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SOME GENERAL HURWITZ-LERCH TYPE ZETA FUNCTIONS ASSOCIATED WITH THE SRIVASTAVA-DAOUST MULTIPLE HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper, we introduce and investigate various properties and relations involving some general families of double and multiple Hurwitz-Lerch type zeta functions which are associated with the Srivastava-Daoust class of hypergeometric functions in two and more variables. Relevant connections with other (known or new) results for functions of the analytic number theory are also considered.
Keywords. Dirichlet-type series; Hurwitz-Lerch type zeta functions; Kampé de Fériet functions; Riemann zeta function; Srivastava-Daoust series.

1. Introduction, Definitions, and Preliminaries

Throughout this paper, we make use of the following standard notations:

$$\mathbb{N} := \{1,2,3,\cdots\}, \quad \mathbb{N}_0 := \{0,1,2,\cdots\} = \mathbb{N} \cup \{0\},$$

and

$$\mathbb{Z}^- := \{-1,-2,-3,\cdots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

Moreover, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of *positive* numbers, and $\mathbb{C}$ denotes the set of complex numbers.

Various important and potentially useful functions in *Analytic Number Theory* include, for example, the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s,a)$,

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which are defined, for $\Re(s) > 1$, by

$$\zeta(s) := \begin{cases} \sum\limits_{n=1}^{\infty} \dfrac{1}{n^s} = \dfrac{1}{1-2^{-s}} \sum\limits_{n=1}^{\infty} \dfrac{1}{(2n-1)^s} & \big(\Re(s) > 1\big) \\[2ex] \dfrac{1}{1-2^{1-s}} \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^s} & \big(\Re(s) > 0;\ s \neq 1\big) \end{cases} \tag{1.1}$$

and

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad \big(\Re(s) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-\big), \tag{1.2}$$

and $\big($for $\Re(s) \leqq 1;\ s \neq 1\big)$ by their *meromorphic* continuations (see, for details, the remarkable works by Titchmarsh [1] and Apostol [2] *as well as* the monumental treatise by Whittaker and Watson [3]; see also [4, Chapter 23], and [5, Chapter 2]).

A substantially more general Dirichlet-type series than those in (1.1) and (1.2) happens to define the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ as follows (see, for example, [6, p. 27. Eq. 1.11 (1)]; see also [5, pp. 121 *et seq.*])

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.3}$$

$$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \ \text{ when } \ |z| < 1;\ \Re(s) > 1 \ \text{ when } \ |z| = 1\big).$$

In fact, just as in the cases of the Riemann Zeta function $\zeta(s)$ and the Hurwitz (or generalized) Zeta function $\zeta(s,a)$, the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be continued *meromorphically* to the whole complex s-plane, except for a simple pole at $s = 1$ with its residue 1 (see also the recent survey-cum-expository review articles [7] and [8] on various widely- and extensively-studied families of the Hurwitz-Lerch and related zeta functions). It is also known that [6, p. 27, Equation 1.11 (3)]

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}\, \mathrm{e}^{-at}}{1 - z\mathrm{e}^{-t}}\, \mathrm{d}t = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}\, \mathrm{e}^{-(a-1)t}}{\mathrm{e}^t - z}\, \mathrm{d}t \tag{1.4}$$

$$\big(\Re(a) > 0;\ \Re(s) > 0 \ \text{ when } \ |z| \leqq 1;\ \Re(s) > 1 \ \text{ when } \ z = 1\big).$$

We now turn to the two-variable Srivastava-Daoust function defined by (see [9], [10], and [11])

$$\begin{aligned} S^{A:B;B'}_{C:D;D'}\binom{z}{w} &= S^{A:B;B'}_{C:D;D'}\left(\begin{matrix} [(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right]; \\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right]; \end{matrix}\ z,w\right) \\ &:= \sum_{m,n=0}^{\infty} \mathscr{H}^{A:B;B'}_{C:D;D'}(m,n)\, \frac{z^m}{m!}\, \frac{w^n}{n!}, \end{aligned} \tag{1.5}$$

where, for convenience,

$$\mathscr{H}^{A:B;B'}_{C:D;D'}(m,n) := \frac{\prod\limits_{j=1}^{A} \Gamma\left(a_j + \theta_j m + \vartheta_j n\right)}{\prod\limits_{j=1}^{C} \Gamma\left(c_j + \delta_j m + \kappa_j n\right)} \frac{\prod\limits_{j=1}^{B} \Gamma\left(b_j + \psi_j m\right)}{\prod\limits_{j=1}^{D} \Gamma\left(d_j + \phi_j m\right)} \frac{\prod\limits_{j=1}^{B'} \Gamma\left(b'_j + \psi'_j n\right)}{\prod\limits_{j=1}^{D'} \Gamma\left(d'_j + \phi'_j n\right)}. \tag{1.6}$$

Here, and elsewhere in this paper, we tacitly assume the following conditions on the coefficients and parameters involved:

$$\theta_j, \vartheta_j \in \mathbb{R}^+ \ \ (j = 1, 2, \cdots, A); \quad \psi_j, \psi_k^{'} \in \mathbb{R}^+ \ \left(j = 1, 2, \cdots, B;\ k = 1, 2, \cdots, B^{'} \right)$$

and

$$\delta_j, \kappa_j \in \mathbb{R}^+ \ \ (j = 1, 2, \cdots, C); \quad \phi_j, \phi_k^{'} \in \mathbb{R}^+ \ \left(j = 1, 2, \cdots, D;\ k = 1, 2, \cdots, D^{'} \right).$$

The convergence conditions of the double hypergeometric series defining the function in (1.5) are given by (see [11, p. 155, Case I])

$$|z| < \infty \qquad \text{and} \qquad |w| < \infty,$$

provided that

$$1 + \sum_{j=1}^{C} \delta_j + \sum_{j=1}^{D} \phi_j - \sum_{j=1}^{A} \theta_j - \sum_{j=1}^{B} \psi_j > 0$$

and

$$1 + \sum_{j=1}^{C} \kappa_j + \sum_{j=1}^{D^{'}} \phi_j^{'} - \sum_{j=1}^{A} \vartheta_j - \sum_{j=1}^{B^{'}} \psi_j^{'} > 0.$$

