J. Nonlinear Var. Anal. 6 (2022), No. 2, pp. 125-137
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.6.2022.2.09

CODERIVATIVES AND AUBIN PROPERTIES OF SOLUTION MAPPINGS FOR PARAMETRIC VECTOR VARIATIONAL INEQUALITY PROBLEMS

XIAOWEI XUE, CHUNMEI LIAO, MINGHUA LI, CHENGWU LU*

The Key Laboratory of Complex Data Analysis and Artificial Intelligence of Chongqing, Chongqing University of Arts and Sciences, Yongchuan, Chongqing, China

Abstract. This paper deals with sensitivity analysis for a parametric vector variational inequality problem in finite dimensional spaces by using advanced tools in modern variational analysis and generalized differentiation. We mainly focus on computing the coderivatives of the solution mapping in the parametric vector variational inequality problem and then apply them to establish verifiable conditions for the Aubin property of the solution mapping.

Keywords. Aubin property; Coderivative; Parametric vector variational inequality; Normal cone; Solution mapping.

1. INTRODUCTION

Let K be a set of R^m, C be a closed and convex cone of R^n, $\mathscr{L}(R^m,R^n)$ be the set of all linear continuous operators from R^m to R^n, and $F: R^m \to \mathscr{L}(R^m,R^n)$ be a vector-valued mapping. When K and F are perturbed by a parameter u in R^l, we consider the parametric vector variational inequality problem (PVVI, for short): find $y \in K(u)$ such that

$$F(u,y)(x-y) \notin -C \setminus \{0\}, \ \forall x \in K(u).$$

As far as we know that the PVVI without parameter u stems from the monograph [1]. In this book, Giannessi introduced some kinds of vector variational inequalities and discussed the relationships between them and other vector optimization problems. After Giannessi's work, many authors devoted themselves to the study of topological properties for PVVI's solution set $S(u)$, which is defined as

$$S(u) := \left\{y \in K(u) | F(u,y)(x-y) \notin -C \setminus \{0\}, \ \forall x \in K(u)\right\}, \text{for each } u \in R^l. \tag{1.1}$$

For more detail, we refer the readers to [2–11].

It is well known that Aubin property of a set-valued mapping is equivalent to the metric regularity as well as to the linear openness of its inverse. These properties play a fundamental role in nonlinear and nonsmooth analysis, optimization, and their applications (e.g., [12–16]). In particular, such properties are very important for studying perturbed optimization problems

*Corresponding author.
E-mail addresses: xuexw1@126.com (X. Xue), liaocm520@163.com (C. Liao), minghuali20021848@163.com (M. Li), l_chengwu@163.com (C. Lu).

Received October 8, 2021; Accepted November 6, 2021.

where they provide some stability results and are related to the regularity of constraint systems. To the best of our knowledge, there are very few results concerning the study of the Aubin property and coderivatives of the solution set mapping for the PVVI. Although such a problem can be transferred to a vector optimization problem by gap functions (cf. [17]), we can not use the results on the sensitivity of parametric vector optimization problems ([18–23]) to derive the Aubin property for the PVVI because the PVVI possesses a more refined structure. Thus, it is interesting to directly study the Aubin property for the PVVI.

The aim of this paper is devoted to the discussion of the coderivatives and Aubin property of the solution mapping for the PVVI. To arrive at this aim, let

$$G(u,y) := \bigcup_{x \in K(u)} F(u,y)(x-y) \tag{1.2}$$

and

$$V(u,y) := \mathrm{Min}_C G(u,y). \tag{1.3}$$

The solution set $S(u)$ can be rewritten as

$$S(u) = \{y \in K(u) | 0 \in V(u,y)\}. \tag{1.4}$$

Base on this formula, we employee the coderivatives, which have been used to discuss sensitivity analysis in optimization theory and applications (cf. [14, 15, 24–26]), to obtain the Aubin property of the solution mapping for the PVVI. We first establish upper estimate and explicit expression for the coderivative of the set-valued mapping G defined in (1.2). We then obtain the coderivative and Aubin property of S defined in (1.4) as V is an general mapping. In order to obtain the coderivatives and Aubin property of the solution mapping for the PVVI, we need to obtain the relation between the coderivatives of G and V. When all these things are done, it is easy to derive the coderivatives and Aubin property of the solution mapping for the PVVI.

The rest of the paper is organized as follows. In Section 2, we recall and discuss some basic definitions from variational analysis for our main results. In Section 3, we establish upper estimate and explicit expression for the coderivative of the set-valued mapping G defined in (1.2). In Section 4, we first study the coderivative and Aubin property of the mapping S defined in (1.4). Then, by using the relation which we constructed between the coderivatives of G and V, we obtain the coderivatives and Aubin property of the solution mapping for the PVVI.

2. PRELIMINARIES

We use standard notations. For all spaces, the norms are always denoted by $\|\cdot\|$, and $\langle \cdot, \cdot \rangle$ denotes the canonical pairing. The closed ball with center x and radius η is denoted by $B_\eta(x)$. The symbol A^T denotes the transpose operator of a linear continuous operator A. Let $F : R^n \rightrightarrows R^s$ be a set-valued mapping. By $\mathrm{dom}F = \{x \in R^n \mid F(x) \neq \emptyset\}$, $\mathrm{gph}F = \{(x,y) \in R^n \times R^s \mid y \in F(x)\}$, and $F^{-1}(y) := \{x \in R^n \mid (x,y) \in gphF\}$, we denote the domain, graph, and inverse of F, respectively. The notation $x_n \overset{S}{\to} x$ means that the sequence x_n is contained in the subset S and converges to x. For a set-valued mapping $F : R^n \rightrightarrows R^s$, the expression

$$\mathrm{Limsup}_{x \to \bar{x}} F(x) = \{y \in R^s \mid \exists \text{ sequences } x_k \to \bar{x},\ y_k \in F(x_k),\ \text{s.t.},\ \ y_k \to y \text{ for all } k \in \mathbb{N}\},$$

signifies the sequential Painlevé Kuratowski upper (outer) limit of F at $\bar{x}$; $\mathbb{N} = \{1, 2, \ldots\}$. The origins of all spaces are denoted by 0.

For any $A \in \mathscr{L}(R^m, R^n)$, we introduce norm

$$\|A\|_{\mathscr{L}} = \sup\{\|A(x)\| \mid \|x\| \leq 1\}.$$

Since R^m and R^n are finite dimension spaces, $\mathscr{L}(R^m, R^n)$ is also finite dimension space with the above norm.

Definition 2.1. Let $F : R^m \to \mathscr{L}(R^m, R^n)$ be a vector-valued function. F is said to be Fréchet differentiable at x_0 if there exists a linear and continuous operator $\Phi : R^m \to \mathscr{L}(R^m, R^n)$ such that

$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - \Phi(x - x_0)\|_{\mathscr{L}}}{\|x - x_0\|} = 0.$$

Obviously, Φ is unique. We denote the derivative Φ of F at x_0 by $\nabla F(x_0)$. If, for any $x \in K \subset R^m$, F is Fréchet differentiable at x, then F is said to be Fréchet differentiable on K. Note that $\nabla F(\cdot) : R^m \to \mathscr{L}(R^m, \mathscr{L}(R^m, R^n))$ is a vector-valued function.

Next, we recall the basic concepts and constructions of variational analysis and generalized differentiation for formulations and justifications of the main results of the paper. Most of the concepts and properties can be found in the recent monographs [16, 27].

Definition 2.2. Let $\Omega \subset R^n$ be a nonempty subset.

