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ON SEMILINEAR FRACTIONAL DIFFERENTIAL INCLUSIONS WITH A NONCONVEX-VALUED RIGHT-HAND SIDE IN BANACH SPACES

VALERI OBUKHOVSKII[1], GARIK PETROSYAN[1,2], CHING-FENG WEN[3,4,*], VLADISLAV BOCHAROV[1]

[1]*Faculty of Physics and Mathematics, Voronezh State Pedagogical University, Voronezh 394043, Russia*
[2]*Research Center of Voronezh State University of Engineering Technologies, Voronezh 394036, Russia*
[3]*Center for Fundamental Science, and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung, 80708, Taiwan*
[4]*Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung, 80708, Taiwan*

Abstract. In this paper, we study the Cauchy problem for a semilinear fractional order differential inclusion with a nonconvex-valued almost lower semicontinuous nonlinearity and a linear closed operator generating a C_0-semigroup in a separable Banach space. By using the fixed point theory for condensing maps, we prove the local and global theorems of the existence of a mild solution to this problem.

Keywords. Almost lower semicontinuous multioperator; Cauchy problem; Caputo fractional derivative; Measure of noncompactness; Semilinear differential inclusion.

1. INTRODUCTION

Recently, the theory of differential equations of fractional order has attracted the attention of a number of researchers thanks to its numerous applications in mathematical physics, engineering, economics, ecology and other branches of natural sciences (see, e.g., [1, 2, 3, 4, 5] and the references therein). Various approaches to the solvability of differential equations and inclusions of a fractional order $q \in (0,1)$ have been developed. The Cauchy type problems for differential equations of fractional order $q \in (0,1)$ were solved in [6, 7, 8, 9, 10, 11], devoted to the study of trajectories of differential inclusions of fractional order $q \in (0,1)$ obeying generalized boundary conditions expressed in the form of operator inclusions. In [12, 13, 14, 15, 16], the solvability of periodic boundary value problems for fractional differential inclusions were studied, and the corresponding results for antiperiodic problems are presented in [17, 18, 19, 20]. Approximations for solutions of fractional differential equations and inclusions were described in [21, 22, 23, 24].

In this paper, we consider the Cauchy problem for a semilinear fractional differential inclusion in a separable Banach space E of the following form:

$$ {}^{C}D_0^q x(t) \in Ax(t) + F(t, x(t)),\ t \in [0,T], \tag{1.1}$$

$$ x(0) = x_0, \tag{1.2}$$

*Corresponding author.
E-mail address: cfwen@kmu.edu.tw (C.-F. Wen).
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where ${}^C D^q$, $0 < q < 1$, is the Caputo fractional derivative, $F : [0,T] \times E \multimap E$ is an almost lower semicontinuous multivalued map, $A : D(A) \subset E \to E$ is a linear closed, not necessarily bounded, operator in E, and $x_0 \in E$. By using the fixed point theory for condensing maps, we prove the local and global theorems on the existence of a mild solution to problem (1.1) - (1.2).

2. Preliminaries

2.1. Fractional integral and derivative.

Definition 2.1. *(See, e.g.,* [4] *and* [3]*).* The fractional integral of order $q \in (0,1)$ of a function $g \in L^1([0,T];E)$ is the function $I_0^q g$ of the following form:

$$I_0^q g(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s)\, ds,$$

where Γ is Euler's gamma-function

$$\Gamma(q) = \int_0^\infty x^{q-1} e^{-x} dx.$$

Definition 2.2. The Caputo fractional derivative of the order $q \in (N-1, N]$ of a function $g \in C^N([0,T];E)$ is the function ${}^C D_0^q g$ of the following form:

$${}^C D_0^q g(t) = \frac{1}{\Gamma(N-q)} \int_0^t (t-s)^{N-q-1} g^{(N)}(s)\, ds.$$

2.2. Multivalued maps.

Let us recall some concepts (see, for example, [25] and [26]).

Let $\mathscr{E}$ be a Banach space. Introduce the following notation:

- $P(\mathscr{E}) = \{A \subseteq \mathscr{E} : A \neq \varnothing\}$ denotes the collection of all non-empty subsets of $\mathscr{E}$;
- $Pb(\mathscr{E}) = \{A \in P(\mathscr{E}) : A \text{ is bounded}\}$;
- $Pv(\mathscr{E}) = \{A \in P(\mathscr{E}) : A \text{ is convex}\}$;
- $C(\mathscr{E}) = \{A \in P(\mathscr{E}) : A \text{ is closed}\}$;
- $K(\mathscr{E}) = \{A \in P(\mathscr{E}) : A \text{ is compact}\}$;
- $Kv(\mathscr{E}) = \{Pv(\mathscr{E}) \cap K(\mathscr{E})\}$ denotes the collection of all non-empty compact and convex subsets of $\mathscr{E}$.

