(S,\bar{x}) = \text{cl}(G^-(S,\bar{x}))$. Firstly, we recall a constraint qualification equivalent to CCQ, namely

Mangasarian-Fromovitz constraint qualification (MFCQ) [23, Definition 3.3], which holds at $\bar{x} \in S$ if there exists $d \in \mathbb{R}^n$ such that $g_i^\circ(\bar{x}, d) < 0$ for all $i \in I(\bar{x})$.

Theorem 3.1. *Let $\bar{x} \in S$. Then MFCQ and CCQ are equivalent at $\bar{x}$.*

Proof. The proof follows easily as we observe that for $d \in \mathbb{R}^n$ if $g_i^\circ(\bar{x}, d) < 0$, for all $i \in I(\bar{x})$, then $\langle \xi, d \rangle < 0$, for all $\xi \in \bigcup_{i \in I(\bar{x})} \partial_c g_i(\bar{x})$, that is, $G^-(S, \bar{x}) \neq \emptyset$. Conversely, any $d \in G^-(S, \bar{x})$ satisfies the condition that $g_i^\circ(\bar{x}, d) < 0$ for all $i \in I(\bar{x})$. □

We next consider the strongest constraint qualification, namely the linear independence constraint qualification (LICQ) [24, Definition 3.10], which holds at $\bar{x} \in S$ if, for every $\xi_i \in \partial_c g_i(\bar{x})$, $i \in I(\bar{x})$, $\{\xi_i : i \in I(\bar{x})\}$ is linearly independent.

We now characterize MFCQ in terms of a weaker form of linearly independent notion given in [36, p. 734]. A set of m vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbb{R}^n$ is said to be positively linearly independent if $\sum_{i=1}^{m} \lambda_i v_i = 0$, for $\lambda_i \geq 0, i \in \{1, 2, \ldots m\}$, implies each $\lambda_i = 0$. The proof can be obtained collectively from Theorem 3.7 and Theorem 3.9 as given in [24].

Theorem 3.2. *Let $\bar{x} \in S$. Then MFCQ holds at $\bar{x}$ if and only if for every $\xi_i \in \partial_c g_i(\bar{x})$, $i \in I(\bar{x})$, the set $\{\xi_i : i \in I(\bar{x})\}$ is positively linearly independent.*

As every linearly independent set is positively linearly independent, in view of Theorem 3.2, we have that the LICQ is stronger than the MFCQ. We now give an example to show that the converse of the above implication fails to hold.

Example 3.4. Let $g_i : \mathbb{R}^2 \to \mathbb{R},\ i = 1, 2$ be functions defined as

$$g_1(x_1, x_2) = x_1 - |x_2|,$$
$$g_2(x_1, x_2) = x_1.$$

At the feasible point $\bar{x} = (0, 0)$, we have $I(\bar{x}) = \{1, 2\}$, $g_1^\circ(\bar{x}, d) = d_1 + |d_2|$, and $g_2^\circ(\bar{x}, d) = d_1$, for all $d = (d_1, d_2) \in \mathbb{R}^2$. Thus, $\partial_c g_1(\bar{x}) = \{(1, t) : -1 \leq t \leq 1\}$ and $\partial_c g_2(\bar{x}) = \{(1, 0)\}$. It can be seen that, for every $\xi_i \in \partial_c g_i(\bar{x})$, $i \in I(\bar{x})$, the set $\{\xi_i : i \in I(\bar{x})\}$ is positively linearly independent. However, this set is not linearly independent for $\xi_1 = \xi_2 = (1, 0)$. Hence, the MFCQ holds at $\bar{x}$, but LICQ fails to hold at $\bar{x}$.

We further define two constraint qualifications, which are weaker than Mangasarian-Fromovitz constraint qualification. They are defined in terms of cones lying between the cone of interior constrained directions and the cone of feasible directions at a given feasible point. We define two cones of directions $G^1(S, \bar{x})$ and $G^2(S, \bar{x})$ at $\bar{x} \in S$ by separating locally pseudolinear and locally pseudoconcave active constraints at $\bar{x}$, respectively. The mathematical formulation of sets $G^1(S, \bar{x})$ and $G^2(S, \bar{x})$ is given as

