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ALGORITHM FOR GENERALIZED HYBRID OPERATORS WITH NUMERICAL ANALYSIS AND APPLICATIONS

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Abstract. The purpose of this paper is to introduce a new three-step iteration procedure for fixed points of generalized hybrid operators. Weak and strong convergence results are obtained. The stability and the data dependence are also obtained for quasi-strictly contractive operators. The split feasibility problem, the problem of maximizing the modulus of a complex polynomial, and the reconstruction of signals using the compressive sensing technique are numerically addressed. Examples are presented to show the efficiency of the iteration procedure.
Keywords. Fixed point; Generalized hybrid operator; Image processing and signal recovery; Qualitative analysis; Split problem.

1. INTRODUCTION

Iterative methods of fixed points are useful tools for various classes of nonlinear problems and their applications, such as split feasibility problems, variational inequalities , equilibrium problems, image recovery, and signal processing; see, e.g., [1, 2, 3, 4, 5, 6, 7] and the references therein. In addition to the class of contractive mappings, various nonlinear operators, such as nonexpansive mappings, asymptotically nonexpansive mappings, nonspreading mappings, (generalized) Suzuki mappings, η-demimetric mappings, strictly pseudocontractive mappings, pseudocontractive mappings, asymptotically pseudocontractive mappings and so on, have been extensively studied. For fixed points of these nonlinear mappings, various iterations were introduced and investigated, such as Mann iteration [8], Halpern iteration, Ishikawa iteration [9], Noor iteration [10]), the S-iteration, the split iterations, the hybrid projection iterations and so on. The design of iterative processes is closely related the involved nonlinear operators The basic analysis usually aims at revealing convergence-related features. In addition, concerns also have been paid to data dependence and stability; see, e.g., [11, 12, 13, 14, 15] and the references therein. For important results concerning nonlinear mappings, fixed point properties, and iteration processes together with their qualitative aspects, we refer to [16, 17, 18, 19, 20] and the references therein.

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Motivated by all these possible research directions, we include them to see a complete picture of an iteration process. We start by initiating a new iterative process, as a basic element for our approach. Furthermore, the analysis is carried out in several directions, the maximum modulus of complex polynomials, the split feasibility problem, and compressive sensing signal reconstruction.

2. Algorithm for Reckoning Fixed Points of Generalized Hybrid Mappings

As a starting point for designing a new iteration scheme, we considered a three-step iteration process of Thakur *et al.* [21]. Another motivation for initiating our new iteration is the MCS procedure, recently introduced by Mouktonglang *et al.* [22]. Following the iteration patterns in [21] and [22], we develop a more versatile three-step iterative process, denoted by $\mathscr{U}_n$ iteration.

Algorithm 2.1 (**$\mathscr{U}_n$ Iteration).** *Let C be a nonempty convex set, and let $T: C \to C$ be a given operator. For an arbitrary initial point $x_0 \in C$, construct the sequence $\{x_n\}$ iteratively by*

$$\begin{cases} z_n = (1-\xi_n)x_n + \xi_n T x_n \\ y_n = T((1-\zeta_n)Tx_n + \zeta_n T z_n) \\ x_{n+1} = (1-\eta_n-\delta_n)Tx_n + \eta_n T y_n + \delta_n T z_n, \end{cases} \tag{2.1}$$

where $\{\xi_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, $\{\delta_n\}$, and $\{\eta_n+\delta_n\}$ are sequences of real numbers in $(0,1)$. Additionally, we further assume that the parametric sequence $\{\xi_n\}$ satisfies $0 < p \le \xi_n \le q < 1$.

2.1. Preliminaries.

Before moving on to our convergence study, we provide a short description for the general setting as well as for some important tools, which are needed throughout our analysis.

Let C be a nonempty subset of a normed space X. A mapping $T: C \to X$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x-y\|$, for all $x, y \in C$. As usual, we denote by $F(T)$ the set of fixed points of T. If the existence of a fixed point is guaranteed, that, is $F(T) \neq \emptyset$, and $\|Tx - p\| \le \|x - p\|$ for all $x \in C$ and $p \in F(T)$, then T is said to be quasi-nonexpansive. It is known that if C is a nonempty, closed, and convex subset of a Banach space and T is quasi-nonexpansive, then $F(T)$ is closed and convex. Recall that T is called firmly nonexpansive if $\|Tx-Ty\|^2 \le \langle x-y, Tx-Ty\rangle$, for all $x, y \in C$. Kohsaka and Takahashi [23] deduced a new general class of nonlinear mappings inspired by the firmly nonexpansive condition. The resulting operators, nonspreading mappings, are defined in Hilbert spaces by the inequality $2\|Tx-Ty\|^2 \le \|Tx-y\|^2 + \|Ty-x\|^2$, for all $x,y \in C$. Furthermore, based on the firmly nonexpansive mapping, Takahashi [24] define the hybrid mappings on Hilbert spaces. It is equivalent to the restriction $3\|Tx-Ty\|^2 \le \|x-y\|^2 + \|Tx-y\|^2 + \|Ty-x\|^2$, for all $x, y \in C$. Note that nonexpansive, nonspreading, and hybrid mappings are independent classes of mappings on Hilbert spaces although not completely disjoint as they contain firmly nonexpansive mappings (see, e.g., [24]). Further, Kocourek *et al.* [25] included all nonexpansive, nonspreading, and hybrid mappings into a new wider class of nonlinear operators, called (α,β)-generalized hybrid mappings. This class of nonlinear operators, which was first introduced in Hilbert spaces, is investigated in Banach space in this paper.

Definition 2.1 ([25])**.** Let C be a nonempty subset of a normed space X. A mapping $T: C \to X$ is said to be generalized hybrid if there exit $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx-Ty\|^2 + (1-\alpha)\|x-Ty\|^2 \le \beta\|Tx-y\|^2 + (1-\beta)\|x-y\|^2, \tag{2.2}$$

for all $x, y \in C$.

We point out that, for $\alpha = 1$ and $\beta = 0$, the class of nonexpansive mappings can be retrieved from condition (2.2): if an (α, β)-generalized hybrid mapping has a fixed point, then it is quasi-nonexpansive. Moreover, a $(2,1)$-generalized hybrid operator is nonspreading and a $\left(\frac{3}{2}, \frac{1}{2}\right)$-generalized hybrid one is hybrid. However, for our purpose, we only relate on the subordination relation existing between (α, β)-generalized hybrid mappings and nonexpansive mappings. Observe that each $(1,0)$-generalized hybrid mapping is nonexpansive, however, the (α, β)-generalized hybrid condition is indeed wider than the class of nonexpansive mappings. This can be seen via the following example.

Example 2.1. Consider the mapping $T\colon [0,1] \to [0,1]$ is defined by $Tx = \frac{3}{4}x^2$. Obviously, $x = 0$ is the fixed point of T in $[0,1]$. Then, T is a generalized hybrid mapping, but not nonexpansive with respect to the usual norm on $\mathbb{R}$.

Proof. To prove that T is generalized hybrid, we set $\alpha = \beta = 4$. Thus, condition (2.2) becomes

$$4(Tx - Ty)^2 - 3(x - Ty)^2 \leq 4(Tx - y)^2 - 3(x - y)^2$$

due to

$$\begin{aligned}
&4(Tx - Ty)^2 - 3(x - Ty)^2 - 4(Tx - y)^2 - 3(x - y)^2 \\
&= 4\left(y - \frac{3}{4}y^2\right)\left(\frac{3}{2}x^2 - \frac{3}{4}y^2 - y\right) - 3\left(y - \frac{3}{4}y^2\right)\left(2x - \frac{3}{4}y^2 - y\right) \\
&\left(y - \frac{3}{4}y^2\right)\left[6x(x-1) - y\left(1 + \frac{3}{4}y\right)\right] \\
&\leq 0,
\end{aligned}$$

for all $x, y \in [0,1]$. To check that T is not nonexpansive, we choose $x = \frac{2}{3}$ and $y = 1$. Thus, $|Tx - Ty| = \frac{5}{12}$ and $|x - y| = \frac{1}{3}$ yielding $|Tx - Ty| > |x - y|$. This contradicts the nonexpansivity. □

A basic property of (α, β)-generalized hybrid mappings is provided by the lemma below in connection with the concept of the demiclosedness. The result was deduced in [25, Lemma 5.1] and further investigated in the setting of Banach spaces with the additional property, the Opial property.

Definition 2.2 ([26])**.** A Banach space X is said to satisfy the Opial property if, for each weakly convergent sequence $\{x_n\}$ in X with the weak limit x, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ holds for all $y \in X$ with $y \neq x$.

Lemma 2.1. *Let $T\colon C \to C$ be an (α, β)-generalized hybrid mapping on a subset C of a Banach space X endowed with the Opial property. If $\{x_n\}$ converges weakly to some $z \in C$ and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tz = z$, that is T is demiclosed at zero.*

In addition to the facts related with the class of (α, β)-generalized hybrid mappings presented above, we list below a series of tools, which are useful for our convergence analysis.

Definition 2.3 ([27]). A normed vector space X is said to be uniformly convex if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that, for all $x, y \in X$ with $\|x\| \le 1$, $\|y\| \le 1$, and $\|x-y\| \ge \varepsilon$, $\left\|\frac{x+y}{2}\right\| \le 1-\delta$.

Lemma 2.2 ([28]). *Let X be a uniformly convex Banach space, and let $\{t_n\}$ be a sequence such that $0 < p \le t_n \le q < 1$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \le r$, $\limsup_{n\to\infty} \|y_n\| \le r$, and $\limsup_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Definition 2.4 ([29]). A mapping $T\colon C \to C$ is said to satisfy condition (I) if there exists a nondecreasing function $f\colon [0,\infty] \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x,Tx) \ge f(d(x,F(T)))$, for all $x \in C$, where $d(x,F(T)) = \inf_{p\in F(T)} d(x,p)$. By $d(x,p)$, we denote the distance between two points $x \in C$ and $p \in F(T)$.

Definition 2.5 ([18]). Let C be a nonempty subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, let $r(x,\{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|$ denote the asymptotic radius of $\{x_n\}$ at x. The asymptotic radius of $\{x_n\}$ relative to C is the real number

$$r(C,\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in C\},$$

and the asymptotic center of $\{x_n\}$ with respect to C is the set

$$A(C,\{x_n\}) = \{x \in C : r(x,\{x_n\}) = r(C,\{x_n\})\}.$$

This definition is due to Edelstein [18] who also proved that, for a nonempty, closed, and convex subset of a uniformly convex Banach space and for each bounded sequence $\{x_n\}$, the set $A(C,\{x_n\})$ is a singleton.