Definition 2.3. [27] Let $(\mathscr{A}, \geq)$ be a partially ordered set. A function $\beta : Pb(\mathscr{E}) \to \mathscr{A}$ is called the measure of noncompactness (MNC) in $\mathscr{E}$ if and only if, for each $\Omega \in Pb(\mathscr{E})$,

$$\beta(\overline{\text{co}}\,\Omega) = \beta(\Omega),$$

where $\overline{\text{co}}\,\Omega$ denotes the closure of the convex hull of Ω.

A measure of noncompactness β is said to be:

1) *monotone* if, for each $\Omega_0, \Omega_1 \in Pb(\mathscr{E})$ with $\Omega_0 \subseteq \Omega_1$, $\beta(\Omega_0) \leq \beta(\Omega_1)$.
2) *nonsingular* if, for each $a \in E$ and each $\Omega \in Pb(\mathscr{E})$, $\beta(\{a\} \cup \Omega) = \beta(\Omega)$.

If $\mathscr{A}$ is a cone in a Banach space, the MNC β is said to be:

3) *regular*, if $\beta(\Omega) = 0$ is equivalent to the relative compactness of $\Omega \in Pb(\mathscr{E})$;
4) *real* if $\mathscr{A}$ is the set of all real numbers $\mathbb{R}$ with the natural ordering.

As the example of a real MNC obeying all above properties, we can consider the Hausdorff MNC $\chi(\Omega)$:

$$\chi(\Omega) = \inf\{\varepsilon > 0, \text{for which } \Omega \text{ has a finite } \varepsilon\text{-net in } \mathscr{E}\}.$$

Notice that the Hausdorff MNC satisfies the semi-homogeneity condition, i.e.:

$$\chi(\lambda\Omega) = |\lambda|\chi(\Omega),$$

for each $\lambda \in \mathbb{R}$ and each $\Omega \in Pb(\mathscr{E})$.

For a set $M \subset \mathscr{E}$, define $\|M\| = \sup_{x\in M} \|x\|_{\mathscr{E}}$. Let X be a metric space, and let Y be a normed space.

Definition 2.4. A multivalued map (multimap) $\mathscr{F} : X \to P(Y)$ is said to be lower semicontinuous (l.s.c.) at a point $x \in X$ if, for every open set $V \subset Y$ such that $\mathscr{F}(x) \cap V \neq \emptyset$, there exists a neighborhood $U(x)$ of x such that $\mathscr{F}(x') \cap V \neq \emptyset$ for all $x' \in U(x)$.

A multimap is lower semicontinuous (l.s.c.) if it is lower semicontinuous at every point $x \in X$.

Definition 2.5. A multimap $\mathscr{F} : [0,T] \times X \to K(Y)$ is said to be almost lower semicontinuous (a.l.s.c.) if there exists a sequence of disjoint compact sets $I_n \subseteq [0,T]$ such that

(i) $meas([0,T] \setminus I) = 0$, where $I = \cup_n I_n$;
(ii) the restriction of $\mathscr{F}$ on each set $J_n = I_n \times Y$ is l.s.c..

Definition 2.6. A multimap $\mathscr{F} : X \to P(Y)$ is called closed if its graph $G_{\mathscr{F}} = \{(x,y) : x \in X, y \in \mathscr{F}(x)\}$ is a closed subset of $X \times Y$.

Definition 2.7. A continuous map $f : X \subseteq \mathscr{E} \to \mathscr{E}$ is called condensing with respect to a MNC β (or β–condensing) if for each bounded set $\Omega \subseteq X$ which is not relatively compact,

$$\beta(f(\Omega)) \ngeq \beta(\Omega).$$

In the sequel we need the following Sadovskii type theorem (see [25] and [27]).

Theorem 2.1. *Let $\mathscr{M}$ be a convex closed bounded subset of $\mathscr{E}$ and $f : \mathscr{M} \to \mathscr{M}$ be a β–condensing map, where β is a monotone nonsingular MNC in $\mathscr{E}$. Then the fixed point set* Fix f *is non-empty.*

2.3. **Measurable multifunctions.** Recall some notions (see, e.g., [25] and [26]). Let E be a separable Banach space.

