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A RELAXED CQ ALGORITHM INVOLVING THE ALTERNATED INERTIAL TECHNIQUE FOR THE MULTIPLE-SETS SPLIT FEASIBILITY PROBLEM

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Abstract. In this paper, we present a relaxed CQ algorithm involving the Armijo-line search and the alternated inertial technique to solve the multiple-sets split feasibility problem in infinite dimensional real Hilbert spaces. We prove the convergence of the sequence generated by our method under some mild assumptions. Finally, numerical simulations show the efficiency of our algorithm.

Keywords. Alternated inertial; Armijo-line search; CQ algorithm; Multiple-sets split feasibility problem.

1. Introduction

In this paper, we focus on the Multiple-Sets Split Feasibility Problem (MSSFP), which is formulated as follows.

$$\text{Find a point } x^* \in C = \bigcap_{i=1}^{t} C_i \text{ such that } Ax^* \in Q = \bigcap_{j=1}^{r} Q_j, \tag{1.1}$$

where $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded and linear operator, $C_i \subset \mathscr{H}_1,\ i = 1,2,\cdots t$, and $Q_j \subset \mathscr{H}_2,\ j = 1,2,\cdots r$ are nonempty, closed, and convex sets, and $\mathscr{H}_1$ and $\mathscr{H}_2$ are real Hilbert spaces. The MSSFP plays an important role in improving the efficiency of image and signal processing and has numerous applications in the fields of medicine, biology, military, image reconstruction, and signal processing; see, e.g., [1]. If $t = r = 1$, it is easy to see that the MSSFP reduces to the Split Feasibility Problem (SFP) as follows.

$$\text{Find a point } x^* \in C \text{ such that } Ax^* \in Q,$$

where $C \subset \mathscr{H}_1$ and $Q \subset \mathscr{H}_2$ are nonempty, closed, and convex sets. The classical ways to solve the SFP are the CQ algorithm introduced by Byrne in [1, 2] and the relaxed CQ algorithm introduced by Yang in [3]. The relaxed CQ algorithm is

$$x^{k+1} = P_{C^k}(x^k - \alpha_k A^*(I - P_{Q^k})Ax^k),$$

where $\alpha_k \in (0, \frac{2}{\|A\|^2})$. The main idea of the relaxation is to construct two series of closed half spaces C^k and Q^k containing C and Q, respectively, on which the projections have closed forms and can be calculated directly.

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For solving the MSSFP, Censor et al. [4] proposed the following algorithm

$$x^{k+1} = P_\Omega\big(x^k - \alpha(\sum_{i=1}^{t} \lambda_i(x^k - P_{C_i}(x^k)) + \sum_{j=1}^{r} \beta_j(Ax^k - P_{Q_j}(Ax^k)))\big),$$

where Ω is an auxiliary closed convex subset, and $0 < \alpha < 2/L$ with $L = \sum_{i=1}^{t} \lambda_i + \|A\|^2 \sum_{j=1}^{r} \beta_j$. The sequence $\{x^k\}$ generated by the algorithm was proved to converge weakly to a solution of the problem.

It is worth noting that, for fixed step sizes, one needs to calculate (or at least estimate) the maximum eigenvalue of $A^T A$, which is also an obstacle for the numerical computation of the above algorithms. Thus, many scholars adopted variable (self-adaptive) step sizes instead of the fixed step sizes. In 2005, Yang [5] proposed the step size in the CQ algorithm

$$\alpha_k = \frac{\rho_k}{\|\nabla f(x^k)\|},$$

where ρ_k satisfies $\rho_k > 0$, $\sum_{n=0}^{\infty} \rho_k = \infty$, and $\sum_{n=0}^{\infty} \rho_k^2 < \infty$, and $f(\cdot) = \frac{1}{2}\|(I - P_Q)A(\cdot)\|^2$. Yang proved the convergence of the iterates in finite dimensional spaces under the appropriate conditions. Based on the relaxed CQ algorithm, López et al. [6] introduced a new choice of the step size sequence

$$\alpha_k = \frac{\rho_k f_k(x^k)}{\|\nabla f_k(x^k)\|^2},$$

where $0 < \rho_k < 4$ and $f_k(\cdot) = \frac{1}{2}\|(I - P_{Q^k})A(\cdot)\|^2$, and proved the weak convergence of the iterative sequence in Hilbert spaces. The advantage of this choice of the step size lies in the fact that neither prior information about the matrix norm of A nor any other conditions on Q and A are required.

Recently, Qu and Xiu in [7] adopted the Armijo-line search to obtain the step size in the CQ algorithm and the relaxed CQ algorithm, and proved the convergence in Euclidean spaces. Gibali et al. [8] extended this result to infinite dimensional Hilbert spaces and prove the weak convergence. Armijo-line search is another kind of self-adaptive strategy and accelerates the convergence rate of the sequence. The inertial terms also can accelerate the convergence rate of sequences. The inertial acceleration was derived from the heavy-ball method of second-order dynamical systems; see Polyak [9]. Subsequently, the method of inertial acceleration has been developed extensively, see [10, 11, 12, 13, 14, 15]. However, the iterate sequences lose their monotonicity due to the inertial steps, which may result in the failure of the convergence. In this case, Mu and Peng [16] introduced the alternated inertial method to the proximal point method for solving the maximal monotone inclusion problem. Their idea is to impose the inertial term only on the odd terms. Then the monotonicity of the even terms of the iterative sequence can be reserved, which is vital for the convergence proof.