2.2. **Convergence results.** In the sequel, fixed points existence results as well as weak and strong convergence results will be phrased and proved for (α,β)-generalized hybrid mappings in connection with $\mathscr{U}_n$ iteration constructed above. The convergence analysis that we aim to perform for this new iteration is in the setting of a uniformly convex Banach space. This ensures the validity of Lemma 2.2 and the conclusion of Edelstein stated above. of our outcomes. Let us first start with a lemma, which play an important role in proving our main results.

Lemma 2.3. *Let C be a nonempty and convex subset of a normed space X, and let $T: C \to C$ be an (α,β)-generalized hybrid mapping with $F(T) \ne \emptyset$. Consider the sequence $\{x_n\}$ generated by $\mathscr{U}_n$ iteration process (2.1). Then $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T)$.*

Proof. Suppose $p \in F(T)$. Since we assume that T has a fixed point, than T is quasi-nonexpansive. Therefore, keeping in mind our iterative scheme but also the convexity of the norm, we have

$$\begin{aligned} \|z_n - p\| &\le (1-\xi_n)\|x_n - p\| + \xi_n \|Tx_n - p\| \\ &\le \|x_n - p\|. \end{aligned} \tag{2.3}$$

Similarly, we have

$$\|y_n - p\| \le \|x_n - p\|. \tag{2.4}$$

From (2.3) and (2.4), we can prove that

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-\eta_n-\delta_n)\|Tx_n-p\|+\eta_n\|Ty_n-p\|+\delta_n\|Tz_n-p\| \\ &\leq (1-\eta_n-\delta_n)\|x_n-p\|+\eta_n\|y_n-p\|+\delta_n\|z_n-p\| \\ &\leq \|x_n-p\|.\end{aligned}$$

We conclude that $\{\|x_n-p\|\}$ is bounded and nonincreasing for all $p\in F(T)$. Therefore, $\lim_{n\to\infty}\|x_n-p\|$ exists. □

Theorem 2.1. *Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space X, and let $T: C\to C$ be an (α,β)-generalized hybrid mapping. Consider the sequence $\{x_n\}$ generated by $\mathscr{U}_n$ iteration process (2.1). Then $F(T)\neq\emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n-x_n\|=0$.*

Proof. We first prove the direct implication. In this respect, suppose $F(T)\neq\emptyset$ and let $p\in F(T)$. From Lemma 2.3, we know that $\lim_{n\to\infty}\|x_n-p\|$ exists and $\{x_n\}$ is bounded. Denote

$$r=\lim_{n\to\infty}\|x_n-p\|. \tag{2.5}$$

From (2.3), we have

$$\limsup_{n\to\infty}\|z_n-p\|\leq\limsup_{n\to\infty}\|x_n-p\|=r. \tag{2.6}$$

Since T is quasi-nonexpansive, we have

$$\limsup_{n\to\infty}\|Tx_n-p\|\leq\limsup_{n\to\infty}\|x_n-p\|=r. \tag{2.7}$$

On the other hand,

$$\begin{aligned}\|x_{n+1}-p\| &\leq (1-\eta_n-\delta_n)\|Tx_n-p\|+\eta_n\|Ty_n-p\|+\delta_n\|Tz_n-p\| \\ &\leq (1-\eta_n-\delta_n)\|x_n-p\|+\eta_n\|x_n-p\|+\delta_n\|z_n-p\| \\ &= \|x_n-p\|-\delta_n\|x_n-p\|+\delta_n\|z_n-p\|,\end{aligned}$$

which further gives that

$$\frac{\|x_{n+1}-p\|-\|x_n-p\|}{\delta_n}\leq\|z_n-p\|-\|x_n-p\|$$

or,

$$\|x_{n+1}-p\|-\|x_n-p\|\leq\frac{\|x_{n+1}-p\|-\|x_n-p\|}{\delta_n}\leq\|z_n-p\|-\|x_n-p\|$$

which leads to $\|x_{n+1}-p\|\leq\|z_n-p\|$. Taking the limit inferior in this inequality and using (2.6), we find that $r\leq\liminf_{n\to\infty}\|z_n-p\|\leq\limsup_{n\to\infty}\|z_n-p\|\leq r$, which shows that $\limsup_{n\to\infty}\|z_n-p\|=r$. Finally, we can explicitly write this last result after the recurrence on $\{z_n\}$

$$\limsup_{n\to\infty}\|z_n-p\|=\limsup_{n\to\infty}\|(1-\xi_n)(x_n-p)+\xi_n(Tx_n-p)\|=r. \tag{2.8}$$

In view of (2.5), (2.7), and (2.8), we find that all the conditions of Lemma 2.2 are guaranteed. Consequently, we have $\lim_{n\to\infty}\|Tx_n-x_n\|=0$, which ends this part of the proof.

We now show the converse statement. In this end, we suppose that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Keeping in mind that T is (α,β)-generalized hybrid, we use relation (2.2) and reconsider it in a more convenient form, that is,

$$\begin{aligned}&\alpha\left(\|Tx_n - Tp\| - \|x_n - Tp\|\right)\left(\|Tx_n - Tp\| + \|x_n - Tp\|\right) + \|x_n - Tp\|^2\\ &\leq \beta\left(\|Tx_n - p\| - \|x_n - p\|\right)\left(\|Tx_n - p\| + \|x_n - p\|\right) + \|x_n - p\|^2.\end{aligned} \tag{2.9}$$

On the other hand, observe that $-\|Tx_n - x_n\| \leq \|Tx_n - Tp\| - \|x_n - Tp\| \leq \|Tx_n - x_n\|$ and $-\|Tx_n - x_n\| \leq \|Tx_n - p\| - \|x_n - p\| \leq \|Tx_n - x_n\|$. Letting $n\to\infty$ and keeping in mind the assumption that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, we obtain

$$\lim_{n\to\infty}\left(\|Tx_n - Tp\| - \|x_n - Tp\|\right) = \lim_{n\to\infty}\left(\|Tx_n - p\| - \|x_n - p\|\right) = 0.$$

We also have

$$\|Tx_n - Tp\| + \|x_n - Tp\| \leq \|Tx_n - x_n\| + 2\|x_n - Tp\|$$

and $\|Tx_n - p\| + \|x_n - p\| \leq \|Tx_n - x_n\| + 2\|x_n - p\|$. Again, using the boundness of $\{x_n\}$, we have $\lim_{n\to\infty}\left(\|Tx_n - Tp\| + \|x_n - Tp\|\right) < \infty$. Simultaneously, we have

$$\lim_{n\to\infty}\left(\|Tx_n - p\| + \|x_n - p\|\right) < \infty.$$

By taking the limit in (2.9), we arrive at

$$\lim_{n\to\infty}\|x_n - Tp\| \leq \lim_{n\to\infty}\|x_n - p\|, \tag{2.10}$$

for all $p \in C$. Let $p \in A(C,\{x_n\})$. Using relation (2.10), we obtain

$$r(Tp,\{x_n\}) = \limsup_{n\to\infty}\|x_n - Tp\| \leq \limsup_{n\to\infty}\|x_n - p\| = r(p,\{x_n\}),$$

which implies that $Tp \in A(C,\{x_n\})$. On the other hand, we have that $A(C,\{x_n\})$ is a singleton. This shows that $p = Tp$ and the proof is complete. □

Theorem 2.2. *Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space X endowed with the Opial property. Let $\{x_n\}$ and T be as in Theorem 2.1 with $F(T) \neq \emptyset$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Put $\omega_w(x_n) = \{z \in C : \exists\{x_{n_k}\} \subset \{x_n\}$ weakly convergent to $z\}$ the weak ω-limit set of the sequence $\{x_n\}$. From Lemma 2.3, we conclude that $\{x_n\}$ is bounded, which implies that $\{x_n\}$ has at least one weakly convergent subsequence. This shows that $\omega_w(x_n)$ a nonempty subset of C. It is to be proved that $\omega_w(x_n)$ consists of exactly one weak limit point. In this respect, we suppose that there exist $z_1, z_2 \in \omega_w(x_n)$, with $z_1 \neq z_2$ such that, for any two arbitrary subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, $\{x_{n_i}\} \rightharpoonup z_1$ and $\{x_{n_j}\} \rightharpoonup z_2$. Since C is closed and convex, then it is weakly closed and convex. Therefore, it contains all the weak limits of all its weakly convergent sequences, which yields $z_1, z_2 \in C$. Since $F(T) \neq \emptyset$, by Theorem 2.1, we have $\lim\limits_{n\to\infty}\|x_{n_i} - Tx_{n_i}\| = 0$. Thus, it follows from Lemma 2.1 that $z_1 = Tz_1$. This means $z_1 \in F(T)$. Following similar arguments for z_2, we find $z_2 \in F(T)$, so $\omega_w(x_n) \subset F(T)$. Observe that X has the Opial property. From Lemma 2.3, we have that $\lim\limits_{n\to\infty}\|x_n - z_1\|$ and $\lim\limits_{n\to\infty}\|x_n - z_2\|$ exist. Hence,

$$\begin{aligned}\lim_{n\to\infty}\|x_n - z_1\| &= \lim_{i\to\infty}\|x_{n_i} - z_1\| < \lim_{i\to\infty}\|x_{n_i} - z_2\| = \lim_{n\to\infty}\|x_n - z_2\|\\ &= \lim_{j\to\infty}\left\|x_{n_j} - z_2\right\| < \lim_{j\to\infty}\left\|x_{n_j} - z_1\right\| = \lim_{n\to\infty}\|x_n - z_1\|.\end{aligned}$$

This obviously leads to a contradiction, so $z_1 = z_2$. Therefore, the subset $\omega_w(x_n)$ consists of exactly one weak limit point. Since all the subsequences of $\{x_n\}$ converge weakly to the same limit point, we conclude that $\{x_n\}$ itself converges weakly to that point, an element of $F(T)$ as $\omega_w(x_n) \subset F(T)$. This completing the proof. □

Theorem 2.3. *Let C be a nonempty, compact, and convex subset of a uniformly convex Banach space X, and let $\{x_n\}$ and T be as in Theorem 2.1. If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Suppose $F(T) \neq \emptyset$. From Theorem 2.1, we have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since subset C is compact, we find that $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges strongly to a point $p \in C$. From relation (2.10), we also have that $\lim_{n\to\infty} \|x_{n_i} - Tp\| \leq \lim_{n\to\infty} \|x_{n_i} - p\|$. Therefore, $\{x_{n_i}\}$ converges strongly to Tp. This yields $Tp = p$, so p is a fixed point of T. By Lemma 2.3, $\lim_{n\to\infty} \|x_n - p\|$ exists. Hence, p is the strong limit of $\{x_n\}$ itself. □