(i): Given $\bar{x} \in \Omega$ and $\varepsilon \geq 0$, the set of ε-*normals* to Ω at $\bar{x} \in \Omega$ is defined by

$$\hat{N}_\varepsilon(\bar{x}, \Omega) = \{x^* \in R^n \mid \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq \varepsilon\}. \tag{2.1}$$

When $\varepsilon = 0$, set (2.1) is a cone that is called the regular normal cone (or the *prenormal cone*) to Ω at $\bar{x}$ and is denoted by $\hat{N}(\bar{x}, \Omega)$. We put $\hat{N}_\varepsilon(\bar{x}, \Omega) = \emptyset$ for all $\varepsilon \geq 0$ if $\bar{x} \notin \Omega$.

(ii): The Mordukhovich normal cone (or basic normal cone, or limiting normal cone) to Ω at $\bar{x} \in \Omega$ is defined through the Painlevé-Kuratowski upper (outer) limit as

$$N(\bar{x}, \Omega) = \text{Limsup}_{x_k \to \bar{x}, \varepsilon_k \to 0_+} \hat{N}_{\varepsilon_k}(x_k, \Omega),$$

that is, $N(\bar{x}, \Omega)$ is the collection of all x^* for which there are sequences $\varepsilon_k \to 0_+$, $x_k \to \bar{x}$, $x_k^* \to x^*$ with $x_k \in \Omega$, and $x_k^* \in \hat{N}_{\varepsilon_k}(x_k, \Omega)$ for all $k \in \mathbb{N}$. Put $N(\bar{x}, \Omega) = \emptyset$ if $\bar{x} \notin \Omega$.

Definition 2.3. Consider a set-valued mapping $\Phi : R^n \rightrightarrows R^s$.

(i): The ε-coderivative $\hat{D}^*_\varepsilon \Phi(\bar{x}, \bar{y})$ at $(\bar{x}, \bar{y}) \in gph\Phi$ is defined through the ε-normal set (2.1) to the graph as

$$\hat{D}^*_\varepsilon \Phi(\bar{x}, \bar{y})(y^*) = \{x^* \in R^n \mid (x^*, -y^*) \in \hat{N}_\varepsilon((\bar{x}, \bar{y}), \text{gph}\Phi)\}. \tag{2.2}$$

When $\varepsilon = 0$, the positive homogeneous set-valued mapping of y^* in (2.2) is called the regular coderivative of Φ at $(\bar{x}, \bar{y})$ and denoted by $\hat{D}^*\Phi(\bar{x}, \bar{y})$.

(ii): The normal (Mordukhovich) coderivative of Φ at $(\bar{x}, \bar{y}) \in gph\Phi$ is

$$D^*\Phi(\bar{x}, \bar{y})(y^*) = \{x^* \in R^n \mid (x^*, -y^*) \in N((\bar{x}, \bar{y}), \text{gph}\Phi)\}.$$

that is, $D^*\Phi(\bar{x}, \bar{y})(y^*)$ is the collection of all x^* for which there are sequences $\varepsilon_k \to 0_+$, $(x_k, y_k) \to (\bar{x}, \bar{y})$, $(x_k^*, y_k^*) \xrightarrow{*} (x^*, y^*)$ with $(x_k, y_k) \in \text{gph}\Phi$ and $x_k^* \in \hat{D}^*_{\varepsilon_k}\Phi(x_k, y_k)(y_k^*)$.

The symbol $D^*\Phi(\bar{x})$ is used when Φ is single-valued at $\bar{x}$ and $\bar{y} = \Phi(\bar{x})$.

We say that Ω is regular at $\bar{x} \in \Omega$ if $N(\bar{x},\Omega) = \hat{N}(\bar{x},\Omega)$ and that Φ is regular at $(\bar{x},\bar{y})$ if $D^*\Phi(\bar{x},\bar{y}) = \hat{D}^*\Phi(\bar{x},\bar{y})$.

The following proposition gives a sufficient condition for the regularity of Φ and special representations of the coderivatives.

Proposition 2.1. [16] *Let $\Phi : R^n \to R^s$ be Fréchet differentiable at $\bar{x}$. Then $\hat{D}^*\Phi(\bar{x})(y^*) = \{(\nabla\Phi(\bar{x}))^T y^*\}$, $\forall y^* \in R^s$. Moreover, if Φ is strictly differentiable at $\bar{x}$, that is, Φ is single-valued around $\bar{x}$ and $\lim_{x,x'\to\bar{x}}[\Phi(x) - \Phi(x') - \nabla\Phi(\bar{x})(x-x')]/\|x-x'\| = 0$, then Φ is regular at $\bar{x}$ and we have $D^*\Phi(\bar{x})(y^*) = \{(\nabla\Phi(\bar{x}))^T y^*\}$, $\forall y^* \in R^s$.*

We also need some Lipschitzian notions in the following analysis.

Definition 2.4. [28] Let $f : R^n \to R^s$ be a single-valued mapping, and let $\bar{x} \in \mathrm{dom} f$. f is said to be locally upper Lipschitzian at $\bar{x}$ if there are numbers $\eta > 0$ and $L > 0$ such that

$$\|f(x) - f(\bar{x})\| \le L\|x - \bar{x}\|, \text{ for all } x \in B_\eta(\bar{x}) \cap \mathrm{dom} f.$$

We say that a set-valued mapping $F : R^n \rightrightarrows R^s$ admits a locally upper Lipschitzian selection at $(\bar{x},\bar{y}) \in gphF$ if there is a single-valued mapping $f : \mathrm{dom}F \to R^s$, which is locally upper Lipschitzian at $\bar{x}$ satisfying $f(\bar{x}) = \bar{y}$ and $f(x) \in F(x)$ for all $x \in \mathrm{dom}F$ in a neighborhood of $\bar{x}$. We say that F admits a locally upper Lipschitzian selection around $(\bar{x},\bar{y}) \in gphF$ if there is a neighborhood U of $(\bar{x},\bar{y})$ such that F admits a local upper Lipschitzian selection at any $(x,y) \in gphF \cap U$.

Definition 2.5. [12] A set-valued mapping F from R^n to R^s is said to be Lipschitz-like (or have the Aubin property) around $(\bar{x},\bar{y}) \in gphF$ if there are a neighborhood U of $\bar{x}$, a neighborhood V of $\bar{y}$, and a constant $l \ge 0$ such that $F(x') \cap V \subset F(x) + l\,\|x' - x\|B$, $\forall x,x' \in U$, where B is the closed unit ball of R^s. The infimum of all such moduli $\{l\}$ is called the exact Lipschitzian bound of F around $(\bar{x},\bar{y})$ and is denoted by $\mathrm{lip}F(\bar{x},\bar{y})$.

The following proposition gives a sufficient and necessary condition for F having the Aubin property and the exact Lipschitzian bound.

Proposition 2.2. [16, 29] *Let F be an arbitrary set-valued mapping with closed graph and $(\bar{x},\bar{y}) \in gphF$. Then the condition $D^*F(\bar{x},\bar{y})(0) = \{0\}$ is necessary and sufficient for F to have the Aubin property around $(\bar{x},\bar{y})$. Moreover, one has the exact Lipschitzian bound of F around $(\bar{x},\bar{y})$*

$$\mathtt{lip}F(\bar{x},\bar{y}) = \|D^*F(\bar{x},\bar{y})\| = \sup\{\|x^*\| \mid x^* \in D^*F(\bar{x},\bar{y})(y^*), \|y^*\| \le 1\}.$$

3. Coderivatives of the Set-Valued Mapping G

From now on, we let $F : R^l \times R^m \to \mathscr{L}(R^m,R^n)$ be a continuous vector-valued mapping. In this section, we will discuss the coderivative properties of G defined in (1.2).

Before stating our main results, we first recall that a set-valued mapping $H : R^m \rightrightarrows R^n$ is said to be inner semicontinuous at $(\hat{x},\hat{y}) \in \mathrm{gph}H$ if, for every sequence $x_k \to \hat{x}$ with $x_k \in \mathrm{dom}H$, there is a sequence $y_k \in H(x_k)$ converging to $\hat{y}$ as $k \to \infty$.