In addition to the monumental works by Srvastava and Daoust (see [9], [10], and [11]), extensive studies of Srivastava-Daoust type hypergeometric functions in two and more variables can be found in (for example) [12], [13], [14], [15], [16], [17], and [18]. Furthermore, from the detailed analysis of the convergence conditions of double hypergeometric series defining the function (1.5), we recall that if

$$1 + \sum_{j=1}^{C} \delta_j + \sum_{j=1}^{D} \phi_j - \sum_{j=1}^{A} \theta_j - \sum_{j=1}^{B} \psi_j = 0$$

and

$$1 + \sum_{j=1}^{C} \kappa_j + \sum_{j=1}^{D^{'}} \phi_j^{'} - \sum_{j=1}^{A} \vartheta_j - \sum_{j=1}^{B^{'}} \psi_j^{'} = 0,$$

then the series in (1.5) converges provided that

$$|z| < \rho \qquad \text{and} \qquad |w| < \rho',$$

where ρ and ρ' are given by Srivastava and Daoust [11, p. 155, Eq. (3.13)] after adjusting the parameters as per the notations used in this paper. On the other hand, if

$$1 + \sum_{j=1}^{C} \delta_j + \sum_{j=1}^{D} \phi_j - \sum_{j=1}^{A} \theta_j - \sum_{j=1}^{B} \psi_j < 0$$

and

$$1 + \sum_{j=1}^{C} \kappa_j + \sum_{j=1}^{D^{'}} \phi_j^{'} - \sum_{j=1}^{A} \vartheta_j - \sum_{j=1}^{B^{'}} \psi_j^{'} < 0,$$

then the double hypergeometric series in (1.5) will be divergent except in the *trivial* case that $z = 0$ and $w = 0$ (see [11, p. 155, Case III]). Furthermore, if we set

$$\mathfrak{G} := \frac{\prod_{j=1}^{C} \kappa_j \prod_{j=1}^{D} \phi_j}{\prod_{j=1}^{A} \vartheta_j \prod_{j=1}^{B} \psi_j} \qquad \text{and} \qquad \mathfrak{H} := \frac{\prod_{j=1}^{C} \delta_j \prod_{j=1}^{D'} \phi_j'}{\prod_{j=1}^{A} \vartheta_j \prod_{j=1}^{B'} \psi_j'},$$

and

$$\Omega := \sum_{j=1}^{A} \vartheta_j - \sum_{j=1}^{C} \kappa_j,$$

then the double hypergeometric series in (1.5) will converge if, in addition to the following inequalities:

$$1 + \sum_{j=1}^{C} \delta_j + \sum_{j=1}^{D} \phi_j - \sum_{j=1}^{A} \theta_j - \sum_{j=1}^{B} \psi_j = 0$$

and

$$1 + \sum_{j=1}^{C} \kappa_j + \sum_{j=1}^{D'} \phi_j' - \sum_{j=1}^{A} \vartheta_j - \sum_{j=1}^{B'} \psi_j' = 0,$$

we have (see [11, pp. 156–157])

$$\Omega > 0 \qquad \text{and} \qquad \left(\frac{|z|}{\mathfrak{G}}\right)^{\frac{1}{\Omega}} + \left(\frac{|w|}{\mathfrak{H}}\right)^{\frac{1}{\Omega}} < 1$$

and

$$\Omega \leqq 0 \qquad \text{and} \qquad \max\left\{\frac{|z|}{\mathfrak{G}}, \frac{|w|}{\mathfrak{H}}\right\} < 1.$$

Analogous convergence conditions for the Srivastava-Daoust hyergeometric functions in n complex variables can be found in [11, p. 157, Section 3].

Remark 1.1. In the special case of (1.5), if

$$\theta_j = \vartheta_j = 1 \;\; (j = 1, 2, \cdots, A); \quad \psi_j = \psi_k' = 1 \;\; \left(j = 1, 2, \cdots, B;\ k = 1, 2, \cdots, B'\right);$$

and

$$\delta_j = \kappa_j = 1 \;\; (j = 1, 2, \cdots, C); \quad \phi_j = \phi_k' = 1 \;\; \left(j = 1, 2, \cdots, D;\ k = 1, 2, \cdots, D'\right),$$

then the two-variable Srivastava-Daoust function

$$S^{A:B;B'}_{C:D;D'}\begin{pmatrix} z \\ w \end{pmatrix},$$

which is defined in (1.5), would reduce immediately to the generalized Kampé de Fériet function

$$F^{A:B;B'}_{C:D;D'}\begin{bmatrix} z \\ w \end{bmatrix},$$

given by (see, for details, [18])

$$S^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):1,1]:[(b):1];\left[\left(b'\right):1\right];\\ [(c):1,1]:[(d):1];\left[\left(d'\right):1\right];\end{matrix}\, z,w\right)$$

$$=\frac{\prod_{j=1}^{A}\Gamma\left(a_j\right)}{\prod_{j=1}^{C}\Gamma\left(c_j\right)}\frac{\prod_{j=1}^{B}\Gamma\left(b_j\right)}{\prod_{j=1}^{D}\Gamma\left(d_j\right)}\frac{\prod_{j=1}^{B'}\Gamma\left(b'_j\right)}{\prod_{j=1}^{D'}\Gamma\left(d'_j\right)}$$

$$\cdot F^{A:B;B'}_{C:D;D'}\left[\begin{matrix}\left(a_j\right)_{1,A}:\left(b_j\right)_{1,B};\left(b'_j\right)_{1,B'};\\ \left(c_j\right)_{1,C}:\left(d_j\right)_{1,D};\left(d'_j\right)_{1,D'};\end{matrix}\, z,w\right]. \tag{1.7}$$

The convergence conditions for the generalized Kampé de Fériet function, which occurs on the right-hand side of (1.7), can indeed be derived fairly easily from the above-detailed conditions for the two-varivable Srivastava-Daoust function in (1.5).

