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A MODEL OF DEFORMATIONS OF A BEAM WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. In this paper, we study a model of deformations of a beam with elastic supports and nonlinear boundary conditions under the influence of an external force. The case when the force can be concentrated at separate points is considered. The minimization problem of the potential energy functional with a constraint on the displacement of the beam end is investigated. The correctness of the model is established, and the necessary and sufficient conditions for the minimum of the potential energy functional are proved.

Keywords. Boundary conditions; Bounded variation; Minimization problem; Potential energy; Stieltjes integral.

1. Introduction

The differential equation

$$\frac{d^2}{dx^2}\left(p\frac{d^2u}{dx^2}\right)+qu=f \tag{1.1}$$

with impulse coefficients and the right-hand side was studied in many works; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9] and the references therein. It usually arises in the problems of various natural science. In [3, 4, 5], (1.1) was investigated by means of Schwarz-Sobolev distributions.

Recently, another research direction, associated with the works of [7, 8, 10], is developing. In the framework of this direction, the equation (1.1) with singularities in the coefficients and the right-hand side is replaced by the equation with measure derivatives

$$((pu''_{xx})'_x)'_\sigma+uQ'_\sigma=F'_\sigma, \tag{1.2}$$

where the measure σ is generated by the increasing function $\sigma(x)$, containing the singularities of the problem. This approach gives the possibility to carry out a pointwise analysis of both

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solutions and relations in (1.2) because Equation (1.2) is defined at each point. However, Equation (1.1) was considered only with linear boundary conditions in [3, 4, 5, 7, 8]. In this paper, we develop a pointwise approach in the case of nonlinear boundary conditions. In this way, we replace Equation (1.2) with the equivalent integro-differential equation

$$\frac{d}{dx}(p\frac{d^2u}{dx^2})(x)+\int_0^x udQ-F(x)=\frac{d}{dx}(p\frac{d^2u}{dx^2})(0)-F(0)$$

and consider the nonlinear boundary value problem for this equation. There are many results devoted to the study of various kinds of nonlinearities described by variational inequalities or sweeping processes; see, e.g., [11, 12, 13, 14, 15] and the references therein. Using variational methods, in the present paper, we investigate a problem with a similar type of nonlinear boundary conditions. Our model is understood in the following sense. Consider the Cartesian coordinate system Oxy, and suppose the beam is located along the segment $[0,l]\in Ox$ (equilibrium position). The first end of the beam is assumed to be rigidly fixed. The second end of the beam is inside the limiter located parallel to the Oy axis.

Suppose the total external load is determined by the function $F(x)$. Under the influence of the external force directed parallel to the Oy axis, the beam deviates from the equilibrium position. We denote by $u(x)$ ($x\in[0,l]$) the function describing the deformations of the beam. The condition of rigid fixation is

$$u(0)=u'(0)=0. \tag{1.3}$$

The restriction on the movement of the second end of the beam can be written as

$$u(l)\in C, \tag{1.4}$$

where $C=[-m,m]$. Depending on the applied external force $u(l)$ either remains an internal point of C or touches the boundary of the set C. As will be shown in Section 3, the potential energy functional for our physical system can be represented as

$$\Phi(u)=\int_0^l \frac{p(x)u''^2(x)}{2}dx+\int_0^l \frac{u^2(x)}{2}dQ(x)-\int_0^l u(x)dF(x). \tag{1.5}$$

Here the function $Q(x)$ describes the elastic response of external, and the properties of the beam material are characterized by the function $p(x)$. The real deformation of the beam $u_0(x)$ is a minimum point of Functional (1.5) with respect to conditions (1.3) and (1.4).

In Section 3, we minimize the energy functional $\Phi(u)$ and establish the necessary condition for the extremum (Theorem 3.1). We obtain that $u_0(x)$ is a solution to the problem

$$\begin{cases} \frac{d}{dx}(p\frac{d^2u}{dx^2})(x)+\int_0^x udQ-F(x)=\frac{d}{dx}(p\frac{d^2u}{dx^2})(0)-F(0), \\ u(0)=u'(0)=0, \\ p(l)u''(l)=0, \\ u(l)\in C, \\ (pu'')'(l)\in N_C(u(l)), \end{cases} \tag{1.6}$$

where $N_C(u(l))$ is an outward normal cone at the point $u(l)$ to the set C, and the integral is understood in the Stieltjes sense.

We suppose that
(i) the functions $p(x)$ and $F(x)$ have bounded variation on $[0,l]$, and $\inf\limits_{x\in[0,l]} p(x) > 0$;
(ii) the function $Q(x)$ does not decrease on the segment $[0,l]$;
(iii) the functions $p(x), Q(x)$, and $F(x)$ are continuous at the points $x = 0$ and $x = l$.
(j) We assume that solutions $u(x)$ belong to the set of absolutely continuous functions with absolutely continuous derivatives on $[0,l]$. Moreover, we assume that the functions pu'' are absolutely continuous and $(pu'')'$ have bounded variation on $[0,l]$.

In Section 4, we prove the theorems of uniqueness (Theorem 4.1) and the existence (Theorem 4.2) of the solution to Problem (1.6). After that we establish the sufficient condition for the minimum of Functional (1.5) (Theorem 4.3) and study the solution dependence on the size of the set C (Theorem 4.4).

The analogues of this problem in the case of string deformations were considered in [16, 17, 18, 19, 20].

2. Preliminaries

In this section, we recall some notions and facts which we will need in the sequel.

The space $BV[0,l]$ (see [18, 21, 22, 23]). This space is defined as the set of functions whose variation

$$V_0^l(w) = \sup_{0\le x_0<x_1<\ldots<x_k\le l} \sum_{i=0}^{k-1} |w(x_{i+1}) - w(x_i)|$$

is bounded. For any $w \in BV[0,l]$, we have $w = w^+ - w^-$, where w^+ and w^- are non-decreasing functions (Jordan decomposition).

For any $w \in BV[0,l]$, at any point $s \in (0,l]$ ($s \in [0,l)$), the left-hand (right-hand) limit exists, that is, $w(s-0) = \lim\limits_{x\to s-0} w(x)$ and $w(s+0) = \lim\limits_{x\to s+0} w(x)$. By a jump of $w(x)$ at a point $x = s$, we mean $\Delta w(s) = w(s+0) - w(s-0)$ ($w(0-0) = w(0)$ and $w(l+0) = w(l)$).

In the sequel, we denote by $S(w)$ the set of points, where the function $w(x)$ is discontinuous. For any function of bounded variation $w(x)$, the set $S(w)$ is at most countable. For $w \in BV[0,l]$, we define the jump function $w_s(x)$ as $w_s(x) = \sum_{s\le x}(w(s+0) - w(s-0))$.

The Stieltjes integral (see [18, 21, 22, 23]). This integral $\int_0^l f(x)dg(x)$ is defined for functions $f(x), g(x)$ ($x \in [0,l]$) by passing to the limit in the integral sums

$$\left(\sum_{i=1}^{n} f(\xi_i)\left(g(x_{i+1}) - g(x_i)\right) \right),$$

when $\max\limits_i |x_{i+1} - x_i| \to 0$.

The Stieltjes integral $\int_0^l f(x)dg(x)$ exists if one of the functions $f(x)$, $g(x)$ is continuous and the other has a bounded variation. For the Stieltjes integral $\int_0^l f(x)dg(x)$, the inequality

$$\left| \int_0^l f(x)dg(x) \right| \le \sup_{x\in[0,l]} |f(x)| V_0^l(g)$$

holds where $g \in BV[0,l]$. We have the formula

$$\int_0^l f(x)dg(x) = \int_0^l f(x)dg_c(x) + \sum_{s\in S(g)} f(s)\Delta g(s),$$

where $g_c(x) = g(x) - g_s(x)$, $f \in C[0,l]$, and $g \in BV[0,l]$.