In 2020, Shehu and Gibali [17] proposed the following alternated inertial relaxed CQ algorithm with the Armijo-line search to solve the SFP

$$\omega^k = \begin{cases} x^k, & k = \text{even}, \\ x^k + \theta_k(x^k - x^{k-1}), & k = \text{odd}, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \bar{x}^k = P_{C^k}(\omega^k - \alpha_k \nabla f_k(\omega^k)), \\ x^{k+1} = P_{C^k}(\omega^k - \alpha_k \nabla f_k(\bar{x}^k)), \end{cases} \tag{1.3}$$

where $\nabla f_k(x^k) = A^*(I - P_{Q^k})Ax^k$, $\{\theta_k\} \subset [0,1)$, $\alpha_k = \gamma l^{m_k}$ with m_k the smallest non-negative integer such that $\alpha_k \|\nabla f_k(\omega^k) - \nabla f_k(\bar{x}^k)\| \le \mu \|\omega^k - \bar{x}^k\|$, $\gamma > 0$, $l \in (0,1)$, and $\mu \in (0,1)$. $\{x^k\}$ was proved to converge weakly to a solution of the SFP.

In this paper, motivated by Xu [18], Qu and Xiu [7], Gibali et al. [8], and Shehu and Gibali [17], we present a modified relaxed CQ algorithm with the Armijo-line search and alternated inertial method to solve the MSSFP (1.1) in real Hilbert spaces. The rest of the paper is arranged as follows. In Section 2, definitions and notions are presented, which are useful for our analysis. In Section 3, we present our algorithm and prove its weak convergence. In Section 4, we present several numerical simulations to show the validity of the modified relaxed CQ algorithm. Section 5, the last section, ends this paper.

2. Preliminaries

In this section, we first define some symbols, and then review some definitions and basic results that are used in this paper. Throughout this paper, $\mathscr{H}$ denotes a real Hilbert space endowed with inner product $\langle \cdot, \cdot \rangle$ and deduced norm $\|\cdot\|$, and I is the identity operator on $\mathscr{H}$. We denote by S the solution set of the MSSFP (1.1). Moreover, $x^k \to x$ ($x^k \rightharpoonup x$) represents that the sequence $\{x^k\}$ converges strongly (weakly) to x. Finally, we denote by $\omega_w(x^k)$ all the weak cluster points of $\{x^k\}$.

An operator $T : \mathscr{H} \to \mathscr{H}$ is said to be nonexpansive if, for all $x, y \in \mathscr{H}$, $\|Tx - Ty\| \le \|x - y\|$; $T : \mathscr{H} \to \mathscr{H}$ is said to be firmly nonexpansive if, for all $x, y \in \mathscr{H}$,

$$||Tx - Ty||^2 \le ||x - y||^2 - ||(I - T)x - (I - T)y||^2,$$

or equivalently

$$||Tx - Ty||^2 \le \langle Tx - Ty, x - y \rangle.$$

It is well known that T is firmly nonexpansive if and only if $I - T$ is firmly nonexpansive.

Let C be a nonempty, closed and convex subset of $\mathscr{H}$. Then the metric projection P_C from $\mathscr{H}$ onto C is defined as

$$P_C(x) = \operatorname*{argmin}_{y \in C} ||x - y||^2, \ x \in \mathscr{H}.$$

The metric projection P_C is a firmly nonexpansive operator.

Recall that a function $f : \mathscr{H} \to \mathbf{R}$ is said to be weakly lower semicontinuous at $\hat{x}$ if x^k converges weakly to $\hat{x}$ implies $f(\hat{x}) \le \liminf_{k \to \infty} f(x^k)$. Let $\varphi : \mathscr{H} \to \mathbf{R}$ be a convex function. The subdifferential of φ at x is defined as

$$\partial \varphi(x) = \{\xi \in H \mid \varphi(y) \ge \varphi(x) + \langle \xi, y - x \rangle, \ \forall y \in \mathscr{H}\}.$$

Lemma 2.1. [19] *Let C be a nonempty, closed, and convex subset of $\mathscr{H}$. Then, for any $x, y \in \mathscr{H}$, $z \in C$, α, $\beta \in \mathbf{R}$, the following assertions hold:*
(i) $\langle x - P_C x, z - P_C x \rangle \le 0$;
(ii) $\|P_C x - z\|^2 \le \|x - z\|^2 - \|P_C x - x\|^2$;
(iii) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle$;
(iv) $\|\alpha x + \beta y\|^2 = \alpha(\alpha + \beta)\|x\|^2 + \beta(\alpha + \beta)\|y\|^2 - \alpha\beta\|x - y\|^2$.

Lemma 2.2. [20] *Let S be a nonempty, closed, and convex subset of $\mathscr{H}$, and let $\{x^k\}$ be a sequence in $\mathscr{H}$ that satisfies the following properties*:
(i) $\lim_{k\to\infty}\|x^k-x\|$ *exists for each* $x\in S$;
(ii) $\omega_w(x^k)\subset S$.
Then $\{x^k\}$ converges weakly to a point in S.