Theorem 3.1. *Let $\hat{u} \in R^l$, $\hat{x},\hat{y} \in K(\hat{u})$, $\hat{z} = F(\hat{u},\hat{y})(\hat{x} - \hat{y}) \in G(\hat{u},\hat{y})$, and*

$$M(u,y,z) = \{x \in K(u) | F(u,y)(x-y) = z\}. \tag{3.1}$$

Suppose that M is inner semicontinuous at $(\hat{u},\hat{y},\hat{z},\hat{x})$ and that

$$Proj_u D^*F(\hat{u},\hat{y})(0)\cap(-D^*K(\hat{u},\hat{x})(0))=\{0\}, \tag{3.2}$$

where $Proj_u D^*F(\hat{u},\hat{y})(0)$ denotes the projection of the set $D^*F(\hat{u},\hat{y})(0)$ on the space R^l. Then, for any $z^*\in R^n$,

$$D^*G(\hat{u},\hat{y},\hat{z})(z^*)\subset D^*F(\hat{u},\hat{y})(z^*(\hat{x}-\hat{y})^T)+(D^*K(\hat{u},\hat{x})(F(\hat{u},\hat{y})^Tz^*),-F(\hat{u},\hat{y})^Tz^*). \tag{3.3}$$

Moreover, if F is local upper Lipschitzian and regular at $(\hat{u},\hat{y})$, K is regular at $(\hat{u},\hat{x})$, and M admits a local upper Lipschitzian selection at $(\hat{u},\hat{y},\hat{z},\hat{x})$, then G is regular at $(\hat{u},\hat{y},\hat{z})$ and, for any $z^*\in R^n$, the converse inclusion of (3.3) holds, that is,

$$D^*G(\hat{u},\hat{y},\hat{z})(z^*)=D^*F(\hat{u},\hat{y})(z^*(\hat{x}-\hat{y})^T)+(D^*K(\hat{u},\hat{x})(F(\hat{u},\hat{y})^Tz^*),-F(\hat{u},\hat{y})^Tz^*).$$

Proof. Let $P(u,y):=K(u)-y$. We first prove that, for any $z^*\in R^n$,

$$D^*G(\hat{u},\hat{y},\hat{z})(z^*)\subset D^*(F,P)(\hat{u},\hat{y},F(\hat{u},\hat{y}),\hat{x}-\hat{y})(z^*(\hat{x}-\hat{y})^T,F(\hat{u},\hat{y})^Tz^*),$$

where the set-valued mapping $(F,P):R^l\times R^m\rightrightarrows\mathscr{L}(R^m,R^n)\times R^m$ is defined by

$$(F,P)(u,y):=F(u,y)\times P(u,y).$$

Let $(u^*,y^*)\in D^*G(\hat{u},\hat{y},\hat{z})(z^*)$. By the definitions of coderivatives and normal cone, there exist $\varepsilon_k\downarrow 0$, $(u_k,y_k,z_k)\xrightarrow{G}(\hat{u},\hat{y},\hat{z})$ and $(u_k^*,y_k^*,z_k^*)\to(u^*,y^*,z^*)$ such that

$$\limsup_{(u_{k_i},y_{k_i},z_{k_i})\xrightarrow{G}(u_k,y_k,z_k)}\frac{\langle u_k^*,u_{k_i}-u_k\rangle+\langle y_k^*,y_{k_i}-y_k\rangle-\langle z_k^*,z_{k_i}-z_k\rangle}{\|u_{k_i}-u_k\|+\|y_{k_i}-y_k\|+\|z_{k_i}-z_k\|}\le\varepsilon_k.$$

Since M is inner semicontinuous at $(\hat{u},\hat{y},\hat{z},\hat{x})$, for the above (u_k,y_k,z_k), there is a sequence $x_k\in K(u_k)$ satisfying $F(u_k,y_k)(x_k-y_k)=z_k$ and $x_k\to\hat{x}$. Thus, for any $(u_{k_i},x_{k_i})\xrightarrow{K}(u_k,x_k),y_{k_i}\to y_k$ with $z_{k_i}=F(u_{k_i},y_{k_i})(x_{k_i}-y_{k_i})$, one has $(u_{k_i},y_{k_i},z_{k_i})\xrightarrow{G}(u_k,y_k,z_k)$, and then

$$\limsup_{(u_{k_i},x_{k_i})\xrightarrow{K}(u_k,x_k),y_{k_i}\to y_k}\frac{\langle u_k^*,u_{k_i}-u_k\rangle+\langle y_k^*,y_{k_i}-y_k\rangle-\langle z_k^*,F(u_{k_i},y_{k_i})(x_{k_i}-y_{k_i})-F(u_k,y_k)(x_k-y_k)\rangle}{\|u_{k_i}-u_k\|+\|y_{k_i}-y_k\|+\|F(u_{k_i},y_{k_i})(x_{k_i}-y_{k_i})-F(u_k,y_k)(x_k-y_k)\|}$$
$$\le\varepsilon_k.$$

Hence,

$$\limsup_{(u_{k_i},x_{k_i})\xrightarrow{K}(u_k,x_k),y_{k_i}\to y_k}\frac{\left\{\begin{array}{r}\langle u_k^*,u_{k_i}-u_k\rangle+\langle y_k^*,y_{k_i}-y_k\rangle-\langle z_k^*,(F(u_{k_i},y_{k_i})-F(u_k,y_k))(x_k-y_k)\rangle\\ -\langle z_k^*,F(u_{k_i},y_{k_i})((x_{k_i}-y_{k_i})-(x_k-y_k))\rangle\end{array}\right\}}{\left\{\begin{array}{r}\|u_{k_i}-u_k\|+\|y_{k_i}-y_k\|+\|(F(u_{k_i},y_{k_i})-F(u_k,y_k))(x_k-y_k)\|\\ +\|F(u_{k_i},y_{k_i})((x_{k_i}-y_{k_i})-(x_k-y_k))\|\end{array}\right\}}$$
$$\le\varepsilon_k.$$

Let $t_k:=\max\{\|x_k-y_k\|,\|F(u_k,y_k)\|_{\mathscr{L}}\}+1$. Then,

$$\limsup_{(u_{k_i},x_{k_i})\xrightarrow{K}(u_k,x_k),y_{k_i}\to y_k}\frac{\left\{\begin{array}{r}\langle u_k^*,u_{k_i}-u_k\rangle+\langle y_k^*,y_{k_i}-y_k\rangle-\langle z_k^*,(F(u_{k_i},y_{k_i})-F(u_k,y_k))(x_k-y_k)\rangle\\ -\langle z_k^*,F(u_{k_i},y_{k_i})((x_{k_i}-y_{k_i})-(x_k-y_k))\rangle\end{array}\right\}}{\|u_{k_i}-u_k\|+\|y_{k_i}-y_k\|+\|F(u_{k_i},y_{k_i})-F(u_k,y_k)\|_{\mathscr{L}}+\|(x_{k_i}-y_{k_i})-(x_k-y_k)\|}$$
$$\le t_k\varepsilon_k.$$

