J. Nonlinear Var. Anal. 6 (2022), No. 2, pp. 101-111
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.6.2022.2.07

# APPROXIMATE ELEMENTS FOR SET OPTIMIZATION PROBLEMS WITH RESPECT TO VARIABLE DOMINATION STRUCTURES

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**Abstract.** We consider set optimization problems governed by the upper set less relation introduced by Kuroiwa. We extend some known results from the literature to a setting that includes (a) variable domination structures and (b) approximate solution concepts and show characterization results by means of inequalities involving duality products. The used notion of optimality is the set approach, and the variable domination structure builds upon comparisons solely dependent on elements in the objective space but the two-argument case. The involved approximate solution concepts build on previous work by Gutiérrez and co-workers.

**Keywords.** Approximate solutions; Optimality; Set optimization; Variable domination structures.

## 1. INTRODUCTION

Set optimization, meaning optimization with a set-valued objective mapping and/or optimality notion introduces through binary relations, called set relations, is a modern dynamic field within mathematics that subsumes scalar and vector optimization, and therefore provides an important extension in optimization theory as a whole. Due to a large number of applications, ranging from duality principles in vector optimization, gap functions for vector variational inequalities, and fuzzy optimization to inverse problems for differential equations and variational inequalities with use cases in image processing, optimal control, viability theory, medicine, biology or in mathematical economics, set optimization has recently expanded as a distinct branch of applied mathematics. As a result, set optimization became a bridge between different areas in optimization.

An important part of set optimization includes comparing sets by means of set relations, which are binary relations among sets. There is a variety of set relations based on convex cones known in the literature (for an overview, see [1, Chapter 2.6.2]), and several authors discussed which set relations are appropriate for certain applications (see [2]). In this paper, we will use a particular set relation, called *upper set less relation* (see Kuroiwa [3, 4]), that is equipped with a variable domination structure, cf. [5].

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Received January 10, 2022; Accepted March 16, 2022.

Going back to Yu [6], variable domination structures generalize the concept of ordering structures in vector optimization and have since been intensely studied in the field of vector optimization. Motivated by applications in medical image registration [5, 7], variable domination structures in vector optimization gained recognition as they allow to introduce a specification of the decision-maker's preferences into the model. Due to these important applications, variable domination structures have gained increasing interest (compare Durea, Strugariu, and Tammer [8] and Bao et al. [9, 10]. Note that Chen et al. [11] considered a vector approach to set optimization with a variable ordering structure. In addition, Bouza and Tammer [12] introduced a nonlinear scalarizing functional to characterize and compute minimal points of a set with respect to a variable domination structure.

Variable domination structures play a crucial role, for example, in medical image registration, which has been used widely in medical treatment, for instance in radiotherapy (treatment verification, treatment planning, and treatment guidance), orthopaedic surgery, and surgical microscope. The problem of image registration is finding a transformation matching two given sets of data (images). The similarity of the transformed data set to the target set can then be measured by several distance measures. As a multitude of measures exist that evaluate distinct characteristics, such as the sum of square differences, mutual information or cross-correlation, it is necessary to decide which distance measure to use. It is well known that different measures can lead to different best transformations. According to [5], some measures fail on special data sets, i.e. they lead to mathematically correct, but useless results. Thus it is important to combine several measures. Possible approaches are a weighted sum of different measures. But difficulties appear, such as badly scaled or nonconvex functions. Instead, Wacker [13] proposed to collect all available distance measures in a vector-valued function and minimizes this function. This leads to a vector-valued optimization problem. This connection with variable domination structures in vector programming has first been analyzed in Wacker [13] and further developed by Eichfelder [7] (see also [5, Section 10.3]). Recently, variable domination structures have been introduced to set optimization problems in [14, 15, 16]. This is particularly useful if uncertainties appear in the objective function, i.e., in the function that comprises the distance measures, for example due to inaccuracies of the data, or movements of the patient during the procedure. Then it is possible to convert the uncertain vector optimization problem into a set optimization problem and compute, for example, *robust* solutions. This is one of the main motivations why recently it has also been of great interest to consider set-valued optimization problems equipped with a variable domination structure by following a set approach. Recently, Köbis [14, 15] and Eichfelder and Pilecka [17] introduced several set relations for the case that the order is given by a cone-valued map. A very general scalarization scheme for solving set optimization problems w.r.t. variable domination structures was proposed in [16] (see also [18]).