Theorem 2.4. *Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space X, and let $\{x_n\}$ and T be as in Theorem 2.1 with $F(T) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a point in $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf_{p\in F(T)} \|x_n - p\|$.*

Proof. The direct implication follows easily, so we only need to focus on proving the converse. Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Lemma 2.3 states the existence of $\lim_{n\to\infty} \|x_n - p\|$ for all $p \in F(T)$. It follows that $\lim_{n\to\infty} d(x_n, F(T))$ exists too. In particular, $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that, for all $n \geq n_\varepsilon$, $d(x_n, F(T)) = \inf_{p\in F(T)} \|x_n - p\| < \frac{\varepsilon}{2}$. Observe that $\{\|x_n - p\|\}$ is nonincreasing. For any $m, n \geq n_\varepsilon$, we have

$$\|x_n - x_m\| \leq \|x_n - p\| + \|x_m - p\| \leq 2\|x_{n_\varepsilon} - p\|,$$

for all $p \in F(T)$. Taking the infimum over $p \in F(T)$, this leads to $\|x_n - x_m\| \leq 2d(x_{n_\varepsilon}, F(T)) < \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence in C. Since C is a nonempty, closed, and convex subset, it follows that $\{x_n\}$ converges strongly to a point $p \in C$. On the other hand, we have $\lim_{n\to\infty} d(x_n, p) = 0$, which yields $d(p, F(T)) = 0$. The set $F(T)$ is also closed, so $p \in F(T)$. Finally, we conclude that $\{x_n\}$ converges strongly to a point $p \in F(T)$. □

The following result proves that $\mathscr{U}_n$ iteration keeps strongly convergent even that the involved (α,β)-generalized hybrid operator T is under condition (I) of Senter and Dotson [29].

Theorem 2.5. *Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space X, and let $\{x_n\}$ and T be as in Theorem 2.1 with $F(T) \neq \emptyset$. If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Since $F(T) \neq \emptyset$, we find from Theorem 2.1 that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since T satisfies condition (I), then $f(d(x_n, F(T))) \leq \|Tx_n - x_n\|$ which, by taking n to infinity, leads to $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Keeping in mind the properties of function f, we further obtain that $d(x_n, F(T)) = 0$. From Theorem 2.4, we obtain the desired result immediately. □

3. Consequences to the Split Feasibility Problem

Further, we construct a projective algorithm based on $\mathscr{U}_n$ iteration to numerically approach the split feasibility problem. We point out that the main results concerning the convergence of this algorithm arise as consequences of the convergence properties of the $\mathscr{U}_n$ iteration with respect to (α,β)-generalized hybrid mappings.

The split feasibility problem (SPF) was introduced by Censor and Elfving [30] and has been extensively studied in the literature due to its various applications in a wide range of inverse problems. Mathematically, the SPF can be described as follows: given C and Q, two nonempty, closed, and convex subsets of two real Hilbert spaces, H_1 and H_2, respectively, find a point $x \in H_1$ such that $x \in C$ and $Ax \in Q$, where $A\colon H_1 \to H_2$ is a bounded linear operator.

In the sequel, we assume that the split feasibility problem is consistent, that is, the solution set

$$\Omega = \{x \in C : Ax \in Q\} = C \cap A^{-1}Q$$

is nonempty.

One of the main advantages in studying the SFP is that if a solution exists, then the SPF can be rephrased as a fixed point problem by setting

$$T = P_C(I - \gamma A^*(I - P_Q)A),$$

where P_C and P_Q are the orthogonal projections onto C and Q, respectively, $\gamma > 0$, and A^* denotes the adjoint operator of A. In particular, if $0 < \gamma < \dfrac{2}{||A||^2}$, then T is a nonexpansive operator. This guarantees that the split feasibility problem can be solved by using fixed point methods.

Byrne's CQ algorithm [31] is the pioneering work for the SFP. Based on Byrne's CQ algorithm, many efficient algorithms were extensively investigated; see, e.g., [32, 33, 34] and the references therein. Recently, Bejenaru and Postolache [35] considered a three-step algorithm to solve the split feasibility problem. They also provided an example to support their theoretical results. The main advantage of the algorithm is that their projective operators interfere only in the final step, resulting in less computations at each iteration. Their resolvent operator on H_1 reads as follows

$$S = I - \frac{2}{||A||^2}A^*(I - P_Q)A.$$

S is nonexpansive and $\Omega = F(P_C S) = F(S) \cap C$ (see [35]).

In what follows, we redefine $\mathscr{U}_n$ iteration by following the same idea in [35] and show it can solve the split feasibility problem under some conditions.

Algorithm 3.1 (Partially Projective $\mathscr{U}_n$ Iteration). *For an arbitrary initial point $x_0 \in C$, the sequence $\{x_0\}$ is generated by*

$$\begin{cases} z_n &= (1-\xi_n)x_n + \xi_n S x_n, \\ y_n &= T((1-\zeta_n)Sx_n + \zeta_n S z_n), \\ x_{n+1} &= P_C\left[(1-\eta_n-\delta_n)Sx_n + \eta_n S y_n + \delta_n S z_n\right], \end{cases}$$

where $\{\xi_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, $\{\delta_n\}$, and $\{\eta_n+\delta_n\}$ are sequences of real numbers in $(0,1)$ with $\{\xi_n\}$ satisfying $0 < p \le \xi_n \le q < 1$.

It is called partially projective $\mathscr{U}_n$ iteration in this paper. Let us analyze the convergence properties of partially projective $\mathscr{U}_n$ for the SFP.

Lemma 3.1. *Let the sequence $\{x_n\}$ be generated by Algorithm 3.1. Then $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in \Omega$.*

Proof. Choose $p \in \Omega = F(S) \cap C$ in the proof of Lemma 2.3 and replace the operator T with S. Keeping in mind that the projection mapping P_C is nonexpansive and $p \in C$ is also a fixed point of P_C, we have the desired conclusion immediately. □

Lemma 3.2. *Let the sequence $\{x_n\}$ be generated by Algorithm 3.1. Then $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$.*

Proof. In the proof of Theorem 2.1, we choose $p \in \Omega = F(S) \cap C$ and consider the operator S instead of T. The proof follows immediately. □

Theorem 3.1. *Let the sequence $\{x_n\}$ be generated by Algorithm 3.1. Then $\{x_n\}$ converges weakly to a point of Ω.*

Proof. From Theorem 2.2, we find $\omega_w(x_n) \subset F(T) \cap C = \Omega$. It is known that Hilbert spaces have the Opial's property. In view of this, the desired result follows similarly. □

Theorem 3.2. *Let the sequence $\{x_n\}$ be generated by Algorithm 3.1. Then, $\{x_n\}$ is strongly convergent to a point in Ω if and only if $\liminf_{n\to\infty} d(x_n, \Omega) = 0$, where $d(x_n, \Omega) = \inf_{p\in\Omega} \|x_n - p\|$.*

Proof. Changing $F(T)$ to $\Omega = F(P_C S)$ in the proof of Theorem 2.4, we have the desired conclusion immediately. □

Theorem 3.3. *Let the sequence $\{x_n\}$ be generated by Algorithm 3.1. If $P_C S$ satisfies condition (I), then $\{x_n\}$ is strongly convergent to a point in Ω.*

Proof. Let us consider the composition mapping $P_C S \colon C \to C$, which is nonexpansive. It follows

$$\|x_n - P_C(Sx_n)\| = \|P_C x_n - P_C(Sx_n)\| \le \|x_n - Sx_n\|,$$

which yields that

$$\lim_{n\to\infty} \|P_C(Sx_n) - x_n\| = 0.$$

Using the proof in Theorem 2.5, we obtain the desired conclusion immediately. □

4. Qualitative Analysis

In addition to the convergence iterative process, stability and data dependence are also under the spotlight of research; see, e.g., [11, 12, 13, 14] and the references therein. In this section, we assess the quality of the $\mathscr{U}_n$ iteration by giving some stability and data dependence results in connection with the class of quasi-strictly contractive mappings.

4.1. **Quasi-strictly contractive mappings.**

Definition 4.1 ([17])**.** Let C be a nonempty subset of a normed space X, and let $\emptyset \neq \mathscr{F} \subseteq C$. Then an operator $T \colon C \to X$ is said to be $\mathscr{F}$-quasi-strictly contractive on C if there exists a constant k with $k \in [0,1)$ such that $\|Tx - Tp\| \le k\|x - p\|$, for all $x \in C$ and $p \in \mathscr{F}$.

Restricting T to be a selfmapping on C, one finds that, for the particular choice $\mathscr{F} = F(T) \neq \emptyset$, $\mathscr{F}$-quasi-strictly contractive condition results in a new class of contractive applications that will be referred to as the class of quasi-strictly contractive mappings from now on.

Definition 4.2. Let C be a nonempty subset of a normed space X, and let $F(T) \neq \emptyset$. Then a mapping $T \colon C \to C$ is said to be quasi-strictly contractive if there exists a constant $k \in [0,1)$ such that $\|Tx - p\| \le k\|x - p\|$, for all $x \in C$ and $p \in F(T)$.

Lemma 4.1. *Let C be a nonempty subset of a normed vector space X, and let $F(T) \neq \emptyset$. If T is quasi-strictly contractive, then T has a unique fixed point.*

Proof. Indeed, if we suppose on the contrary and consider $p, p' \in F(T)$ with $p \neq p'$, then,

$$\left\|p-p'\right\| \leq \|Tp-p\| + \left\|p-Tp'\right\| = \left\|p-Tp'\right\| \leq k\left\|p-p'\right\|,$$

which implies $\|p-p'\| = 0$, which yields a contradiction. □

Remark 4.1. If the existence of a fixed point is guaranteed, the contractive condition in Definition 4.2 determines a new class of nonlinear mappings which extends not only contraction mappings, but also many known contractive conditions like those defined by Imoru and Olatinwo [14] (which also includes Berinde, Kannan and Chatterjea operators. Furthermore, Ciric mappings are also partially covered by the quasi-strictly contractive condition; see [36].

4.2. **Stability and data dependence.** Before we formulate the qualitative computational results for $\mathscr{U}_n$ iteration (2.1), we first prove its strong convergence to the unique fixed point of a quasi-strictly contractive mapping.