Remark 1.2. Various specialized and confluent cases of both the generalized Kampé de Fériet function and Srivastava-Daoust functions in two and more variables are potentially useful several fields of science and engineering (see, for example, [1], [12], [13], [16], and [17]). Moreover, in a recent work, Pathan and Kumar [19] gave a representation of a multi-parameter Mittag-Leffler function in terms of the Srivstava-Daoust function in (1.5) and used it in their analysis of the multivariable Cauchy residue theorem. On the other hand, some authors (see [20], [21], and [22]) used the Srivastava-Daoust function (1.5) in fractional calculus and in various other physical problems.

In the existing literature, one can find the studies of several general families of the Hurwitz-Lerch zeta functions and their applications (see, for details, [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], and [33]; see also the above-cited survey-cum-expository review articles [7] and [8] on various widely- and extensively-studied families of the Hurwitz-Lerch and related zeta functions). Motivated by these developments, we first introduce an integral involving the Srivastava-Daoust function (1.5), and then establish several results related to the Hurwitz-Lerch type zeta functions in two variables and the Srivastava-Daoust series in the Srivastava-Daoust class of hypergeometric functions of two variables.

2. A Family of Two-Variable Hurwitz-Lerch Zeta Type Functions

We begin this section by recalling the general Wright function $\mathfrak{E}_{\alpha,\beta}(\phi;z)$ emerged from a systematic study of the asymptotic expansion of the following Taylor-Maclaurin series, which obviously provides a rather deep generalization of the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ (see [34, p. 424]):

$$\mathfrak{E}_{\alpha,\beta}(\phi;z):=\sum_{n=0}^{\infty}\frac{\phi(n)}{\Gamma(\alpha n+\beta)}\,z^n \qquad \left(\alpha,\beta\in\mathbb{C};\ \Re(\alpha)>0\right), \tag{2.1}$$

where $\phi(\tau)$ is a function of τ satisfying suitable conditions. For a reasonably detailed historical background and other details about the following interesting unification of the definition

(2.1) and some multi-parameter extensions of several functions occurring in Analytic Number Theory, the reader is referred, for example, to the recent works [35], [36], and [37]:

$$\mathscr{E}_{\alpha,\beta}(\varphi;z,s,a) := \sum_{n=0}^{\infty} \frac{\varphi(n)}{(n+a)^s\,\Gamma(\alpha n+\beta)}\, z^n \qquad \big(\alpha,\beta \in \mathbb{C};\ \Re(\alpha)>0\big),$$

where, for a suitably-restricted function φ, the parameters α, β, s, and a are appropriately constrained. In fact, in its specialized case that

$$\varphi(n) = \Gamma(\alpha n+\beta) \qquad (n \in \mathbb{N}_0),$$

the Srivastava function $\mathscr{E}_{\alpha,\beta}(\varphi;z,s,a)$ would reduce immediately to the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined by (1.3).

We now introduce the following family of two-variable Hurwitz-Lerch type zeta functions by using the Srivastava-Daoust double hypergeometric function in (1.5):

$$\begin{aligned} {}^{\sigma}S^{A:B;B'}_{C:D;D'}(z,w) &= {}^{\sigma}S^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\ z,w\right)\\ &:= \sum_{m,n=0}^{\infty} \mathscr{H}^{A:B;B'}_{C:D;D'}(m,n)\,\frac{z^m}{m!}\,\frac{w^n}{n!}\,\frac{1}{(m+n+1)^{\sigma}}, \end{aligned} \tag{2.2}$$

where the coefficients $\mathscr{H}^{A:B;B'}_{C:D;D'}(m,n)$ are given by (1.6), $\Re(\sigma)>0$, $|z|<1$, and $|w|<1$. More generally, we have

$$\begin{aligned} {}^{\sigma}_{\omega}S^{A:B;B'}_{C:D;D'}(z,w) &= {}^{\sigma}_{\omega}S^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\ z,w\right)\\ &:= \sum_{m,n=0}^{\infty} \mathscr{H}^{A:B;B'}_{C:D;D'}(m,n)\,\frac{z^m}{m!}\,\frac{w^n}{n!}\,\frac{1}{(m+n+\omega)^{\sigma}}, \end{aligned} \tag{2.3}$$

where, as above, the coefficients $\mathscr{H}^{A:B;B'}_{C:D;D'}(m,n)$ are given by (1.6), $\Re(\sigma)>0$, $\Re(\omega) \geqq 1+\Re(\sigma)$, $|z|<1$, and $|w|<1$. Clearly, upon comparing the definitions (2.2) and (2.3), we have

$$\lim_{\omega\to 1} {}^{\sigma}_{\omega}S^{A:B;B'}_{C:D;D'}(z,w) = {}^{\sigma}S^{A:B;B'}_{C:D;D'}(z,w).$$

Theorem 2.1. *Let* $\Re(\sigma)>0$, $\Re(\omega) \geqq 1+\Re(\sigma)$, $|z|<1$, *and* $|w|<1$. *If the parameters in* (2.3) *satisfy the following inequalities:*

$$\Omega_1 := \sum_{j=1}^{C} \delta_j \log\delta_j + \sum_{j=1}^{D} \phi_j \log\phi_j - \sum_{j=1}^{A} \theta_j \log\theta_j - \sum_{j=1}^{B} \psi_j \log\psi_j > 1 \tag{2.4}$$

and

$$\Omega_2 := \sum_{j=1}^{C} \kappa_j \log\kappa_j + \sum_{j=1}^{D'} \phi'_j \log\phi'_j - \sum_{j=1}^{A} \vartheta_j \log\vartheta_j - \sum_{j=1}^{B'} \psi'_j \log\psi'_j > 1, \tag{2.5}$$

then the series in (2.3) *is convergent under the conditions given by*

$$1+\sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\phi_j-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\psi_j=0$$

and

$$1+\sum_{j=1}^{C}\kappa_j+\sum_{j=1}^{D'}\phi_j'-\sum_{j=1}^{A}\vartheta_j-\sum_{j=1}^{B'}\psi_j'=0.$$