## 2. Preliminaries and Notations

Throughout this work, unless stated otherwise, we consider a set-valued optimization problem in the following setting: Let $X$ be a real linear space, $Y$ a real linear space with zero element 0, and let a set-valued mapping $F\colon S \subseteq X \rightrightarrows Y$ (the objective map that is to be minimized) be given. $S$ denotes the feasible region of the set optimization problem, which might be defined via inequality constraints, might restrict feasible points to a discrete set, or be not present at all. In any case, the mapping $F$ acting on all elements of $S$ constitutes a *family of sets* $\mathscr{A}$, whose elements are equipped with some order, abstractly described via a set relation $\preceq$, which is a binary relation among sets in $\overline{\mathscr{P}}(Y)$ which represents the *power set* of $Y$. By $\mathscr{P}(Y)$, we denote the *power set* of $Y$ without the empty set, i.e., $\mathscr{P}(Y) := \{A \subseteq Y \mid A \text{ is nonempty}\}$.

For two elements $A$, $B$ of $\mathscr{P}(Y)$, we denote the sum of sets by

$$A+B := \{a+b \mid a \in A,\ b \in B\}.$$

The set $C \subseteq Y$ is a *cone* if for all $c \in C$ and $\lambda \geq 0$, $\lambda c \in C$ holds true. The cone $C$ is *convex* if $C+C \subseteq C$. We say that a set $C$ is *proper* (or *nontrivial*) if $C \neq \{0\}$ and $C \neq Y$. The cone $C$ is *pointed* of $C \cap (-C) = \{0\}$ holds. We call the cone $C$ *reproducing* if $C - C = Y$.

The dual cone to a cone $C$ is denoted by

$$C^* := \{y^* \in Y^* \mid \forall\, c \in C:\ y^*(c) \geq 0\}.$$

The quasi-interior of $C^*$ is defined as

$$C^\# := \{y^* \in Y^* \mid \forall\, c \in C \setminus \{0\}:\ y^*(c) > 0\}.$$

Fixed cones are usually the mathematical entities used to provide the “order” in vector/set optimization. This is achieved by defining to vectors $y_1$, $y_2 \in Y$ to be in relation to each other if $y_1 \in y_2 - C$. On the one hand, this notion was extended to set optimization by using comparisons among set elements giving rise to concepts like the upper set less relation [3]. On the other hand, more flexibility for complex modeling tasks and decision processes in applications can be realized if the restriction to fixed orderings, maybe even the ordering solely by cones, is dropped, leading to *variable domination structures*, cf. [7]. Here, one is confronted with a choice of what the ordering structure $C$ is allowed to depend on; it can either be dependent on arguments within $X$, or the elements of the image space $Y$. In this work, we go for the latter approach. More specifically, the ordering set is allowed to (a) not be a cone and (b) depend on both $y_1$ and $y_2$, the elements in the image/objective space being compared with each other. We will use the notation $D(y_1, y_2)$ for this set-valued mapping from now on and provide further notes whenever ‘cone-like’ prerequisites (e.g. $0 \in D(y_1, y_2)$) for this mapping are necessary.

We begin with the definition of variable domination structures in set optimization based on the upper set less relation by Kuroiwa ([3, 4]):

**Definition 2.1** (Variable Domination Structures in Set Optimization, [15])**.** Let $A, B$ be nonempty subsets of a real linear space $Y$, and let $D\colon Y \times Y \rightrightarrows Y$. Then we define the *variable upper set less relation* by

$$A \preceq^u_{D,v} B :\Longleftrightarrow \forall\, a \in A,\ \exists\, b \in B:\ a \in b - D(a,b). \tag{2.1}$$

Some properties of the variable upper set less relation are collected below.