$$G^1(S, \bar{x}) := \Big\{ d \in \mathbb{R}^n : \quad \langle \xi, d \rangle \leq 0, \text{ for all } \xi \in \bigcup_{i \in L(\bar{x})} \partial_c g_i(\bar{x});$$
$$\langle \xi, d \rangle < 0, \text{ for all } \xi \in \bigcup_{i \in I(\bar{x}) \setminus L(\bar{x})} \partial_c g_i(\bar{x}) \Big\},$$

$$G^2(S,\bar{x}) := \Big\{ d \in \mathbb{R}^n : \quad \langle \xi, d \rangle \leq 0, \text{ for all } \xi \in \bigcup_{i \in M(\bar{x})} \partial_c g_i(\bar{x}); \\ \langle \xi, d \rangle < 0, \text{ for all } \xi \in \bigcup_{i \in I(\bar{x}) \setminus M(\bar{x})} \partial_c g_i(\bar{x}) \Big\},$$

$$\text{where} \quad L(\bar{x}) := \{ i \in I(\bar{x}) : g_i \text{ is locally pseudolinear at } \bar{x} \} \text{ and}$$
$$M(\bar{x}) := \{ i \in I(\bar{x}) : g_i \text{ is locally pseudoconcave at } \bar{x} \}.$$

We now redefine second Arrow-Hurwicz-Uzawa constraint qualification [19, Definition 3.2] in terms of the cone $G^2(S,\bar{x})$.

Definition 3.4. Second Arrow-Hurwicz-Uzawa constraint qualification (SAHUCQ) holds at $\bar{x} \in S$ if $G^2(S,\bar{x}) \neq \emptyset$.

Similarly, a notion of second Abadie constraint qualification has been considered in [1, p. 993] by partitioning the active constraints into two components, namely linear and nonlinear at $\bar{x}$. We extend this definition to nonsmooth setting where the division is based on pseudolinearity instead of linearity at $\bar{x}$.

Definition 3.5. Second Abadie constraint qualification (SACQ) holds at $\bar{x} \in S$ if $G^1(S,\bar{x}) \neq \emptyset$.

Theorem 3.3. *Let $\bar{x} \in S$. Then*

$$G^-(S,\bar{x}) \subseteq G^1(S,\bar{x}) \subseteq G^2(S,\bar{x}) \subseteq D(S,\bar{x}).$$

Proof. As a locally pseudolinear function is locally pseudoconcave, therefore

$$G^-(S,\bar{x}) \subseteq G^1(S,\bar{x}) \subseteq G^2(S,\bar{x}).$$

For $0 \neq d \in G^2(S,\bar{x})$ and $i \in I(\bar{x}) \setminus M(\bar{x})$, let $V = \{\xi \in \mathbb{R}^n : \langle \xi, d \rangle < 0\}$. Then $\partial_c g_i(\bar{x}) \subseteq V$. Since $\partial_c g_i(\bar{x})$ is a compact set and V is open, there exists $\varepsilon_i > 0$ such that $\partial_c g_i(\bar{x}) + B_{\varepsilon_i}(0) \subseteq V$. By the upper semicontinuity of $\partial_c g_i$ at $\bar{x}$, there exists $\delta_i > 0$ such that

$$\partial_c g_i(x) \subseteq \partial_c g_i(\bar{x}) + B_{\varepsilon_i}(0) \subseteq V, \text{ for all } x \in B_{\delta_i}(\bar{x}). \tag{3.1}$$

Let $t_{0i} = \dfrac{\delta_i}{2\|d\|}$. Clearly, $\bar{x} + td \in B_{\delta_i}(\bar{x})$, for all $t \in [0, t_{0i}]$. Consider any $t \in [0, t_{0i}]$. Then, by Theorem 2.1, there exist $t_i^{'} \in (0,t)$ and $\xi_i^{'} \in \partial_c g_i(\bar{x} + t_i^{'} d)$ such that $g_i(\bar{x} + td) - g_i(\bar{x}) = \langle \xi_i^{'}, d \rangle$. Thus, it follows from (3.1) that $g_i(\bar{x} + td) < 0$ for all $t \in [0, t_{0i}]$.