The measure transformation theorem (see [18, 21, 22, 23]). *For any $\sigma \in BV[0,l]$ and continuous functions $w(x)$, $\varphi(x)$, we have*

$$\int_0^l \varphi(x)d\mu(x) = \int_0^l \varphi(x)w(x)d\sigma(x),$$

where $\mu(x) = \int_0^x w(t)d\sigma(t) + const.$

Absolutely continuous function (see [18, 21, 22, 23]). The function u is said to be absolutely continuous on $[0,l]$ if, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for any finite or countable system of pairwise non intersecting intervals (α_i, β_i) with a sum of lengths less than δ, i.e.,

$$\sum_i (\beta_i - \alpha_i) < \delta$$

the inequality

$$\sum_i |(u(\beta_i) - u(\alpha_i))| < \varepsilon$$

holds.

The Lebesgue theorem states that the derivative $w = u'$ of the absolutely continuous function is integrable and

$$\int_0^x w(x)dx = u(x) - u(0).$$

Notice that any absolutely continuous function on $[0,l]$ has bounded variation on $[0,l]$.

Integro - differential equation. The equation

$$\frac{d}{dx}(p\frac{d^2u}{dx^2})(x) + \int_0^x udQ - F(x) = \frac{d}{dx}(p\frac{d^2u}{dx^2})(0) - F(0) \tag{2.1}$$

is an analogue of the Euler — Poisson equation for our problem. We need to explain a set, which x belongs to, so that Equation (2.1) has the correct meaning at singular points. As in [7, 8, 10, 18, 23], we introduce a special "extension" of the segment $[0,l]$. We will denote it by $\overline{[0,l]}_\sigma$. Let us describe this construction in an exact way.

Let S be the set of all points at which the functions $p(x), Q(x)$, and $F(x)$ have non-zero jumps. Consider the set $[0,l]\setminus S$, and introduce on this set the metric $r(x,y) = |\sigma(x) - \sigma(y)|$, where

$$\sigma(x) = x + Q(x) + p^+(x) + p^-(x) + F^+(x) + F^-(x), \tag{2.2}$$

$p^+(x)$, $p^-(x)$, $F^+(x)$, and $F^-(x)$ are non-decreasing functions from Jordan decompositions of bounded variation functions $p(x)$ and $F(x)$. These functions can be chosen so that $\sigma(x)$ has discontinuities only at points from the set S. If $S \neq \emptyset$, then the metric space $[0,l]\setminus S$ is not complete. Its metric completion coincides with $\overline{[0,l]}_\sigma$ (up to isomorphism) and induces a topology on $\overline{[0,l]}_\sigma$. We will denote the elements appearing during the completion instead of each point $s \in S$ as $s-0$ and $s+0$, respectively. For any $x < s$, we have $s-0 > x$. For any $x > s$, we have $s+0 < x$. Thus, on $\overline{[0,l]}_\sigma$, each point s of the discontinuity of the functions p, Q, and F is replaced by a pair of points $\{s-0, s+0\}$. We will define at these points the functions p, Q, and F by limit values, i.e., we suppose $p(s \pm 0) = \lim_{x \to s \pm 0} p(x)$, $Q(s \pm 0) = \lim_{x \to s \pm 0} Q(x)$ and $F(s \pm 0) = \lim_{x \to s \pm 0} F(x)$. We suppose that x in Equation (2.1) belongs to $\overline{[0,l]}_\sigma$. Thus, we do not allow x to take a value in the set S. The functions p, Q, and F become continuous on $\overline{[0,l]}_\sigma$. The continuity of the functions u allows us to preserve the usual Stieltjes meaning for the integral term in (2.1) at $x = s-0$ and $x = s+0$. Thus, we consider Equation (2.1) in two layers: the lower level is for $x \in [0,l]$ when speaking about the solutions $u(x)$ themselves (under the integral sign), and the second level is for the values x in (2.1), where $x \in \overline{[0,l]}_\sigma$.

The Equation (2.1) implies the equality

$$(pu'')'(s+0) - (pu'')'(s-0) + u(s)(Q(s+0) - Q(s-0)) = F(s+0) - F(s-0),$$

where $s \in S$. Here the left and right derivatives coincide with the limit values of the derivative, that is, $(pu'')'_-(s) = (pu'')'(s-0)$ and $(pu'')'_+(s) = (pu'')'(s+0)$.

We will use the following notions and theorems from [7].

Theorem 2.1. *For any real numbers u_0, u_1, u_2, u_3, and for any point $x_0 \in \overline{[0,l]}_\sigma$, there is a unique solution to the problem*

$$\begin{cases} \dfrac{d}{dx}(p\dfrac{d^2u}{dx^2})(x) + \displaystyle\int_0^x u dQ - F(x) = \dfrac{d}{dx}(p\dfrac{d^2u}{dx^2})(0) - F(0), \\ u(x_0) = u_0, \\ u'(x_0) = u_1, \\ u''(x_0) = u_2, \\ (pu'')'(x_0) = u_3. \end{cases}$$

Consider the homogeneous equation

$$\frac{d}{dx}(p\frac{d^2u}{dx^2})(x) + \int_0^x u dQ = \frac{d}{dx}(p\frac{d^2u}{dx^2})(0). \tag{2.3}$$

It follows from Theorem 2.1 that the space of solutions to Equation (2.3) has dimension 4.

Let us define an analogue $W(x)$ of the Wronski determinant on the set $\overline{[0,l]}_\sigma$.

$$W(x) = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) & \varphi_3(x) & \varphi_4(x) \\ \varphi_1'(x) & \varphi_2'(x) & \varphi_3'(x) & \varphi_4'(x) \\ \varphi_1''(x) & \varphi_2''(x) & \varphi_3''(x) & \varphi_4''(x) \\ (p\varphi_1'')'(x) & (p\varphi_2'')'(x) & (p\varphi_3'')'(x) & (p\varphi_4'')'(x) \end{vmatrix}, \tag{2.4}$$

where $\varphi_i(x)(i = 1,2,3,4)$ are solutions to homogeneous equation (2.3).

Theorem 2.2. *Let $\varphi_i(x)(i = 1,2,3,4)$ be solutions to homogeneous equation (2.3). The following conditions are equivalent:*
1) there is a point $x_0 \in \overline{[0,l]}_\sigma$ such that $W(x_0) = 0$;

2) $W(x) \equiv 0$ on the set $\overline{[0,l]}_\sigma$;
3) the functions $\varphi_1(x), \varphi_2(x), \varphi_3(x)$, and $\varphi_4(x)$ are linearly dependent on $\overline{[0,l]}_\sigma$.

Theorem 2.3. *Let $\varphi_1(x), \varphi_2(x), \varphi_3(x)$, and $\varphi_4(x)$ be solutions to homogeneous equation (2.3). Then $(pW)(x) \equiv const$ on the set $\overline{[0,l]}_\sigma$.*

Let $C \subset H$ be a closed convex set, where H is a Hilbert space.

Normal cone (see [12]). The set

$$N_C(x) = \{\xi \in H : \langle \xi, c - x\rangle \le 0 \quad \forall c \in C\}$$

denotes the outward normal cone to C at x, where $x \in C$.

Notice that if x is an interior point of C, then $N_C(x) = \{0\}$. In the present paper, $H = R^1$, norm and scalar product are Euclidean.