**Proposition 2.1** ([15, Proposition 2.7])**.** *Let $D : Y \times Y \rightrightarrows Y$ be a set-valued map such that, for every $y_1, y_2 \in Y$, $D(y_1, y_2)$ is a cone. Then the following assertions hold true.*

*(1) The relation $\preceq^u_{D,v}$ defined in (2.1) is reflexive.*

*(2) The relation $\preceq^u_{D,v}$ defined in (2.1) is transitive if, for all $y_1, y_2, y_3 \in Y$ and for every $d_1 \in D(y_1, y_3)$, $d_2 \in D(y_3, y_2)$, it holds*

$$D(y_1, y_1 + d_1) + D(y_2 - d_2, y_2) \subseteq D(y_1, y_2).$$

(3) *Suppose that, for every* $y_1, y_2 \in Y$, $D(y_1,y_2) = D(y_2,y_1)$. *Furthermore, for* $A, B \in \mathscr{P}(Y)$, *suppose that*

$$D[A\times B] := \bigcup_{a\in A,\ b\in B} D(a,b)$$

*is a convex cone. Then*

$$A \preceq^u_{D,v} B \text{ and } B \preceq^u_{D,v} A \Longrightarrow A - D[A\times B] = B - D[A\times B].$$

(4) *(Compatibility with nonnegative scalar multiplication). Let* $A, B \in \mathscr{P}(Y)$. *Then*

$$A \preceq^u_{D,v} B,\ \lambda > 0 \implies \forall\, a\in A,\ \exists\, b\in B:\ \lambda a \in \{\lambda b\} - D(\lambda a, \lambda b)$$

*holds if*

$$\forall\, a\in A,\ \forall\, b\in B,\ \forall\, \lambda > 0:\ D(a,b) \subseteq D(\lambda a, \lambda b).$$

(5) *(Compatibility with addition). Let* $A, B, C, E \in \mathscr{P}cal(Y)$. *Then*

$$A \preceq^u_{D,v} B,\ C \preceq^u_{D,v} E \implies A + C \preceq^u_{D,v} B + E$$

*holds if*

$$\forall\, a\in A,\ \forall\, b\in B,\ \forall\, c\in C,\ \forall\, e\in E:\ D(a,b) + D(c,e) \subseteq D(a+c, b+e).$$

**Remark 2.1** ([15, Remark 2.8])**.** Let $D: Y\times Y \rightrightarrows Y$ be a set-valued map such that, for every $y_1, y_2 \in Y$, $D(y_1,y_2)$ is a cone. Then the subadditivity property on the whole space $Y$

$$\forall\, y_1, y_2, y_3, y_4 \in Y:\ D(y_1,y_2) + D(y_3,y_4) \subseteq D(y_1+y_3, y_2+y_4) \tag{2.2}$$

implies that $D$ is a constant map (compare [5, Lemma 3.23]): Set $y_1 = -y_3$ and $y_2 = -y_4$ in (2.2). Then $D(y_1,y_2) + D(-y_1,-y_2) \subseteq D(0,0)$ for every $y_1, y_2 \in Y$. Since $D(-y_1,-y_2)$ is a cone, it holds $0 \in D(-y_1,-y_2)$, and this yields $D(y_1,y_2) \subseteq D(0,0)$ for every $y_1, y_2 \in Y$. On the other hand, (2.2) implies that $D(y_1,y_2) + D(0,0) \subseteq D(y_1,y_2)$, and hence $D(0,0) \subseteq D(y_1,y_2)$. Hence, we obtain $D(y_1,y_2) = D(0,0)$ for all $y_1, y_2 \in Y$.

**Remark 2.2** ([15, Remark 2.9])**.** Let $A, B \in \mathscr{P}(Y)$ be given, and let $D_1, D_2 : Y\times Y \rightrightarrows Y$. Suppose that, for all $a \in A$ and $b \in B$, $D_1(a,b) \subseteq D_2(a,b)$ is satisfied. Then the implication $A \preceq^u_{D_1,v} B \Longrightarrow A \preceq^u_{D_2,v} B$ holds.

**Remark 2.3.** Within this work, we restrict the analysis to the upper set less relation, which is an established notion within set optimization with fixed cones and sometimes referred to as the 'pessimistic approach'. Note, however, that many alternatives exist in the literature and much of the mentioned analysis directly translates to these cases. In particular, we mention the *lower set less relation* $\preceq^l_{D,v}$, which can be based on the according 'basic' (i.e. non variable) domination structure

$$A, B \in \mathscr{A} : A \preceq^l_C B \iff B \subseteq A + C,$$

and often understood as the 'optimistic' counterpart to $\preceq^u_D$.