Next, assume that $0 \neq d \in G^2(S,\bar{x})$ and $i \in M(\bar{x})$. We claim that there exists $t_{1i} > 0$ such that $g_i(\bar{x} + td) \leq 0$ for all $t \in [0, t_{1i}]$. Otherwise, for every $n \in \mathbb{N}$, there exists $0 < t_n < \dfrac{1}{n}$ with $g_i(\bar{x} + t_n d) > 0 = g_i(\bar{x})$. Then, by local pseudoconcavity of g_i at $\bar{x}$, $t_{n_0} g_i^{\circ}(\bar{x}, -d) < 0$ for some $n_0 \in \mathbb{N}$. This implies that there exists $\xi_i \in \partial_c g_i(\bar{x})$ such that $\langle \xi_i, d \rangle > 0$, which is a contradiction to the fact that $d \in G^2(S,\bar{x})$.

Now, for $i \notin I(\bar{x})$, by the continuity of g_i at $\bar{x}$, there exists $t_{2i} > 0$ such that $g_i(\bar{x} + td) < 0$ for all $t \in [0, t_{2i}]$. Define $\bar{t} := \min\{t_{0i}, \text{for } i \in I(\bar{x}) \setminus M(\bar{x});\ t_{1i}, \text{for } i \in M(\bar{x});\ t_{2i}, \text{for } i \notin I(\bar{x})\}$. Then $g_i(\bar{x} + td) \leq 0$, for all $t \in [0, \bar{t}]$ and $i \in I$, and hence $d \in D(S,\bar{x})$. $\square$

Remark 3.2. From the above theorem, it is clear that the constraint qualifications SACQ and SAHUCQ lie in Level 4. Moreover, by noting that $\mathrm{cl}(G^2(S,\bar{x})) = G(S,\bar{x})$ provided $G^2(S,\bar{x}) \neq \emptyset$, we obtain

$$\text{CCQ} \implies \text{SACQ} \implies \text{SAHUCQ} \implies \text{ZCQ}.$$

We now give examples to justify that the converse implications may fail to hold.

Example 3.5. Let $g_i : \mathbb{R} \to \mathbb{R}$, $i = 1,2$, be functions defined as

$$g_1(x) = 3x + |x|,$$
$$g_2(x) = -x.$$

Then at $\bar{x} = 0$, $I(\bar{x}) = \{1,2\}$ and g_i is locally pseudolinear at $\bar{x}$ for $i \in I(\bar{x})$. Also, $g_1^\circ(\bar{x},d) = 3d + |d|$ and $g_2^\circ(\bar{x},d) = -d$, for all $d \in \mathbb{R}$. Thus, $G^1(S,\bar{x}) = \{0\}$ whereas $G^-(S,\bar{x}) = \emptyset$, and hence SACQ holds at $\bar{x}$ while CCQ fails to hold at $\bar{x}$.

Example 3.6. Let $g_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1,2$ be functions defined as

$$g_1(x_1,x_2) = x_1 - |x_2|,$$
$$g_2(x_1,x_2) = -x_1.$$

At the feasible point $\bar{x} = (0,0)$, we have $I(\bar{x}) = \{1,2\}$ and the constraints g_1 and g_2 are concave and linear at $\bar{x}$, respectively. Also, $g_1^\circ(\bar{x},d) = d_1 + |d_2|$ and $g_2^\circ(\bar{x},d) = -d_1$ for all $d = (d_1,d_2) \in \mathbb{R}^2$. Hence, $G^2(S,\bar{x}) = \{(0,0)\}$ whereas $G^1(S,\bar{x}) = \emptyset$, which gives that SAHUCQ holds at $\bar{x}$ but SACQ fails to hold at $\bar{x}$.

The previous examples justify that the converse implications fail to hold among constraint qualifications belonging to same level. We now establish implications among constraint qualifications in different levels and provide counter examples wherever needed. Recall that in Remark 3.2, SAHUCQ at Level 4 implies ZCQ at Level 3. The following example shows that the converse of this implication may not hold.

Example 3.7. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function given by

$$g(x_1,x_2) = |x_1| - |x_2|.$$

At the feasible point $\bar{x} = (0,0)$, we observe that g is not locally pseudoconcave at $\bar{x}$ and $g^\circ(\bar{x},d) = |d_1| + |d_2|$ for all $d = (d_1,d_2) \in \mathbb{R}^2$. It can be seen easily that $G(S,\bar{x}) = \{(0,0)\}$, $G^2(S,\bar{x}) = \emptyset$, and $D(S,\bar{x}) = S$. Hence, ZCQ holds at $\bar{x}$ but SAHUCQ fails to hold at $\bar{x}$.