Furthermore, under the above-mentioned conditions, for Ω_1 *and* Ω_2 *defined by* (2.4) *and* (2.5), *respectively, the series in* (2.3) *converges when*

$$\sum_{j=1}^{C}\delta_j\log\delta_j+\sum_{j=1}^{D}\phi_j\log\phi_j-\sum_{j=1}^{A}\theta_j\log\theta_j-\sum_{j=1}^{B}\psi_j\log\psi_j=0$$

and

$$\sum_{j=1}^{C}\kappa_j\log\kappa_j+\sum_{j=1}^{D'}\phi_j'\log\phi_j'-\sum_{j=1}^{A}\vartheta_j\log\vartheta_j-\sum_{j=1}^{B'}\psi_j'\log\psi_j'=0$$

under appropriately-constrained values of the arguments z *and* w.

Proof. Just as we indicated in the case of the convergence of the Srivastava-Daoust double hypergeometric series in (1.5), the convergence conditions, which are asserted by Theorem 2.1, would follow by applying Horn's theorem on convergence of double hypergeometric series. We choose here to leave the details involved as an exercise for the interested reader (see, for details, [11]; see also [38]). □

Our next result (Theorem 2.2 below) provides an integral representation for the function ${}_{\omega}^{\sigma}S_{C:D;D'}^{A:B;B'}(z,w)$ defined by (2.3).

Theorem 2.2. *Under the conditions stated in Theorem 2.1 together with the constraints on* Ω_1 *and* Ω_2 *defined by* (2.4) *and* (2.5), *respectively, the following integral representation of the function* ${}_{\omega}^{\sigma}S_{C:D;D'}^{A:B;B'}(z,w)$, *which is given by* (2.3), *holds true:*

$$\begin{aligned}{}_{\omega}^{\sigma}S_{C:D;D'}^{A:B;B'}(z,w)&={}_{\omega}^{\sigma}S_{C:D;D'}^{A:B;B'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\ z,w\right)\\&=\frac{1}{\Gamma(\sigma)}\int_0^\infty t^{\sigma-1}\,\mathrm{e}^{-\omega t}\\&\quad\cdot S_{C:D;D'}^{A:B;B'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\ z\mathrm{e}^{-t},w\mathrm{e}^{-t}\right)\mathrm{d}t,\end{aligned}\tag{2.6}$$

provided that the integral in (2.6) *is convergent.*

Proof. The derivation of the integral representation (2.6) would run parallel to that of the well-known result (1.4) for the familiar Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined by (1.3). Indeed, for convenience, we denote by $\Lambda(z,w)$ the right-hand side of the formula (2.6). Then, on replacing the two-variable Srivastava-Daoust function by its double series given by (1.5), we find under the hypotheses of Theorem 2.2 that

$$\begin{aligned}\Lambda(z,w) &:= \frac{1}{\Gamma(\sigma)} \int_0^\infty t^{\sigma-1}\, \mathrm{e}^{-\omega t} \\ &\cdot S^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right]; \\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\ ze^{-t}, we^{-t}\right)\mathrm{d}t \\ &= \sum_{m,n=0}^\infty \mathscr{H}^{A:B;B'}_{C:D;D'}(m,n)\,\frac{z^m}{m!}\,\frac{w^n}{n!} \\ &\qquad\qquad \cdot \frac{1}{\Gamma(\sigma)}\int_0^\infty t^{\sigma-1}\,\mathrm{e}^{-(m+n+\omega)t}\,\mathrm{d}t \\ &= \sum_{m,n=0}^\infty \mathscr{H}^{A:B;B'}_{C:D;D'}(m,n)\,\frac{z^m}{m!}\,\frac{w^n}{n!}\,\frac{1}{(m+n+\omega)^\sigma},\end{aligned} \tag{2.7}$$

where we used the formula for the Laplace transform of a power function.

We now interpret the second member of (2.7) by means of the definition (1.5) of the two-variable Srivastava-Daoust function. We thus find from (2.7) that

$$\begin{aligned}\Lambda(z,w) &= {}^{\sigma}_{\omega}S^{A:B;B'}_{C:D;D'}(z,w) \\ &= {}^{\sigma}_{\omega}S^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right]; \\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\ z,w\right),\end{aligned}$$

which is precisely the left-hand side of (2.6). Our demonstration of Theorem 2.2 is thus completed under the stated conditions. $\square$

Remark 2.1. Many special cases and consequences of the definition (2.3) can be deduced fairly easily. For example, if we replace the parameter ω by $\frac{\omega}{2}$, we obtain the following relation:

$${}^{\sigma}_{\frac{\omega}{2}}S^{A:B;B'}_{C:D;D'}(z,w) = 2^\sigma \sum_{m,n=0}^\infty \mathscr{H}^{A:B;B'}_{C:D;D'}(m,n)\,\frac{z^m}{m!}\,\frac{w^n}{n!}\,\frac{1}{[2(m+n)+\omega]^\sigma}.$$

We next recall that

$$(1-z)^{-\lambda} = \sum_{n=0}^\infty (\lambda)_n \frac{z^n}{n!} \qquad (|z|<1;\ \lambda\in\mathbb{C}),$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined (for $\lambda,\nu\in\mathbb{C}$) by

$$(\lambda)_n := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N}),\end{cases}$$

in which it is assumed tacitly that the Γ-quotient exists and understood conditionally the $(0)_0 := 1$. We thus find the following series expansion connecting the functions defined by (2.2) and (2.3).