Definition 2.8. For a given $p \geq 1$, a multifunction $G : [0,T] \to K(E)$ is said to be:

- L^p–integrable if it admits an L^p–Bochner integrable selection, i.e., there exists a function $g \in L^p([0,T];E)$ such that $g(t) \in G(t)$ for a.e. $t \in [0,T]$;
- L^p–integrably bounded if there exists a function $\xi \in L^p([0,T])$ such that

$$\|G(t)\| \leq \xi(t)$$

for a.e. $t \in [0,T]$.

The set of all L^p–integrable selections of a multifunction $G : [0,T] \to K(E)$ is denoted by $\mathscr{S}^p_G$.

A multifunction $G : [0,T] \to K(E)$ is called measurable if, for every open subset $V \subset E$, the set $G^{-1}(V)$ is Lebesgue measurable. Every multimap $\mathscr{F} : [0,T] \times E \to K(E)$ generates a correspondence assigning to every function $q : [0,T] \to E$ the multifunction $\Phi : [0,T] \to P(E)$ defined by the formula

$$\Phi(t) = \mathscr{F}(t,q(t)).$$

If, for every measurable function q, the multifunction Φ is measurable, then a multimap $\mathscr{F}$ is called superpositionally measurable.

Lemma 2.1. *(see [25] and [26]) If a multimap $\mathscr{F} : [0,T] \times E \to P(E)$ is a.l.s.c., then it is superpositionally measurable.*

Lemma 2.2. *(see [25], Theorem 4.2.1) Let a sequence of functions $\{\xi_n\} \subset L^1([0,T];E)$ be L^1–integrably bounded. Suppose that*

$$\chi(\{\xi_n\}(t)) \leq \alpha(t) \quad a.e.\ t \in [0,T],$$

for all $n = 1,2,...$, where $\alpha \in L^1_+([0,T])$. Then, for every $\delta > 0$, there exist a compact set $K_\delta \subset E$, a set $m_\delta \subset [0,T]$ of a Lebesgue measure $m_\delta < \delta$, and a set of functions $G_\delta \subset L^1([0,T];E)$ with values in K_δ, such that, for every $n \geq 1$, there exists a function $b_n \in G_\delta$ for which

$$\|\xi_n(t) - b_n(t)\|_E \leq 2\alpha(t) + \delta, \qquad t \in [0,T] \setminus m_\delta.$$

Moreover, the sequence $\{b_n\}$ may be chosen so that $b_n \equiv 0$ on m_δ and this sequence is weakly compact.

3. Main Results

Let E is a separable Banach space and a multimap $F : [0,T] \times E \to K(E)$ be such that:

$(F1)$ $F : [0,T] \times E \to K(E)$ is a.l.s.c.;

$(F2)$ for each $r > 0$, there exists a function $\omega_r \in L^\infty([0,T])$ such that, for each $x \in E$ with $\|x\| \leq r$,

$$\|F(t,x)\| \leq \omega_r(t)$$

for a.e. $t \in [0,T]$;

$(F3)$ there exists a function $\mu \in L^\infty([0,T])$ such that, for each bounded set $Q \subset E$,

$$\chi(F(t,Q)) \leq \mu(t)\chi(Q),$$

for a.e. $t \in [0,T]$, where χ is the Hausdorff MNC in E.

On a linear operator A, we pose the following condition:

(A) $A : D(A) \subset E \to E$ is a linear closed operator in E generating a C_0–semigroup $\{U(t)\}_{t\geq 0}$.

Denote $M = \sup\{\|U(t)\|; t \in [0;T]\}$.

For $x \in C([0,\tau];E), 0 < \tau \leq T$, consider the multifunction:

$$\Phi_F : [0,\tau] \to K(E), \qquad \Phi_F(t) = F(t,x(t)).$$

From above conditions $(F1)-(F2)$ and Lemma 2.1, it follows that the multifunction Φ_F is measurable and L^p–integrable for each $p \geq 1$. Then, the superposition multioperator $\mathscr{P}^\infty_F : C([0,\tau];E) \multimap L^\infty([0,\tau];E)$ given as:

$$\mathscr{P}^\infty_F(x) = \mathscr{S}^\infty_{\Phi_F},$$

is well defined.