3. Algorithm and its Convergence

In this section, we present our main algorithm and prove its weak convergence. The sets C_i and Q_j are expressed by

$$C_i=\{x\in\mathscr{H}_1 \mid c_i(x)\le 0\} \text{ and } Q_j=\{y\in\mathscr{H}_2 \mid q_j(y)\le 0\},$$

where $c_i:\mathscr{H}_1\to\mathbf{R}$ $(i=1,2,\cdots,t)$, and $q_j:\mathscr{H}_2\to\mathbf{R}$ $(j=1,2,\cdots,r)$ are convex functions. Define two sets at point x^k by

$$C_i^k=\{x\in\mathscr{H}_1 \mid c_i(x^k)+\langle\xi_i^k,x-x^k\rangle\le 0\}$$

and

$$Q_j^k=\{y\in\mathscr{H}_2 \mid q_j(Ax^k)+\langle\eta_j^k,y-Ax^k\rangle\le 0\},$$

where $\xi_i^k\in\partial c_i(x^k)$ and $\eta_j^k\in\partial q_j(Ax^k)$. We see that C_i^k, $i=1,2,\cdots,t$, and Q_j^k, $j=1,2,\cdots,r$ are half-spaces such that $C_i\subset C_i^k$ and $Q_j\subset Q_j^k$ for all $k\ge 1$. Define the function f_k by

$$f_k(x)=\frac{1}{2}\sum_{j=1}^{r}\beta_j\|(I-P_{Q_j^k})Ax\|^2,$$

where $\beta_j>0$. Then the function $f_k(x)$ is convex and differentiable with gradient $\nabla f_k(x)=\sum_{j=1}^{r}\beta_jA^*(I-P_{Q_j^k})Ax$, and the Lipschitz constant of $\nabla f_k(x)$ is $L=\|A\|^2\sum_{j=1}^{r}\beta_j$.

For the proposed algorithm, we assume that the following three assumptions hold.

(A1) The solution set S of the MSSFP (1.1) is nonempty.

(A2) The functions $c_i:\mathscr{H}_1\to\mathbf{R}$ and $q_j:\mathscr{H}_2\to\mathbf{R}$ are convex and weakly lower semicontinuous functions.

(A3) For any $x\in\mathscr{H}_1$ and $y\in\mathscr{H}_2$, at least one subgradient $\xi_i\in\partial c_i(x)$ and $\eta_j\in\partial q_j(y)$ can be calculated. The subdifferential ∂c_i and ∂q_j are bounded on the bounded sets.

Now we give our algorithm to solve the MSSFP (1.1).

Algorithm 3.1. Given constants $\gamma>0$, $l\in(0,1)$, and $\mu\in(0,1)$, choose the parameter θ_k such that

$$0\le\theta_k\le\theta<\frac{1-\mu}{1+\mu}.$$

Let x^1, x^2 be arbitrarily chosen. For $k=1,2,\cdots$, compute

$$\omega^k=\begin{cases}x^k, & k=\text{even},\\ x^k+\theta_k(x^k-x^{k-1}), & k=\text{odd}.\end{cases}$$

Compute $\bar{x}^k=P_{C_{[k]}^k}(\omega^k-\alpha_k\nabla f_k(\omega^k))$, where $[k]=k \bmod t$ and $\alpha_k=\gamma l^{m_k}$ with m_k the smallest non-negative integer such that

$$\alpha_k\|\nabla f_k(\omega^k)-\nabla f_k(\bar{x}^k)\|\le\mu\|\omega^k-\bar{x}^k\|. \tag{3.1}$$

Construct the next iterate x^{k+1} by $x^{k+1} = P_{C^k_{[k]}}(\omega^k - \alpha_k \nabla f_k(\bar{x}^k))$.

Lemma 3.1. [7] *The Armijo-line search terminates after a finite number of steps. In addition,* $\frac{\mu l}{L} \leq \alpha_k \leq \gamma$ *for all* $k \geq 1$, *where* $L = \|A\|^2 \sum_{j=1}^r \beta_j$.

The weak convergence of Algorithm 3.1 is established below.

Lemma 3.2. *Let* $\{x^k\}$ *be the sequence generated by Algorithm 3.1 and the assumptions (A1), (A2), and (A3) hold. Then* $\{x^{2k}\}$ *is Fejér monotone with respect to S, i.e.,*

$$\|x^{2k+2} - x^*\| \leq \|x^{2k} - x^*\|, \text{ where } x^* \in S.$$

Proof. Let $x^* \in S$. Note that $C \subset C_i \subset C_i^k$ and $Q \subset Q_j \subset Q_j^k$, $i = 1,2,\cdots,t$, $j = 1,2,\cdots,r$, $k = 1,2,\cdots$. Thus $x^* = P_C(x^*) = P_{C_i}(x^*) = P_{C_i^k}(x^*)$ and $Ax^* = P_Q(Ax^*) = P_{Q_j}(Ax^*) = P_{Q_j^k}(Ax^*)$. It follows that $f_k(x^*) = 0$ and $\nabla f_k(x^*) = 0$. Following Lemma 2.1 (ii), we have

$$\begin{aligned}
\|x^{2k+2} - x^*\|^2 &= \|P_{C^{2k+1}_{[2k+1]}}(\omega^{2k+1} - \alpha_{2k+1}\nabla f_{2k+1}(\bar{x}^{2k+1})) - x^*\|^2 \\
&\leq \|\omega^{2k+1} - \alpha_{2k+1}\nabla f_{2k+1}(\bar{x}^{2k+1}) - x^*\|^2 \\
&\quad - \|x^{2k+2} - \omega^{2k+1} + \alpha_{2k+1}\nabla f_{2k+1}(\bar{x}^{2k+1})\|^2 \\
&= \|\omega^{2k+1} - x^*\|^2 - 2\alpha_{2k+1}\langle \nabla f_{2k+1}(\bar{x}^{2k+1}), \omega^{2k+1} - x^*\rangle \\
&\quad - 2\alpha_{2k+1}\langle \nabla f_{2k+1}(\bar{x}^{2k+1}), x^{2k+2} - \omega^{2k+1}\rangle - \|x^{2k+2} - \omega^{2k+1}\|^2 \\
&= \|\omega^{2k+1} - x^*\|^2 - 2\alpha_{2k+1}\langle \nabla f_{2k+1}(\bar{x}^{2k+1}), x^{2k+2} - x^*\rangle - \|x^{2k+2} - \omega^{2k+1}\|^2.
\end{aligned} \tag{3.2}$$