Regarding the definition of optimality in set optimization, we follow the *set approach*, i.e. a concept that solely relies on comparisons among entire sets within the family of possible images, noting that there do exist other concepts in the literature as well; see, e.g., [1]. The optimality notion can then be implemented via the following definition.

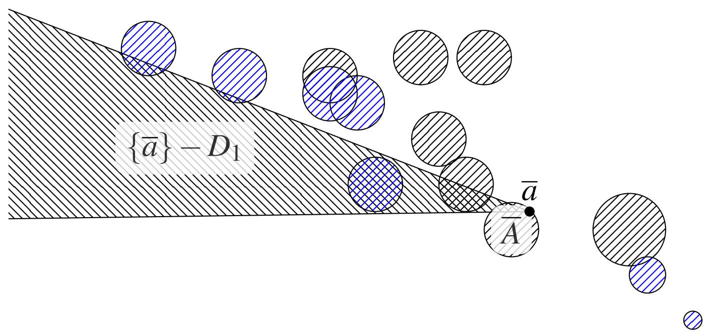


FIGURE 1. Let $\mathscr{A}$ be comprised of several sets, which are closed balls in $\mathbb{R}^2$ of different radii. $\overline{A} \in \mathscr{A}$ is *not* a minimal element of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$, where, for all $A \subset Y$, $A \neq \overline{A}$, $a \in A$, $\overline{a} \in \overline{A}$, $D(\overline{a}, a) = D_1$, and $\mathbb{R}^2_+$ else. The blue sets are all minimal element of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$. Note that a constant comparison structure via $D(a,b) \equiv \mathbb{R}^2$ would render $\overline{A}$ a part of $\mathscr{A}_{\preceq^u_{D,v}}$.

**Definition 2.2** (Optimality Notion in Set Optimization with Variable Domination Structures, [19])**.** Let $\mathscr{A}$ be a family of nonempty subsets of the real linear space $Y$ and let the set relation $\preceq^u_{D,v}$ on the power set of $Y$ be given. Then $\overline{A} \in \mathscr{A}$ is called a *minimal element* of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$ if

$$A \preceq^u_{D,v} \overline{A},\ A \in \mathscr{A} \quad \Longrightarrow \quad \overline{A} \preceq^u_{D,v} A\,.$$

The set of all minimal elements of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$ is denoted by $\mathscr{A}_{\preceq^u_{D,v}}$.

Figure 1 visualizes the optimality notion in set optimization with variable domination structures. This concepts enables the decision maker to include as much information as possible in the model, instead of following a fixed minimization concept.

It is well known that the existence of minimal elements can only be usually guaranteed under additional assumptions (for an existence result of minimal elements with fixed ordering structure; see, for example, [20]). Because the set $\mathscr{A}_{\preceq^u_{D,v}}$ may be empty, it is necessary to introduce a weaker notion of minimality. For this reason, we introduce three new notions of approximate minimality, which have been derived from the definition of approximate elements of set optimization problems in [21], where a different set relation was used and the domination structure was fixed.

**Definition 2.3.** Let $\mathscr{A}$ be a family of elements of $\mathscr{P}(Y)$, $D, H \in \mathscr{P}(Y)$, and $D, H \neq Y$.

(a) $\overline{A} \in \mathscr{A}$ is called an $H^1$–**approximate minimal element** of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$ if

$$A \preceq^u_{D,v} \overline{A},\ A \in \mathscr{A} \quad \Longrightarrow \quad \overline{A} \preceq^u_{D,v} A + H\,.$$

The set of all $H^1$–approximate minimal elements of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$ will be denoted by $\mathscr{A}_{H^1, \preceq^u_{D,v}}$.

(b) $\overline{A} \in \mathscr{A}$ is called an $H^2$**–approximate minimal element** of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$ if

$$A + H \preceq^u_{D,v} \overline{A},\ A \in \mathscr{A} \quad \Longrightarrow \quad \overline{A} \preceq^u_{D,v} A + H\,.$$

We call the set of all $H^2$–approximate minimal elements of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$ $\mathscr{A}_{H^2,\preceq^u_{D,v}}$.