3. The Variational Motivation of Our Approach

Let us explain the representation (1.5) of the potential energy functional for our model of beam deformations. The beam is located along the segment $[0,l]$ of the Ox axis. Under the influence of external force, the beam is deformed in the vertical plane. Applying the considerations of Euler, let us assume that the beam is vertically stratified into elementary flat layers. There is (this is the standard assumption) a layer, which under deformation does not change its length in any area. This layer is called neutral. Intersection of the neutral layer with the longitudinal section of the beam forms a neutral line. The deviation of the neutral line from the equilibrium position is called the deformation of the beam. We denote it by $u(x)$, where $x \in [0,l]$. Consider the case of small deformations. For zero deformation of the beam, the vertical gap occupied by the beam from the neutral line is denoted by $[-h_0, h_1]$. Consider an elementary layer between h and $h+\Delta h$. Denote by $dS(h)$ the cross-sectional area of this layer, where $S(h)$ is cross-sectional area of the beam at the point x between the levels from $-h_0$ to h_1. Considering a piece of the beam on the interval $[x, x+\Delta x)$, this layer will have an initial length $l(h) = \Delta h$. Resistance to tensile or compression of this layer exerts according to Hooke's law with a force equal to $kdl(h)$, where $dl(h)$ is the elongation of the layer under consideration, and k is the coefficient of elasticity equal to

$$k = \frac{E(h)dS(h)}{\Delta x},$$

where $E(h)$ is Young's modulus of the material of the layer. The work to overcome the linear force $k\tau$ on the interval $0 \le \tau \le \tau_0$ is $k\frac{\tau_0^2}{2}$. Thus the energy $dV(h)$ accumulated by an elementary layer (of level h), when the length changes by $dl(h)$, equals

$$dV(h) = k\frac{(dl(h))^2}{2} = \frac{E(h)dS(h)}{\Delta x}\frac{(dl(h))^2}{2}.$$

Let us find the elongation $dl(h)$ of the given layer. It is determined by its distance h from the neutral line and the angle $\Delta\Theta$ between the transverse sections at the points x and $x+\Delta x$. Up to small values of the higher order $dl(h) = h\Delta\Theta$, due to which the energy, accumulated by the elementary layer at the h level is

$$dV(h) = \frac{E(h)dS(h)h^2}{2}\frac{(\Delta\Theta)^2}{\Delta x}.$$

The integration of the latter value over the thickness of the beam in the range from $-h_0$ to h_1 leads to the energy accumulated by a section of the beam on the interval $[x, x+\Delta x)$. It is determined by the value

$$\frac{p(x)}{2}\frac{(\Delta\Theta)^2}{\Delta x}, \tag{3.1}$$

where $p(x) = \int\limits_{-h_0}^{h_1} E(h)h^2 dS(h)$. The angle between the normals is the same as the angle between the tangents. Thus $d\Theta(x) = du'(x)$. The latter leads (3.1) to the form $\frac{p(x)u''^2(x)}{2}\Delta x$. Consequently, the bending energy of the beam is $\int\limits_0^l \frac{p(x)(u''(x))^2}{2}dx$.

Consider a partition of the segment $[0,l]$ by points $0 = x_0 < x_1 < .... < x_n = l$. Let us choose arbitrary numbers $\xi_i \in [x_i, x_{i+1})$ on each section. We denote $\Delta x_i = x_{i+1} - x_i$. We will assume that $\Delta x_i = x_{i+1} - x_i$ are small. The total external load in the range from 0 to x is determined by the function $F(x)$. Thus the force $F(x_{i+1}) - F(x_i)$ acts on the interval $[x_i, x_{i+1})$. The movement in this section is assumed to be equal to $u(\xi_i)$. The corresponding work is equal to

$$u(\xi_i)(F(x_{i+1}) - F(x_i)).$$

Thus the work of the external force on $[0,l]$ equals $\int\limits_0^l u(x)dF(x)$. Notice that the function $F(x)$ will be discontinuous at the points, where concentrated forces are applied. The elastic response of the external environment is described by the function $Q(x)$. According to Hooke's law, when the element $[x_i, x_{i+1})$ is deflected at a distance h, the elastic reaction force is equal to $h(Q(x_{i+1}) - Q(x_i))$. So the work on overcoming this force, when h changes from zero to $u(\xi_i)$ is equal to

$$\left(\int\limits_0^{u(\xi_i)} h dh\right)(Q(x_{i+1}) - Q(x_i)) = \frac{u^2(\xi_i)}{2}(Q(x_{i+1}) - Q(x_i)),$$

and for the whole $[0,l]$, we have the integral $\int\limits_0^l \frac{u^2}{2}dQ$. This means that if the beam made a deformation determined by the function $u(x)$, then its potential energy would be

$$\Phi(u) = \int\limits_0^l \frac{p(x)u''^2(x)}{2}dx + \int\limits_0^l \frac{u^2(x)}{2}dQ(x) - \int\limits_0^l u(x)dF(x).$$

We assume that conditions (i), (ii), and (iii) are satisfied. Denote by E the class of absolutely continuous on $[0,l]$ functions with absolutely continuous derivatives. Moreover, we assume that the functions u'' have bounded variation on $[0,l]$.

We will consider the functional $\Phi(u)$ on the set of functions $u \in E$ satisfying (1.3) and (1.4), i.e.,

$$u(0) = u'(0) = 0, \quad u(l) \in C.$$

According to the Hamilton – Lagrange principle, the real deformation $u_0(x)$ minimizes the potential energy functional Φ with respect to conditions (1.3) and (1.4), i.e.,

$$u_0 \to \min_{u(0)=u'(0)=0, u(l)\in C} \Phi(u).$$

Let us consider $h \in E$ satisfying the conditions

$$h(0) = h'(0) = 0, \quad h(l) = h'(l) = 0 \tag{3.2}$$

and consider the function $u(x) = u_0(x) + \lambda h(x)$, where λ is a real number. We have $u \in E$, $u(0) = u'(0) = 0$, and $u(l) = u_0(l) \in C$. Thus $\Phi(u_0) \le \Phi(u_0 + \lambda h)$. Fixing h, let us define the function $\varphi_h(\lambda) = \Phi(u_0 + \lambda h)$. Then the inequality $\varphi_h(0) \le \varphi_h(\lambda)$ holds for all real λ. According to Fermat's theorem, we have $\frac{d}{d\lambda}\varphi_h(\lambda)|_{\lambda=0} = 0$. The last equality has the form

$$\int_0^l p u_0'' h'' dx + \int_0^l u_0 h dQ - \int_0^l h dF = 0.$$

Denote by

$$g(x) = \int_0^x u_0 dQ. \tag{3.3}$$

Then

$$\int_0^l p u_0'' dh' + \int_0^l h dg - \int_0^l h dF = 0,$$

and

$$p(l)u_0''(l)h'(l) - p(0)u_0''(0)h'(0) - \int_0^l h' d(pu_0'') + h(l)g(l) - h(0)g(0)$$
$$- \int_0^l g h' dx - h(l)F(l) + h(0)F(0) + \int_0^l F h' dx = 0.$$

Since we consider the case, when Conditions (3.2) are satisfied, we have

$$\int_0^l h' d(pu_0'') + \int_0^l g h' dx - \int_0^l F h' dx = 0.$$

Denote by $\tilde{g}(x) = \int_0^x g(s)ds$, $\tilde{F}(x) = \int_0^x F(s)ds$. Thus

$$\int_0^l h' d((pu_0'') + \tilde{g} - \tilde{F}) = 0, \tag{3.4}$$

which holds for any function $h \in E$ satisfying (3.2).

Lemma 3.1. *Suppose that $G(x)$ is a bounded variation function, where $x \in [0,l]$. Assume, for any $h \in E$, satisfying (3.2), we have*

$$\int_0^l h' dG = 0. \tag{3.5}$$

Then there exist numbers a, b such that the identity

$$G(x-0) = G(x+0) \equiv ax + b$$

holds for all $x \in (0,l)$.