Lemma 2.1 (iii) indicates that

$$\begin{aligned}
\|x^{2k+2} - \omega^{2k+1}\|^2 = \|x^{2k+2} - \bar{x}^{2k+1}\|^2 + \|\bar{x}^{2k+1} - \omega^{2k+1}\|^2 \\
+ 2\langle x^{2k+2} - \bar{x}^{2k+1}, \bar{x}^{2k+1} - \omega^{2k+1}\rangle.
\end{aligned} \tag{3.3}$$

Observe that

$$\begin{aligned}
&2\alpha_{2k+1}\langle \nabla f_{2k+1}(\bar{x}^{2k+1}), x^{2k+2} - x^*\rangle \\
&= 2\alpha_{2k+1}\langle \nabla f_{2k+1}(\bar{x}^{2k+1}), x^{2k+2} - \bar{x}^{2k+1}\rangle + 2\alpha_{2k+1}\langle \nabla f_{2k+1}(\bar{x}^{2k+1}), \bar{x}^{2k+1} - x^*\rangle.
\end{aligned} \tag{3.4}$$

Putting (3.3) and (3.4) into (3.2), we obtain

$$\begin{aligned}
\|x^{2k+2} - x^*\|^2 \leq \|\omega^{2k+1} - x^*\|^2 &- 2\alpha_{2k+1}\langle \nabla f_{2k+1}(\bar{x}^{2k+1}), \bar{x}^{2k+1} - x^*\rangle \\
&- 2\langle \bar{x}^{2k+1} - \omega^{2k+1} + \alpha_{2k+1}\nabla f_{2k+1}(\bar{x}^{2k+1}), x^{2k+2} - \bar{x}^{2k+1}\rangle \\
&- \|x^{2k+2} - \bar{x}^{2k+1}\|^2 - \|\bar{x}^{2k+1} - \omega^{2k+1}\|^2.
\end{aligned} \tag{3.5}$$

Since $I-P_{Q_j^{2k+1}}$ is firmly nonexpensive and $\nabla f_{2k+1}(x^*)=0$, we find from Lemma 3.1 that

$$\begin{aligned}
&2\alpha_{2k+1}\langle \nabla f_{2k+1}(\bar{x}^{2k+1}),\bar{x}^{2k+1}-x^*\rangle \\
&=2\alpha_{2k+1}\langle \sum_{j=1}^{r}\beta_j A^*(I-P_{Q_j^{2k+1}})A\bar{x}^{2k+1}-\sum_{j=1}^{r}\beta_j A^*(I-P_{Q_j^{2k+1}})Ax^*,\bar{x}^{2k+1}-x^*\rangle \\
&=2\alpha_{2k+1}\sum_{j=1}^{r}\beta_j\langle (I-P_{Q_j^{2k+1}})A\bar{x}^{2k+1}-(I-P_{Q_j^{2k+1}})Ax^*,A\bar{x}^{2k+1}-Ax^*\rangle \\
&\geq 2\frac{\mu l}{L}\sum_{j=1}^{r}\beta_j\|(I-P_{Q_j^{2k+1}})A\bar{x}^{2k+1}\|^2.
\end{aligned} \tag{3.6}$$

Based on Lemma 2.1 (i) and the definition of $\bar{x}^{2k+1}$, we see that

$$\langle \bar{x}^{2k+1}-\omega^{2k+1}+\alpha_{2k+1}\nabla f_{2k+1}(\omega^{2k+1}),x^{2k+2}-\bar{x}^{2k+1}\rangle \geq 0.$$

This together with (3.1) yields that

$$\begin{aligned}
&-2\langle \bar{x}^{2k+1}-\omega^{2k+1}+\alpha_{2k+1}\nabla f_{2k+1}(\bar{x}^{2k+1}),x^{2k+2}-\bar{x}^{2k+1}\rangle \\
&\leq 2\langle \omega^{2k+1}-\bar{x}^{2k+1}-\alpha_{2k+1}\nabla f_{2k+1}(\bar{x}^{2k+1}),x^{2k+2}-\bar{x}^{2k+1}\rangle \\
&\quad +2\langle \bar{x}^{2k+1}-\omega^{2k+1}+\alpha_{2k+1}\nabla f_{2k+1}(\omega^{2k+1}),x^{2k+2}-\bar{x}^{2k+1}\rangle \\
&=2\alpha_{2k+1}\langle \nabla f_{2k+1}(\omega^{2k+1})-\nabla f_{2k+1}(\bar{x}^{2k+1}),x^{2k+2}-\bar{x}^{2k+1}\rangle \\
&\leq 2\alpha_{2k+1}\|\nabla f_{2k+1}(\omega^{2k+1})-\nabla f_{2k+1}(\bar{x}^{2k+1})\|\|x^{2k+2}-\bar{x}^{2k+1}\| \\
&\leq \alpha_{2k+1}^2\|\nabla f_{2k+1}(\omega^{2k+1})-\nabla f_{2k+1}(\bar{x}^{2k+1})\|^2+\|x^{2k+2}-\bar{x}^{2k+1}\|^2 \\
&\leq \mu^2\|\omega^{2k+1}-\bar{x}^{2k+1}\|^2+\|x^{2k+2}-\bar{x}^{2k+1}\|^2.
\end{aligned} \tag{3.7}$$