Definition 3.1. A mild solution of Cauchy problem (1.1) - (1.2) on an interval $[0,\tau],\ \tau\in(0,T]$ is a function $x\in C([0,\tau];E)$, which can be represented as:

$$x(t)=\mathscr{G}(t)x_0+\int_0^t(t-s)^{q-1}\mathscr{T}(t-s)\phi(s)ds,\qquad t\in[0,\tau],$$

where $\phi\in\mathscr{P}_F^\infty(x)$,

$$\mathscr{G}(t)=\int_0^\infty\xi_q(\theta)U(t^q\theta)d\theta,\qquad \mathscr{T}(t)=q\int_0^\infty\theta\xi_q(\theta)U(t^q\theta)d\theta,$$

$$\xi_q(\theta)=\frac{1}{q}\theta^{-1-\frac{1}{q}}\Psi_q(\theta^{-1/q}),$$

and

$$\Psi_q(\theta)=\frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q),\theta\in\mathbb{R}^+.$$

Remark 3.1. [9] $\xi_q(\theta)\geq0$, $\int_0^\infty\xi_q(\theta)\,d\theta=1$, and $\int_0^\infty\theta\xi_q(\theta)\,d\theta=\frac{1}{\Gamma(q+1)}$.

Lemma 3.1. [9] *The operator functions $\mathscr{G}$ and $\mathscr{T}$ possess the following properties:*

1) *for each $t\in[0,T]$, $\mathscr{G}(t)$ and $\mathscr{T}(t)$ are linear bounded operators. More precisely, for each $x\in E$,*

$$\|\mathscr{G}(t)x\|_E\leq M\|x\|_E,\tag{3.1}$$

and

$$\|\mathscr{T}(t)x\|_E\leq\frac{qM}{\Gamma(1+q)}\|x\|_E;\tag{3.2}$$

2) *the operator functions $\mathscr{G}(\cdot)$ and $\mathscr{T}(\cdot)$ are strongly continuous, i.e. functions $t\in[0,T]\to\mathscr{G}(t)x$ and $t\in[0,T]\to\mathscr{T}(t)x$ are continuous for each $x\in E$.*

To search for mild solutions of problem (1.1) - (1.2), consider the map

$$S:L^\infty([0,\tau];E)\to C([0,\tau];E),$$

and

$$S(\phi)(t)=\int_0^t(t-s)^{q-1}\mathscr{T}(t-s)\phi(s)ds.$$

Lemma 3.2. [6] *The operator S obeys the following properties:*

(S_1) *if $\frac{1}{q}<p<\infty$, then there exists a constant $C>0$ such that*

$$\|S(\xi)(t)-S(\eta)(t)\|_E^p\leq C^p\int_0^t\|\xi(s)-\eta(s)\|_E^p\,ds,\qquad \xi,\eta\in L^p([0,\tau];E);$$

(S_2) *for each compact set $K\subset E$ and bounded sequence $\{\eta_n\}\subset L^\infty([0,\tau];E)$ such that $\{\eta_n(t)\}\subset K$ for a.e. $t\in[0,\tau]$, the weak convergence $\eta_n\rightharpoonup\eta_0$ in $L^1([0,\tau];E)$ implies the convergence $S(\eta_n)\to S(\eta_0)$ in $C([0,\tau];E)$.*

Consider the multioperator $G:C([0,\tau];E)\multimap C([0,\tau];E)$, given in the following way:

$$G(x)=g_0+S\circ\mathscr{P}_F^\infty(x),\qquad t\in[0,\tau],$$

where the function $g_0(t)=\mathscr{G}(t)x_0$.

It is clear that a function $x\in C([0,\tau];E)$ is a mild solution of problem (1.1) - (1.2) on the interval $[0,\tau]$ if it is a fixed point $x\in G(x)$ of the multioperator G.

We need the following notion and results.

Definition 3.2. (see [28] and [25], Definition 5.5.1) A non-empty subset $\mathscr{M} \subset L^p([0,\tau];E)$, $p \geq 1$, is said to be decomposable if for every $f, g \in \mathscr{M}$ and each measurable subset m in $[0,\tau]$:

$$f \cdot \kappa_m + g \cdot \kappa_{[0,\tau]\setminus m} \in \mathscr{M},$$

where κ is a characteristic function of a set.

It is clear that the superposition multioperator $\mathscr{P}_F^\infty$ has closed and decomposable values, and moreover it is l.s.c. (see [25, Section 5.5], or [26, Theorem 1.5.36]).

The collection of all non-empty closed decomposable subsets of the space $L^p([0,\tau];E)$ will be denoted by $D(L^p([0,\tau];E))$. The following analogue of the Michael selection theorem which is due to Fryszkowski–Bressan–Colombo holds true (see [28] and [29]).