Theorem 4.1. *Let C be a nonempty convex subset of a normed space X, and let $T : C \to C$ be a quasi-strictly contractive mapping with constant k. Let the sequence $\{x_n\}$ be generated by $\mathscr{U}_n$ iteration. Then $\{x_n\}$ converges strongly to the unique fixed point of T.*

Proof. Since T is quasi-strictly contractive, there exists a unique fixed point p of T. Keeping in mind the iteration function of $\mathscr{U}_n$ procedure, we find

$$\|z_n - p\| \quad \leq \quad \|x_n - p\|. \tag{4.1}$$

By using the relation (4.1) previously obtained, we obtain

$$\|y_n - p\| \quad \leq \quad \|x_n - p\|. \tag{4.2}$$

In view of (4.1) and (4.2), we have $\|x_{n+1} - p\| \leq k\|x_n - p\|$. It follows that

$$\|x_{n+1} - p\| \leq k^{n+1}\|x_0 - p\|.$$

Letting $n \to \infty$ and recalling the fact that $k \in [0,1)$, we have $\lim_{n\to\infty}\|x_{n+1} - p\| = 0$, which completes the desired conclusion. □

Next, we consider a stability result for $\mathscr{U}_n$ iteration in the context of quasi-strictly contractive mappings. For this, we recall that an iterative process which converges to a unique fixed point is said to be stable if it does not gradually magnify approximation errors, which occur at each step of the iteration. This ensures that, at the end of the procedure, the numerical value of the fixed point does not widely differ from its real one. Pioneering results related to numerical stability were obtained by Ostrowski [11] and subsequently extended by Harder and Hicks [12]. Also, Osilike [13] defined a weaker concept of stability than the one developed by Ostrowski, which named the almost-stability.

Definition 4.3 ([13])**.** Let X be a normed space, and let T a selfmap of X. Suppose that $f(T, x_n)$ with $x_0 \in X$ defines some iteration procedure involving T, which yields a sequence of points $\{x_n\}$ in X. Further, suppose that $F(T) \neq \emptyset$ and that $\{x_n\}$ converges to a fixed point p of T. Let $\{u_n\}$ be an arbitrary sequence in X, and define $\varepsilon_n = \|u_{n+1} - f(T, u_n)\|$ for $n = 0, 1, \ldots$. If

$\sum_{n=0}^{\infty} \varepsilon_n < \infty$ implies that $\lim_{n\to\infty} u_n = p$, then the iteration procedure $f(T,x_n)$ is said to be almost T-stable, or almost stable with respect to T.

If we replace condition $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ in Definition 4.3 by $\lim_{n\to\infty} \varepsilon_n = 0$, then we find the stability definition of Ostrowski. Therefore, it is clear that any stable iterative procedure is also almost-stable while the reverse is not generally true. For our stability result, we further consider the concept of almost-stability via the following technical lemma.

Lemma 4.2 ([16]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative numbers and let $0 \le q < 1$ such that $a_{n+1} \le qa_n + b_n$. for all $n \ge 0$. If $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Theorem 4.2. *Let C be a nonempty convex subset of a normed space X, and let $T : C \to C$ be a quasi-strictly contractive mapping with constant k. Then the iteration procedure $\mathscr{U}_n$ is almost T-stable.*

Proof. let us denote by p the unique fixed point of T. Let $\{x_n\}$ be the sequence generated by the iteration procedure $\mathscr{U}_n$ with an initial $x_0 \in C$. Then $x_n \to p$ as n goes to infinity. Suppose that $\{u_n\}$ is an arbitrary sequence in C and define a sequence $\{\varepsilon_n\}$ by

$$\varepsilon_n = \|u_{n+1} - (1-\eta_n-\delta_n)Tu_n - \eta_n Tv_n - \delta_n Tw_n\|,$$

where $v_n = T((1-\zeta_n)Tu_n + \zeta_n Tw_n)$ and $w_n = (1-\xi_n)u_n + \xi_n Tu_n$. Assume that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. Let us prove that $\lim_{n\to\infty} u_n = p$. In view of (4.1) and (4.2), we obtain

$$\begin{aligned}
\|u_{n+1}-p\| &\le \|u_{n+1}-x_{n+1}\| + \|x_{n+1}-p\| \\
&\le \varepsilon_n + (1-\eta_n-\delta_n)\|Tu_n - Tx_n\| + \eta_n\|Tv_n - Ty_n\| \\
&\quad + \delta_n\|Tw_n - Tz_n\| + \|x_{n+1}-p\| \\
&\le \varepsilon_n + (1-\eta_n-\delta_n)(\|Tu_n-p\| + \|Tx_n-p\|) + \eta_n(\|Tv_n-p\| + \|Ty_n-p\|) \\
&\quad + \delta_n(\|Tw_n-p\| + \|Tz_n-p\|) + \|x_{n+1}-p\| \\
&\le \varepsilon_n + k(1-\eta_n-\delta_n)(\|u_n-p\| + \|x_n-p\|) + k\eta_n(\|v_n-p\| + \|y_n-p\|) \\
&\quad + k\delta_n(\|w_n-p\| + \|z_n-p\|) + \|x_{n+1}-p\| \\
&\le \varepsilon_n + k(1-\eta_n-\delta_n)(\|u_n-p\| + \|x_n-p\|) + k\eta_n(\|u_n-p\| + \|x_n-p\|) \\
&\quad + k\delta_n(\|u_n-p\| + \|x_n-p\|) + \|x_{n+1}-p\| \\
&= \varepsilon_n + k\|u_n-p\| + k\|x_n-p\| + \|x_{n+1}-p\|.
\end{aligned}$$

In view of Lemma 4.2, let us denote $a_n = \|u_n - p\|$, $q = k$, $k \in [0,1)$, and $b_n = \varepsilon_n + k\|x_n - p\| + \|x_{n+1} - p\|$. From Theorem 4.1, we have $\lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|x_n - p\| = 0$. It follows that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$, which implies $\lim_{n\to\infty} \varepsilon_n = 0$. Therefore, by taking the limit in the above expression of the sequence $\{b_n\}$, we find $\lim_{n\to\infty} b_n = 0$. By using Lemma 4.2, we find that $\lim_{n\to\infty} u_n = p$, which completes the proof. □

Next, a data dependence result is considered for $\mathscr{U}_n$ iteration for the same class of quasi-strictly contractive mappings. The motivation for such an approach is mostly given by memory limitations which arise when numerical algorithms are run through various programmes. Because of such limitations, in practice, we can only deal with approximations instead of theoretical values. This causes various errors that are absorbed into a perturbed operator which replaces the real one. A standard definition for such an operator is given below.

Definition 4.4 ([16])**.** Let $T, \tilde{T} : X \to X$ be two mappings. We say that $\tilde{T}$ is an approximate mapping of T if, for some $\varepsilon > 0$, called maximum admissible error, $\left\|Tx - \tilde{T}x\right\| < \varepsilon$ for all $x \in X$.

The data dependence analysis that we will perform aims to answer the following question: to what extent the usage of an approximating mapping instead of the theoretical one will affect the deviation between the output value of the fixed point and its actual value? The expected answer would be that minor perturbations of T will keep the solution close enough to the theoretical one, and let us say under an onset tolerance. In this situation, the iterative procedure could be considered data independent. Obviously, the error would be minimal if the procedure depends only on the initial estimation and not on the operator itself. A formal statement regarding data dependence analysis is included in the definition below.

Definition 4.5 ([37])**.** Let X be a normed space, and let T be a selfmapping with a fixed point p. Let $\tilde{T}$ be an approximate mapping of T with the maximum admissible error $\varepsilon > 0$ admitting a fixed point $\tilde{p}$. Assume that f defines some iteration procedure such that, for $x_0 \in X$, the sequences $x_{n+1} = f(T, x_n)$ and $\tilde{x}_{n+1} = f(\tilde{T}, \tilde{x}_n)$ converge to p and $\tilde{p}$, respectively. We call the iteration procedure defined by f data independent if $\|\tilde{p} - p\| \to 0$ as $\varepsilon \to 0$.

The following result provides not only a data independence result of $\mathscr{U}_n$ iteration, but also the estimates of the approximation error, that is, the deviation between the theoretical fixed point, p of T, and the disturbed one, $\tilde{p}$ of $\tilde{T}$.

Theorem 4.3. *Let C be a nonempty convex subset of a normed space X, and let $T : C \to C$ be a quasi-strictly contractive mapping with constant k. Let $\tilde{T}$ be an approximate mapping of T with the maximum admissible error ε. Let $\{x_n\}$ be the sequence generated by $\mathscr{U}_n$ iteration and define its approximate iterative sequence $\{\tilde{x}_n\}$ as follows*

$$\begin{cases} \tilde{z}_n &= (1-\xi_n)\tilde{x}_n + \xi_n \tilde{T}\tilde{x}_n \\ \tilde{y}_n &= \tilde{T}((1-\zeta_n)\tilde{T}\tilde{x}_n + \zeta_n \tilde{T}\tilde{z}_n) \\ \tilde{x}_{n+1} &= (1-\eta_n-\delta_n)\tilde{T}\tilde{x}_n + \eta_n \tilde{T}\tilde{y}_n + \delta_n \tilde{T}\tilde{z}_n, \end{cases}$$

If $Tp = p$ and $\tilde{T}\tilde{p} = \tilde{p}$ such that $\lim_{n\to\infty} \tilde{x}_n = \tilde{p}$, then

$$\|\tilde{p} - p\| \le \frac{k^3 + k^2 + 2k + 1}{1-k}\varepsilon. \tag{4.3}$$

Proof. We start with the following inequality

$$\begin{aligned} \|\tilde{z}_n - p\| &\le (1-\xi_n)\|\tilde{x}_n - p\| + \xi_n \left\|\tilde{T}\tilde{x}_n - p\right\| \\ &\le (1-\xi_n)\|\tilde{x}_n - p\| + \xi_n \left\|\tilde{T}\tilde{x}_n - T\tilde{x}_n\right\| + \xi_n \|T\tilde{x}_n - p\| \\ &\le [1-\xi_n(1-k)]\|\tilde{x}_n - p\| + \xi_n\varepsilon. \end{aligned} \tag{4.4}$$

Similarly, we conclude from (4.4) that

$$\|\tilde{y}_n - p\| \le (1 + k + k^2\zeta_n\xi_n)\varepsilon + k^2[1-\zeta_n\xi_n(1-k)]\|\tilde{x}_n - p\|. \tag{4.5}$$