(c) $\overline{A} \in \mathscr{A}$ is called an $H^3$**–approximate minimal element** of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$ if

$$A + H \not\preceq^u_{D,v} \overline{A}\,, \text{for all } A \in \mathscr{A}\,.$$

We call the set of all $H^3$–approximate minimal elements of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$ $\mathscr{A}_{H^3,\preceq^u_{D,v}}$.

**Remark 2.4.**
- Note that the above notions of approximate minimality using variable domination structures reduce to the corresponding notions of approximate minimality in set optimization with fixed ordering structure, as given in [21], if the ordering structure $\preceq^u_{D,v}$ is replaced by $\preceq^l_D$, as outlined above.
- Notice that Definition 2.3 (b), is equivalent to

$$\nexists A \in \mathscr{A} \text{ with } A + H \preceq^u_{D,v} \overline{A} \text{ and } \overline{A} \not\preceq^u_{D,v} A + H.$$

  Therefore, we see that $\mathscr{A}_{H^3,\preceq^u_{D,v}} \subseteq \mathscr{A}_{H^2,\preceq^u_{D,v}}$.
- If $H = \{0\}$, then Definition 2.3 (a) and (b) coincide with Definition 2.2.
- In case the family $\mathscr{A}$ and $H$ consist of single-valued sets, Definition 2.3 is closely related to well-known notions of *approximate efficiency*. For example, if $H = \{\varepsilon\}$ and $D \subset Y$ is a constant nonempty set, then set $\mathscr{A}_{H^1,\preceq^u_{D,v}}$ coincides with a notion of approximate solution of vector optimization problems due to White (see [22]). Furthermore, if $0 \in D$, $D$ is pointed (i.e., $D \cap (-D) = \{0\}$) and $H = \{\varepsilon\}$, the concept of $H^2$-approximate minimality introduced in Definition 2.3 (b) coincides with the concept introduced by Kutateladze in [23], the most popular notion of approximate efficient solutions in vector optimization. Indeed, from Definition 2.3 (b) for this special case, we obtain that $\bar{y} \in \mathscr{A}_{H^2,\preceq^u_{D,v}}$ if

$$\{y\} + \{\varepsilon\} \preceq^u_{D,v} \{\bar{y}\},\ y \in \mathscr{A} \quad \Longrightarrow \quad \{\bar{y}\} \preceq^u_{D,v} \{y\} + \{\varepsilon\},$$

  i.e., taking into account the definition of $\preceq^u_{D,v}$

$$\{\bar{y}\} \subseteq \{y\} + \{\varepsilon\} + D,\ y \in \mathscr{A} \quad \Longrightarrow \quad \{y\} + \{\varepsilon\} \subseteq \{\bar{y}\} + D.$$

  Since $D$ is pointed, it holds that $\bar{y} \in \mathscr{A}_{H^2,\preceq^u_{D,v}}$ if

$$y \in \bar{y} - \varepsilon - D,\ y \in \mathscr{A} \quad \Longrightarrow \quad y = \bar{y} - \varepsilon.$$

Under certain assumptions, the set of $H^1$-approximate minimal element of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$ is a subset of $H^2$-approximate minimal elements of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$:

**Proposition 2.2.** *Assume that, for all $a_1, a_2 \in Y$ and $h \in H$, $H + D(a_1 + h, a_2) \subseteq D(a_1, a_2)$. Then $\mathscr{A}_{H^1,\preceq^u_{D,v}} \subseteq \mathscr{A}_{H^2,\preceq^u_{D,v}}$.*

*Proof.* Let $\overline{A} \in \mathscr{A}_{H^1,\preceq^u_{D,v}}$, i.e., the following implication holds for all $A \in \mathscr{A}$

$$\forall a \in A \exists \overline{a} \in \overline{A} : a \in \overline{a} - D(a, \overline{a}) \Longrightarrow \forall \overline{a} \in \overline{A} \exists a^h \in A + H : \overline{a} \in a^h - D(\overline{a}, a^h). \tag{2.3}$$