Since F is continuous, for the above ε_k, there exists $N_0 \in \mathbb{N}$ such that $\|F(u_{k_i}, y_{k_i}) - F(u_k, y_k)\|_{\mathscr{L}} \leq \varepsilon_k$ whenever $i > N_0$. Therefore

$$\limsup_{(u_{k_i},x_{k_i})\xrightarrow{K}(u_k,x_k),y_{k_i}\to y_k} \frac{\left\{\begin{array}{r}\langle u_k^*, u_{k_i}-u_k\rangle + \langle y_k^*, y_{k_i}-y_k\rangle - \langle z_k^*, (F(u_{k_i},y_{k_i}) - F(u_k,y_k))(x_k-y_k)\rangle \\ -\langle z_k^*, F(u_k,y_k)((x_{k_i}-y_{k_i})-(x_k-y_k))\rangle\end{array}\right\}}{\|u_{k_i}-u_k\| + \|y_{k_i}-y_k\| + \|F(u_{k_i},y_{k_i}) - F(u_k,y_k)\|_{\mathscr{L}} + \|(x_{k_i}-y_{k_i})-(x_k-y_k)\|}$$
$$\leq \limsup_{(u_{k_i},x_{k_i})\xrightarrow{K}(u_k,x_k),y_{k_i}\to y_k} t_k\varepsilon_k + \|z_k^*\|\cdot\|F(u_{k_i},y_{k_i}) - F(u_k,y_k)\|_{\mathscr{L}}$$
$$\leq \varepsilon_k^{'},$$

where $\varepsilon_k^{'} := (t_k + \|z_k^*\|)\varepsilon_k \to 0$ as $k \to +\infty$. Thus,

$$\limsup_{(u_{k_i},x_{k_i})\xrightarrow{K}(u_k,x_k),y_{k_i}\to y_k} \frac{\left\{\begin{array}{r}\langle u_k^*, u_{k_i}-u_k\rangle + \langle y_k^*, y_{k_i}-y_k\rangle - \langle z_k^*(x_k-y_k)^T, F(u_{k_i},y_{k_i}) - F(u_k,y_k)\rangle \\ -\langle F(u_k,y_k)^T z_k^*, (x_{k_i}-y_{k_i})-(x_k-y_k)\rangle\end{array}\right\}}{\|u_{k_i}-u_k\| + \|y_{k_i}-y_k\| + \|F(u_{k_i},y_{k_i}) - F(u_k,y_k)\|_{\mathscr{L}} + \|(x_{k_i}-y_{k_i})-(x_k-y_k)\|}$$
$$\leq \varepsilon_k^{'}.$$

This means that $(u^*, y^*) \in D^*(F,P)(\hat{u},\hat{y},F(\hat{u},\hat{y}),\hat{x}-\hat{y})(z^*(\hat{x}-\hat{y})^T, F(\hat{u},\hat{y})^T z^*)$. Observe that the identity mapping is continuously Fréchet differentiable. [16, Theorem 1.62] ensures that

$$D^*P(\hat{u},\hat{y},\hat{x}-\hat{y})(F(\hat{u},\hat{y})^T z^*) = (D^*K(\hat{u},\hat{x})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*).$$

So, condition (3.2) ensures that $D^*F(\hat{u},\hat{y})(0) \cap (-D^*P(\hat{u},\hat{y},\hat{x}-\hat{y})(0)) = \{0\}$. Similar to the proof of [30, Corollary 31], we can show that

$$D^*(F,P)(\hat{u},\hat{y},F(\hat{u},\hat{y}),\hat{x}-\hat{y})(y_1^*,y_2^*) \subset D^*F(\hat{u},\hat{y})(y_1^*) + D^*P(\hat{u},\hat{y},\hat{x}-\hat{y})(y_2^*)$$

holds for any $(y_1^*, y_2^*) \in (\mathscr{L}(R^m,R^n), R^m)$, so (3.3) holds.

Now we prove the converse inclusion. Let $(u_1^*, y^*) \in D^*F(\hat{u},\hat{y})(z^*(\hat{x}-\hat{y})^T)$ and $u_2^* \in D^*K(\hat{u},\hat{x})(F(\hat{u},\hat{y})^T z^*)$. By the regularity assumption of F and K, we obtain

$$\limsup_{(u,y)\xrightarrow{F}(\hat{u},\hat{y})} \frac{\langle u_1^*, u-\hat{u}\rangle + \langle y^*, y-\hat{y}\rangle - \langle z^*(\hat{x}-\hat{y})^T, F(u,y)-F(\hat{u},\hat{y})\rangle}{\|u-\hat{u}\| + \|y-\hat{y}\| + \|F(u,y)-F(\hat{u},\hat{y})\|_{\mathscr{L}}} \leq 0$$

and

$$\limsup_{(u,x)\xrightarrow{K}(\hat{u},\hat{x})} \frac{\langle u_2^*, u-\hat{u}\rangle - \langle F(\hat{u},\hat{y})^T z^*, (x-\hat{x})\rangle}{\|u-\hat{u}\| + \|x-\hat{x}\|} \leq 0.$$

Then, for any $\varepsilon > 0$, one can find $\eta > 0$ such that

$$\begin{aligned}&\langle u_1^*, u-\hat{u}\rangle + \langle y^*, y-\hat{y}\rangle - \langle z^*(\hat{x}-\hat{y})^T, F(u,y)-F(\hat{u},\hat{y})\rangle \\ &\leq \varepsilon(\|u-\hat{u}\| + \|y-\hat{y}\| + \|F(u,y)-F(\hat{u},\hat{y})\|_{\mathscr{L}})\end{aligned}$$

and

$$\langle u_2^*, u-\hat{u}\rangle - \langle F(\hat{u},\hat{y})^T z^*, (x-\hat{x})\rangle \leq \varepsilon(\|u-\hat{u}\| + \|x-\hat{x}\|),$$

for all $u \in B_\eta(\hat{u})$, $y \in B_\eta(\hat{y})$, and $x \in B_\eta(\hat{x})$. Since M admits a local upper Lipschitzian selection at $(\hat{u},\hat{y},\hat{z},\hat{x})$, for any $(u,y,z) \xrightarrow{\mathrm{gph}G} (\hat{u},\hat{y},\hat{z})$, there are a constant $t > 0$ and $x \in M(u,y,z)$ such that $\|x-\hat{x}\| \leq t(\|u-\hat{u}\| + \|y-\hat{y}\| + \|z-\hat{z}\|)$. Furthermore, the locally upper Lipschitzian assumption of F ensures that $\|F(u,y)-F(\hat{u},\hat{y})\|_{\mathscr{L}} \leq l(\|u-\hat{u}\|+\|y-\hat{y}\|)$ for some $l>0$ whenever $(u,y) \to (\hat{u},\hat{y})$. Thus, for any $(u,y,z) \xrightarrow{\text{gph}G} (\hat{u},\hat{y},\hat{z})$, we have

$$\begin{aligned}
&\langle u_1^*+u_2^*, u-\hat{u}\rangle + \langle y^* - F(\hat{u},\hat{y})^T z^*, y-\hat{y}\rangle - \langle z^*, z-\hat{z}\rangle \\
&= \langle u_1^*+u_2^*, u-\hat{u}\rangle + \langle y^*, y-\hat{y}\rangle - \langle z^*, F(\hat{u},\hat{y})(y-\hat{y})\rangle - \langle z^*, F(u,y)(x-y) - F(\hat{u},\hat{y})(\hat{x}-\hat{y})\rangle \\
&= \langle u_1^*, u-\hat{u}\rangle + \langle y^*, y-\hat{y}\rangle - \langle z^*(\hat{x}-\hat{y})^T, F(u,y)-F(\hat{u},\hat{y})\rangle \\
&\quad + \langle u_2^*, u-\hat{u}\rangle - \langle F(\hat{u},\hat{y})^T z^*, (x-\hat{x})\rangle - \langle z^*, [F(u,y)-F(\hat{u},\hat{y})][(x-y)-(\hat{x}-\hat{y})]\rangle \\
&\leq \varepsilon(\|u-\hat{u}\|+\|y-\hat{y}\|+\|F(u,y)-F(\hat{u},\hat{y})\|_{\mathscr{L}}) + \varepsilon(\|u-\hat{u}\|+\|x-\hat{x}\|) \\
&\quad - \langle z^*, [F(u,y)-F(\hat{u},\hat{y})][(x-y)-(\hat{x}-\hat{y})]\rangle \\
&\leq \varepsilon(\|u-\hat{u}\|+\|y-\hat{y}\|+\|F(u,y)-F(\hat{u},\hat{y})\|_{\mathscr{L}}) + \varepsilon(\|u-\hat{u}\|+\|x-\hat{x}\|) \\
&\quad + l\|z^*\|(\|u-\hat{u}\|+\|y-\hat{y}\|)\|(x-y)-(\hat{x}-\hat{y})\| \\
&\leq (l+1)\varepsilon(\|u-\hat{u}\|+\|y-\hat{y}\|) + (t+1)\varepsilon(\|u-\hat{u}\|+\|y-\hat{y}\|+\|z-\hat{z}\|) \\
&\quad + l(t+1)\|z^*\|(\|u-\hat{u}\|+\|y-\hat{y}\|+\|z-\hat{z}\|)^2 \\
&= ((l+t+2)\varepsilon + l(t+1)\|z^*\|(\|u-\hat{u}\|+\|y-\hat{y}\|+\|z-\hat{z}\|))(\|u-\hat{u}\|+\|y-\hat{y}\|+\|z-\hat{z}\|).
\end{aligned}$$