Theorem 2.3. *Under the hypotheses of Theorem 2.1, it is asserted that*

$$
\begin{aligned}
{}_{\omega}^{\sigma}S_{C:D;D'}^{A:B;B'}(z,w) &= {}_{\omega}^{\sigma}S_{C:D;D'}^{A:B;B'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\; z,w\right)\\
&= \sum_{r=0}^{\infty}(-1)^{r}\,(\sigma)_{r}\;\frac{(\omega-1)^{r}}{r!}\;{}^{\sigma+r}S_{C:D;D'}^{A:B;B'}(z,w),
\end{aligned}
\tag{2.8}
$$

provided that each member of (2.8) *exists.*

Theorem 2.4. *Under the hypotheses of Theorem 2.1, the following inequality holds true*:

$$
\begin{aligned}
&\left|\frac{1}{(\omega-1)}\left({}_{\omega}^{\sigma}S_{C:D;D'}^{A:B;B'}(z,w)-{}^{\sigma}S_{C:D;D'}^{A:B;B'}(z,w)\right)\right|\\
&\qquad<\left|\frac{\partial}{\partial\omega}\left\{{}_{\omega}^{\sigma}S_{C:D;D'}^{A:B;B'}(z,w)\right\}\right|,
\end{aligned}
\tag{2.9}
$$

provided that each member of (2.9) *exists.*

Proof. If we operate upon both sides of the integral representation (2.6) by the partial derivative operator $\frac{\partial}{\partial\omega}$, we find that

$$
\begin{aligned}
\frac{\partial}{\partial\omega}\left\{{}_{\omega}^{\sigma}S_{C:D;D'}^{A:B;B'}(z,w)\right\} &= -\frac{1}{\Gamma(\sigma)}\int_{0}^{\infty}\mathrm{e}^{-\omega t}\,t^{\sigma}\\
&\quad\cdot S_{C:D;D'}^{A:B;B'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\; z\mathrm{e}^{-t},w\mathrm{e}^{-t}\right)\mathrm{d}t\\
&= -\sigma\sum_{m,n=0}^{\infty}\mathscr{H}_{C:D;D'}^{A:B;B'}(m,n)\,\frac{z^{m}}{m!}\,\frac{w^{n}}{n!}\,\frac{1}{[(\omega-1)+m+n+1]^{\sigma+1}}.
\end{aligned}
$$

Now, by applying the relationship (2.8) asserted by Theorem 2.3, we have

$$
\begin{aligned}
&\frac{\partial}{\partial\omega}\left\{{}_{\omega}^{\sigma}S_{C:D;D'}^{A:B;B'}(z,w)\right\}\\
&\quad=\sum_{r=0}^{\infty}(-1)^{r+1}\,(\sigma)_{r+1}\;\frac{(\omega-1)^{r}}{r!}\;{}^{\sigma+r+1}S_{C:D;D'}^{A:B;B'}(z,w)\\
&\quad=\frac{1}{(\omega-1)}\sum_{r=1}^{\infty}(-1)^{r}\,(\sigma)_{r}\;\frac{(\omega-1)^{r}}{(r-1)!}\;{}^{\sigma+r}S_{C:D;D'}^{A:B;B'}(z,w),
\end{aligned}
$$

which leads us to the following inequality:

$$
\begin{aligned}
&\left|\frac{\partial}{\partial\omega}\left\{{}_{\omega}^{\sigma}S_{C:D;D'}^{A:B;B'}(z,w)\right\}\right|\\
&\quad>\left|\left(\frac{1}{(\omega-1)}\sum_{r=0}^{\infty}(-1)^{r}\,(\sigma)_{r}\;\frac{(\omega-1)^{r}}{r!}\;{}^{\sigma+r}S_{C:D;D'}^{A:B;B'}(z,w)\right.\right.\\
&\qquad\left.\left.-\frac{1}{(\omega-1)}\;{}^{\sigma}S_{C:D;D'}^{A:B;B'}(z,w)\right)\right|.
\end{aligned}
\tag{2.10}
$$

Thus, by applying Theorem 2.3, the above inequality (2.10) immediately gives the inequality (2.9) asserted by Theorem 2.4. $\square$

Finally, the following result involves an extension of the function (1.5), which is associated with the Euler-Zagier type sum.

Theorem 2.5. *Let $\Re\left(\omega_j\right) > 1+\Re\left(\sigma_j\right)$, $j=1,2$, and $\Re(\mu)>0$, and suppose that*

$$
\begin{aligned}
&{}^{\sigma_1,\sigma_2}_{\omega_1,\omega_2}S^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\ \mu;z,w\right)\\
&\quad:=\sum_{m=0}^{\infty}\frac{z^m}{m!\ (m+\omega_1)^{\sigma_1}}\sum_{n=0}^{\infty}\mathscr{H}^{A:B;B'}_{C:D;D'}(m,n)\,\frac{w^n}{n!\ (m+\mu n+\omega_2)^{\sigma_2}},
\end{aligned}\tag{2.11}
$$

where the coefficients $\mathscr{H}^{A:B;B'}_{C:D;D'}(m,n)$ are given by (1.6). Then the following inequality holds true:

$$
\begin{aligned}
&\left|{}^{\sigma_1,\sigma_2}_{\omega_1,\omega_2}S^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\ \mu;z,w\right)\right|<\left|S^{A+1:B+1;B'}_{C+1:D+1;D'}\right.\\
&\left.\left(\begin{matrix}[(a):\theta,\vartheta],[2:1,\mu]:[(b):\psi],[2:1];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa],[1+\Re(\sigma_2):1,\mu]:[(d):\phi],[1+\Re(\sigma_1):1];\left[\left(d'\right):\phi'\right];\end{matrix}\ z,w\right)\right|,
\end{aligned}\tag{2.12}
$$

provided that each member of inequality (2.12) exists.