Suppose that $\overline{A} \notin \mathscr{A}_{H^2,\preceq^u_{D,v}}$. Then there exists some $A \in \mathscr{A}$ with the property that

$$\forall a^h \in A+H \exists \overline{a} \in \overline{A} : a^h \in \overline{a} - D(a^h,\overline{a}) \wedge \neg\Big(\forall \overline{a} \in \overline{A} \exists a^h \in A+H : \overline{a} \in a^h - D(\overline{a},a^h)\Big). \quad (2.4)$$

The first expression in (2.4) leads to

$$\forall a+h \in A+H \exists \overline{a} \in \overline{A} : a \in -h+\overline{a} - D(a+h,\overline{a}) \subseteq \overline{a} - H - D(a+h,\overline{a}) \subseteq \overline{a} - D(a,\overline{a}).$$

Using (2.3), we obtain

$$\forall \overline{a} \in \overline{A} \exists a^h \in A+H : \overline{a} \in a^h - D(\overline{a},a^h),$$

a contradiction to (2.4). □

Similarly, one can prove that the set of minimal elements of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$ is a subset of the set of $H^1$-approximate minimal element of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$.

**Proposition 2.3.** *Assume that* $0 \in H$ *and, for all* $a_1, a_2 \in Y$ *and* $h \in H$, $D(a_1,a_2) \subseteq D(a_1,a_2+h)$. *Then* $\mathscr{A}_{\preceq^u_{D,v}} \subseteq \mathscr{A}_{H^1,\preceq^u_{D,v}}$.

**Remark 2.5.**
- The inclusion $D(a_1+h,a_2) \subseteq D(a_1,a_2)$ holds, for example, if $D(\cdot,a_2)$ is constant for each $a_2 \in Y$.
- Likewise, the inclusion $D(a_1,a_2) \subseteq D(a_1,a_2+h)$ hold under the assumption that $D(a_1,\cdot)$ is constant.

Both prerequisites mirror special cases of optimization concepts with variable domination structures which have been studied in great detail for the case of vector optimization. Note that $0 \in H$ is a quite natural assumption as approximate elements should allow for returning to 'classical' solution concepts. However, note also that just $0 \in H$ did not suffice for the case of variable domination structures: The point $a_2+h$ might include a completely different domination structure than just $a_2$ and the optimality notion is based on comparisons for the sets $\bar{A}$ and $A+H$.

## 3. Linear Scalarization Results

In this section, we will give alternative formulations of the fulfillment of the set relation $A \preceq^u_{D,v} B$ for two nonempty sets $A, B \subset Y$ in terms of inequalities.

Let us define

$$\overline{D} := \bigcap_{y_1,y_2 \in Y} D(y_1,y_2)$$

and

$$\widetilde{D} := \bigcup_{y_1,y_2 \in Y} D(y_1,y_2).$$

Now we set

$$\overline{D}^* := \bigcap_{y_1,y_2 \in A} D^*(y_1,y_2), \quad (3.1)$$

$$\overline{D}^\# := \bigcap_{y_1,y_2 \in A} D^\#(y_1,y_2) \quad (3.2)$$

and assume that $\overline{D}^*$ and $\overline{D}^\#$ are cones. Note that we had not assumed originally that, for any $y_1, y_2 \in Y$, $D(y_1,y_2)$ is a cone, however, if they are cones, then, naturally, $\overline{D}^*$ and $\overline{D}^\#$ are, too. Furthermore, we will use the dual cone of the intersection of all ordering sets as

$$(\overline{D})^* := \left( \bigcap_{y_1,y_2 \in A} D(y_1,y_2) \right)^*$$

and the dual cone of the union of all ordering sets as

$$(\widetilde{D})^* := \left( \bigcup_{y_1,y_2 \in A} D(y_1,y_2) \right)^*,$$

in accordance with the above defintions.

We have the following proposition which establishes a separation result for characterizing $\preceq^u_{D,v}$.