Proof. Since $h'(0) = h'(l) = 0$, with respect to (3.5), we have $\int\limits_0^l h' dG = -\int_0^l G dh' = 0$. Since (3.2) is satisfied, for any numbers a and b, the equality $\int\limits_0^l (ax+b)dh' = 0$ holds. Thus $\int\limits_0^l (G(x) - (ax+b))dh' = 0$. Then, for

$$a = \frac{6l\int\limits_0^l G(s)ds - 12\int\limits_0^l\int\limits_0^t G(s)dsdt}{l^3},$$

$$b = \frac{6\int\limits_0^l\int\limits_0^t G(s)dsdt - 2l\int\limits_0^l G(s)ds}{l^2},$$

and

$$h(x) = \int\limits_0^x\int\limits_0^t (G(s) - as - b)\,dsdt,$$

we obtain $\int\limits_0^l (G(x) - (ax+b))^2 dx = 0$. Thus, for almost all x, we have $G(x) = ax+b$. Denote by S^* the set of points $x \in [0,l]$ such that $G(x) \neq ax+b$. Consider an arbitrary point $s \in [0,l)$. Since $G \in BV[0,l]$, there exists a sequence of points $s_n \in [0,l]\backslash S^*$ such that $s_n \to s+0$ and $G(s_n) \to G(s+0)$. Thus $G(s+0) = as+b$. Similarly, $G(s-0) = as+b$. □

This lemma allows us to define the function $G(x)$ on the segment $[0,l]$ by continuity. Applying Lemma 3.1 to Equality (3.4), we obtain $(pu_0'')(x) + \tilde{g}(x) - \tilde{F}(x) = ax+b$, i.e.,

$$(pu_0'')(x) = \int\limits_0^x F(s)ds + ax + b - \int\limits_0^x g(s)ds. \tag{3.6}$$

Hence the function (pu_0'') is absolutely continuous and

$$(pu_0'')'(x) + \int\limits_0^x u_0 dQ = F(x) + a. \tag{3.7}$$

We can rewrite Equality (3.7) as

$$\frac{d}{dx}(p\frac{d^2u}{dx^2})(x) + \int\limits_0^x udQ - F(x) = \frac{d}{dx}(p\frac{d^2u}{dx^2})(0) - F(0).$$

Let us consider functions $h \in E$ such that $h(0) = h'(0) = h(l) = 0$ and the function $u(x) = u_0(x) + \lambda h(x)$, where $\lambda \in R$. Notice that $u \in E$, $u(0) = u'(0) = 0$, $u(l) = u_0(l) \in C$. Then

$$\Phi(u_0) \leq \Phi(u_0 + \lambda h).$$

Define the function $\varphi_h(\lambda) = \Phi(u_0 + \lambda h)$. Then, for all real numbers λ, the inequality $\varphi_h(0) \leq \varphi_h(\lambda)$ holds. Thus $\frac{d}{d\lambda}\varphi_h(\lambda)|_{\lambda=0} = 0$, and we have the equality

$$p(l)u_0''(l)h'(l) - \int_0^l h'd(pu_0'') - \int_0^l gh'dx + \int_0^l Fh'dx = 0,$$

where $g(x)$ is defined by (3.3). With respect to (3.6), we have

$$p(l)u_0''(l)h'(l) - \int_0^l h'(F(x) + a - g(x))dx - \int_0^l gh'dx + \int_0^l Fh'dx = 0.$$

Since $h(l) = h(0) = 0$, we obtain $p(l)u_0''(l)h'(l) = 0$. Hence

$$p(l)u_0''(l) = 0. \tag{3.8}$$

Let us fix any element $c \in C$. Consider the function $h \in E$ satisfying conditions $h(0) = h'(0) = 0$, $h(l) = c - u_0(l)$. Suppose $u(x) = u_0(x) + \lambda h(x)$. Notice that $u \in E$ and $u(0) = u'(0) = 0$. We have $u(l) = u_0(l) + \lambda h(l) = u_0(l) + \lambda(c - u_0(l)) = \lambda c + (1 - \lambda)u_0(l)$. Since $c \in C$, $u_0(l) \in C$, the set C is convex, for all $\lambda \in [0,1]$, we have $u(l) \in C$. Thus $\Phi(u_0) \leq \Phi(u_0 + \lambda h)$. Fixing h, let us define the function $\varphi_h(\lambda) = \Phi(u_0 + \lambda h)$, where $\lambda \in [0,1]$. Hence $\varphi_h(0) \leq \varphi_h(\lambda)$. Then, for the right derivative, we have the inequality $\frac{d^+}{d\lambda}\varphi_h(\lambda)|_{\lambda=0} \geq 0$, i.e.,

$$\begin{aligned} &p(l)u_0''(l)h'(l) - p(0)u_0''(0)h'(0) - \int_0^l h'd(pu_0'') + h(l)g(l) - h(0)g(0) \\ &- \int_0^l gh'dx - h(l)F(l) + h(0)F(0) + \int_0^l Fh'dx \geq 0, \end{aligned} \tag{3.9}$$

where $g(x)$ is defined by (3.3). Using conditions $h(0) = h'(0) = 0$, (3.6), and (3.8), Inequality (3.9) has the form $h(l)(g(l) - F(l) - a) \geq 0$. According to (3.7), $a = (pu_0'')'(l) + g(l) - F(l)$, and we obtain $-(pu_0'')'(l)h(l) \geq 0$, i.e., $-(pu_0'')'(l)(c - u_0(l)) \geq 0$, where $c \in C$. Thus $(pu_0'')'(l) \in N_C(u_0(l))$.

We have proved the following theorem.

Theorem 3.1. *Assume that u_0 minimizes the potential energy functional $\Phi(u)$ given by (1.5) with respect to conditions $u(0) = u'(0) = 0, u(l) \in C$. Then u_0 is a solution to Problem (1.6), i.e.,*

$$\begin{cases} \dfrac{d}{dx}(p\dfrac{d^2u}{dx^2})(x) + \displaystyle\int_0^x udQ - F(x) = \dfrac{d}{dx}(p\dfrac{d^2u}{dx^2})(0) - F(0), \\ u(0) = 0, u'(0) = 0, \\ p(l)u''(l) = 0, \\ u(l) \in C, \\ (pu'')'(l) \in N_C(u(l)), \end{cases}$$

where $x \in \overline{[0,l]}_\sigma$, and $\sigma(x)$ is defined by (2.2).

4. MAIN RESULTS

In the sequel, we assume that conditions (i), (ii), (iii), and (j) are satisfied. Consider Problem (1.6).

Definition 4.1. By a solution to Problem (1.6), we mean a function u, satisfying Equation (2.1) for all $x \in \overline{[0,l]}_\sigma$ and satisfying conditions $u(0)=u'(0)=0$, $p(l)u''(l)=0$, $u(l)\in C$, and $(pu'')'(l)\in N_C(u(l))$.