Now, we have

$$
\begin{aligned}
& \|x_{n+1}-\tilde{x}_{n+1}\| \\
\leq\ & (1-\eta_n-\delta_n)\left\|Tx_n-\tilde{T}\tilde{x}_n\right\|+\eta_n\left\|Ty_n-\tilde{T}\tilde{y}_n\right\|+\delta_n\left\|Tz_n-\tilde{T}\tilde{z}_n\right\| \\
\leq\ & (1-\eta_n-\delta_n)\left(\|Tx_n-T\tilde{x}_n\|+\left\|T\tilde{x}_n-\tilde{T}\tilde{x}_n\right\|\right) \\
& +\eta_n\left(\|Ty_n-T\tilde{y}_n\|+\left\|T\tilde{y}_n-\tilde{T}\tilde{y}_n\right\|\right)+\delta_n\left(\|Tz_n-T\tilde{z}_n\|+\left\|T\tilde{z}_n-\tilde{T}\tilde{z}_n\right\|\right) \\
\leq\ & (1-\eta_n-\delta_n)\|Tx_n-T\tilde{x}_n\|+\eta_n\|Ty_n-T\tilde{y}_n\|+\delta_n\|Tz_n-T\tilde{z}_n\|+\varepsilon \\
\leq\ & (1-\eta_n-\delta_n)(\|Tx_n-p\|+\|T\tilde{x}_n-p\|)+\eta_n(\|Ty_n-p\|+\|T\tilde{y}_n-p\|) \\
& +\delta_n(\|Tz_n-p\|+\|T\tilde{z}_n-p\|)+\varepsilon \\
\leq\ & k(1-\eta_n-\delta_n)(\|x_n-p\|+\|\tilde{x}_n-p\|)+k\eta_n(\|y_n-p\|+\|\tilde{y}_n-p\|) \\
& +k\delta_n(\|z_n-p\|+\|\tilde{z}_n-p\|)+\varepsilon.
\end{aligned}
$$

From (4.1), (4.2), (4.4), and (4.5), we have

$$
\begin{aligned}
& \|x_{n+1}-\tilde{x}_{n+1}\| \\
\leq\ & k(1-\eta_n-\delta_n)(\|x_n-p\|+\|\tilde{x}_n-p\|)+k\eta_n(\|y_n-p\|+\|\tilde{y}_n-p\|) \\
& +k\delta_n(\|z_n-p\|+\|\tilde{z}_n-p\|)+\varepsilon \\
\leq\ & k\|x_n-p\|+k(1-\eta_n-\delta_n)\|\tilde{x}_n-p\|++\eta_n k^3[1-\zeta_n\xi_n(1-k)]\|\tilde{x}_n-p\| \\
& +k\delta_n[1-\xi_n(1-k)]\|\tilde{x}_n-p\|+\left[k\eta_n(1+k+k^2\zeta_n\xi_n)+k\delta_n\xi_n+1\right]\varepsilon \\
\leq\ & k\|x_n-p\|+k(1-\eta_n-\delta_n)\|\tilde{x}_n-\tilde{p}\|+k(1-\eta_n-\delta_n)\|\tilde{p}-p\| \\
& +\eta_n k^3[1-\zeta_n\xi_n(1-k)]\|\tilde{x}_n-\tilde{p}\|+\eta_n k^3[1-\zeta_n\xi_n(1-k)]\|\tilde{p}-p\| \\
& +k\delta_n[1-\xi_n(1-k)]\|\tilde{x}_n-\tilde{p}\|+k\delta_n[1-\xi_n(1-k)]\|\tilde{p}-p\| \\
& +\left[k\eta_n(1+k+k^2\zeta_n\xi_n)+k\delta_n\xi_n+1\right]\varepsilon \\
\leq\ & k\|x_n-p\|+k(1-\eta_n-\delta_n)\|\tilde{x}_n-\tilde{p}\|+\eta_n k^3[1-\zeta_n\xi_n(1-k)]\|\tilde{x}_n-\tilde{p}\| \\
& +k\delta_n[1-\xi_n(1-k)]\|\tilde{x}_n-\tilde{p}\|+k\|\tilde{p}-p\|+\left(k^3+k^2+2k+1\right)\varepsilon. \qquad (4.6)
\end{aligned}
$$

Observe that the norm is continuous and the assumption that $\lim_{n\to\infty}\tilde{x}_{n+1}=\lim_{n\to\infty}\tilde{x}_n=\tilde{p}$. Letting $n\to\infty$ in inequality (4.6) and using the result provided by Theorem 4.1, we obtain (4.3). □

We remark that the results presented in Theorem 4.1, Theorem 4.2, and Theorem 4.3 are independent from the choice of $\{\xi_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, and $\{\delta_n\}$.

5. Applications and Numerical Simulations

This section is devoted to the application of $\mathscr{U}_n$ iteration to the modulus maximization of complex polynomials over the unit disk and polynomiographic representations, solution points searches for the split feasibility problem, and original signal recovery from compressive measurements.

5.1. **Local maxima of the polynomial modulus over the unit disc.** The Maximum Modulus Principle - MMP - states that, for a holomorphic function (such as a polynomial), the maximum value of its modulus is exclusively attained at a boundary point of the domain. If we consider a nonconstant complex polynomial $p(z)$ over the unit disk, then, according to the MMP, the maximum modulus of p is $\|p\|_\infty=\max\{|p(z)|: z\in\mathbb{C},\ |z|\leq 1\}$. Recently, Kalantari [38] presented a necessary and sufficient condition for a point to be a local maxima for a polynomial modulus over the unit disk based on the negation of a Newton direction at the point.

Theorem 5.1 ([38])**.** *Let $p(z)$ be a nonconstant polynomial. Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$. A point $z_* \in D$ is a local maximum of $|p(z)|$ over D if and only if*

$$z_* = \left(\frac{p(z_*)}{p'(z_*)}\right) / \left(\left|\frac{p(z_*)}{p'(z_*)}\right|\right). \tag{5.1}$$

Kalantari [38] changed the optimization problem of polynomial modulus into a fixed point problem. In fact, Kalantari [38] emphasized that problem (5.1) can be solved by finding the fixed points of a operator

$$F(z) = \left(\frac{p(z)}{p'(z)}\right) / \left(\left|\frac{p(z)}{p'(z)}\right|\right),$$

via Picard iteration

$$z_0 \in \mathbb{C}, \quad z_{n+1} = F(z_n), \qquad n = 0, 1, \ldots.$$

Moreover, he considered (5.1) in the form of a pseudo-polynomial equation $G(z) = 0$, where $G(z) = p(z)\,|p'(z)| - zp'(z)\,|p(z)|$, and investigated a so-called pseudo-Newton method. For an arbitrary $z_0 \in \mathbb{C}$, construct $\{z_n\}$ iteratively by the formula $z_{n+1} = z_n - \frac{G_n(z_n)}{G_n'(z_n)}$, where

$$G_n(z) = p(z)\left|p'(z_n)\right| - zp'(z)\,|p(z_n)|.$$

Obviously, if we denote

$$N_n(z) = z - \frac{G_n(z)}{G_n'(z)},$$

then $z_{n+1} = N_n(z_n)$. Moreover, he investigated this method for the modulus maxima of several polynomials and visually studied the dynamics of the procedure through the technique of polynomiography. We recall that a polynomiograph is a two-dimensional image which visually illustrates the root-finding process of a certain polynomial through colorful images. Also, Kalantari pointed out the possibility of defining more general pseudo-methods (called MMP methods) using high-order polynomial root-finding iterations from the Basic Family. Adapting Basic Family of iterations for the sequence of functions $\{G_n(z)\}$, we can form the MMP-Basic Family of iterations. The first three members are

$$\begin{aligned}
&\text{(Pseudo-Newton)} & B_{2,n}(z) &= z - \frac{G_n(z)}{G_n'(z)},\\
&\text{(Pseudo-Halley)} & B_{3,n}(z) &= z - \frac{2G_n'(z)G_n(z)}{2G_n'(z)^2 - G_n''(z)G_n(z)},\\
& & B_{4,n}(z) &= z - \frac{6G_n'(z)^2G_n(z) - 3G_n''(z)G_n(z)^2}{G_n'''(z)G_n(z)^2 + 6G_n'(z)^3 - 6G_n''(z)G_n'(z)G_n(z)}.
\end{aligned}$$

Based on Kalantari [38], Gdawiec and Kotarski [39] coupled high-order MMP methods (derived from Householder iteration, Basic Family iterations, Parametric Basic Family iterations, Euler-Schröder Family iterations) with some iteration procedures (such as Mann [8], Ishikawa [9], and Noor [10]). They obtained new iterative algorithms for the modulus maxima the polynomials and provided several polynomiographs to illustrate the behaviour of such algorithms. A new approach of the same problem was investigated in [37] where two Newton-like methods were introduced as new root-finding algorithms for complex polynomials together with sufficient convergence conditions. They extended the results presented in [39] by analyzing stability and data dependence concepts with respect to a new modulus maxima-finding iteration for complex polynomials.

In what follows, we carry on the studies in [39] and [37] and illustrate that $\mathscr{U}_n$ iteration can be applied for the same problem of computing local maxima for complex polynomials modulus over the unit disk. If we denote the MMP iteration function by T_n and replace the Picard iteration with $\mathscr{U}_n$ iteration scheme (2.1), then we find modified $\mathscr{U}_n$ iteration.

Algorithm 5.1 (**Modified $\mathscr{U}_n$ Iteration**). *For an arbitrary initial point $z_0 \in \mathbb{C}$, the sequence $\{z_n\}$ is generated iteratively by*

$$\begin{cases} w_n &= (1-\xi_n)z_n + \xi_n T_n z_n \\ v_n &= T_n((1-\zeta_n)T_n z_n + \zeta_n T_n w_n) \\ z_{n+1} &= (1-\eta_n-\delta_n)T_n z_n + \eta_n T_n v_n + \delta_n T_n w_n, \end{cases} \tag{5.2}$$

where $\{\xi_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$, $\{\delta_n\}$, and $\{\eta_n+\delta_n\}$ are sequences of real numbers in $(0,1)$.

Set $\xi_n = \xi = 0.5$, $\zeta_n = \zeta = 0.5$, $\eta_n = \eta = 0.3$, and $\delta_n = \delta = 0.2$. The stopping criterion is the one given by the standard convergence test $|z_{n+1} - z_n| < \varepsilon$, where the admissible error was set to $\varepsilon = 0.001$. The tables below gather the approximate solutions from each iteration step, starting from three distinct initials $z_0 \in \{0.5+0.5i, -0.3+0.3i, 0.7-0.7i\} \subset D$ and different choices of the MMP-iteration function from the MMP-Basic Family. The results from Tables 1, 2, and 3, unveil that the maxima are attained in $0.8660+0.5000i$, $-0.8660+0.5000i$, and $-i$.