Lemma 3.3. *Let X be a separable metric space. Then each lower semicontinuous multimap $\mathscr{F} : X \to D(L^1([0,\tau];E))$ admits a continuous selection.*

Applying the last result to the multimap $\mathscr{P}_F^\infty$, we see that it has a continuous selection $p : C([0,\tau];E) \to L^\infty([0,\tau];E)$. Then a map $g : C([0,\tau];E) \to C([0,\tau];E)$

$$g(x)(t) = \mathscr{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)p(x)(s)ds$$

is a continuous selection of the multimap G and its fixed points $Fix\, g \subset Fix\, G$. Therefore, it is sufficient to prove the existence of fixed points of map g.

Lemma 3.4. *Let $\Omega \subset C([0,\tau];E)$ be a non-empty set and $\Omega(t)$ be a relatively compact subset of E for each $t \in [0,\tau]$. Then a set of functions*

$$\left\{(S \circ p)(\Omega) = \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)p(x)(s)ds : x \in \Omega\right\}$$

is equicontinuous.

Proof. Let us fix $\varepsilon > 0$. If we take $t_1, t_2 \in [0,\tau]$ such that $0 < t_1 < t_2 \leq \tau \leq T$, then, for arbitrary $f \in p(\Omega)$,

$$\begin{aligned}
&\left\|S(f)(t_2) - S(f)(t_1)\right\|_E \\
&\leq \left\|\int_0^{t_2} (t_2-s)^{q-1}\mathscr{T}(t_2-s)f(s)ds - \int_0^{t_1}(t_1-s)^{q-1}\mathscr{T}(t_1-s)f(s)ds\right\|_E \\
&\leq \left\|\int_{t_1}^{t_2} (t_2-s)^{q-1}\mathscr{T}(t_2-s)f(s)ds\right\|_E \\
&\quad + \left\|\int_0^{t_1}\Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s)\Big)f(s)ds\right\|_E \\
&= Z_1 + Z_2,
\end{aligned}$$

where

$$Z_1 = \left\|\int_{t_1}^{t_2}(t_2-s)^{q-1}\mathscr{T}(t_2-s)f(s)ds\right\|_E,$$

and

$$Z_2 = \left\| \int_0^{t_1} \left((t_2-s)^{q-1} \mathscr{T}(t_2-s) - (t_1-s)^{q-1} \mathscr{T}(t_1-s) \right) f(s) ds \right\|_E .$$

By using Lemma 3.1 and condition $(F2)$, we can take $\delta_1 > 0$ such that the condition $|t_2 - t_1| < \delta_1$ for $f \in p(\Omega)$ implies the following estimate:

$$Z_1 \leq \frac{qM \|\omega_{r_\Omega}\|_\infty}{\Gamma(1+q)} \frac{(t_2-t_1)^q}{q} < \frac{\varepsilon}{6}.$$

To estimate Z_2, we choose

$$d < \delta_2 := \left[\frac{\frac{\varepsilon}{6}\Gamma(1+q)}{M \|\omega_{r_\Omega}\|_\infty (2^q+1)} \right]^{\frac{1}{q}}.$$

Then, for $t_1 < d$ and $t_2 - t_1 < d$,

$$\begin{aligned} Z_2 &\leq \int_0^{t_1} (t_2-s)^{q-1} \|\mathscr{T}(t_2-s)\| \cdot \|f(s)\| ds + \int_0^{t_1} (t_1-s)^{q-1} \|\mathscr{T}(t_1-s)\| \cdot \|f(s)\| ds \\ &\leq \int_0^{t_2} (t_2-s)^{q-1} \|\mathscr{T}(t_2-s)\| \cdot \|f(s)\| ds + \int_0^{t_1} (t_1-s)^{q-1} \|\mathscr{T}(t_1-s)\| \cdot \|f(s)\| ds \\ &\leq \frac{M \|\omega_{r_\Omega}\|_\infty}{\Gamma(1+q)} (2^q+1) d^q < \frac{\varepsilon}{6}. \end{aligned}$$

For $t_1 > d$, we have

$$\begin{aligned} Z_2 &\leq \left\| \int_0^{t_1-d} \left((t_2-s)^{q-1} \mathscr{T}(t_2-s) - (t_1-s)^{q-1} \mathscr{T}(t_1-s) \right) f(s) ds \right\|_E \\ &\quad + \left\| \int_{t_1-d}^{t_1} \left((t_2-s)^{q-1} \mathscr{T}(t_2-s) - (t_1-s)^{q-1} \mathscr{T}(t_1-s) \right) f(s) ds \right\|_E \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 = \left\| \int_0^{t_1-d} \left((t_2-s)^{q-1} \mathscr{T}(t_2-s) - (t_1-s)^{q-1} \mathscr{T}(t_1-s) \right) f(s) ds \right\|_E ,$$

and

$$I_2 = \left\| \int_{t_1-d}^{t_1} \left((t_2-s)^{q-1} \mathscr{T}(t_2-s) - (t_1-s)^{q-1} \mathscr{T}(t_1-s) \right) f(s) ds \right\|_E .$$