**Proposition 3.1.** *For two sets $A, B \in \mathscr{P}(Y)$ and $y^* \in (\widetilde{D})^*$, we have*

$$A \preceq^u_{D,v} B \quad \Longrightarrow \quad \sup_{a\in A} y^*(a) \le \sup_{b \in B} y^*(b).$$

*Conversely, let $Y$ be a real locally convex space. If $B - \overline{D}$ is closed and convex and $y^* \in (\overline{D})^*$, then*

$$A \preceq^u_{D,v} B \quad \Longleftarrow \quad \sup_{a\in A} y^*(a) \le \sup_{b \in B} y^*(b).$$

*Proof.* Let $y^* \in (\widetilde{D})^*$. Then,

$$\begin{aligned} A \preceq^u_{D,v} B \iff & \quad \forall a \in A\ \exists b \in B:\ a \in b - D(a,b) \\ \Longrightarrow & \ \forall a \in A\ \exists b \in B:\ a \in b - \widetilde{D} \\ \iff & \quad \forall a \in A\ \exists b \in B:\ y^*(a) \le y^*(b) \\ \Longrightarrow & \quad \sup_{a\in A} y^*(a) \le \sup_{b\in B} y^*(b). \end{aligned}$$

Now, let $B - \overline{D}$ be closed and convex, $y^* \in (\overline{D})^*$ and $\sup_{a\in A} y^*(a) \le \sup_{b\in B} y^*(b)$. Suppose that $A \not\preceq^u_{D,v} B$. Then there exists some $a \in A$ s. t. for all $b \in B$, $a \notin b - D(a,b)$. This implies

$$\exists a \in A\ \ \forall b \in B: \quad a \notin b - \overline{D},$$

leading to

$$\exists a \in A: \quad a \notin B - \overline{D}.$$

As $B - \overline{D}$ is assumed to be closed and convex, we obtain the existence of an $\ell \in Y^* \setminus \{0\}$ and some $\alpha > 0$ s. t. $\ell(a) > \alpha \ge \ell(b) - \ell(\overline{d})$ for all $b \in B$ and $\overline{d} \in \overline{D}$. We obtain that $\ell \in (\overline{D})^*$, as $\ell \notin (\overline{D})^*$ would yield the existence of an $l \in (\overline{D})^*$ with $\ell(l) < 0$, contradicting the before mentioned inequality. This yields $\sup_{a\in A} \ell(a) > \sup_{b\in B} \ell(b)$, in contradiction to the assumption. □

The above result already indicates that it will be rather challenging to obtain a sufficiently large proportion of the approximate minimal elements via linear scalarization, as the fulfillment of the set relation $A \preceq^u_{D,v} B$ and the inequality $\sup_{a\in A} y^*(a) \le \sup_{b\in B} y^*(b)$ are generally not equivalent. We collect several characterizations for approximate minimality by linear scalarization below.

**Theorem 3.1.** *(1) Let $y^* \in (\overline{D})^*$ and suppose that, for each $A \in \mathscr{A}$, $A - \overline{D}$ is closed and convex and $0 \in H$. If for all $A \in \mathscr{A}$,*

$$\sup_{\overline{a}\in\overline{A}} y^*(\overline{a}) \leq \sup_{a\in A} y^*(a),$$

*then $\overline{A}$ is an $H^1$-minimal element of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$.*

*(2) Let $y^* \in (\widetilde{D})^* \setminus \{0\}$. If for all $A \in \mathscr{A} \setminus \overline{A}$,*

$$\sup_{\overline{a}\in\overline{A}} y^*(\overline{a}) < \sup_{a\in A} y^*(a),$$

*then $\overline{A}$ is an $H^1$-minimal element of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$.*

*(3) Let $y^* \in (\widetilde{D})^* \setminus \{0\}$. If for all $A \in \mathscr{A}$,*

$$\sup_{\overline{a}\in\overline{A}+H} y^*(\overline{a}) < \sup_{a\in A} y^*(a), \tag{3.3}$$

*then $\overline{A}$ is an $H^2$-minimal element of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$.*

*(4) Let $y^* \in (\widetilde{D})^* \setminus \{0\}$. If for all $A \in \mathscr{A} \setminus \overline{A}$,*

$$\sup_{\overline{a}\in\overline{A}} y^*(\overline{a}) < \sup_{a^h\in A+H} y^*(a^h),$$

*then $\overline{A}$ is an $H^2$-minimal element of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$.*

*Proof.* (1) Suppose that $\overline{A}$ is not an $H^1$-minimal element of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$. This means that there exists some $A \in \mathscr{A}$ s. t.

$$A \preceq^u_{D,v} \overline{A} \quad \text{and} \quad \overline{A} \not\preceq^u_{D,v} A+H\,. \tag{3.4}$$

Let $y^* \in (\overline{D})^*$. According to Proposition 3.1, the second relation in (3.4) implies $\sup_{\overline{a}\in\overline{A}} y^*(\overline{a}) > \sup_{a^h\in A+H} y^*(a^h) \geq \sup_{a\in A} y^*(a)$, as $0 \in H$ and therefore $A \subseteq A+H$. We get the desired contradiction.