Since $\varepsilon>0$ is chosen arbitrarily, we have

$$\limsup_{(u,y,z)\xrightarrow{\text{gph}G}(\hat{u},\hat{y},\hat{z})} \frac{\langle u_1^*+u_2^*, u-\hat{u}\rangle + \langle y^* - F(\hat{u},\hat{y})^T z^*, y-\hat{y}\rangle - \langle z^*, z-\hat{z}\rangle}{\|u-\hat{u}\|+\|y-\hat{y}\|+\|z-\hat{z}\|} \leq 0.$$

By the definition of regular coderivative, $(u_1^*+u_2^*, y^*-F(\hat{u},\hat{y})^T z^*) \in \hat{D}^*G(\hat{u},\hat{y},\hat{z})(z^*)$ and then $(u_1^*+u_2^*, y^*-F(\hat{u},\hat{y})^T z^*) \in D^*G(\hat{u},\hat{y},\hat{z})(z^*)$. So we have the converse inclusion relation

$$\begin{aligned}
&D^*F(\hat{u},\hat{y})(z^*(\hat{x}-\hat{y})^T) + (D^*K(\hat{u},\hat{x})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*) \subset \hat{D}^*G(\hat{u},\hat{y},\hat{z})(z^*) \\
&\subset D^*G(\hat{u},\hat{y},\hat{z})(z^*),
\end{aligned}$$

which implies the regularity of G. This completes the proof. □

If F is Lipschitzian around $(\hat{u},\hat{y})$, then $D^*F(\hat{u},\hat{y})(0) = \{(0,0)\}$, and thus condition (3.2) holds trivially. So, the following result is a consequence of Theorem 3.1.

Corollary 3.1. *Let $\hat{u} \in R^l$, $\hat{x},\hat{y} \in K(\hat{u})$, and $\hat{z} = F(\hat{u},\hat{y})(\hat{x}-\hat{y}) \in G(\hat{u},\hat{y})$. Suppose that F is Lipschitzian around $(\hat{u},\hat{y})$, and M is inner semicontinuous at $(\hat{u},\hat{y},\hat{z},\hat{x})$. Then, for any $z^* \in R^n$,*

$$D^*G(\hat{u},\hat{y},\hat{z})(z^*) \subset D^*F(\hat{u},\hat{y})(z^*(\hat{x}-\hat{y})^T) + (D^*K(\hat{u},\hat{x})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*).$$

Moreover, if F is regular at $(\hat{u},\hat{y})$, K is regular at $(\hat{u},\hat{x})$, and M admits a local upper Lipschitzian selection at $(\hat{u},\hat{y},\hat{z},\hat{x})$, then G is regular at $(\hat{u},\hat{y},\hat{z})$ and, for any $z^* \in R^n$,

$$D^*G(\hat{u},\hat{y},\hat{z})(z^*) = D^*F(\hat{u},\hat{y})(z^*(\hat{x}-\hat{y})^T) + (D^*K(\hat{u},\hat{x})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*).$$

If, moreover, F is strictly differentiable at $(\hat{u},\hat{y})$, then, by Proposition 2.1, F is regular at $(\hat{u},\hat{y})$ and $D^*F(\hat{u},\hat{y})(z^*) = (\nabla_u F(\hat{u},\hat{y})^T z^*, \nabla_y F(\hat{u},\hat{y})^T z^*)$. Thus, we have the following corollary.

Corollary 3.2. *Let $\hat{u} \in R^l$, $\hat{x},\hat{y} \in K(\hat{u})$, and$\hat{z} = F(\hat{u},\hat{y})(\hat{x}-\hat{y}) \in G(\hat{u},\hat{y})$. Suppose that F is strictly differentiable at $(\hat{u},\hat{y})$, and M is inner semicontinuous at $(\hat{u},\hat{y},\hat{z},\hat{x})$. Then, for any $z^* \in R^n$,*

$$\begin{aligned}
&D^*G(\hat{u},\hat{y},\hat{z})(z^*) \\
\subset\ &\left(\nabla_u F(\hat{u},\hat{y})^T(z^*(\hat{x}-\hat{y})^T) + D^*K(\hat{u},\hat{x})(F(\hat{u},\hat{y})^T z^*), \nabla_y F(\hat{u},\hat{y})^T(z^*(\hat{x}-\hat{y})^T) - F(\hat{u},\hat{y})^T z^*\right).
\end{aligned}$$

Moreover, if K is regular at $(\hat{u},\hat{x})$, and M admits a local upper Lipschitzian selection at $(\hat{u},\hat{y},\hat{z},\hat{x})$, then G is regular at $(\hat{u},\hat{y},\hat{z})$ and, for any $z^* \in R^n$,

$$\begin{aligned} & D^*G(\hat{u},\hat{y},\hat{z})(z^*) \\ = \ & \left(\nabla_u F(\hat{u},\hat{y})^T (z^*(\hat{x}-\hat{y})^T) + D^*K(\hat{u},\hat{x})(F(\hat{u},\hat{y})^T z^*), \nabla_y F(\hat{u},\hat{y})^T (z^*(\hat{x}-\hat{y})^T) - F(\hat{u},\hat{y})^T z^*\right). \end{aligned}$$

4. Coderivative and Aubin Property of the Solution Mapping

In this section, we discuss the coderivative and Aubin property of the solution mapping. We first discuss the mapping S, which is defined in (1.4).

Theorem 4.1. *Let $\hat{y} \in S(\hat{u})$. Suppose that K and V have locally closed graph around $(\hat{u},\hat{y})$ and $(\hat{u},\hat{y},0)$, and regular at these points, respectively. Furthermore, assume that*

$$0 \in D^*V(\hat{u},\hat{y},0)(z^*) \Rightarrow z^* = 0 \tag{4.1}$$

and, for any $z^* \in R^n$,

$$D^*V(\hat{u},\hat{y},0)(z^*) \cap (-N((\hat{u},\hat{y}),gphK)\backslash\{0\}) = \emptyset, \tag{4.2}$$

which are equivalent to

$$[(u^*,y^*) \in D^*V(\hat{u},\hat{y},0)(z^*), -u^* \in D^*K(\hat{u},\hat{y})(y^*)] \Rightarrow (u^*,y^*,z^*) = 0. \tag{4.3}$$

Then S is regular at $(\hat{u},\hat{y})$ and, for any $y^* \in R^m$,

$$D^*S(\hat{u},\hat{y})(y^*) = \{u^* \in R^l | \exists z^* \text{such that } (u^*,-y^*) \in D^*V(\hat{u},\hat{y},0)(z^*) + N((\hat{u},\hat{y}),gphK)\}. \tag{4.4}$$