Remark 3.3. As $D(S,\bar{x}) \subseteq A(S,\bar{x})$, we have the following relations between constraint qualifications in Level 2 and Level 3.

(i) ZCQ $\implies$ KTCQ;
(ii) CQ4 $\implies$ CQ2;
(iii) CQ5 $\implies$ AHUCQ.

We next give an example to show that converse of above implications may fail to hold.

Example 3.8. Let $g_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1,2$ be functions defined as

$$g_1(x_1,x_2) = x_1^2 - |x_2|,$$
$$g_2(x_1,x_2) = -x_1^2 + |x_2|.$$

At the feasible point $\bar{x}=(0,0)$, we have $I(\bar{x})=\{1,2\}$, $A(S,\bar{x})=\{(t,0):t\in\mathbb{R}\}$, and $D(S,\bar{x})=\{(0,0)\}$. Also, as $g_1^\circ(\bar{x},d)=g_2^\circ(\bar{x},d)=|d_2|$, for all $d=(d_1,d_2)\in\mathbb{R}^2$, we have $G(S,\bar{x})=A(S,\bar{x})$. Hence, all the constraint qualifications lying in Level 2 hold at $\bar{x}$, while none of the constraint qualifications in Level 3 are satisfied at $\bar{x}$.

Remark 3.4. On the basis of the inclusion relation $A(S,\bar{x})\subseteq T(S,\bar{x})$, we have the following obvious relations among the constraint qualifications in Level 1 and Level 2.

(i) KTCQ $\implies$ ACQ;
(ii) CQ2 $\implies$ CQ1;
(iii) AHUCQ $\implies$ GCQ.

The following example shows that converse of above implications may fail to hold.

Example 3.9. Let $g:\mathbb{R}\to\mathbb{R}$ be a function given as

$$g(x)=\begin{cases}0, & \text{if } x\geq 2 \text{ or } x\leq 0,\\ -x^2+\dfrac{16}{2^{2n}}, & \text{if } \dfrac{3}{2^n}\leq x\leq\dfrac{4}{2^n},\ n=1,2,\ldots,\\ \dfrac{7}{5}\left(x^2-\dfrac{16}{2^{2(n+1)}}\right), & \text{if } \dfrac{4}{2^{(n+1)}}\leq x\leq\dfrac{3}{2^n},\ n=1,2,\ldots.\end{cases}$$

The feasible region S is given by $S=(-\infty,0]\cup\left\{\dfrac{4}{2^{(n+1)}}:n=1,2,\ldots\right\}\cup[2,+\infty)$. At the feasible point $\bar{x}=0$, we have $A(S,\bar{x})=(-\infty,0]$ and $T(S,\bar{x})=\mathbb{R}$. Also, $g^\circ(\bar{x},d)=0$, for all $d\in\mathbb{R}$, which implies that $G(S,\bar{x})=\mathbb{R}$. Hence, ACQ, CQ1, and GCQ hold at $\bar{x}$ but neither KTCQ, CQ2 nor AHUCQ hold at $\bar{x}$.

We summarize the inter-relations between all the constraint qualifications studied so far and arrange them corresponding to the level they belong.