	MMP-method			
No. steps	$B_{2,n}$	$B_{3,n}$	$B_{4,n}$	$B_{5,n}$
1	$0.7675+0.3789i$	$0.7781+0.4362i$	$0.7672+0.4285i$	$0.7688+0.4284i$
2	$0.8166+0.4838i$	$0.8319+0.4814i$	$0.8270+0.4788i$	$0.8275+0.4791i$
3	$0.8527+0.4907i$	$0.8551+0.4936i$	$0.8535+0.4926i$	$0.8536+0.4927i$
4	$0.8616+0.4976i$	$0.8626+0.4980i$	$0.8621+0.4977i$	$0.8621+0.4978i$
5	$0.8647+0.4992i$	$0.8650+0.4994i$	$0.8648+0.4993i$	$0.8648+0.4993i$
6	$0.8656+0.4998i$	$0.8657+0.4998i$	$0.8657+0.4998i$	$0.8657+0.4998i$
7	$0.8659+0.4999i$			

TABLE 1. Pseudo-iteration of $z_0 = 0.5+0.5i$

	MMP-method			
No. steps	$B_{2,n}$	$B_{3,n}$	$B_{4,n}$	$B_{5,n}$
1	$-0.6405+0.2986i$	$-0.5784+0.3445i$	$-0.5802+0.3262i$	$-0.5813+0.3291i$
2	$-0.7571+0.4581i$	$-0.7323+0.4224i$	$-0.7299+0.4221i$	$-0.7311+0.4225i$
3	$-0.8341+0.4780i$	$-0.8145+0.4703i$	$-0.8139+0.4698i$	$-0.8144+0.4701i$
4	$-0.8553+0.4942i$	$-0.8487+0.4900i$	$-0.8485+0.4899i$	$-0.8487+0.4900i$
5	$-0.8628+0.4981i$	$-0.8605+0.4968i$	$-0.8605+0.4968i$	$-0.8605+0.4968i$
6	$-0.8650+0.4994i$	$-0.8643+0.4990i$	$-0.8643+0.4990i$	$-0.8643+0.4990i$
7	$-0.8657+0.4998i$	$-0.8655+0.4997i$	$-0.8655+0.4997i$	$-0.8655+0.4997i$
8		$-0.8659+0.4999i$	$-0.8659+0.4999i$	$-0.8659+0.4999i$

TABLE 2. Pseudo-iteration of $z_0 = -0.3+0.3i$

	MMP-method			
No. steps	$B_{2,n}$	$B_{3,n}$	$B_{4,n}$	$B_{5,n}$
1	$-0.4772-0.5844i$	$0.4243-1.7658i$	$-0.1134-1.0940i$	$0.0438-1.5999i$
2	$0.1630-0.6298i$	$-0.0479-1.0518i$	$0.0126-1.0314i$	$-0.0149-1.0110i$
3	$-0.0559-0.8283i$	$0.0051-1.0158i$	$-0.0013-1.0092i$	$0.0015-1.0034i$
4	$0.0117-0.9389i$	$-0.0005-1.0047i$	$0.0001-1.0028i$	$-0.0002-1.0010i$
5	$-0.0016-0.9804i$	$0.0001-1.0014i$	$-1.0008i$	$-1.0003i$
6	$0.0002-0.9939i$	$-1.0004i$	$-1.0003i$	
7	$-0.0000-0.9981i$			
8	$-0.9994i$			
9	$-0.9998i$			

TABLE 3. Pseudo-iteration of $z_0 = 0.7-0.7i$

In the sequel, we use the modified $\mathscr{U}_n$ iteration with the same MMP iteration functions selected above to generate visualizations for the maximum modulus search process. In this view, we adopted the technique of polynomiography. More precisely, we evaluated the performance of the algorithm by considering the unit disk enclosed in a square set to $[-4,4]\times[-4,4]$, which was represented graphically by a bidimensional image equipped with a color palette. The procedure used involves applying the algorithm simultaneously to all points in the selected area and based on their orbits' behaviour, associating to each point a colour coding. The rule for assigning colours states that each initial receives a hue by linearly mapping the number of iterations performed until an exit criterion is satisfied onto the colour from the corresponding position in the palette. In addition to the exit criterion used for the previous experiment, we established an extra stopping condition: if the given accuracy is not attained after 30 iterative steps, the algorithm breaks. This ensures that, while running the algorithm, slow or nonconvergent points will not launch the process into large-time consuming or possibly infinite loops. We specify that for such points we supplemented the right-sided color bar with an additional colour, i.e.

the white one. The polynomiographs for the $\mathscr{U}_n$ iteration coupled with the first four iteration functions from MMP-Basic Family are given by Figures 4-3. The maxima obtained from the previous experiment are marked with a red bullet on the unit disk of each polynomiograph.

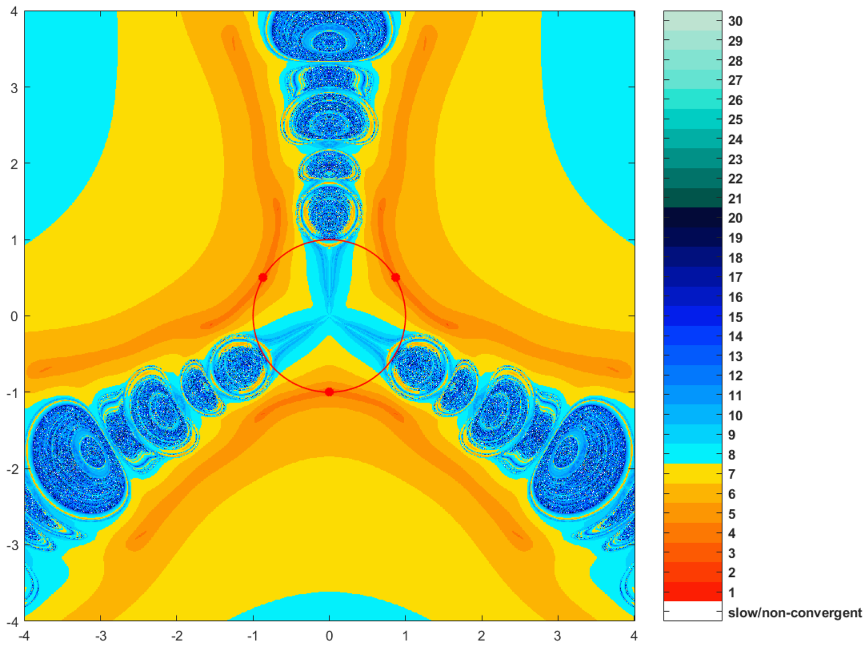

FIGURE 1. Polynomiograph for modified $\mathscr{U}_n$-process based on pseudo-Newton method

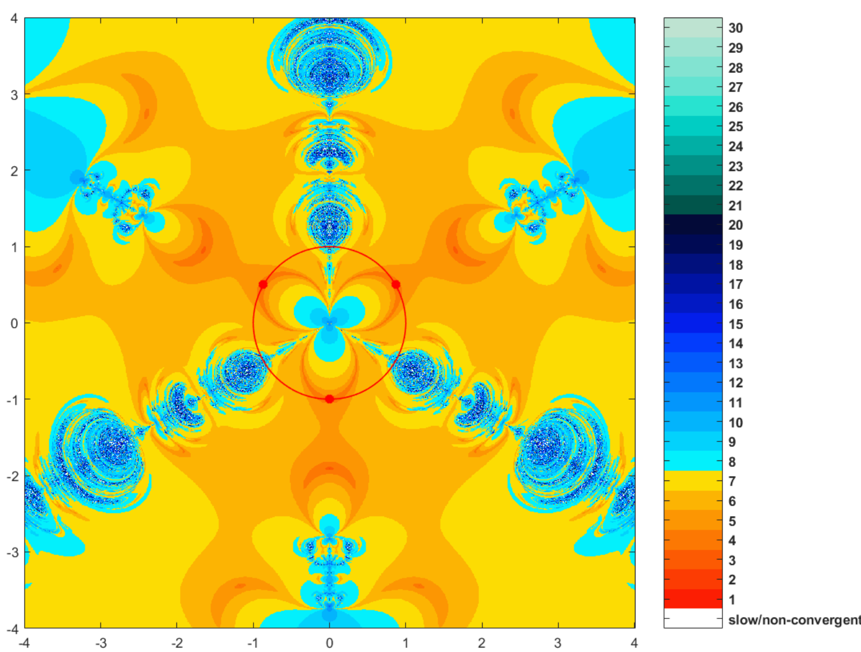

FIGURE 2. Polynomiograph for modified $\mathscr{U}_n$-process based on pseudo-Halley method

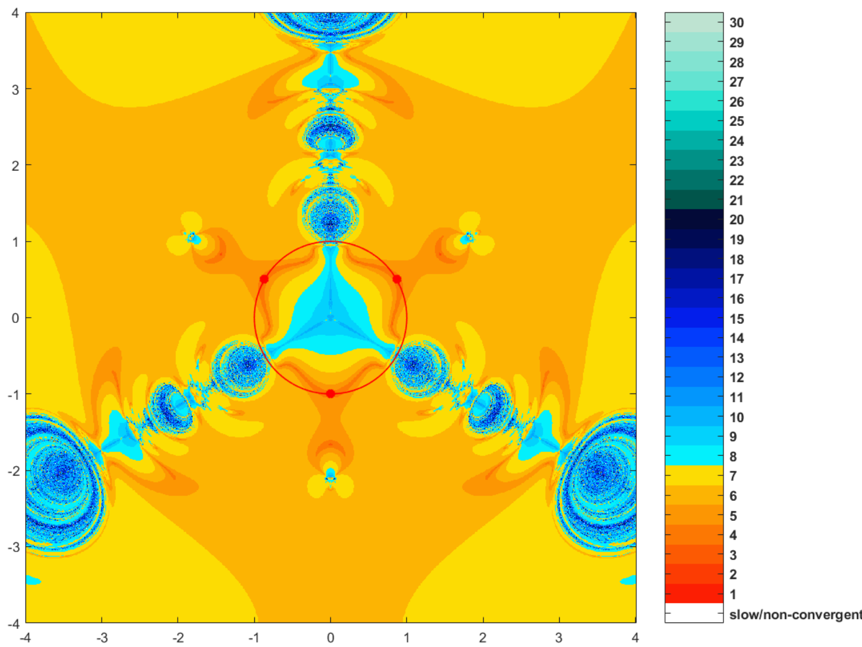

FIGURE 3. Polynomiograph for modified $\mathscr{U}_n$-process based on $B_{5,n}$ method