Combining (3.5), (3.6), and (3.7), we obtain

$$\begin{aligned}
\|x^{2k+2}-x^*\|^2 \leq \|\omega^{2k+1}-x^*\|^2-(1-\mu^2)\|\bar{x}^{2k+1}-\omega^{2k+1}\|^2 \\
-2\frac{\mu l}{L}\sum_{j=1}^{r}\beta_j\|(I-P_{Q_j^{2k+1}})A\bar{x}^{2k+1}\|^2.
\end{aligned} \tag{3.8}$$

Thanks to the definition of ω^{2k+1} and Lemma 2.1 (iv), we obtain that

$$\begin{aligned}
\|\omega^{2k+1}-x^*\|^2 &= \|(1+\theta_{2k+1})(x^{2k+1}-x^*)-\theta_{2k+1}(x^{2k}-x^*)\|^2 \\
&=(1+\theta_{2k+1})\|x^{2k+1}-x^*\|^2-\theta_{2k+1}\|x^{2k}-x^*\|^2 \\
&\quad +\theta_{2k+1}(1+\theta_{2k+1})\|x^{2k+1}-x^{2k}\|^2.
\end{aligned} \tag{3.9}$$

Using the similar arguments in (3.8), one arrives at

$$\|x^{2k+1}-x^*\|^2 \leq \|x^{2k}-x^*\|^2-2\frac{\mu l}{L}\sum_{j=1}^{r}\beta_j\|(I-P_{Q_j^{2k}})A\bar{x}^{2k}\|^2-(1-\mu^2)\|\bar{x}^{2k}-x^{2k}\|^2. \tag{3.10}$$

Substituting (3.9) and (3.10) into (3.8), we obtain that

$$\begin{aligned}
&\|x^{2k+2}-x^*\|^2 \\
&\leq \|x^{2k}-x^*\|^2 - 2\frac{\mu l}{L}(1+\theta_{2k+1})\sum_{j=1}^{r}\beta_j\|(I-P_{Q_j^{2k}})A\bar{x}^{2k}\|^2 \\
&\quad -(1+\theta_{2k+1})(1-\mu^2)\|\omega^{2k}-\bar{x}^{2k}\|^2 + \theta_{2k+1}(1+\theta_{2k+1})\|x^{2k+1}-x^{2k}\|^2 \\
&\quad -(1-\mu^2)\|\bar{x}^{2k+1}-\omega^{2k+1}\|^2 - 2\frac{\mu l}{L}\sum_{j=1}^{r}\beta_j\|(I-P_{Q_j^{2k+1}})A\bar{x}^{2k+1}\|^2.
\end{aligned} \tag{3.11}$$

Observe that

$$\begin{aligned}
&\|x^{2k+1}-x^{2k}\| \\
&\leq \|x^{2k+1}-\bar{x}^{2k}\| + \|\bar{x}^{2k}-x^{2k}\| \\
&= \|P_{C_{[2k]}^{2k}}(\omega^{2k}-\alpha_{2k}\nabla f_{2k}(\bar{x}^{2k})) - P_{C_{[2k]}^{2k}}(\omega^{2k}-\alpha_{2k}\nabla f_{2k}(\omega^{2k}))\| + \|\bar{x}^{2k}-x^{2k}\| \\
&\leq \|\omega^{2k}-\alpha_{2k}\nabla f_{2k}(\bar{x}^{2k}) - \omega^{2k}+\alpha_{2k}\nabla f_{2k}(\omega^{2k})\| + \|\bar{x}^{2k}-x^{2k}\| \\
&= \alpha_{2k}\|\nabla f_{2k}(\omega^{2k})-\nabla f_{2k}(\bar{x}^{2k})\| + \|\bar{x}^{2k}-x^{2k}\| \\
&\leq \mu\|\omega^{2k}-\bar{x}^{2k}\| + \|\bar{x}^{2k}-x^{2k}\| \\
&= (1+\mu)\|\bar{x}^{2k}-\omega^{2k}\|.
\end{aligned} \tag{3.12}$$

From the choice of parameter θ_k and the assumption of μ, we conclude that

$$\begin{aligned}
&(1+\theta_{2k+1})(1-\mu^2) - \theta_{2k+1}(1+\theta_{2k+1})(1+\mu)^2 \\
&= (1+\theta_{2k+1})(1+\mu)\big[(1-\mu)-\theta_{2k+1}(1+\mu)\big] > 0.
\end{aligned}$$

This together with (3.11) and (3.12) yields that

$$\begin{aligned}
&\|x^{2k+2}-x^*\|^2 \\
&\leq \|x^{2k}-x^*\|^2 - 2\frac{\mu l}{L}\sum_{j=1}^{r}\beta_j(1+\theta_{2k+1})\|(I-P_{Q_j^{2k}})A\bar{x}^{2k}\|^2 \\
&\quad -\Big[(1+\theta_{2k+1})(1-\mu^2)-\theta_{2k+1}(1+\theta_{2k+1})(1+\mu)^2\Big]\|x^{2k}-\bar{x}^{2k}\|^2 \\
&\quad -(1-\mu^2)\|\bar{x}^{2k+1}-\omega^{2k+1}\|^2 - 2\frac{\mu l}{L}\sum_{j=1}^{r}\beta_j\|(I-P_{Q_j^{2k+1}})A\bar{x}^{2k+1}\|^2 \\
&\leq \|x^{2k}-x^*\|^2.
\end{aligned} \tag{3.13}$$

The desired result is obtained. □

Lemma 3.3. *Let assumptions (A1), (A2), and (A3) hold. Let $\{x^k\}$ be a sequence generated by Algorithm 3.1. Then* $\lim_{k\to\infty}\|(I-P_{Q_j^{2k}})A\bar{x}^{2k}\| = 0$, $\lim_{k\to\infty}\|\bar{x}^{2k}-x^{2k}\| = 0$, *and* $\lim_{k\to\infty}\|x^{2k+1}-x^{2k}\| = 0$.