Proof. Our demonstration of the inequality (2.12), which is asserted by Theorem 2.5, is based upon the lines of the proof of Theorem 2.1. The details involved are being omitted here. $\square$

If, in the definition (2.11), we set $\sigma_1\to 0$ and $\mu=1$, we obtain the following limit case in terms of the function in (1.5):

$$
\begin{aligned}
&\lim_{\sigma_1\to 0}\left\{{}^{\sigma_1,\sigma_2}_{\omega_1,\omega_2}S^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\ 1;z,w\right)\right\}\\
&\quad=\sum_{n=0}^{\infty}\mathscr{H}^{A:B;B'}_{C:D;D'}(m,n)\,\frac{z^m}{m!}\,\frac{w^n}{n!\ (m+n+\omega_2)^{\sigma_2}}\\
&\quad={}^{\sigma_2}_{\omega_2}S^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\ z,w\right).
\end{aligned}
$$

In the particular case when

$$
\mathscr{H}^{A:B;B'}_{C:D;D'}(m,n)=\mathscr{H}^{0:1;1}_{0:0;0}(m,n)=m!\,n!,
$$

we have another Euler-Zagier type sum in the form:

$$
{}_{\omega_1,\omega_2}^{\sigma_1,\sigma_2}S^{0:1;1}_{0:0;0}\left(\begin{matrix}[-:-,-]:[1:1];[1:1];\\ [-:-,-]:[-:-];[-:-];\end{matrix}\mu;z,w\right)
= \sum_{m=0}^{\infty}\frac{z^m}{(m+\omega_1)^{\sigma_1}}\sum_{n=0}^{\infty}\frac{w^n}{(m+\mu n+\omega_2)^{\sigma_2}},
$$

which, for $z=w=1$ and $\omega_1=\omega_2=\omega$, yields the Matsumoto type double zeta function $\zeta_2(\sigma_1,\sigma_2;\omega,\mu)$ given by (see, for details, [39])

$$
\begin{aligned}
{}_{\omega_1,\omega_2}^{\sigma_1,\sigma_2}S^{0:1;1}_{0:0;0}&\left(\begin{matrix}[-:-,-]:[1:1];[1:1];\\ [-:-,-]:[-:-];[-:-];\end{matrix}\mu;1,1\right)\\
&= \sum_{m=0}^{\infty}\frac{1}{(m+\omega)^{\sigma_1}}\sum_{n=0}^{\infty}\frac{1}{(m+\mu n+\omega)^{\sigma_2}}\\
&=: \zeta_2(\sigma_1,\sigma_2;\omega,\mu).
\end{aligned}
$$

3. Special Cases and Consequences

In this section, by suitably specializing the various parameters, which are involved in the results obtained in Section 2, we derive several other relations associated with the zeta and related functions.

First of all, upon setting

$$
\theta_j=\vartheta_j=1\;\;(j=1,2,\cdots,A);\quad \psi_j=\psi_k^{'}=1\;\;\left(j=1,2,\cdots,B;\;k=1,2,\cdots,B^{'}\right);
$$

and

$$
\delta_j=\kappa_j=1\;\;(j=1,2,\cdots,C);\quad \phi_j=\phi_k^{'}=1\;\;\left(j=1,2,\cdots,D;\;k=1,2,\cdots,D^{'}\right)
$$

in (1.5), we have the following relation with the two-variable Hurwitz-Lerch zeta function considered by Choi *et al.* [40]:

$$
\begin{aligned}
&\frac{\prod_{j=1}^{C}\Gamma(c_j)\prod_{j=1}^{D}\Gamma(d_j)\prod_{j=1}^{D^{'}}\Gamma\left(d_j^{'}\right)}{\prod_{j=1}^{A}\Gamma(a_j)\prod_{j=1}^{B}\Gamma(b_j)\prod_{j=1}^{B^{'}}\Gamma\left(b_j^{'}\right)}\;{}_{\omega}^{\sigma}S^{A:B;B^{'}}_{C:D;D^{'}}\left(\begin{matrix}[(a):1,1]:[(b):1];\left[\left(b^{'}\right):1\right];\\ [(c):1,1]:[(d):1];\left[\left(d^{'}\right):1\right];\end{matrix}z,w\right)\\
&\qquad = \Phi^{A:B;B^{'}}_{C:D;D^{'}}\left(\begin{matrix}(a_A):(b_B);\left(b^{'}_{B^{'}}\right);\\ (c_A):(d_B);\left(d^{'}_{B^{'}}\right);\end{matrix}z,w,\sigma,\omega\right),
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
&\Phi^{A:B;B^{'}}_{C:D;D^{'}}\left(\begin{matrix}(a_A):(b_B);\left(b^{'}_{B^{'}}\right);\\ (c_A):(d_B);\left(d^{'}_{B^{'}}\right);\end{matrix}z,w,\sigma,\omega\right)\\
&\qquad := \sum_{m,n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m+n}}{\prod_{j=1}^{C}(c_j)_{m+n}}\frac{\prod_{j=1}^{B}(b_j)_m}{\prod_{j=1}^{D}(d_j)_m}\frac{\prod_{j=1}^{B^{'}}\left(b_j^{'}\right)_n}{\prod_{j=1}^{D^{'}}\left(d_j^{'}\right)_n}\frac{z^m\,w^n}{m!\,n!\,(m+n+\omega)^{\sigma}},
\end{aligned}
$$

provided that all of the applicable conditions, which are listed in Theorem 2.1 and Remark 1.1, are satisfied.