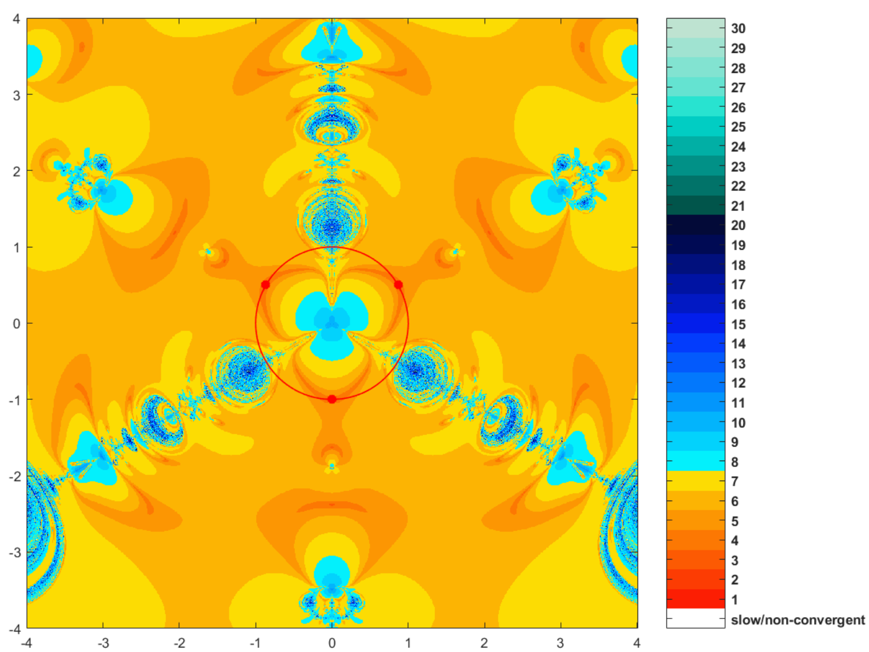

FIGURE 4. Polynomiograph for modified $\mathscr{U}_n$-process based on $B_{5,n}$ method

Behind their appearance, polynomiographs encode information about the convergence speed of the algorithm. More precisely, each pixel centered in a starting point provides through the intensity of its color, how many iterations are necessary for its orbit to get into the neighborhood of a zero. Beyond all of this, the symmetrical structure of the polynomiographs and their ability to be reconstructed by simply running a code, prefigures the possibility for them to be integrated into the methods of image compression.

5.2. **Split feasibility problem.** In view of the SPF convergence analysis we provided, we aim to illustrate the efficiency of partially projective $\mathscr{U}_n$ iteration in determining the solution set for the split feasibility problem by applying to the following example.

Example 5.1. Consider $H_1 = \mathbb{R}^2$, $H_2 = \mathbb{R}^3$, $C = \{x \in \mathbb{R}^2 : \|x\| \leq 2\}$, and $Q = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$. The projection mappings onto the subsets C and Q are defined by

$$P_C\colon \mathbb{R}^2 \to C, \quad P_C(x) = \begin{cases} x, & \|x\| \leq 2 \\ \dfrac{2}{\|x\|}x, & \|x\| > 2; \end{cases} \qquad P_Q\colon \mathbb{R}^3 \to Q, \quad P_Q(x) = \begin{cases} x, & \|x\| \leq 1 \\ \dfrac{1}{\|x\|}x, & \|x\| > 1. \end{cases}$$

Let $A\colon H_1 \to H_2$ be a bounded linear operator defined by

$$A(x_1, x_2) = \left(\frac{2}{3}x_2, -2x_1 + \frac{2}{3}x_2, 2x_1 + \frac{2}{3}x_2\right).$$

The associated matrix of A is

$$^tA = \begin{pmatrix} 0 & -2 & 2 \\ \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

and the spectral norm $\|A\| = 2\sqrt{2}$. Since the selected domain for our example is $\mathbb{R}^2$, we can analyze the performance of partially projective $\mathscr{U}_n$ iteration by using a similar technique to the one previously adopted to generate polynomiographs.

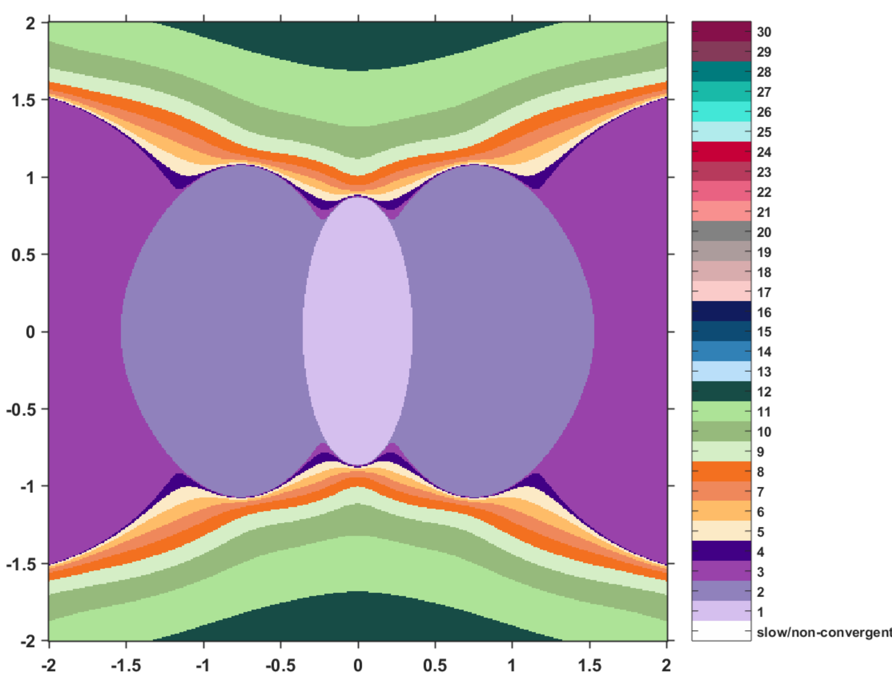

FIGURE 5. Solution set of partially projective $\mathscr{U}_n$ iteration

In this way, rather than having a view on how the algorithm works just for some initial, we have an overview on how the algorithm acts for the entire domain. Clearly, this approach will lead to an approximate visualisation of the SFP solution set. We kept the same inputs as for the previous example, i.e. $\xi_n = \xi = 0.5$, $\zeta_n = \zeta = 0.5$, $\eta_n = \eta = 0.3$, and $\delta_n = \delta = 0.2$, the exit criterion to $\|x_{n+1} - x_n\| < \varepsilon$, the admissible error set to $\varepsilon = 0.001$, and the maximum number of iterations to be performed to 30 steps. The square which generates the image is limited from $\mathbb{R}^2$ to $[-2,2] \times [-2,2]$ as the first iterate will bring x_1 into the subset C. In addition, the procedure runs the same as for the case of polynomiography, resulting the above exhibited graphical representation. Analysing Figure 5, the first thing to be pointed out is that the central elliptical disk is light-purple colored. Checking the right-sided colorbar, this means that the points enclosed in this disk satisfy the exit condition after simply one iteration. Therefore, the elliptical disk symbolizes the approximate representation of the SFP solution set. Further, one can observe that most of the points surrounding the elliptical disk take the second and third hue of purple from the palette. This means that they need simply two or three iteration steps to meet the exit criterion. Moreover, the area below and above the solution set uses colors corresponding up to only 12 iterations, while no point is white colored. Thus, we conclude that the procedure has good performance for the example under analysis.

5.3. **Compressive sensing signal reconstruction.** Compressive sensing (CS) is a technique in signal recovery, used for reconstructing a signal based on a greatly reduced number of measurements. The compressive sensing relies on the assumption that the original signal attains very much close to or equal to zero coefficients. Let $x \in \mathbb{R}^N$ be a given signal, which has less than K nonzero components. The number K, which is by assumption significantly less than N is called the sparsity of x. Usually, the sparsity is measured by using the l_0-norm, that is, $K = \|x\|_0 = \text{card}\{\text{supp}(x)\}$. However, practical reasons make the l_1-norm a better tool in numerical simulations, therefore a frequently used approach on sparsity relies on the assumption $\|x\|_1 \leq K$. Let us suppose that a number of measurements are performed, resulting a sample vector y (for instance, M out of the N components of the signal are determined). In CS scenario, we should fully recover the entire signal even if we do not have a full set of signal samples, that is, $M < N$. The mathematical model for the measurement procedure could be written as $y = Ax$, where $A \in \mathbb{R}^{M \times N}$ will be referred to as CS matrix. Moreover, sometimes the measurements are corrupted by a stochastic or deterministic unknown error (a noise) ε. This makes the observed sample vector $\bar{y}$ slightly different from the theoretical sample y, that is, $\bar{y} = Ax + \varepsilon$. All these make the reconstruction process of x equivalent to the problem: find $x \in \mathbb{R}^N$ such that $Ax = \bar{y} - \varepsilon, \quad \|x\|_1 \leq K$. A classic approach to this problem relies on the LASSO (least absolute shrinkage and selection operator) technique. A possible phrase is in terms of the following optimization problem:

$$\min_{x \in \mathbb{R}^N} \left\{ \frac{1}{2} \|Ax - y\|_2^2 \right\}, \quad \text{subject to } \|x\|_1 \leq K.$$

Furthermore, the issue above could be rephrased as a fixed point problem in connection with the nonexpansive operator $Tx = P_D\left(x - \gamma A^T(Ax - y)\right)$, where $D = \{x \in \mathbb{R}^N : \|x\|_1 \leq K\}$ and $0 < \gamma < \dfrac{2}{\|A\|^2}$ is a fixed step-size.

Now, we use the $\mathscr{U}_n$ procedure previously described with the nonexpansive mapping T by taking $\gamma = \frac{1}{\|A\|^2}$. The numerical simulation below is to prove that the algorithm is able to provide a good reconstruction of the original signal. For this, we set the signal space dimension $N = 2^{12}$, the sample space dimension $M = 2^{11}$, and the maximal sparsity $K = 2^7$. We generate a random signal x and a random sensing matrix A, computing the theoretical sample data y. We also assume that a possible noise ε could alter the precision of the measurements. We set this noise as being normally distributed with standard deviation $\sigma = 0.01$. Now, going backward from the measured vector, we apply the $\mathscr{U}_n$ procedure to recover an approximation for the original signal. We choose the initial value $x_0 = A^T y$ and we set two stopping criteria: the mean squared error $MSE_n = \frac{1}{N}\|x_{n+1} - x_n\|_2^2 \leq 10^{-10}$ or $n = 150$. Moreover, we run the procedure with the same inputs as for the previous examples, i.e., $\xi_n = \xi = 0.5$, $\zeta_n = \zeta = 0.5$, $\eta_n = \eta = 0.3$, and $\delta_n = \delta = 0.2$.