Take a small d such that

$$I_2 \leq \frac{M \|\omega_{r_\Omega}\|_\infty d^q (2+2^q)}{\Gamma(1+q)} < \frac{\varepsilon}{6}.$$

Since $\Omega(t)$ is a relatively compact set for each $t \in [0,\tau]$, we have that $\chi_E(\Omega(t)) \equiv 0$. By Lemma 2.2 for every $\delta_3 > 0$, there exist a compact set $K_{\delta_3} \subset E$, a set $m_{\delta_3} \subseteq [0,\tau]$ of Lebesgue's measure $mes(m_{\delta_3}) < \delta_3$, and a set of functions $\Delta \subset L^1([0,\tau];E)$ with values in K_{δ_3} such that there exists a function $b \in \Delta$ for which

$$\|f(t) - b(t)\|_E \leq \delta_3, t \in [0,\tau] \setminus m_{\delta_3}. \tag{3.3}$$

Moreover, the function $b \in \Delta$ may be chosen so that $b(t) \equiv 0$ on m_{δ_3} and the set Δ is weakly compact in $L^1([0,\tau];E)$. Then, for I_1, the following estimate holds:

$$
\begin{aligned}
I_1 \leq & \left\| \int_0^{t_1-d} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) (f(s)-b(s)+b(s))\,ds \right\|_E \\
\leq & \left\| \int_0^{t_1-d} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) (f(s)-b(s))\,ds \right\|_E \\
& + \left\| \int_0^{t_1-d} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) b(s)ds \right\|_E \\
\leq & \left\| \int_{[0,t_1-d]\setminus m_{\delta_3}} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) (f(s)-b(s))\,ds \right\|_E \\
& + \left\| \int_{[0,t_1-d]\cap m_{\delta_3}} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) (f(s)-b(s))\,ds \right\|_E \\
& + \left\| \int_{[0,t_1-d]\setminus m_{\delta_3}} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) b(s)ds \right\|_E \\
& + \left\| \int_{[0,t_1-d]\cap m_{\delta_3}} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) b(s)ds \right\|_E \\
= & \left\| \int_{[0,t_1-d]\setminus m_{\delta_3}} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) (f(s)-b(s))\,ds \right\|_E \\
& + \left\| \int_{[0,t_1-d]\cap m_{\delta_3}} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) (f(s)-b(s))\,ds \right\|_E \\
& + \left\| \int_{[0,t_1-d]\setminus m_{\delta_3}} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) b(s)ds \right\|_E \\
= & N_1 + N_2 + N_3,
\end{aligned}
$$

where

$$
N_1 = \left\| \int_{[0,t_1-d]\setminus m_{\delta_3}} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) (f(s)-b(s))\,ds \right\|_E,
$$

$$
N_2 = \left\| \int_{[0,t_1-d]\cap m_{\delta_3}} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) (f(s)-b(s))\,ds \right\|_E,
$$

and

$$
N_3 = \left\| \int_{[0,t_1-d]\setminus m_{\delta_3}} \Big((t_2-s)^{q-1}\mathscr{T}(t_2-s) - (t_1-s)^{q-1}\mathscr{T}(t_1-s) \Big) b(s)ds \right\|_E.
$$

By using (3.3), we may find small $\delta_3 > 0$ so that $mes(m_{\delta_3}) < 2\frac{\varepsilon}{6}d^{1-q}$ yields $N_1 < \frac{\varepsilon}{6}$, and $N_2 < \frac{\varepsilon}{6}$. Recall that the functions from Δ take their values in K_{δ_3} that implies $\Delta \subset L^\infty([0,\tau];E)$. Then by using Lemma 3.2, we can choose $\delta_4 > 0$ such that $|t_2 - t_1| < \delta_4$ yields $N_3 < \frac{\varepsilon}{6}$. So, for each $\varepsilon > 0$, we may choose $\delta = min\{\delta_1, \delta_2, \delta_3, \delta_4\}$ such that

$$\begin{aligned}\left\| S(f)(t_2) - S(f)(t_1) \right\|_E &\le Z_1 + Z_2 \le Z_1 + I_1 + I_2 \\ &\le Z_1 + I_2 + N_1 + N_2 + N_3 \\ &< \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \varepsilon\end{aligned}$$

for each $f \in p(\Omega)$ and $|t_2 - t_1| < \delta$, i.e., the set $(S \circ p)(\Omega)$ is equicontinuous. □