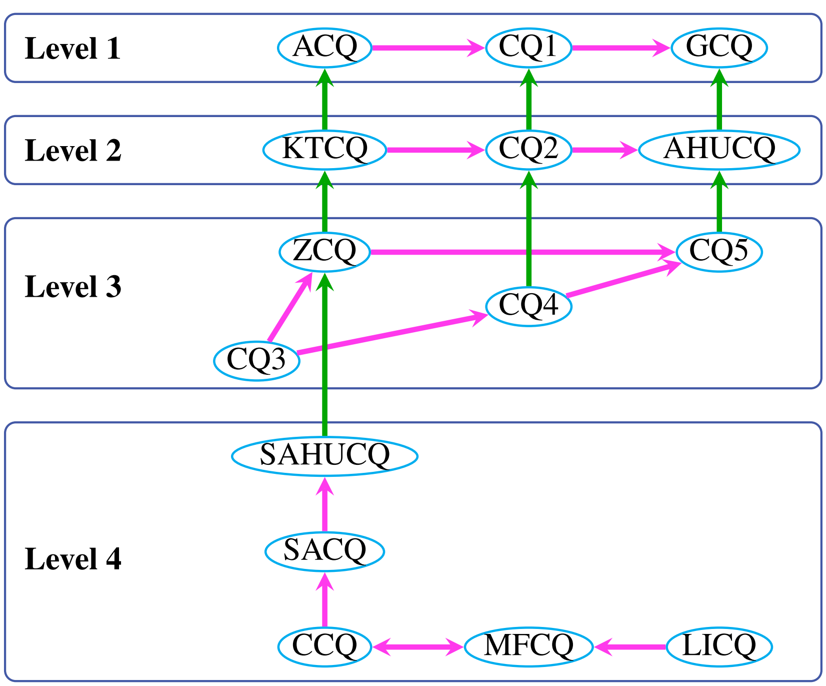

FIGURE 2. Inter-relations among constraint qualifications corresponding to different levels

Based on the implications studied in this section and Theorem 2.3, we have the following KKT necessary optimality condition.

Theorem 3.4. *If $\bar{x}$ is an optimal solution of (P) and any of constraint qualification holds at $\bar{x}$ and* $\operatorname{cone}(B(\bar{x}))$ *is closed, then there exist $\lambda_i \geq 0$, $i \in I(\bar{x})$ such that*

$$0 \in \partial_c f(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \partial_c g_i(\bar{x}),$$

provided that the function $d \mapsto f^\circ(\bar{x}, d)$ is concave when GCQ, CQ1, CQ2, AHUCQ, CQ4, or CQ5 holds.

Remark 3.5. In view of Remark 2.2, the closedness hypothesis of $\operatorname{cone}(B(\bar{x}))$ in the above theorem follows from the constraint qualifications, namely LICQ, MFCQ, and CCQ.

4. Constraint Qualifications under Semilocal Convexity Assumption

Convexity plays an important role in the study of optimality conditions of an optimization problem. Both smooth and nonsmooth optimization problems are extensively studied in the literature under the convexity assumption (see, for example, [10, 37–39]). It is observed that convexity is usually required locally around the point under consideration. Thus, researchers considered the generalizations of convexity for which some important results of convexity hypothesis remain valid locally around that point. One such generalizations is the notion of semilocal convexity considered by Ewing [40]. In fact, it is known from [40, Theorem 1.1] that a local optimal solution is global provided the function is semilocally convex at that optimal point. This has motivated the study of optimality conditions using semilocal convexity assumption. One may refer to [41, 42] for more details.

In this section, we study the constraint qualifications considered in the previous section under an additional assumption of semilocal convexity. We propose to show that the tangent cone at a feasible point is the closure of the cone of feasible directions at that point under semilocal convexity assumption. For this, we only require semilocal convexity of the active constraints at that point.

Definition 4.1. [40, pp. 202–203] Let X be a nonempty subset of $\mathbb{R}^n$.

(i) The set X is locally star-shaped at $\bar{x} \in X$ if for every $x \in X$, there exists a maximal positive number $a(\bar{x}, x) \leq 1$ such that $\bar{x} + t(x - \bar{x}) \in X$, for all $0 < t < a(\bar{x}, x)$.

(ii) A function $f : X \to \mathbb{R}$ is said to be semilocally convex at $\bar{x} \in X$, where X is locally star-shaped at $\bar{x}$, if for every $x \in X$, there exists a maximal positive number $e(\bar{x}, x) \leq a(\bar{x}, x)$ such that

$$f((1-t)\bar{x} + tx) \leq (1-t) f(\bar{x}) + t f(x), \quad \text{for all } 0 < t < e(\bar{x}, x).$$

If $a(\bar{x}, x) = 1$, for all $x \in X$, then X is referred to as star-shaped at $\bar{x}$ and if $e(\bar{x}, x) = 1$, for all $x \in X$, then f is referred to as convex at $\bar{x}$. We now give an example of a function defined on a locally star-shaped set which is semilocally convex at a point but is not convex at that point.

Example 4.1. Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{|x_1|} + \sqrt{|x_2|} \leq 1\}$ and $f : X \to \mathbb{R}$ be defined as $f(x) = \min\{4(x_1^2 + x_2^2) - 1, 0\}$. Clearly, X is locally star-shaped (in fact, star-shaped) at $\bar{x} = (0, 0)$ and f is semilocally convex at $\bar{x}$ but not convex at that point.