Theorem 4.1. *If a solution to Problem (1.6) exists, then it is unique.*

Proof. Assume that $u_1(x)$ and $u_2(x)$ are solutions to Problem (1.6). Then $w(x)=u_1(x)-u_2(x)$ satisfies the equation

$$\frac{d}{dx}(p\frac{d^2w}{dx^2})(x)+\int\limits_0^x wdQ=\frac{d}{dx}(p\frac{d^2w}{dx^2})(0), \tag{4.1}$$

and satisfies the conditions $w(0)=w'(0)=0, p(l)w''(l)=0$. For all $c\in C$, we have $(pu_1'')'(l)(c-u_1(l))\le 0$ and $(pu_2'')'(l)(c-u_2(l))\le 0$. Since $u_1(l)\in C$, $u_2(l)\in C$, we have $(pu_1'')'(l)(u_2(l)-u_1(l))\le 0$ and $(pu_2'')'(l)(u_1(l)-u_2(l))\le 0$. Hence $(pw'')'(l)w(l)\ge 0$. From Equality (4.1), we have $\int_0^l wd(pw'')'+\int_0^l w^2dQ=0$. Thus

$$w(l)(pw'')'(l)-\int_0^l (pw'')'w'dx+\int_0^l w^2dQ=0,$$

i.e.,

$$w(l)(pw'')'(l)+\int_0^l pw''^2dx+\int_0^l w^2dQ=0.$$

Hence $w\equiv 0$. □

Theorem 4.2. *Let $\varphi_1,\varphi_2,\varphi_3$, and φ_4 be a system of solutions to homogeneous equation (4.1) and satisfy the conditions:*

$$\begin{cases}\varphi_1(0)=1,\\ \varphi_1'(0)=0,\\ \varphi_1''(l)=0,\\ (p\varphi_1'')'(l)=0;\end{cases}\quad \begin{cases}\varphi_2(0)=0,\\ \varphi_2'(0)=1,\\ \varphi_2''(l)=0,\\ (p\varphi_2'')'(l)=0;\end{cases}\quad \begin{cases}\varphi_3(0)=0,\\ \varphi_3'(0)=0,\\ \varphi_3''(l)=1,\\ (p\varphi_3'')'(l)=0;\end{cases}\quad \begin{cases}\varphi_4(0)=0,\\ \varphi_4'(0)=0,\\ \varphi_4''(l)=0,\\ (p\varphi_4'')'(l)=1.\end{cases}$$

If $\left|\frac{1}{p(0)W(0)}(\varphi_1(l)\int\limits_0^l\Delta_1(s)dF(s)+\varphi_2(l)\int\limits_0^l\Delta_2(s)dF(s))\right|<m$*, then the solution to Problem (1.6) is*

$$u(x)=\frac{1}{p(0)W(0)}(\varphi_1(x)\int\limits_0^x\Delta_1(s)dF(s)+\varphi_2(x)\int_0^x\Delta_2(s)dF(s)+$$

$$+\varphi_3(x)\int\limits_x^l\Delta_3(s)dF(s)+\varphi_4(x)\int_x^l\Delta_4(s)dF(s)), \tag{4.2}$$

where

$$\Delta_1(s)=-\begin{vmatrix}\varphi_2(s)&\varphi_3(s)&\varphi_4(s)\\ \varphi_2'(s)&\varphi_3'(s)&\varphi_4'(s)\\ p(s)\varphi_2''(s)&p(s)\varphi_3''(s)&p(s)\varphi_4''(s)\end{vmatrix};\Delta_2(s)=\begin{vmatrix}\varphi_1(s)&\varphi_3(s)&\varphi_4(s)\\ \varphi_1'(s)&\varphi_3'(s)&\varphi_4'(s)\\ p(s)\varphi_1''(s)&p(s)\varphi_3''(s)&p(s)\varphi_4''(s)\end{vmatrix};$$

$$\Delta_3(s)=\begin{vmatrix}\varphi_1(s)&\varphi_2(s)&\varphi_4(s)\\ \varphi_1'(s)&\varphi_2'(s)&\varphi_4'(s)\\ p(s)\varphi_1''(s)&p(s)\varphi_2''(s)&p(s)\varphi_4''(s)\end{vmatrix};\Delta_4(s)=-\begin{vmatrix}\varphi_1(s)&\varphi_2(s)&\varphi_3(s)\\ \varphi_1'(s)&\varphi_2'(s)&\varphi_3'(s)\\ p(s)\varphi_1''(s)&p(s)\varphi_2''(s)&p(s)\varphi_3''(s)\end{vmatrix}$$

If $\frac{1}{p(0)W(0)}(\varphi_1(l)\int\limits_0^l \Delta_1(s)dF(s)+\varphi_2(l)\int\limits_0^l \Delta_2(s)dF(s)) \geq m$*, then the solution to Problem (1.6) is*

$$\begin{aligned} u(x) = & \frac{m\varphi_4(x)}{\varphi_4(l)} + \frac{1}{p(0)W(0)}\Big(\varphi_1(x)\int\limits_0^x \Delta_1(s)dF(s) \\ & + \varphi_2(x)\int\limits_0^x \Delta_2(s)dF(s) + \varphi_3(x)\int\limits_x^l \Delta_3(s)dF(s) \\ & + \varphi_4(x)\int\limits_x^l \Delta_4(s)dF(s)\Big) - \frac{\varphi_4(x)\varphi_1(l)}{p(0)W(0)\varphi_4(l)}\int\limits_0^l \Delta_1(s)dF(s) \\ & - \frac{\varphi_2(l)\varphi_4(x)}{p(0)W(0)\varphi_4(l)}\int\limits_0^l \Delta_2(s)dF(s). \end{aligned} \tag{4.3}$$

If $\frac{1}{p(0)W(0)}(\varphi_1(l)\int\limits_0^l \Delta_1(s)dF(s)+\varphi_2(l)\int\limits_0^l \Delta_2(s)dF(s)) \leq -m$*, then the solution to Problem (1.6) is*

$$\begin{aligned} u(x) = & \frac{-m\varphi_4(x)}{\varphi_4(l)} + \frac{1}{p(0)W(0)}\Big(\varphi_1(x)\int\limits_0^x \Delta_1(s)dF(s) \\ & + \varphi_2(x)\int\limits_0^x \Delta_2(s)dF(s) + \varphi_3(x)\int\limits_x^l \Delta_3(s)dF(s) \\ & + \varphi_4(x)\int\limits_x^l \Delta_4(s)dF(s)\Big) - \frac{\varphi_4(x)\varphi_1(l)}{p(0)W(0)\varphi_4(l)}\int\limits_0^l \Delta_1(s)dF(s) \\ & - \frac{\varphi_2(l)\varphi_4(x)}{p(0)W(0)\varphi_4(l)}\int\limits_0^l \Delta_2(s)dF(s), \end{aligned}$$

where the Wronskian W *is defined by (2.4).*

Proof. Let us show that the system of functions φ_1, φ_2, φ_3, and φ_4 exists. Consider the problem

$$\begin{cases} (pw'')'(x) + \int\limits_0^x w dQ = (pw'')'(0), \\ w(0) = 0, w'(0) = 0, \\ w''(l) = 0, \\ (pw'')'(l) = 0. \end{cases} \tag{4.4}$$

Note that Problem (4.4) has only the zero solution. Indeed, it follows from the equation in (4.4) that

$$\int_0^l w d(pw'')' + \int_0^l w^2 dQ = 0.$$

Integrating the first integral by parts two times, we obtain

$$\int_0^l pw''^2 dx + \int_0^l w^2 dQ = 0,$$

and hence $w \equiv 0$. Consider the following problems

$$\begin{cases} (pw_1'')'(x) + \int\limits_0^x w_1 dQ = (pw_1'')'(0), \\ w_1(0) = 1, w_1'(0) = 0, \\ w_1''(0) = 0, (pw_1'')'(0) = 0, \end{cases}$$

$$\begin{cases} (pw_2'')'(x) + \int\limits_0^x w_2 dQ = (pw_2'')'(0), \\ w_2(0) = 0, w_2'(0) = 1, \\ w_2''(0) = 0, (pw_2'')'(0) = 0, \end{cases}$$

$$\begin{cases} (pw_3'')'(x) + \int\limits_0^x w_3 dQ = (pw_3'')'(0), \\ w_3(0) = 0, w_3'(0) = 0, \\ w_3''(0) = 1, (pw_3'')'(0) = 0, \end{cases}$$

and

$$\begin{cases} (pw_4'')'(x) + \int\limits_0^x w_4 dQ = (pw_4'')'(0), \\ w_4(0) = 0, w_4'(0) = 0, \\ w_4''(0) = 0, (pw_4'')'(0) = 1. \end{cases}$$