Proof. Since $\{x^{2k}\}$ is Fejér monotone, one has that $\lim_{k\to\infty}\|x^{2k}-x^*\|$ exists. Letting $k\to\infty$ in (3.13), one arrives at

$$\begin{aligned}
&2\frac{\mu l}{L}\sum_{j=1}^{r}\beta_j(1+\theta_{2k+1})\|(I-P_{Q_j^{2k}})A\bar{x}^{2k}\|^2+(1-\mu^2)\|\bar{x}^{2k+1}-\omega^{2k+1}\|^2\\
&\quad+\left[(1+\theta_{2k+1})(1-\mu^2)-\theta_{2k+1}(1+\theta_{2k+1})(1+\mu)^2\right]\|x^{2k}-\bar{x}^{2k}\|^2\\
&\quad+2\frac{\mu l}{L}\sum_{j=1}^{r}\beta_j\|(I-P_{Q_j^{2k+1}})A\bar{x}^{2k+1}\|^2\\
&\le\|x^{2k}-x^*\|^2-\|x^{2k+2}-x^*\|^2\to 0.
\end{aligned}$$

Thus $\lim_{k\to\infty}\|\bar{x}^{2k+1}-\omega^{2k+1}\|=0$. From the assumptions on θ_k, we have $(1+\theta_k)\nrightarrow 0$ as $k\to\infty$. For every $j=1,2,\cdots,r$, one has $\lim_{k\to\infty}\|(I-P_{Q_j^{2k+1}})A\bar{x}^{2k+1}\|=0$ and $\lim_{k\to\infty}\|(I-P_{Q_j^{2k}})A\bar{x}^{2k}\|=0$. In view of $(1+\theta_{2k+1})(1-\mu^2)-\theta_{2k+1}(1+\theta_{2k+1})(1+\mu)^2\nrightarrow 0$, we have $\lim_{k\to\infty}\|\bar{x}^{2k}-x^{2k}\|=0$, which together with (3.12) that $\lim_{k\to\infty}\|x^{2k+1}-x^{2k}\|=0$. □

Theorem 3.1. *Let assumptions (A1), (A2), and (A3) hold. Let $\{x^k\}$ be a sequence generated by Algorithm 3.1. Then $\{x^k\}$ converges weakly to a point in S.*

Proof. From Lemma 3.2, we have that $\lim_{k\to\infty}\|x^{2k}-x^*\|$ exists. Thus $\{x^{2k}\}$ is bounded, which indicates that $\omega_w(x^{2k})$ is nonempty. Let $\hat{x}\in\omega_w(x^{2k})$. It follows that there exists a subsequence $\{x^{2k_n}\}$ of $\{x^{2k}\}$ such that $x^{2k_n}\rightharpoonup\hat{x}$.

Next, we prove that $\hat{x}$ is a solution of the MSSFP (1.1), which asserts that $\omega_w(x^{2k})\subset S$. In fact, since $x^{2k_n+1}\in C_{[2k_n]}^{2k_n}$, we conclude from the definition of $C_{[2k_n]}^{2k_n}$ that

$$c_{[2k_n]}(x^{2k_n})+\langle\xi_{[2k_n]}^{2k_n},x^{2k_n+1}-x^{2k_n}\rangle=c_{[2k_n]}(\omega^{2k_n})+\langle\xi_{[2k_n]}^{2k_n},x^{2k_n+1}-\omega^{2k_n}\rangle\le 0,\tag{3.14}$$

where $\xi_{[2k_n]}^{k_n}\in\partial c_{[2k_n]}(x^{2k_n})$. For every $i=1,2,\cdots,t$, choose a subsequence $\{k_{n_s}\}\subset\{k_n\}$ such that $[k_{n_s}]=i$. Thus (3.14) is reduced to

$$c_i(x^{2k_{n_s}})+\langle\xi_i^{2k_{n_s}},x^{2k_{n_s}+1}-x^{2k_{n_s}}\rangle\le 0.\tag{3.15}$$

Due to the assumption (A3) on the boundedness of ∂c_i, Lemma 3.3 and (3.15), there exists a constant M_1 such that

$$\begin{aligned}
c_i(x^{2k_{n_s}})=c_i(\omega^{2k_{n_s}})&\le\langle\xi_i^{2k_{n_s}},x^{2k_n}-x^{2k_{n_s}+1}\rangle\\
&\le M_1\|x^{2k_{n_s}}-x^{2k_{n_s}+1}\|\to 0
\end{aligned}\tag{3.16}$$

as $s\to\infty$. Based on the weak lower semicontinuity of the convex function c_i, we deduce from (3.16) that $c_i(\hat{x})\le\liminf_{s\to\infty}c_i(x^{2k_{n_s}})\le 0$, i.e., $\hat{x}\in C=\bigcap_{i=1}^{t}C_i$.