Let us now consider (P) with an additional abstract set constraint.

$$\begin{aligned} &\text{Minimize} \quad f(x) \\ &\text{subject to} \quad g_i(x) \leq 0, \quad i = 1, 2, \ldots, m, \\ &\qquad\qquad\quad x \in X, \end{aligned} \tag{P$'$}$$

where $X \subseteq \mathbb{R}^n$ is a locally star-shaped set and f and g_i, $i = 1, 2, \ldots, m$, are real valued locally Lipschitz functions defined on $\mathbb{R}^n$. Let the feasible set of (P$'$) be S', that is,

$$S' := \{x \in X : g_i(x) \leq 0, i \in I\}.$$

Clearly, (P$'$) and S' reduce to (P) and S, respectively when $X = \mathbb{R}^n$.

The following result states that the semilocal convexity of the active constraints at $\bar{x} \in S'$ ensures local star-shapedness of the feasible set at $\bar{x}$, which can be proved on the similar lines of Lemma 4.1 in [42].

Theorem 4.1. *If g_i, $i \in I(\bar{x})$, is semilocally convex at $\bar{x} \in S'$, then S' is locally star-shaped at $\bar{x}$.*

It is evident from the following example that the feasible set S' need not be locally star-shaped when the hypothesis of the above theorem fails to hold.

Example 4.2. Let X be the set as considered in Example 4.1 and $g : X \to \mathbb{R}$ be defined as

$$g(x_1, x_2) = \begin{cases} x_1(1 - x_2), & \text{if } x_1 \geq 0, \\ -x_1(1 + x_2), & \text{if } x_1 < 0. \end{cases}$$

Then the feasible region $S' = \{(0,t) \in X : -1 \leq t \leq 1\} \cup \{(1,0), (-1,0)\}$ is not locally star-shaped at $\bar{x} = (0,0)$. Also, observe that the function g does not satisfy semilocal convexity condition at $\bar{x}$ as $g(\bar{x}) = 0$ and $g(tx) > tg(x)$, for $x = (x_1, x_2) \in X$ with $x_i > 0$, $i = 1, 2$, and for all $0 < t < 1$.

We next show that the tangent cone is the closure of the cone of feasible directions if the feasible set is locally star-shaped. Before that we have the following two theorems.

Theorem 4.2. *If X is locally star-shaped at $\bar{x} \in X$ and $\mathbb{R}_+(A) = \{ta : t \geq 0, a \in A\}$ for any set A in $\mathbb{R}^n$, then*

$$\mathbb{R}_+(X - \bar{x}) \subseteq T(X, \bar{x}).$$

Proof. It is enough to show that $X - \bar{x} \subseteq T(X, \bar{x})$. Let $x \in X$. Since X is locally star-shaped at $\bar{x}$, there exists a maximal positive number $a(\bar{x}, x) \leq 1$ such that $\bar{x} + t(x - \bar{x}) \in X$, for all $0 < t < a(\bar{x}, x)$. Define d_k and t_k, $k \in \mathbb{N}$, as $d_k = x - \bar{x}$ and $t_k = \dfrac{a(\bar{x}, x)}{k+1}$. Then $d_k \to x - \bar{x}$, $t_k \downarrow 0$, and $\bar{x} + t_k d_k = \bar{x} + \dfrac{a(\bar{x}, x)}{k+1}(x - \bar{x}) \in X$, for all $k \in \mathbb{N}$, which implies that $x - \bar{x} \in T(X, \bar{x})$. □

The following theorem is a more general form of the result in [31, Proposition 5.2.1].

Theorem 4.3. *Let X be a nonempty subset of $\mathbb{R}^n$ and $\bar{x} \in X$. Then*

$$T(X, \bar{x}) \subseteq \mathrm{cl}(\mathbb{R}_+(X - \bar{x})).$$

Proof. Let $d \in T(X, \bar{x})$. Then there exist $d_k \in \mathbb{R}^n$ and $t_k \in \mathbb{R}$ for $k \in \mathbb{N}$ with $d_k \to d$ and $t_k \downarrow 0$ such that $\bar{x} + t_k d_k \in X$. Thus $d_k \in \mathbb{R}_+(X - \bar{x})$, for all $k \in \mathbb{N}$. Hence $d \in \mathrm{cl}(\mathbb{R}_+(X - \bar{x}))$. □

The following corollary is immediate from Theorems 4.1–4.3.