Proof. Let g be defined by $g(u,y) := (u,y,0)$. Then the graph of the solution mapping S can be represented as

$$gphS = g^{-1}(\Theta) \cap gphK, \ \Theta = gphV.$$

Let Δ_Θ denote the indicator function of Θ. Observe that Δ_Θ is regular at $(\hat{u},\hat{y},0)$ if V is regular at $(\hat{u},\hat{y},0)$. Meanwhile, since V has locally closed graph around $(\hat{u},\hat{y},0)$, Δ_Θ^{-1} and $g^{-1}(\Theta)$ have locally closed graph around $(0,\hat{u},\hat{y},0)$ and $(\hat{u},\hat{y},0,\hat{u},\hat{y})$, respectively. Using the chain rule of [16, Theorem 3.13(iii)], one has $g^{-1}(\Theta)$ is regular at $(\hat{u},\hat{y})$ and

$$\begin{aligned} N((\hat{u},\hat{y}),g^{-1}(\Theta)) &= D^*\Delta_{g^{-1}(\Theta)}(\hat{u},\hat{y})(y^*) = D^*\Delta_\Theta \circ g(\hat{u},\hat{y})(y^*) \\ &= D^*g(\hat{u},\hat{y}) \circ D^*\Delta_\Theta(g(\hat{u},\hat{y}))(y^*) = D^*g(\hat{u},\hat{y}) \circ N((\hat{u},\hat{y},0),gphV) \\ &= \{(u^*,-y^*) \in R^l \times R^m | \exists z^* \text{such that } (u^*,-y^*) \in D^*V(\hat{u},\hat{y},0)(z^*)\} \end{aligned} \tag{4.5}$$

provided that

$$N((\hat{u},\hat{y},0),gphV) \cap (-D^*g^{-1}(\hat{u},\hat{y},0)(0)) = \{0\}. \tag{4.6}$$

By the definition of g, we can easily obtain that

$$(u^*,y^*,z^*) \in D^*g^{-1}(\hat{u},\hat{y},0)(0) \Rightarrow (u^*,y^*) = 0.$$

So, qualification condition (4.6) is equivalent to (4.1).

On the other hand, using (4.5), one can check the equivalence of (4.2) and

$$N((\hat{u},\hat{y}),g^{-1}(\Theta)) \cap (-N((\hat{u},\hat{y}),gphK)) = \{0\}.$$

By [16, Theorem 3.4], we have

$$N((\hat{u},\hat{y}),gphS) = N((\hat{u},\hat{y}),g^{-1}(\Theta)) + N((\hat{u},\hat{y}),gphK).$$

Substituting equality (4.5) into the above equation, we justify representation (4.4) and the regularity of S at $(\hat{u},\hat{y})$ under the assumptions. The equivalence between qualification conditions (4.1), (4.2) and qualification condition (4.3) is obvious. This completes the proof. □

Remark 4.1. If $K(u)=R^m$, then qualification condition (4.2) holds naturally, and then the coderivative of S turns into

$$D^*S(\hat{u},\hat{y})(y^*)=\{u^*\in R^l|\exists z^* \textit{such that } (u^*,-y^*)\in D^*V(\hat{u},\hat{y},0)(z^*)\}.$$

Many authors have discussed the case when $K(u)=R^m$, and V is represented as sum of two maps; see, e.g. [16, Theorem 4.44], [31] and the references therein. In these papers, more strict qualification conditions were given to obtain the coderivative of S since the conditions therein still need to ensure that the coderivatives of V can be represented as the sum of coderivatives of two maps.

Theorem 4.2. *Let $\hat{y}\in S(\hat{u})$. Assume that*

$$(u^*,0)\in D^*V(\hat{u},\hat{y},0)(z^*)+N((\hat{u},\hat{y}),gphK)\Rightarrow u^*=z^*=0, \tag{4.7}$$

(4.2) holds for any $z^*\in R^n$, and that K and V have locally closed graph around $(\hat{u},\hat{y})$ and $(\hat{u},\hat{y},0)$, and regular at these points, respectively. Then S is regular at $(\hat{u},\hat{y})$ and admits the Aubin property around $(\hat{u},\hat{y})$ with the exact bound formula

$$\texttt{lip}S(\hat{u},\hat{y})=\sup\{\|u^*\|\mid\exists z^* \textit{such that } (u^*,-y^*)\in D^*V(\hat{u},\hat{y},0)(z^*)+N((\hat{u},\hat{y}),gphK),\|y^*\|\le 1\}.$$

Proof. Observe that condition (4.7) implies that qualification condition (4.1) holds. Then Theorem 4.1 ensures that S is regular at $(\hat{u},\hat{y})$, and that $D^*S(\hat{u},\hat{y})$ is computed by the formula therein. Thus

$$D^*S(\hat{u},\hat{y})(0)=\{u^*\in R^l|\exists z^* \textit{such that } (u^*,0)\in D^*V(\hat{u},\hat{y},0)(z^*)+N((\hat{u},\hat{y}),gphK)\}.$$

Hence condition (4.7) implies that $D^*S(\hat{u},\hat{y})(0)=\{0\}$. So, [16, Theorem 4.10] ensures the Aubin property of S around $(\hat{u},\hat{y})$ and the exact bound formula. □

Now we turn to the coderivative and Aubin property of the solution mapping (1.1) for the PVVI. In order to do it, we need to give the relation between the coderivatives of G and V ,which is defined in (1.3). Recall that Min_C denotes the set of the minimum points with respect to C in the sense of:

$$z\in Min_C A\Leftrightarrow\nexists\, z'\in A \text{ such that } z-z'\in C\backslash\{0\}.$$

Proposition 4.1. *Let $\hat{z}\in V(\hat{u},\hat{y})$. Assume that $K(\hat{u})$ is a compact set, $F(\hat{u},\hat{y})$ has full column rank, and that $\hat{z}$ is an isolated point of $G(\hat{u},\hat{y})$. If there exists a neighborhood U of $\hat{u}$ such that $K(U)\subset K(\hat{u})$, then, for any $z^*\in R^n$, $\hat{D}^*V(\hat{u},\hat{y},\hat{z})(z^*)=\hat{D}^*G(\hat{u},\hat{y},\hat{z})(z^*)$ and $D^*V(\hat{u},\hat{y},\hat{z})(z^*)=D^*G(\hat{u},\hat{y},\hat{z})(z^*)$.*