We represent the solution to the equation from (4.4) in the form $w = d_1w_1 + d_2w_2 + d_3w_3 + d_4w_4$ and substitute it into the conditions (4.4). Since Problem (4.4) has only zero solution, the determinant of the matrix of this system is not equal to zero, i.e.

$$\begin{vmatrix} w_1(0) & w_2(0) & w_3(0) & w_4(0) \\ w_1'(0) & w_2'(0) & w_3'(0) & w_4'(0) \\ w_1''(l) & w_2''(l) & w_3''(l) & w_4''(l) \\ (pw_1'')'(l) & (pw_2'')'(l) & (pw_3'')'(l) & (pw_4'')'(l) \end{vmatrix} \neq 0. \tag{4.5}$$

We represent $\varphi_1(x)$ in the form $\varphi_1(x) = c_1w_1(x) + c_2w_2(x) + c_3w_3(x) + c_4w_4(x)$. Substituting into the conditions for $\varphi_1(x)$, we obtain a system for determining c_1, c_2, c_3, and c_4

$$\begin{cases} c_1w_1(0) + c_2w_2(0) + c_3w_3(0) + c_4w_4(0) = 1, \\ c_1w_1'(0) + c_2w_2'(0) + c_3w_3'(0) + c_4w_4'(0) = 0, \\ c_1w_1''(l) + c_2w_2''(l) + c_3w_3''(l) + c_4w_4''(l) = 0, \\ c_1(pw_1'')'(l) + c_2(pw_2'')'(l) + c_3(pw_3'')'(l) + c_4(pw_4'')'(l) = 0. \end{cases}$$

By virtue of (4.5), the latter system has a unique solution. The existence of the functions φ_2, φ_3, and φ_4 is proved in the similar way. It is easy to see that the system of functions φ_1, φ_2, φ_3, and φ_4 is linearly independent, therefore it is a fundamental system of solutions to homogeneous equation (2.3). According to Theorem 2.2, we have $p(0)W(0) \neq 0$. Consider the case

$$\left| \frac{1}{p(0)W(0)} (\varphi_1(l) \int\limits_0^l \Delta_1(s) dF(s) + \varphi_2(l) \int\limits_0^l \Delta_2(s) dF(s)) \right| < m.$$

Let us show that function (4.2) is the solution to Problem (1.6). We must check that (4.2) satisfies conditions (j). Notice that the integrals on the right-hand side of (4.2) are defined since $\Delta_i(s)$ are continuous. From the representation of $u(x)$ for all points $x \in S$, we have

$$\begin{aligned}
\Delta u(x) = \frac{1}{p(0)W(0)} \Big(& \varphi_1(x+0) \int_0^{x+0} \Delta_1(s)dF(s) \\
& + \varphi_2(x+0) \int_0^{x+0} \Delta_2(s)dF(s) + \varphi_3(x+0) \int_{x+0}^{l} \Delta_3(s)dF(s) \\
& + \varphi_4(x+0) \int_{x+0}^{l} \Delta_4(s)dF(s) - \varphi_1(x-0) \int_0^{x-0} \Delta_1(s)dF(s) \\
& - \varphi_2(x-0) \int_0^{x-0} \Delta_2(s)dF(s) - \varphi_3(x-0) \int_{x-0}^{l} \Delta_3(s)dF(s) - \varphi_4(x-0) \int_{x-0}^{l} \Delta_4(s)dF(s) \Big) \\
& = \frac{\Delta F(x)}{p(0)W(0)} \Big(\varphi_1(x)\Delta_1(x) + \varphi_2(x)\Delta_2(x) - \varphi_3(x)\Delta_3(x) - \varphi_4(x)\Delta_4(x) \Big) \\
= 0,
\end{aligned}$$

due to

$$\begin{aligned}
& \varphi_1(x)\Delta_1(x) + \varphi_2(x)\Delta_2(x) - \varphi_3(x)\Delta_3(x) - \varphi_4(x)\Delta_4(x) \\
& = - \begin{vmatrix} \varphi_1(x) & \varphi_2(x) & \varphi_3(x) & \varphi_4(x) \\ \varphi_1(x) & \varphi_2(x) & \varphi_3(x) & \varphi_4(x) \\ \varphi_1'(x) & \varphi_2'(x) & \varphi_3'(x) & \varphi_4'(x) \\ (p\varphi_1'')(x) & (p\varphi_2'')(x) & (p\varphi_3'')(x) & (p\varphi_4'')(x) \end{vmatrix} = 0.
\end{aligned} \tag{4.6}$$

Hence the function $u(x)$ can be determined by continuity on the segment $[0,l]$. Let us show that $u(x)$ is absolutely continuous. For any $\alpha, \beta \in \overline{[0,l]}_\sigma$, we have

$$\begin{aligned}
& u(\beta) - u(\alpha) \\
& = \frac{1}{p(0)W(0)} \Big((\varphi_1(\beta) - \varphi_1(\alpha)) \int_0^\beta \Delta_1(s)dF(s) + (\varphi_2(\beta) - \varphi_2(\alpha)) \int_0^\beta \Delta_2(s)dF(s) \\
& + (\varphi_3(\beta) - \varphi_3(\alpha)) \int_\beta^l \Delta_3(s)dF(s) + (\varphi_4(\beta) - \varphi_4(\alpha)) \int_\beta^l \Delta_4(s)dF(s) \\
& + \int_\alpha^\beta (\varphi_1(\alpha) - \varphi_1(s))\Delta_1(s)dF(s) + \int_\alpha^\beta (\varphi_2(\alpha) - \varphi_2(s))\Delta_2(s)dF(s) \\
& + \int_\alpha^\beta (\varphi_3(s) - \varphi_3(\alpha))\Delta_3(s)dF(s) + \int_\alpha^\beta (\varphi_4(s) - \varphi_4(\alpha))\Delta_4(s)dF(s) \\
& + \int_\alpha^\beta (\varphi_1(s)\Delta_1(s) + \varphi_2(s)\Delta_2(s) - \varphi_3(s)\Delta_3(s) - \varphi_4(s)\Delta_4(s))dF(s) \Big).
\end{aligned}$$

According to (4.6),

$$\int_\alpha^\beta (\varphi_1(s)\Delta_1(s) + \varphi_2(s)\Delta_2(s) - \varphi_3(s)\Delta_3(s) - \varphi_4(s)\Delta_4(s))dF(s) = 0,$$

and we obtain that the function $u(x)$ is absolutely continuous. Let us show that

$$u'(x) = \frac{1}{p(0)W(0)}\big(\varphi_1'(x)\int_0^x \Delta_1(s)dF(s) + \varphi_2'(x)\int_0^x \Delta_2(s)dF(s) + \varphi_3'(x)\int_x^l \Delta_3(s)dF(s) + \varphi_4'(x)\int_x^l \Delta_4(s)dF(s)\big). \tag{4.7}$$