On the other hand, since $I-P_{Q_j^{2k_n}}$ is nonexpansive, and A is a bounded and linear operator, we obtain from Lemma 3.3 that

$$\begin{aligned}
\|(I-P_{Q_j^{2k_n}})Ax^{2k_n}\|&\le\|(I-P_{Q_j^{2k_n}})Ax^{2k_n}-(I-P_{Q_j^{2k_n}})A\bar{x}^{2k_n}\|+\|(I-P_{Q_j^{2k_n}})A\bar{x}^{2k_n}\|\\
&\le\|Ax^{2k_n}-A\bar{x}^{2k_n}\|+\|(I-P_{Q_j^{2k_n}})A\bar{x}^{2k_n}\|\\
&\le\|A\|\|x^{2k_n}-\bar{x}^{2k_n}\|+\|(I-P_{Q_j^{2k_n}})A\bar{x}^{2k_n}\|\to 0
\end{aligned}\tag{3.17}$$

as $n \to \infty$. Since $P_{Q_j^{2k_n}}(Ax^{2k_n}) \in Q_j^{2k_n}$, we have

$$q_j(Ax^{2k_n}) + \langle \eta_j^{2k_n}, P_{Q_j^{2k_n}}(Ax^{2k_n}) - Ax^{2k_n} \rangle \leq 0, \tag{3.18}$$

where $\eta_j^{2k_n} \in \partial q_j(Ax^{2k_n})$. From (A3), (3.17), and (3.18), there exists a constant M_2 such that

$$q_j(Ax^{2k_n}) \leq \|\eta_j^{2k_n}\| \|(I - P_{Q_j^{2k_n}})Ax^{2k_n}\| \leq M_2 \|(I - P_{Q_j^{2k_n}})Ax^{2k_n}\| \to 0$$

as $n \to \infty$. Thus $q_j(A\hat{x}) \leq \liminf_{n\to\infty} q_j(Ax^{2k_n}) \leq 0$, which means that $A\hat{x} \in Q = \bigcap_{j=1}^r Q_j$. Hence, $\hat{x} \in S$, and $\{x^{2k}\}$ is weakly convergent to a solution in S due to Lemma 2.2.

Next, we prove that the sequence of odd terms $\{x^{2k+1}\}$ also converges weakly to x^*. In fact, for all $z \in \mathscr{H}$, Lemma 3.3 guarantees that

$$\begin{aligned} | \langle x^{2k+1} - x^*, z \rangle | &\leq | \langle x^{2k} - x^*, z \rangle | + | \langle x^{2k+1} - x^{2k}, z \rangle | \\ &\leq | \langle x^{2k} - x^*, z \rangle | + \|x^{2k+1} - x^{2k}\| \|z\| \to 0, \ k \to \infty. \end{aligned}$$

Thus $x^{2k+1} \rightharpoonup x^* \in S$. Therefore, we conclude that $\{x^k\}$ converges weakly to a solution of the MSSFP (1.1). □

4. Numerical Experiments

In this section, we present several examples and compare Algorithm 3.1, Yang's algorithm in [3] (fixed step size) and Chen et al.'s algorithm in [21]. Chen et al.'s algorithm is an Armijo-line search relaxed CQ algorithm without inertial, that is,

$$\begin{cases} \bar{x}^k = P_{C_{[k]}^k}(x^k - \alpha_k \nabla f_k(x^k)), \\ x^{k+1} = P_{C_{[k]}^k}(x^k - \alpha_k \nabla f_k(\bar{x}^k)), \end{cases}$$

where $\nabla f_k(x^k) = \sum_{j=1}^r \beta_j A^*(I - P_{Q_j^k})Ax^k$, $\alpha_k = \gamma l^{m_k}$ with m_k the smallest non-negative integer such that $\alpha_k \|\nabla f_k(x^k) - \nabla f_k(\bar{x}^k)\| \leq \mu \|x^k - \bar{x}^k\|$, $\gamma > 0$, $l \in (0,1)$, and $\mu \in (0,1)$.

Numerical results show that imposing alternated inertial terms can accelerate the convergence rate of the iterative sequence. The codes are written in Matlab 2016a and run on Inter(R) Core(TM) i7-8550U CPU @ 1.80GHz 2.00GHz, RAM 8.00GB.

Example 4.1. Consider the following LASSO problem [22]

$$\min \left\{ \frac{1}{2} \|Ax - b\|_2^2 \mid x \in \mathbf{R}^n, \|x\|_1 \leq \varepsilon \right\},$$

where $A \in \mathbf{R}^{m\times n}$, $m < n$, $b \in \mathbf{R}^m$, and $\varepsilon > 0$. The matrix A is generated from a standard normal distribution with mean zero and unit variance. The true sparse signal x^* is generated from uniformly distribution in the interval $[-2,2]$ with random p position nonzero, while the rest is kept zero. The sample data $b = Ax^*$. For the considered MSSFP, let $r = t = 1$, $C = \{x \mid \|x\|_1 \leq \varepsilon\}$, and $Q = \{b\}$. The objective function is defined as $f(x) = \dfrac{1}{2}\|Ax - b\|_2^2$. Let $\alpha_k = \gamma l^{m_k}$, $\gamma = 1$, $l = \frac{1}{2}$, $\mu = \frac{1}{2}$, $\theta_k = \frac{1}{4}$, and $x^1 = x^2$. Take $\|x^k - x^*\| < 10^{-4}$ as the stopping criterion. We report the final error between the reconstructed signal and the true signal.