Corollary 4.1. *If g_i, $i \in I(\bar{x})$, is semilocally convex at $\bar{x} \in S'$, then*

$$T(S',\bar{x}) = \text{cl}(\mathbb{R}_+(S' - \bar{x})).$$

In the next theorem, we establish the relation between the tangent cone and the cone of feasible directions.

Theorem 4.4. *If g_i, $i \in I(\bar{x})$, is semilocally convex at $\bar{x} \in S'$, then*

$$T(S',\bar{x}) = \text{cl}(D(S',\bar{x})).$$

Proof. Let $d \in T(S',\bar{x})$. By Corollary 4.1, it follows that $d \in \text{cl}(\mathbb{R}_+(S' - \bar{x}))$. Then there exists $d_k \in \mathbb{R}_+(S' - \bar{x})$, $k \in \mathbb{N}$ such that $d_k \to d$. As $d_k \in \mathbb{R}_+(S' - \bar{x})$, without loss of generality, we assume that there exists $t_k > 0$ such that $d_k = t_k(x_k - \bar{x})$, for some $x_k \in S'$. By Theorem 4.1, we obtain that there exists a minimal positive number $a_k(\bar{x},x_k) \leq 1$ such that $\bar{x} + t(x_k - \bar{x}) \in S'$, for all $0 < t < a_k(\bar{x},x_k)$. Clearly, $\bar{x} + \frac{t}{t_k} d_k \in S'$, for all $0 < t < a_k(\bar{x},x_k)$. If we take $\bar{t}_k = \frac{a_k(\bar{x},x_k)}{2t_k}$, then $\bar{x} + \lambda d_k \in S'$, for all $\lambda \in [0,\bar{t}_k]$. Thus, $d_k \in D(S',\bar{x})$, which implies that $d \in \text{cl}(D(S',\bar{x}))$. □

It is evident from the counter examples discussed in Section 3 that the implication relations among various constraint qualifications in Figure 2 can be strict. We now provide a sufficient condition to validate the reverse implications. For this purpose, we examine relative strength of the constraint qualifications for the problem (P′), with $X = \mathbb{R}^n$, under the assumption of semilocal convexity on active constraints. Recall that both tangent cone and cone of attainable directions at a feasible point are closed. Then Theorem 2.2 and Theorem 4.4 collectively result into the following implications.

Theorem 4.5. *Let $\bar{x} \in S$. If for every $i \in I(\bar{x})$, g_i is semilocally convex at $\bar{x}$, then*

(i) *ACQ $\implies$ KTCQ $\implies$ ZCQ;*
(ii) *CQ1 $\implies$ CQ2 $\implies$ CQ4;*
(iii) *GCQ $\implies$ AHUCQ $\implies$ CQ5.*

Consequently, by above theorem and the hierarchy of constraint qualifications illustrated in Figure 2, we infer that the constraint qualifications ACQ, KTCQ, and ZCQ are equivalent under semilocal convexity assumption of the active constraints. Similarly, the constraint qualifications in (ii) or in (iii) of Theorem 4.5 are equivalent under the same assumption on the active constraints.

5. Conclusions

In the literature, extensive and systematic study of constraint qualifications has been carried out for smooth optimization problems. However, such a study has not been explored so far in the nonsmooth case. In this paper, we categorized various constraints qualifications into four levels and also introduced some related constraints qualifications for a nonsmooth problem with the involved functions being locally Lipschitz continuous. The relative strengths of these constraint qualifications were illustrated through a schematic diagram and numerous examples were given in support of the strict implications. We further proved that some local approximations of the feasible set, namely the tangent cone, the cone of attainable directions, and the closure of the cone of feasible directions are identical under semilocal convexity assumption of the constraint functions which leads to the equivalence among some constraint qualifications.

Acknowledgments

The research work of the first author was supported by the Council of Scientific and Industrial Research (CSIR), National R & D Organisation (File No: 09/045(1560)/2018-EMR-I), and the second author was supported by the research grant under the Faculty Research Programme of the IoE Scheme of University of Delhi, Delhi, India.

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