Next, since an empty product is conventionally interpreted as 1, in the further special case of the relation (3.1) when

$$A = C = B = B^{'} = 1 \quad \text{and} \quad D = D^{'} = 0,$$

we are led to another relation with the Hurwitz-Lerch zeta function considered by Choi and Parmar [41] in the form given by (see also [42])

$$\begin{aligned}
&\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(b')}\, {}_{\omega}^{\sigma}S^{1:1;1}_{1:0;0}\left(\begin{matrix}[a:1,1]:[b:1];\left[b^{'}:1\right]; \\ [c:1,1]:[-:-];[-:-];\end{matrix}\, z,w\right) \\
&\qquad = \Phi_{a,b,b',c}(z,w,\sigma,\omega) \\
&\qquad := \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{z^m\, w^n}{m!\, n!\, (m+n+\omega)^{\sigma}},
\end{aligned}$$

where it is assumed that all of the applicable conditions, which are listed in Theorem 2.1 and Remark 1.1, are satisfied.

Many other (known or new) special cases and consequences of the definitions and the results, which we have presented in the preceding sections, can also be deduced.

4. A Multiple Hurwitz-Lerch Zeta Function Based Upon the Srivastava-Daoust Hypergeometric Series in Several Variables

In this section, we first present a further generalization of the double Hurwitz-Lerch zeta function (1.5) by applying the Srivastava-Daoust hypergeometric series in several variables.

For $\Re(\sigma) > 0$ and $\Re(\omega) \geqq 1 + \Re(\sigma)$, a family of multiple Hurwitz-Lerch zeta functions, which are associated with the Srivastava-Daoust hypergeometric series in n variables (see, for details, [10]; see also [16] and [38]), is defined by

$$\begin{aligned}
&{}_{\omega}^{\sigma}F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}(z_1,\cdots,z_n) \\
&\quad = {}_{\omega}^{\sigma}F^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}} \\
&\qquad\qquad \left(\begin{matrix}\left[(a):\theta^{(1)},\cdots,\theta^{(n)}\right]:\left[\left(b^{(1)}\right):\psi^{(1)}\right];\cdots;\left[\left(b^{(n)}\right):\psi^{(n)}\right]; \\ \left[(c):\delta^{(1)},\cdots,\delta^{(n)}\right]:\left[\left(d^{(1)}\right):\phi^{(1)}\right];\cdots;\left[\left(d^{(n)}\right):\phi^{(n)}\right];\end{matrix}\, z_1,\cdots,z_n\right) \\
&\quad := \sum_{m_1,\cdots,m_n=0}^{\infty} \mathscr{K}^{A:B^{(1)};\cdots;B^{(n)}}_{C:D^{(1)};\cdots;D^{(n)}}(m_1,\cdots,m_n) \\
&\qquad\qquad \cdot \frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!}\, \frac{1}{(m_1+\cdots+m_n+\omega)^{\sigma}},
\end{aligned} \tag{4.1}$$

where, for convenience,

$$\mathscr{K}_{C:D^{(1)};\cdots;D^{(n)}}^{A:B^{(1)};\cdots;B^{(n)}}(m_1,\cdots,m_n)$$
$$:= \frac{\prod_{j=1}^{A}(a_j)_{\theta_j^{(1)}m_1+\cdots+\theta_j^{(n)}m_n}}{\prod_{j=1}^{C}(a_j)_{\delta_j^{(1)}m_1+\cdots+\delta_j^{(n)}m_n}} \frac{\prod_{j=1}^{B^{(1)}}\left(b_j^{(1)}\right)_{\psi_j^{(1)}m_1}}{\prod_{j=1}^{D^{(1)}}\left(d_j^{(1)}\right)_{\phi_j^{(1)}m_1}} \cdots \frac{\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{\psi_j^{(n)}m_n}}{\prod_{j=1}^{D^{(n)}}\left(d_j^{(n)}\right)_{\phi_j^{(n)}m_n}}.$$

The multiple series in (4.1) converges for

$$|z_1| < 1, \cdots, |z_n| < 1,$$

provided that

$$\sum_{j=1}^{C}\delta_j^{(\ell)} + \sum_{j=1}^{D^{(\ell)}}\phi_j^{(\ell)} - \sum_{j=1}^{A}\theta_j^{(\ell)} - \sum_{j=1}^{B^{(\ell)}}\psi_j^{(\ell)} + 1 = 0 \qquad (\forall\, \ell = 1,\cdots,n).$$

In the particular case when $n = 2$, upon comparing the definitions in (4.1) and (1.5), we have following relation:

$${}_{\omega}^{\sigma}S_{C:D;D'}^{A:B;B'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\, z,w\right)$$
$$= \frac{\prod_{j=1}^{A}\Gamma(a_j)}{\prod_{j=1}^{C}\Gamma(c_j)} \frac{\prod_{j=1}^{B}\Gamma(b_j)}{\prod_{j=1}^{D}\Gamma(d_j)} \frac{\prod_{j=1}^{B'}\Gamma\left(b_j'\right)}{\prod_{j=1}^{D'}\Gamma\left(d_j'\right)}$$
$$\cdot {}_{\omega}^{\sigma}F_{C:D;D'}^{A:B;B'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];\left[\left(b'\right):\psi'\right];\\ [(c):\delta,\kappa]:[(d):\phi];\left[\left(d'\right):\phi'\right];\end{matrix}\, z,w\right).$$