Now, let us consider the conditions under which the operator g is condensing. Introduce in the space $C([0,\tau];E)$ the measure of noncompactness

$$\nu : Pb(C([0,\tau];E)) \to \mathbb{R}^2_+$$

with the values in the cone $\mathbb{R}^2_+$ defined as

$$\nu(\Omega) = (\varphi(\Omega), mod_C(\Omega)),$$

where $\varphi(\Omega)$ is the module of fiber noncompactness

$$\varphi(\Omega) = \sup_{t\in[0,\tau]} \chi(\{y(t) : y \in \Omega\})$$

and the second component is the equicontinuity module which is given as

$$mod_C(\Omega) = \lim_{\delta\to 0} \sup_{y\in\Omega} \max_{|t_1-t_2|\le\delta} \|y(t_1) - y(t_2)\|.$$

It is known (see [25]) that the MNC ν is monotone, nonsingular, algebraically semiadditive, and regular.

We need the following assertion, which follows from [6].

Lemma 3.5. *Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $L^{\infty}([0,\tau];E)$ such that*

$$\chi(\{f_n(t)\}) \le v(t) \text{ a.e. } t \in [0,\tau],$$

where $v \in L^{\infty}_+(0,\tau)$. Then

$$\chi(\{Sf_n(t)\}) \le 2\frac{qM}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} v(s)ds.$$

Now, let us formulate the following condition under which the operator g is condensing.

Lemma 3.6. *Under conditions (A), $(F1)$ - $(F3)$, and condition*

(C) $$l := \frac{2MT^q}{\Gamma(1+q)}\|\mu\|_{\infty} < 1$$

the operator g is ν-condensing.

Proof. Let $\Omega \subset C([0,\tau];E)$ be a nonempty bounded set, $\|\Omega\| \le r_{\Omega}$, and

$$\nu(g(\Omega)) \ge \nu(\Omega). \tag{3.4}$$

We show that Ω is a relatively compact set. It is clear that it is sufficient to prove the assertion for map $S \circ p$. Since the MNC ν is nonsingular we have

$$\nu((S \circ p)(\Omega)) \ge \nu(\Omega). \tag{3.5}$$

From (3.5), it follows that

$$\varphi((S\circ p)(\Omega)) \geq \varphi(\Omega). \tag{3.6}$$

Applying regularity condition $(F3)$, we have for $0 \leq s \leq \tau \leq T$ the following estimate:

$$\chi_E(p(\Omega)(s)) = \chi_E(\{f(s) : f \in p(\Omega)\}) \leq \mu(s)\cdot\chi_E(\{x(s) : x \in \Omega\}) \leq \mu(s)\varphi(\Omega).$$

Now, by using this estimate and Lemma 3.5, we obtain

$$\chi_E\left((S\circ p)(\Omega)(t)\right) \leq \frac{2qM}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\mu(s)ds\cdot\varphi(\Omega) \leq \frac{2MT^q}{\Gamma(1+q)}\|\mu\|_\infty\cdot\varphi(\Omega).$$

Hence

$$\varphi\left((S\circ p)(\Omega)\right) \leq l\varphi(\Omega), \tag{3.7}$$

with $l < 1$. Comparing inequalities (3.6) and (3.7), we have

$$\varphi(\Omega) = 0.$$

Now, from (3.5) we have

$$mod_C((S\circ p)(\Omega)) \geq mod_C(\Omega). \tag{3.8}$$

From Lemma 3.4, we know that the set $(S\circ p)(\Omega)$ is equicontinuous, and then

$$mod_C((S\circ p)(\Omega)) = 0.$$

It follows that $mod_C(\Omega) = 0$. Therefore, $\nu(\Omega) = (0,0)$. We conclude that Ω is relatively compact set, which yields that the operator g is condensing w.r.t. the MNC ν. □

Now we can prove the local existence theorem for Cauchy problem (1.1) - (1.2).

Theorem 3.1. *Under conditions (A), $(F1)-(F3)$, there exists $\tau \in (0,T]$ such that the set of mild solutions of Cauchy problem (1.1) - (1.2) $\Sigma_{x_0}^F[0,\tau]$ on the interval $[0,\tau]$ is a non-empty subset of the space $C([0,\tau];E)$.*

Proof. Take a number $r > 0$. Since the family of operators $\mathscr{G}(t)$ is equicontinuous, we may choose $0 < \tau_1 < T$ such that

$$\|(\mathscr{G}(t) - \mathscr{G}(0))x_0\|_E \leq r/2 \text{ for all } t \in [0,\tau_1]. \tag{3.9}$$

Let $\overline{B}_r(\mathscr{G}(0)x_0) \subset E$ be a closed ball and $R = \|\mathscr{G}(0)x_0\|_E + r$. Take $\tau_2 \in (0,T]$ such that

$$\frac{M\|\omega_R\|_\infty\tau_2^q}{\Gamma(1+q)} \leq r/2, \tag{3.10}$$

where M is the constant from condition (A), and ω_R is the function from condition $(F2)$.