Proof. First, we prove that, under the assumption conditions, $(u_k,y_k,z_k)\xrightarrow{\text{gph}G}(\hat{u},\hat{y},\hat{z})$ if and only if $(u_k,y_k,z_k)\xrightarrow{\text{gph}V}(\hat{u},\hat{y},\hat{z})$. To do it, we need only to verfity the necessity part as the sufficiency part is trivial (noting that $gphV\subseteq gphG$). Assume that there exists a sequence $(u_k,y_k,z_k)\to(\hat{u},\hat{y},\hat{z})$ with $(u_k,y_k,z_k)\in \text{gph}G\setminus\text{gph}V$. So there exists $z'_k\in G(u_k,y_k)$ satisfying $z'_k\le_{C\backslash\{0\}} z_k$. By the construction of G, there are $\hat{x}\in K(\hat{u})$ and $x_k,x'_k\in K(u_k)\subset K(\hat{u})$ such that $\hat{z}=F(\hat{u},\hat{y})(\hat{x}-\hat{y})$, $z_k=F(u_k,y_k)(x_k-y_k)$, and $z'_k=F(u_k,y_k)(x'_k-y_k)$. Since $u_k\to\hat{u}$ and $K(\hat{u})$ is a compact set, we assume without loss of generality that $x'_k \to x' \in K(\hat{u})$ and thus $z'_k \to z' = F(\hat{u},\hat{y})(x'-\hat{y}) \in G(\hat{u},\hat{y})$. By $z'_k \leq_{C\setminus\{0\}} z_k$, we have $z' \leq \hat{z}$ and then $\hat{z} = z'$ since $\hat{z} \in V(\hat{u},\hat{y})$. Let $\tilde{z}_k := F(\hat{u},\hat{y})(x_k-\hat{y})$ and $\tilde{z}'_k := F(\hat{u},\hat{y})(x'_k-\hat{y})$. Observe that $\tilde{z}_k, \tilde{z}'_k \in G(\hat{u},\hat{y})$ and $\tilde{z}_k, \tilde{z}'_k \to \hat{z}$. Since $x_k \neq x'_k$ and $F(\hat{u},\hat{y})$ has full column rank, it is impossible that both $\tilde{z}_k \equiv \hat{z}$ and $\tilde{z}'_k \equiv \hat{z}$ hold. So we have found a sequence $\tilde{z}_k$ (or $\tilde{z}'_k$) in $G(\hat{u},\hat{y})$ converging to $\hat{z}$. This contradicts the fact that $\hat{z}$ is an isolated point of $G(\hat{u},\hat{y})$. Thus, $(u_k,y_k,z_k) \overset{\text{gph}G}{\longrightarrow} (\hat{u},\hat{y},\hat{z})$ if and only if $(u_k,y_k,z_k) \overset{\text{gph}V}{\longrightarrow} (\hat{u},\hat{y},\hat{z})$. By the definition of regular coderivative, we have that, for any $z^* \in R^n$, $\hat{D}^*V(\hat{u},\hat{y},\hat{z})(z^*) = \hat{D}^*G(\hat{u},\hat{y},\hat{z})(z^*)$.

Now, for the above (u_k,y_k,z_k), let $(u_{k_i},y_{k_i},z_{k_i}) \to (u_k,y_k,z_k)$. We claim that $(u_{k_i},y_{k_i},z_{k_i}) \overset{\text{gph}G}{\longrightarrow} (u_k,y_k,z_k)$ if and only if $(u_{k_i},y_{k_i},z_{k_i}) \overset{\text{gph}V}{\longrightarrow} (u_k,y_k,z_k)$. If not, since $(u_k,y_k,z_k) \to (\hat{u},\hat{y},\hat{z})$, we can choose a subsequence $(u_{k_{i(k)}},y_{k_{i(k)}},z_{k_{i(k)}})$ such that

$$(u_{k_{i(k)}},y_{k_{i(k)}},z_{k_{i(k)}}) \to (\hat{u},\hat{y},\hat{z})$$

and

$$(u_{k_{i(k)}},y_{k_{i(k)}},z_{k_{i(k)}}) \in \text{gph}G \setminus \text{gph}V.$$

From the above proof, this is impossible. So, by the definitions of coderivatives, we directly have $D^*V(\hat{u},\hat{y},\hat{z})(z^*) = D^*G(\hat{u},\hat{y},\hat{z})(z^*)$. This completes the proof. □

Now it is easy to derive from Theorems 4.1 and 4.2 the corresponding results on coderivatives and Aubin property of the solution mapping for the initial vector variational inequality problem under consideration.

Theorem 4.3. *Let $\hat{u} \in R^l$, $\hat{y} \in S(\hat{u})$, and $0 \in V(\hat{u},\hat{y})$. Suppose that all the assumptions of Proposition 4.1 hold for $\hat{z} = 0$. Assume that the mapping M defined in (3.1) admits a local upper Lipschitzian selection at $(\hat{u},\hat{y},\hat{0},\hat{y})$, the qualification condition (3.2) holds, and K and V are locally closed-graph around $(\hat{u},\hat{y})$ and $(\hat{u},\hat{y},0)$, respectively. Moreover, suppose that the following conditions hold:*

(a):

$$\left.\begin{array}{l} (u^*, F(\hat{u},\hat{y})^T z^*) \in D^*F(\hat{u},\hat{y})(0) \\ -u^* \in D^*K(\hat{u},\hat{y})(F(\hat{u},\hat{y})^T z^*) \end{array}\right\} \Rightarrow z^* = 0; \tag{4.8}$$

(b): *for any $z^* \in R^n$,*

$$(D^*F(\hat{u},\hat{y})(0) + (D^*K(\hat{u},\hat{y})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*)) \cap (-N((\hat{u},\hat{y}),gphK) \setminus \{0\}) = \emptyset; \tag{4.9}$$

(c): *K is regular at $(\hat{u},\hat{y})$, and F is local upper Lipschitzian and regular at $(\hat{u},\hat{y})$.*

Then S is regular at $(\hat{u},\hat{y})$ and, for any $y^* \in R^m$,

$$\begin{aligned} D^*S(\hat{u},\hat{y})(y^*) = \{u^* \in U^* | \exists z^* \textit{such that } (u^*,-y^*) \in D^*F(\hat{u},\hat{y})(0) \\ +(D^*K(\hat{u},\hat{y})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*) + N((\hat{u},\hat{y}),gphK)\}. \end{aligned} \tag{4.10}$$

If condition (4.8) is replaced by

$$\begin{aligned} (u^*,0) \in D^*F(\hat{u},\hat{y})(0) + (D^*K(\hat{u},\hat{y})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*) + N((\hat{u},\hat{y}),gphK) \\ \Rightarrow u^* = z^* = 0, \end{aligned} \tag{4.11}$$

then S is regular at $(\hat{u},\hat{y})$ and admits the Aubin property around $(\hat{u},\hat{y})$ with the exact bound formula

$$\begin{aligned}\mathtt{lip}S(\hat{u},\hat{y}) = \sup\{\|u^*\| \mid \exists z^* \textit{such that } (u^*,-y^*) \in D^*F(\hat{u},\hat{y})(0)\\ +(D^*K(\hat{u},\hat{y})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*) + N((\hat{u},\hat{y}), gphK), \|y^*\| \le 1\}.\end{aligned} \tag{4.12}$$

Proof. It is obvious that all conditions of Theorem 3.1 and Proposition 4.1 hold. Furthermore, since $F(\hat{u},\hat{y})$ has full column rank, we have $0 \in V(\hat{u},\hat{y})$ only when $\hat{x} = \hat{y}$. So, V is regular at $(\hat{u},\hat{y},0)$ and, for any $z^* \in R^n$,

$$D^*V(\hat{u},\hat{y},\hat{z})(z^*) = D^*F(\hat{u},\hat{y})(0) + (D^*K(\hat{u},\hat{y})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*).$$

Using the above formula, we can easily check that qualifications (4.8) and (4.9) are equivalent to (4.1) and (4.2), respectively. By Theorem 4.1, S is regular at $(\hat{u},\hat{y})$ and (4.10) holds. Evidently, by the formula of D^*V, (4.11) is equivalent to (4.7). Theorem 4.2 ensures that S is regular at $(\hat{u},\hat{y})$ and admits the Aubin property around $(\hat{u},\hat{y})$ with the exact bound formula (4.12). This completes the proof. □

Corollary 4.1. *Let $\hat{u} \in R^l$, $\hat{y} \in S(\hat{u})$, and $0 \in V(\hat{u},\hat{y})$. Suppose that all the assumptions of Proposition 4.1 hold for $\hat{z} = 0$. Assume that the mapping M defined in (3.1) admits a local upper Lipschitzian selection at $(\hat{u},\hat{y},\hat{0},\hat{y})$, qualification condition (3.2) holds, and K and V are locally closed-graph around $(\hat{u},\hat{y})$ and $(\hat{u},\hat{y},0)$, respectively. Moreover, suppose that the following conditions hold:*

(a): *for any $z^* \in R^n$,*

$$\big(D^*K(\hat{u},\hat{y})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*\big) \cap (-N((\hat{u},\hat{y}), gphK) \setminus \{0\}) = \emptyset;$$