If, in the definition (4.1), we set

$$\theta_j^{(\ell)} = \psi_k^{(\ell)} = 1 \ \left(j = 1,2,\cdots,A;\ k = 1,2,\cdots,B^{(\ell)};\ \ell = 1,2,3,\cdots,n\right),$$

$$\delta_j^{(\ell)} = 1 \ \ (j = 1,2,\cdots,C;\ \ell = 1,2,3,\cdots,n) \qquad \text{and} \qquad D^{(1)} = \cdots = D^{(n)} = 0,$$

then we have the following multiple Hurwitz-Lerch type zeta function:

$$\begin{aligned}
&{}_{\omega}^{\sigma}F^{A:B^{(1)};\cdots;B^{(n)}}_{C:0;\cdots;0}\left(\begin{matrix}[(a):1,\cdots,1]:\left[\left(b^{(1)}\right):1\right];\cdots;\left[\left(b^{(n)}\right):1\right];\\ [(c):1,\cdots,1]:[-:-];\cdots;[-:-];\end{matrix}\, z_1,\cdots,z_n\right)\\
&\qquad := \sum_{m_1,\cdots,m_n=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{m_1+\cdots+m_n}}{\prod_{j=1}^{C}(c_j)_{m_1+\cdots+m_n}}\,\frac{\prod_{j=1}^{B^{(1)}}\left(b_j^{(1)}\right)_{m_1}}{m_1!}\cdots\frac{\prod_{j=1}^{B^{(n)}}\left(b_j^{(n)}\right)_{m_n}}{m_n!}\\
&\qquad\qquad\cdot\frac{z_1^{m_1}\cdots z_n^{m_n}}{(m_1+\cdots+m_n+\omega)^{\sigma}},
\end{aligned} \tag{4.2}$$

where, for convergence,

$$|z_1|<1,\cdots,|z_n|<1,$$

provided that $\Re(\sigma)>0$, $\Re(\omega)\geqq 1+\Re(\sigma)$ and $C-A-B^{(\ell)}+1=0$, $\forall\ell=1,\cdots,n)$. For $C=A=B^{(\ell)}=1$ $(\forall\,\ell=1,\cdots,n)$, we have the following further multiple Hurwitz-Lerch zeta type function contained in the definition (4.2):

$$\begin{aligned}
&{}_{\omega}^{\sigma}F^{A:B^{(1)};\cdots;B^{(n)}}_{C:0;\cdots;0}\left(\begin{matrix}[(a):1,\cdots,1]:\left[\left(b^{(1)}\right):1\right];\cdots;\left[\left(b^{(n)}\right):1\right];\\ [(c):1,\cdots,1]:[-:-];\cdots;[-:-];\end{matrix}\, z_1,\cdots,z_n\right)\\
&\quad=\Phi^{(n)}\left(a,b^{(1)},\cdots,b^{(n)};c;z_1,\cdots,z_n,\sigma,\omega\right),
\end{aligned}$$

where

$$\begin{aligned}
&\Phi^{(n)}\left(a,b^{(1)},\cdots,b^{(n)};c;z_1,\cdots,z_n,\sigma,\omega\right)\\
&\qquad:=\sum_{m_1,\cdots,m_n=0}^{\infty}\frac{(a_j)_{m_1+\cdots+m_n}}{(c_j)_{m_1+\cdots+m_n}}\,\frac{\left(b_j^{(1)}\right)_{m_1}}{m_1!}\cdots\frac{\left(b_j^{(n)}\right)_{m_n}}{m_n!}\,\frac{z_1^{m_1}\cdots z_n^{m_n}}{(m_1+\cdots+m_n+\omega)^{\sigma}},
\end{aligned}$$

it being assumed, for convergence, that

$$|z_1|<1,\cdots,|z_n|<1,$$

$\Re(\sigma)>0$, $\Re(\omega)\geqq 1+\Re(\sigma)$.

It is not difficult to apply the methods and techniques, which we have used in the preceding sections, in order to derive the corresponding results for the family of multiple Hurwitz-Lerch zeta type functions of this section. As an example, here we give an integral representation which corresponds to that in Theorem 2.2.

Theorem 4.1. *Let $\Re(\sigma)>0$ and $\Re(\omega)\geqq 1+\Re(\sigma)$. Suppose that $|z_1|<1,\cdots,|z_n|<1$ and*

$$1+\sum_{j=1}^{C}\delta_j^{(\ell)}+\sum_{j=1}^{D^{(\ell)}}\phi_j^{(\ell)}-\sum_{j=1}^{A}\theta_j^{(\ell)}-\sum_{j=1}^{B^{(\ell)}}\psi_j^{(\ell)}=0\qquad(\forall\,\ell=1,\cdots,n).$$

Then the following integral representation holds true for the multiple Hurwitz-Lerch type function in (4.1):

$$
{}_{\omega}^{\sigma}F_{C:D^{(1)};\cdots;D^{(n)}}^{A:B^{(1)};\cdots;B^{(n)}}(z_1,\cdots,z_n)=\frac{1}{\Gamma(\sigma)}\int_0^\infty t^{\sigma-1}\,\mathrm{e}^{-\omega t}\,F_{C:D^{(1)};\cdots;D^{(n)}}^{A:B^{(1)};\cdots;B^{(n)}}
$$
$$
\left(\begin{matrix}\left[(a):\theta^{(1)},\cdots,\theta^{(n)}\right]:\left[\left(b^{(1)}\right):\psi^{(1)}\right];\cdots;\left[\left(b^{(n)}\right):\psi^{(n)}\right];\\ \left[(c):\delta^{(1)},\cdots,\delta^{(n)}\right]:\left[\left(d^{(1)}\right):\phi^{(1)}\right];\cdots;\left[\left(d^{(n)}\right):\phi^{(n)}\right];\end{matrix}\; z_1\mathrm{e}^{-t},\cdots,z_n\mathrm{e}^{-t}\right)\mathrm{d}t,\quad (4.3)
$$

provided that the integral in (4.3) *is convergent.*

Proof. Our demonstration of Theorem 4.1 follows essentially the same lines as in the proof of Theorem 2.2, so we skip the details involved. □

5. Concluding Remarks and Observations

The present investigation is motivated by the various families of Hurwitz-Lerch zeta type functions which have appeared in the existing literature in Analytic Number Theory. Here, in this paper, we unified and generalized many of these Hurwitz-Lerch zeta type functions by making use of the Srivastava-Daoust hypergeometric series in two and more variables. The properties and relations, which we presented in this article, include the existence and convergence conditions based upon Horn's theorem on convergence of double series, integral representations using the Eulerian integral for the Gamma function, and so on. Our main results and their special cases and consequences, given in this paper, are potentially useful in scientific problems and related computational work.

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