From Lemma 3.6, we know that the operator g is ν-condensing. Take $\tau = \min(\tau_1,\tau_2)$ and consider the ball $\overline{B}_r(x^0) \subset C([0,\tau];E)$, where x^0 is the function identically equal to $\mathscr{G}(0)x_0$. We show that the operator g transforms the ball $\overline{B}_r(x^0)$ into itself. In fact, if $x \in \overline{B}_r(x^0)$, then $\|x\|_{C([0,\tau];E)} \leq R$ for all $t \in [0,\tau]$. From condition $(F2)$, we have

$$\|f(t)\|_E \leq \omega_R(t),\ \text{a.e. } t \in [0,\tau],$$

for $f = p(x)$.

Now, for $y = g(x)$, we have

$$y(t) = \mathscr{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathscr{T}(t-s)f(s)ds.$$

By using (3.9), (3.10), and Lemma 3.1, we have the following estimate:

$$\|y(t)-x_0\|_E \le \|(\mathscr{G}(t)-\mathscr{G}(0))x_0\|_E + \int_0^t (t-s)^{q-1}\|\mathscr{T}(t-s)\|_{L(E)}\|f(s)\|_E\,ds$$
$$\le \|(\mathscr{G}(t)-\mathscr{G}(0))x_0\|_E + \frac{M\|\omega_R\|_\infty \tau^q}{\Gamma(1+q)} \le r/2+r/2 \le r,$$

from which it follows that $y \in \overline{B}_r(x^0)$. Now we can apply Theorem 2.1. □

Now, let us prove the global existence result.

Theorem 3.2. *Under conditions (A), $(F1)$, and $(F3)$, suppose that condition $(F'2)$ has the following form:*

$(F'2)$ *there exists a function $\alpha \in L^\infty_+([0,T])$ such that*

$$\|F(t,x)\|_E \le \alpha(t)(1+\|x(t)\|_E)\ \text{for a.e. } t\in[0,T].$$

If

$$\frac{2MT^q}{\Gamma(1+q)}k < 1,$$

where $k = \max\{\|\alpha\|_\infty, \|\mu\|_\infty\}$, and functions α and μ are from conditions $(F'2)$ and $(F3)$, respectively, then problem (1.1) - (1.2) has a mild solution.

Proof. Taking an arbitrary $x \in C([0,T];E)$, we have for $t\in[0,T]$ the following estimate:

$$\|g(x)(t)\|_E \le \|\mathscr{G}(t)x_0\|_E + \int_0^t (t-s)^{q-1}\|\mathscr{T}(t-s)\|_{L(E)}\|p(x)(s)\|_E\,ds$$
$$\le M\|x_0\|_E + \frac{Mq}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\alpha(t)(1+\|x(s)\|_E)ds$$
$$\le M\|x_0\|_E + \frac{Mq\|\alpha\|_\infty}{\Gamma(1+q)}\left(\int_0^t (t-s)^{q-1}ds + \int_0^t (t-s)^{q-1}\|x(s)\|_E\,ds\right)$$
$$\le M\|x_0\|_E + \frac{Mq\|\alpha\|_\infty}{\Gamma(1+q)}\frac{T^q}{q} + \|x\|_{C([0,T];E)}\frac{Mq\|\alpha\|_\infty}{\Gamma(1+q)}\frac{T^q}{q}$$
$$\le a + c\|x\|_{C([0,T];E)},$$

where

$$a = M\|x_0\|_E + \frac{M\|\alpha\|_\infty T^q}{\Gamma(1+q)},\ c = \frac{2MT^q}{\Gamma(1+q)}k.$$

So, if we take $R \ge \frac{a}{1-c}$, then the inequality $\|x\|_{C([0,T];E)} \le R$ implies $\|g(x)\|_{C([0,T];E)} \le R$. Therefore, the operator g transforms the closed ball $\overline{B}_R(0) \subset C([0,T];E)$ into itself. Now, since g is condensing, by Theorem 2.1, it has a fixed point, which is a mild solution of problem (1.1) - (1.2). □

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