(b): *K is regular at $(\hat{u},\hat{x})$, and F is strict differentiable at $(\hat{u},\hat{y})$.*

Then S is regular at $(\hat{u},\hat{y})$ and, for any $y^* \in R^m$,

$$\begin{aligned}D^*S(\hat{u},\hat{y})(y^*) = \big\{u^* \in U^* | \exists z^* \textit{such that } (u^*,-y^*) \;\in\; & (D^*K(\hat{u},\hat{y})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*)\\ & + N((\hat{u},\hat{y}), gphK)\big\}.\end{aligned}$$

In addition, if

$$(u^*,0) \in (D^*K(\hat{u},\hat{y})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*) + N((\hat{u},\hat{y}), gphK) \Rightarrow u^* = z^* = 0,$$

then S is regular at $(\hat{u},\hat{y})$ and admits the Aubin property around $(\hat{u},\hat{y})$ with the exact bound formula

$$\begin{aligned}\mathtt{lip}S(\hat{u},\hat{y}) = \sup \big\{\|u^*\| \mid \exists z^* \textit{such that } (u^*,-y^*) \;\in\; & \big(D^*K(\hat{u},\hat{y})(F(\hat{u},\hat{y})^T z^*), -F(\hat{u},\hat{y})^T z^*\big)\\ & + N((\hat{u},\hat{y}), gphK), \|y^*\| \le 1\big\}.\end{aligned}$$

Proof. Observe that F is strict differentiable at $(\hat{u},\hat{y})$. By Proposition 2.1, F is regular at $(\hat{u},\hat{y})$ and $D^*F(\hat{u},\hat{y})(z^*) = (\nabla_u F(\hat{u},\hat{y})^T z^*, \nabla_y F(\hat{u},\hat{y})^T z^*)$. So qualification conditions (3.2) and (4.8) hold. The result directly follows Corollary 3.2 and Theorem 4.3. □

Acknowledgments

The first author was supported by Natural Science Foundation of Chongqing (cstc2019jcyj-msxmX0443), the Science and Technology Research Program of Chongqing Municipal Education Commission (KJQN202101328), and the Scientific Research Foundation of Chongqing

University of Arts and Sciences (R2019FSC16). The second author was supported by the Basic Science and Frontier Technology Research of Yongchuan (Grant number: YCSTC 2018nb1401). The third author was supported by the Foundation for High-level Talents of Chongqing University of Art and Sciences (Grant numbers: P2017SC01). The authors would like to thank the anonymous referee for the valuable comments and suggestions, which helped to improve the paper.

REFERENCES

[1] F. Giannessi, Theorems of the alternative, quadratic programs and complementary problems, in: R.W. Cottle, F. Giannessi, J.L. Lions (Eds.), Variational Inequalities and Complementary Problems, Wiley, New York, 1980.
[2] S.J. Li, G.Y. Chen, K.L. Teo, On the stability of generalized vector quasivariational inequality problems, J. Optim. Theory Appl. 113 v, 283-295.
[3] S.J. Li, H. Yan, G.Y. Chen, Differential and sensitivity properties of gap functions for vector variational inequalities, Math. Methods Oper. Res. 57 (2003), 377-391.
[4] L.Q. Anh, P.Q. Khanh, Semicontinuity of the solution set of parametric multivalued vector quasiequilibrium problems, J. Math. Anal. Appl. 294 (2004), 699-711.
[5] Y.H. Cheng, D.L. Zhu, Global stability results for the weak vector variational inequality, J. Global Optim. 32 (2005), 543-550.
[6] K.W. Meng, S.J. Li, Differential and sensitivity properties of gap functions for Minty vector variational inequalities, J. Math. Anal. Appl. 337 (2008), 386-398.
[7] S.J. Li, C.R. Chen, Stability of weak vector variational inequality, Nonlinear Anal. 70 (2009), 1528-1535.
[8] M.M. Wong, Lower semicontinuity of the solution map to a parametric vector variational inequality, J. Global Optim. 46 (2010), 435-446.
[9] M.H. Li, S.J. Li, Second-order differential and sensitivity properties of weak vector variational inequalities, J. Optim. Theory Appl., 144 (2010), 76-87.
[10] S. J. Li, X. W. Xue, Sensitivity analysis of gap functions for vector variational inequality via coderivatives, Optimization, 63 (2014), 1075-1098.
[11] X.W. Xue, Y. Zhang, Coderivatives of gap function for Minty vector variational inequality, J. Inequal. Appl., 2015 (2015), 285.
[12] J.P. Aubin, Lipschitz behavior of solutions to convex minimization problems, Math. Oper. Res. 9 (1984), 87-111.
[13] J.P. Penot, Metric regularity, openness and Lipschitzian behavior of multifunctions, Nonlinear Anal. 13 (1989), 629-643.
[14] J.P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhauser, Boston, 1990.
[15] R.T. Rockafellar, R.J-B. Wets, Variational Analysis. Springer-Verlag, Berlin, 1998.
[16] B.S. Mordukhovich, Variational Analysis and Generalized Differentiation vol. I: Basic Theory, vol. II: Applications, Springer-Verlag, Berlin, 2006.
[17] G.Y. Chen, C.J. Goh, X.Q. Yang, On gap functions for vector variational inequalities, in Vector Variational Inequality and Vector Equilibria. Mathematical Theories, Vols. 55-70, F. Giannessi, ed., Kluwer Academic, Boston, 2000.
[18] T. Tanino, Stability and sensitivity analysis in convex vector optimization, SIAM J. Control Optim. 26 (1988), 521-536.
[19] D.S. Shi, Contingent derivative of the perturbation map in multiobjective optimization, J. Optim. Theory Appl. 70 (1991), 385-396.
[20] H. Kuk, T. Tanino, M. Tanaka, Sensitivity analysis in vector optimization, J. Optim. Theory Appl. 89 (1996), 713-730.
[21] E.M. Bednarczuk, W. Song, Contingent epiderivative and its applications to set valued maps, Control Cybernet. 27 (1998), 375-386.

[22] N.Q. Huy, B.S. Mordukhovich, J.C. Yao, Coderivatives of frontier and solution maps in parametric multiobjective optimization, Taiwanese J. Math. 12 (2008), 2083-2111.

[23] X.W. Xue, S.J. Li, C.M. Liao, J.C. Yao, Sensitivity analysis of parametric vector set-valued optimization problems via coderivatives, Taiwanese J. Math. 15 (2011), 2533-2554.

[24] A.B. Levy, Nonsingularity conditions for multifunctions, Set-Valued Anal. 7 (1999), 89-99.

[25] A.B. Levy, B.S. Mordukhovich, Coderivatives in parametric optimization, Math. Program. Ser. A, 99 (2004), 311-327.

[26] B.S. Mordukhovich, N.M. Nam, N.D. Yen, Subgradients of marginal functions in parametric mathematical programming, Math. Program. 116 (2009), 369-396.

[27] B.S. Mordukhovich, Variational Analysis and Applications, Springer-Verlag, Berlin, 2018.

[28] S.M. Robinson, Generalized equations and their solutions, I: Basic theory, Math. Program. Study 10 (1979), 128-141.

[29] B.S. Mordukhovich, Sensitivity analysis in nonsmooth optimization, in Theoretical Aspects of Industrial Design, edited by D. A. Field and V. Komkov, SIAM Proc. Appl. Math. 58, pp.32-46, Philadelphia, Pennsylvania, 1992.

[30] S.J. Li, J-P. Penot, X.W. Xue, Codifferential calculus, Set-Valued Var. Anal. 19 (2011), 505-536.

[31] B.S. Mordukhovich, N.M. Nam, Variational analysis of extended generalized equations via coderivative calculus in asplund spaces, J. Math. Anal. Appl. 350 (2009), 663-679.