(2) Suppose that $\overline{A}$ is not an $H^1$-minimal element of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$. Let $y^* \in (\widetilde{D})^* \setminus \{0\}$. From the first relation in (3.4), we obtain with the help of Proposition 3.1 that $\sup_{a\in A} y^*(a) \leq \sup_{\overline{a}\in\overline{A}} y^*(\overline{a})$, a contradiction.

(3) Suppose the opposite. Then there exists some $A \in \mathscr{A}$ s. t.

$$A+H \preceq^u_{D,v} \overline{A} \quad \text{and} \quad \overline{A} \not\preceq^u_{D,v} A+H\,. \tag{3.5}$$

We get from the first relation in (3.5) by Proposition 3.1 that $\sup_{a^h\in A+H} y^*(a^h) \leq \sup_{\overline{a}\in\overline{A}} y^*(\overline{a})$. Because $0 \in H$, $A \subseteq A+H$ holds. Thus, $\sup_{a\in A} y^*(a) \leq \sup_{a^h\in A+H} y^*(a^h) \leq \sup_{\overline{a}\in\overline{A}} y^*(\overline{a})$, in contradiction to (3.3).

(4) Suppose that $\overline{A}$ is not an $H^2$-minimal element of $\mathscr{A}$ w.r.t. $\preceq^u_{D,v}$. Then there exists some $A \in \mathscr{A}$ s.t. (3.5) holds. The first relation in (3.5) implies by Proposition 3.1 that $\sup_{a^h\in A+H} y^*(a^h) \leq \sup_{\overline{a}\in\overline{A}} y^*(\overline{a})$, a contradiction.

□

**Remark 3.1.** Note that (1) in Theorem 3.1 includes a (rather strong) convexity and closedness assumption whereas (2), (3), and (4) do not rely on this on the cost of having very strong *strict* inequality premises. Given (in light of the prospect of numerical algorithms for these problems) that the intended conditions derived here are based on dual arguments, i.e. linear functionals, this was, however, to be expected. The starting point of the analysis was the family of image sets $\mathscr{A}$ after all and we did not impose further assumptions concerning the objective $F$ of the original set optimization problem. One should also keep in mind that the statements in Theorem 3.1 classify approximate solutions. In real-world applications, solution strategies for these problems might therefore often be used more as a pre-processing step to filter out certain possible values for the design variables $x$.

In that light, one might use (3) and (4) of Theorem 3.1 also for the pre-approximation of $H^3$-minimal elements as

$$\mathscr{A}_{H^3,\preceq^u_{D,v}} \subseteq \mathscr{A}_{H^2,\preceq^u_{D,v}}.$$

**Remark 3.2.** The involved quantities in the characterizing inequalities in Theorem 3.1 are of the form

$$\sup_{b\in B} y^*(b)$$

for $b \in \bar{\mathscr{P}}(Y)$ and $y^* \in (\tilde{D})^*$. In practical applications, set optimization problems are most often formulated in a finite-dimensional objective space $Y$. In this case, the domination structure can often be (sufficiently well) described by a series of matrix products or, more generally, by characterization in the form of block norms, see [24].

Going a step beyond the purely duality-based characterization, the relations introduced in Theorem 3.1 for *one* element of the dual cone can be generalized by considering maxima and/ or minima over several of these dual elements. The mathematical structure of those comparison measures (i.e. scalarizations) are then functionals of Drummond Svaiter [25] and/or Gerstewitz-type; see [26].

**Acknowledgments**

The work of M.A. Köbis was carried out during the tenure of an ERCIM 'Alain Bensoussan' Fellowship Programme of the author at the Norwegian University of Science and Technology, Norway.

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