Denote by $\Delta_\varepsilon z = z(x+\varepsilon) - z(x+0)$, where $\varepsilon > 0$. Let us prove the statement for the right derivative (for the left derivative the proof is similar). We have

$$\frac{\Delta_\varepsilon u}{\varepsilon} = \frac{1}{p(0)W(0)}\Big(\frac{\Delta_\varepsilon \varphi_1}{\varepsilon}\int_0^{x+\varepsilon} \Delta_1 dF + \frac{\Delta_\varepsilon \varphi_2}{\varepsilon}\int_0^{x+\varepsilon} \varphi_2 dF + \frac{\Delta_\varepsilon \varphi_3}{\varepsilon}\int_{x+\varepsilon}^{l} \Delta_3 dF + \frac{\Delta_\varepsilon \varphi_4}{\varepsilon}\int_{x+\varepsilon}^{l} \Delta_4 dF + \int_{x+0}^{x+\varepsilon}\frac{\varphi_1(x)\Delta_1(s)+\varphi_2(x)\Delta_2(s)-\varphi_3(x)\Delta_3(s)-\varphi_4(x)\Delta_4(s)}{\varepsilon}dF\Big).$$

Let us show that

$$\lim_{\varepsilon\to 0^+}\int_{x+0}^{x+\varepsilon}\frac{\varphi_1(x)\Delta_1(s)+\varphi_2(x)\Delta_2(s)-\varphi_3(x)\Delta_3(s)-\varphi_4(x)\Delta_4(s)}{\varepsilon}dF = 0. \tag{4.8}$$

Observe that

$$\frac{1}{\varepsilon}\Big|\int_{x+0}^{x+\varepsilon}(\varphi_1(x)\Delta_1(s)+\varphi_2(x)\Delta_2(s)-\varphi_3(x)\Delta_3(s)-\varphi_4(x)\Delta_4(s))\,dF(s)\Big| \le \frac{\sup\limits_{x+0\le s\le x+\varepsilon}|\varphi_1(x)\Delta_1(s)+\varphi_2(x)\Delta_2(s)-\varphi_3(x)\Delta_3(s)-\varphi_4(x)\Delta_4(s)|}{\varepsilon}V_{x+0}^{x+\varepsilon}(F).$$

Let τ be a point at which the continuous function

$$|\varphi_1(x)\Delta_1(s)+\varphi_2(x)\Delta_2(s)-\varphi_3(x)\Delta_3(s)-\varphi_4(x)\Delta_4(s)|$$

reaches its maximum on the compact set $[x+0, x+\varepsilon]$. Then the inequality

$$\begin{aligned}&\frac{\max\limits_{x+0\le s\le x+\varepsilon}|\varphi_1(x)\Delta_1(s)+\varphi_2(x)\Delta_2(s)-\varphi_3(x)\Delta_3(s)-\varphi_4(x)\Delta_4(s)|}{\varepsilon}\\ &\le |\Delta_1(\tau)|\left|\frac{\varphi_1(x)-\varphi_1(\tau)}{\varepsilon}\right| + |\Delta_2(\tau)|\left|\frac{\varphi_2(x)-\varphi_2(\tau)}{\varepsilon}\right|\\ &\quad + |\Delta_3(\tau)|\left|\frac{\varphi_3(x)-\varphi_3(\tau)}{\varepsilon}\right| + |\Delta_4(\tau)|\left|\frac{\varphi_4(x)-\varphi_4(\tau)}{\varepsilon}\right|\\ &\quad + \frac{1}{\varepsilon}|\Delta_1(\tau)\varphi_1(\tau)+\Delta_2(\tau)\varphi_2(\tau)-\Delta_3(\tau)\varphi_3(\tau)-\Delta_4(\tau)\varphi_4(\tau)|\end{aligned}$$

holds. Since

$$\frac{1}{\varepsilon}|\varphi_i(x)-\varphi_i(\tau)| \le \frac{1}{\varepsilon}\Big|\int_x^\tau \varphi_i'(s)ds\Big| \le \frac{1}{\varepsilon}\int_x^{x+\varepsilon}|\varphi_i'(s)|ds \le c_i,$$

$i = 1,2,3,4$ and (4.6), we obtain that the fraction

$$\frac{\max\limits_{x+0\le s\le x+\varepsilon} |\varphi_1(x)\Delta_1(s)+\varphi_2(x)\Delta_2(s)-\varphi_3(x)\Delta_3(s)-\varphi_4(x)\Delta_4(s)|}{\varepsilon},$$

where $\varepsilon > 0$, is bounded. Since $V_{x+0}^{x+\varepsilon}(F) \to 0$ as $\varepsilon \to 0^+$, we obtain Equality (4.8). Thus Representation (4.7) is true. Similarly, it is proved that the functions $u'(x)$ and $(pu'')(x)$ are absolutely continuous, and

$$u''(x) = \frac{1}{p(0)W(0)}\big(\varphi_1''(x)\int\limits_0^x \Delta_1(s)dF(s)+\varphi_2''(x)\int_0^x \Delta_2(s)dF(s)$$
$$+\varphi_3''(x)\int\limits_x^l \Delta_3(s)dF(s)+\varphi_4''(x)\int_x^l \Delta_4(s)dF(s)\big),$$

and

$$(pu'')'(x) = \frac{1}{p(0)W(0)}\big((p\varphi_1'')'(x)\int\limits_0^x \Delta_1(s)dF(s)+(p\varphi_2'')'(x)\int_0^x \Delta_2(s)dF(s)$$
$$+(p\varphi_3'')'(x)\int\limits_x^l \Delta_3(s)dF(s)+(p\varphi_4'')'(x)\int_x^l \Delta_4(s)dF(s)\big).$$

Let us show that function (4.2) is the solution to Equation (2.1). Let us consider the integral $\int_0^x udQ$. Note that

$$\int_0^x udQ = \frac{1}{p(0)W(0)}\big(\int_0^x \varphi_1(s)\int_0^s \Delta_1(t)dF(t)dQ(s)+\int_0^x \varphi_2(s)\int_0^s \Delta_2(t)dF(t)dQ(s)$$
$$+\int_0^x \varphi_3(s)\int_s^l \Delta_3(t)dF(t)dQ(s)+\int_0^x \varphi_4(s)\int_s^l \Delta_4(t)dF(t)dQ(s)\big).$$

Changing the limits of integration, we have

$$\int_0^x \varphi_1(s)\int_0^s \Delta_1(t)dF(t)dQ(s)+\int_0^x \varphi_2(s)\int_0^s \Delta_2(t)dF(t)dQ(s)$$
$$=\int_0^x \Delta_1(t)((-(p\varphi_1'')'(x)+(p\varphi_1'')'(t))dF(t)+\int_0^x \Delta_2(t)((-(p\varphi_2'')'(x)+(p\varphi_2'')'(t))dF(t)$$

and

$$\int_0^x \varphi_3(s)\int_s^l \Delta_3(t)dF(t)dQ(s)+\int_0^x \varphi_4(s)\int_s^l \Delta_4(t)dF(t)dQ(s)$$
$$=\int_0^x \Delta_3(t)((-(p\varphi_3'')'(t)+(p\varphi_3'')'(0))dF(t)+\int_x^l \Delta_3(t)((-(p\varphi_3'')'(x)+(p\varphi_3'')'(0))dF(t)$$
$$+\int_0^x \Delta_4(t)((-(p\varphi_4'')'(t)+(p\varphi_4'')'(0))dF(t)+\int_x^l \Delta_4(t)((-(p\varphi_4'')'(x)+(p\varphi_4'')'(0))dF(t).$$

It follows that

$$\begin{aligned}
&(pu'')'(x)+\int_0^x udQ \\
&=\frac{1}{p(0)W(0)}\Big(\int_0^x (\Delta_1(t)(p\varphi_1'')'(t)+\Delta_2(t)(p\varphi_2'')'(t)-\Delta_3(t)(p\varphi_3'')'(t) \\
&\quad -\Delta_4(t)(p\varphi_4'')'(t))dF(t)+(p\varphi_3'')'(0)\int_0^l \Delta_3(t)dF(t)+(p\varphi_4'')'(0)\int_0^l \Delta_4(t)dF(t) \\
&=\frac{1}{p(0)W(0)}\int_0^x p(t)W(t)dF(t)+(pu'')'(0).
\end{aligned}$$