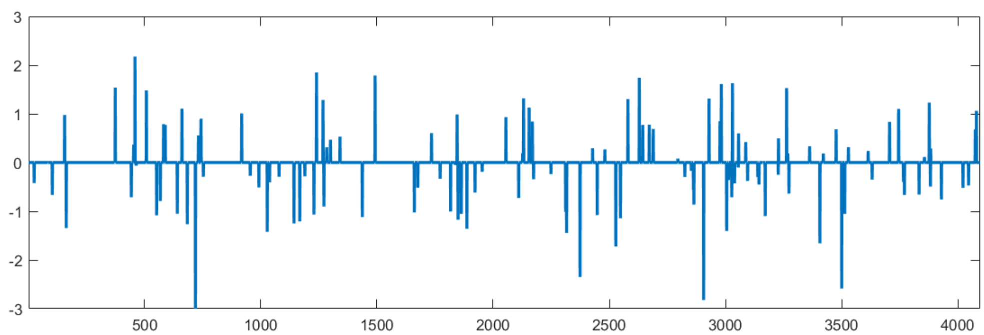

FIGURE 6. The original signal

Analyzing Figures 6-8 and comparing the image of the original signal with the image of the recovered signal, we find that they are very similar. Just a small noise can be noticed for the recovered signal as we follow the horizontal axis. Such a noise is however anticipated for a numerical simulation, as we do not recover the exact original signal, but a very precise approximation of it. Consequently, we can state that the algorithm is efficient.

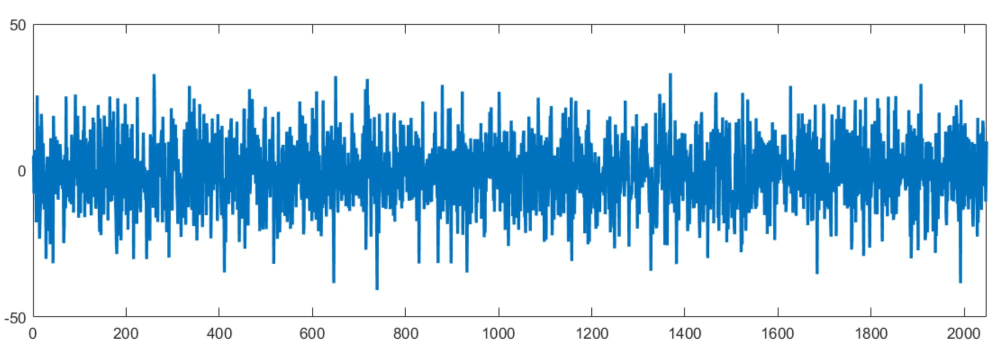

FIGURE 7. The measured sample vector

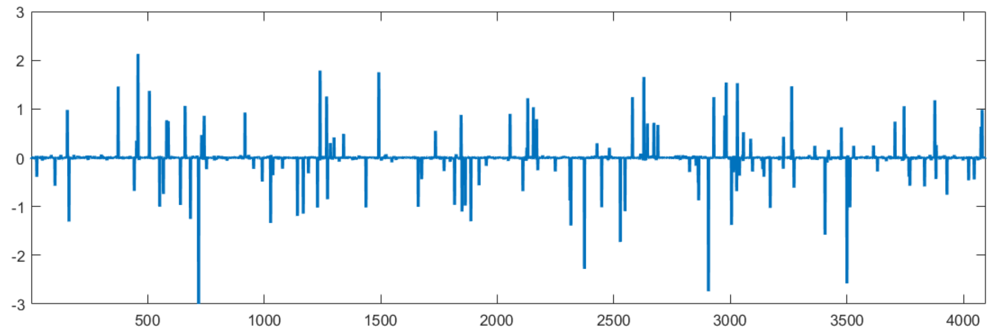

FIGURE 8. The recovered signal by $\mathscr{U}_n$ proocedure

6. CONCLUSION

In this paper, we studied a new three-step iteration procedure. We started with the fixed point problem for generalized hybrid mappings on uniformly convex Banach spaces. The theoretical approach included the convergence analysis. With the quasi-strictly contractive mappings, we provided the results of stability and data dependence. In addition, we approached three important issues: (1) The maximum modulus of complex polynomials, (2) The split feasibility problem, and (3) Compressive sensing signal reconstruction.

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REFERENCES

[1] L. Liu, S.Y. Cho, J.C. Yao, Convergence analysis of an inertial Tseng's extragradient algorithm for solving pseudomonotone variational inequalities and applications, J. Nonlinear Var. Anal. 5 (2021), 627-644.
[2] L.C. Ceng, A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems, Fixed Point Theory 21 (2020), 93-108.
[3] L. Liu, X. Qin, Strong convergence theorems for solving pseudo-monotone variational inequality problems and applications, Optimization, doi: 10.1080/02331934.2021.190564.
[4] O. Chadli, A Tikhonov-type regularization for Brezis pseudomonotone equilibrium problems in Banach spaces, J. Appl. Numer. Optim. 3 (2021), 285-295.
[5] L.V. Nguyen, Q.H. Ansari, X. Qin, Weak sharpness and finite convergence for solutions of nonsmooth variational inequalities in Hilbert spaces, Appl. Math. Optim. 84 (2021), 807-828.
[6] Y. Shehu, C. Izuchukwu, X. Qin, J.C. Yao, Strongly convergent inertial extragradient type methods for equilibrium problems, Appl. Anal. doi: 10.1080/00036811.2021.2021187.
[7] S. Zeng, S. Migorski, Z. Liu, J.C. Yao, Convergence of a generalized penalty method for variational-hemivariational inequalities, Commun. Nonlinear Sci. Numer. Simul. 92 (2021), 105476.
[8] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953), 506-510.
[9] S. Ishikawa, Fixed points by a new iteration method, Proc. Amer. Math. Soc. 44 (1974), 147-150.
[10] M.A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251 (2000), 217-229.
[11] A.M. Ostrowski, The round-off stability of iterations, Z. Angew. Math. Mech. 47 (1967), 77-81.
[12] A.M. Harder, T.L. Hicks, A stable iteration procedure for nonexpansive mappings, Math. Japon. 33 (1988), 687-692.

[13] M.O. Osilike, Stability of the Mann and Ishikawa iteration procedures for ϕ-strong pseudocontractions and nonlinear equations of the ϕ-strongly accretive type, J. Math. Anal. Appl. 227 (1998), 319-334.
[14] C.O. Imoru, M.O. Olatinwo, On the stability of Picard and Mann iteration processes, Carpathian J. Math. 19 (2003), 155-160.
[15] Y. Shehu, J.C. Yao, Rate of convergence for inertial iterative method for countable family of certain quasi-nonexpansive mappings, J. Nonlinear Convex Anal. 21 (2020), 533-541.
[16] V. Berinde, Iterative Approximation of Fixed Points, Springer, Berlin, 2007.
[17] O. Scherzer, Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems, J. Math. Anal. Appl. 194 (1995), 911-933.
[18] M. Edelstein, Fixed points theorems in uniformly convex Banach spaces, Proc. Amer. Math. Soc. 44 (1974), 369-374.
[19] B.A.B. Dehaish, A regularization projection algorithm for various problems with nonlinear mappings in Hilbert spaces. J. Inequal. Appl. 2015 (2015), Article ID 51.
[20] J. Fan, L. Liu, X. Qin, A subgradient extragradient algorithm with inertial effects for solving strongly pseudomonotone variational inequalities, Optimization, 69 (2020), 2199-2215.
[21] B.S. Thakur, D. Thakur, M. Postolache, A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings, Appl. Math. Comput. 275 (2016), 147-155.
[22] T. Mouktonglang, P. Charoensawan, R. Suparatulatorn, A novel iterative scheme of operators with property (E) and its applications, U. Politeh. Buch. Ser A 8 (2021), 45-54.
[23] F. Kohsaka, W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, Arch. Math. 91 (2008), 166-177.
[24] W. Takahashi, Fixed point theorems for new nonlinear mappings in a Hilbert space, J. Nonlinear Convex Anal. 11 (2010), 79-88.
[25] P. Kocourek, W. Takahashi, J-C. Yao, Fixed point theorems and weak convergence theorems for generalized hybird mappings in Hilbert spaces, Taiwanese J. Math. 14 (2010), 2497-2511.
[26] Z. Opial, Weak convergence of the sequence of succesive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 596-597.
[27] K. Goebel, W.A. Kirk, Topic in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.
[28] J. Schu, Weak and strong convergence of fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. 43 (1991), 153-159.
[29] H.F. Senter, W.G. Dotson, Approximating fixed points of nonexpansive mappings. Proc. Amer. Math. Soc. 44 (1974), 375-380.
[30] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algorithms 8 (1994), 221-239.
[31] C. Byrne, A unified tratament of some iterative algorithms in signal processing and image reconstruction, Inverse Probl. 20 (2004), 103-120.
[32] F. Wang, H.K. Xu, Approximating curve and strong convergence of the CQ algorithm for the split feasibility problem J. Inequal. Appl. 2010 (2010), Art. ID 102085.
[33] X. Qin, J.C. Yao, A viscosity iterative method for a split feasibility problem, J. Nonlinear Convex Anal. 20 (2019), 1497-1506.
[34] J.N. Ezeora, R.C. Ogbonna, An inertial algorithm for approximating solutions of split feasibility problems in real Banach spaces, Appl. Set-Valued Anal. Optim. 3 (2021), 171-131.
[35] A. Bejenaru, M. Postolache, Partially projective algorithm for the split feasibility problem with visualization of the solution set, Symmetry 12 (2020), Art. ID 608.
[36] L.B. Ciric, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45 (1974), 267-273.
[37] G.I. Usurelu, A. Bejenaru, M. Postolache, Newton-like methods and polynomiographic visualization of modified Thakur processes, Int. J. Comput. Math. 98 (2020), 1049-1068.
[38] B. Kalantari, A necessary and sufficient condition for local maxima of polynomial modulus over unit disc, arXiv:1605.00621, (2016).
[39] K. Gdawiec, W. Kotarski, Polynomiography for the polynomial infinity norm via Kalantari's formula and nonstandard iterations, Appl. Math. Comput. 307 (2017), 17-30.