According to Theorem 2.3 $p(t)W(t)=p(0)W(0)$, we obtain

$$(pu'')'(x)+\int_0^x udQ=F(x)-F(0)+(pu'')'(0)$$

as required. Let us show that the function $u(x)$ satisfies the conditions of Problem (1.6). It is easy to see that $u(0)=u'(0)=p(l)u''(l)=0$. Since

$$|u(l)|=\Big|\frac{1}{p(0)W(0)}(\varphi_1(l)\int_0^l \Delta_1(s)dF(s)+\varphi_2(l)\int_0^l \Delta_2(s)dF(s))\Big|<m$$

we have $u(l)\in C$ and

$$(pu'')'(l)=\frac{1}{p(0)W(0)}((p\varphi_1'')'(l)\int_0^l \Delta_1(s)dF(s)+(p\varphi_2'')'(l)\int_0^l \Delta_2(s)dF(s))=0$$

as required. Consider the case

$$\frac{1}{p(0)W(0)}(\varphi_1(l)\int_0^l \Delta_1(s)dF(s)+\varphi_2(l)\int_0^l \Delta_2(s)dF(s))\geq m.$$

As it is similar to the first case, one can prove that the function defined by Equality (4.3) is a solution to Equation (2.1) and satisfies $u(0)=u'(0)=p(l)u''(l)=0$. Since $u(l)=m$, we have $u(l)\in C=[-m,m]$. Let us show that $(pu'')'(l)\in N_C(u(l))$, i.e., $(pu'')'(l)\geq 0$. Notice that $\varphi_4(l)<0$. Since

$$(p\varphi_4'')'(x)+\int_0^x \varphi_4 dQ=(p\varphi_4'')'(0),$$

we have

$$\int_0^l \varphi_4 d(p\varphi_4'')'+\int_0^l \varphi_4^2 dQ=0,$$

i.e.,

$$\varphi_4(l)(p\varphi_4'')'(l)-\varphi_4(0)(p\varphi_4'')'(0)-\int_0^l (p\varphi_4'')'\varphi_4' dx+\int_0^l \varphi_4^2 dQ=0.$$

Using the conditions for $\varphi_4(x)$, we obtain

$$\varphi_4(l)=-(\int_0^l \varphi_4^2 dQ+\int_0^l p\varphi_4''^2 dx).$$

If $\varphi_4(l) = 0$, then $\int_0^l \varphi_4^2 dQ + \int_0^l p\varphi_4''^2 dx = 0$ and $\varphi_4 \equiv 0$, but this contradicts the condition $(p\varphi_4'')'(l) = 1$. Hence $\varphi_4(l) < 0$. Then, for $(pu'')'(l)$,

$$(pu'')'(l) = \frac{1}{\varphi_4(l)}\big(m - \frac{\varphi_1(l)}{p(0)W(0)}\int_0^l \Delta_1(s)dF(s) - \frac{\varphi_2(l)}{p(0)W(0)}\int_0^l \Delta_2(s)dF(s)\big) \geq 0$$

as required. The case that

$$\frac{1}{p(0)W(0)}(\varphi_1(l)\int_0^l \Delta_1(s)dF(s) + \varphi_2(l)\int_0^l \Delta_2(s)dF(s)) \leq -m,$$

can be considered similarly. □

Theorem 4.3. *Let $u_0(x)$ be the solution to Problem (1.6). Then $u_0(x)$ minimizes the potential energy functional $\Phi(u)$ defined by (1.5) with respect to conditions $u(0) = u'(0) = 0$ and $u(l) \in C$.*

Proof. Let us show that, for any $u \in E$ with $u(0) = u'(0) = 0$ and $u(l) \in C$, the inequality $\Phi(u) - \Phi(u_0) \geq 0$ holds. We have $u(x) = u_0(x) + h(x)$, where $h(x) = u(x) - u_0(x)$. Notice that $h \in E$ and $h(0) = h'(0) = 0$. Thus

$$\begin{aligned}\Phi(u_0 + h) - \Phi(u_0) &= -h(l)(pu_0'')'(l) + \int_0^l \frac{ph''^2}{2}dx + \int_0^l \frac{h^2}{2}dQ \\ &= -(pu_0'')'(l)(u(l) - u_0(l)) + \int_0^l \frac{ph''^2}{2}dx + \int_0^l \frac{h^2}{2}dQ.\end{aligned}$$

Since $u(l) \in C$ and $(pu_0'')'(l) \in N_C(u_0(l))$, we have

$$-(pu_0'')'(l)(u(l) - u_0(l)) \geq 0.$$

Since $p(x) > 0$, $Q(x)$ does not decrease on $[0,l]$, we obtain $\Phi(u_0 + h) - \Phi(u_0) \geq 0$ as required. □

Theorem 4.4. *If $m \to 0$, then the solution $u_m(x)$ to Problem (1.6) tends to the solution to the problem*

$$\begin{cases} \frac{d}{dx}(p\frac{d^2u}{dx^2})(x) + \int_0^x udQ - F(x) = \frac{d}{dx}(p\frac{d^2u}{dx^2})(0) - F(0), \\ u(0) = u'(0) = 0, \\ p(l)u''(l) = 0, \\ u(l) = 0 \end{cases} \tag{4.9}$$

uniformly on $\overline{[0,l]}_\sigma$.

Proof. Since $m \to 0$, we have

$$\Big|\frac{1}{p(0)W(0)}(\varphi_1(l)\int_0^l \Delta_1(s)dF(s) + \varphi_2(l)\int_0^l \Delta_2(s)dF(s))\Big| \geq m.$$

Then

$$\Big|u_m(x) - \frac{1}{p(0)W(0)}\Big(\varphi_1(x)\int_0^x \Delta_1(s)dF(s) + \varphi_2(x)\int_0^x \Delta_2(s)dF(s)$$

$$+\varphi_3(x)\int_x^l \Delta_3(s)dF(s) + \varphi_4(x)\int_x^l \Delta_4(s)dF(s)\Big) + \frac{\varphi_4(x)\varphi_1(l)}{p(0)W(0)\varphi_4(l)}\int_0^l \Delta_1(s)dF(s)$$

$$+\frac{\varphi_2(l)\varphi_4(x)}{p(0)W(0)\varphi_4(l)}\int_0^l \Delta_2(s)dF(s)\Big|$$

$$= \left|\frac{m\varphi_4(x)}{\varphi_4(l)}\right| \le cm \to 0.$$

Thus

$$u_m(x) \rightrightarrows u^*(x) = \frac{1}{p(0)W(0)}\Big(\varphi_1(x)\int_0^x \Delta_1(s)dF(s)$$

$$+\varphi_2(x)\int_0^x \Delta_2(s)dF(s) + \varphi_3(x)\int_x^l \Delta_3(s)dF(s)$$

$$+\varphi_4(x)\int_x^l \Delta_4(s)dF(s)\Big) - \frac{\varphi_4(x)\varphi_1(l)}{p(0)W(0)\varphi_4(l)}\int_0^l \Delta_1(s)dF(s)$$

$$-\frac{\varphi_2(l)\varphi_4(x)}{p(0)W(0)\varphi_4(l)}\int_0^l \Delta_2(s)dF(s).$$

Similarly to Theorem 4.2, we can obtain that $u^*(x)$ is the solution to Problem (4.9). □

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