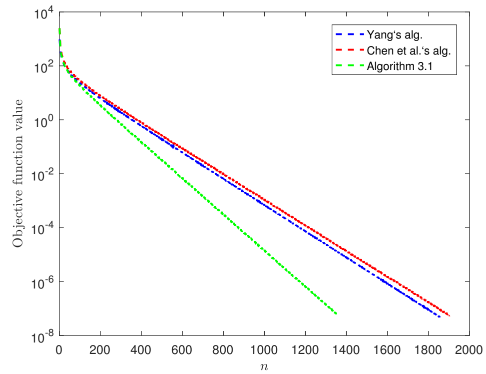


FIGURE 1. The objective function value versus the iteration number.

We compare Algorithm 3.1 with Chen et al.'s algorithm in [21], and Yang's algorithm in [3]. Let $\alpha_k = \dfrac{1}{\|A\|^2}$ in Yang's algorithm [3]. The results are reported in Table 1. Figure 1 shows the objective function values versus iteration number when $m = 240$, $n = 1024$, and $p = 30$.

TABLE 1. Comparison of Algorithm 3.1, Chen et al.'s algorithm, and Yang's algorithm.

m	n	p		Algorithm 3.1	Chen et al.'s alg.	Yang's alg.
120	512	15	No. of Iter	2618	3662	3597
			cpu(time)	5.2134	6.9993	1.0764
240	1024	30	No. of Iter	1353	1903	1859
			cpu(time)	6.1794	8.5984	1.4351
480	2048	60	No. of Iter	1991	2817	2782
			cpu(time)	36.0100	51.0228	8.5558
720	3072	90	No. of Iter	2133	2949	2714
			cpu(time)	118.0682	148.5158	17.6724

From Table 1 and Figure 1, we know that the alternated inertial term can improve the convergence of the algorithm in both CPU time and number of the iterates. However, in most cases, the relaxed CQ algorithm with fixed step size has more advantages than Armijo-line search because Yang's algorithm uses less CPU time.

We also measure the restoration accuracy by means of the mean squared error, i.e., MSE$=(1/k)\|x^* - x^k\|$. Figure 2 shows a comparison of the accuracy of the recovered signals when $m = 1440$, $n = 6144$, and $p = 180$. Given the same number of iterations, the recovered signals generated by Algorithm 3.1 in this paper outperform the ones generated by Yang's algorithm and Chen et al.'s algorithm in restoration accuracy. Thus, imposing alternated inertial terms accelerates the convergence rate and accuracy of signal recovery. However, Yang's algorithm obviously has more advantages in CPU time than the Armijo-line search relaxed CQ algorithm.

Example 4.2. [23] Take $\mathscr{H}_1 = \mathscr{H}_2 = \mathbf{R}^3$, $r = t = 2$, $\beta_1 = \beta_2 = \frac{1}{2}$, and $\alpha_k = \gamma l^{m_k}$ for all $k \geq 1$, where $\gamma = 1$, $l = \frac{1}{2}$, $\mu = \frac{1}{2}$. Define

$$C_1 \quad = \quad \left\{ x = (x_1, x_2, x_3)^T \in \mathbf{R^3} \mid x_1 + x_2^2 + 2x_3 \leq 0 \right\},$$

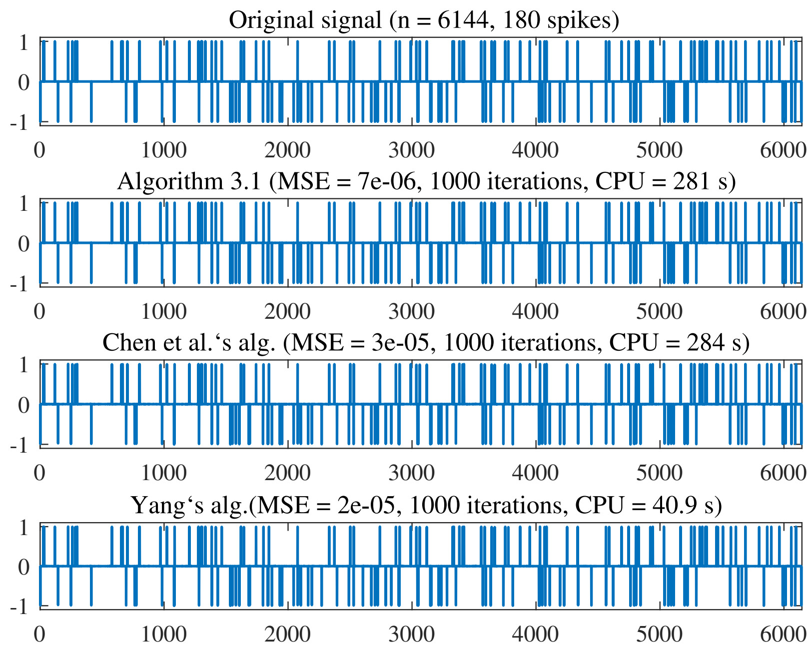


FIGURE 2. Comparison of signal processing.

$$C_2 = \left\{x=(x_1,x_2,x_3)^T \in \mathbf{R^3} \mid \frac{x_1^2}{16}+\frac{x_2^2}{9}+\frac{x_3^2}{4}-1 \le 0\right\},$$

$$Q_1 = \left\{x=(x_1,x_2,x_3)^T \in \mathbf{R^3} \mid x_1^2+x_2-x_3 \le 0\right\},$$

$$Q_2 = \left\{x=(x_1,x_2,x_3)^T \in \mathbf{R^3} \mid \frac{x_1^2}{4}+\frac{x_2^2}{4}+\frac{x_3^2}{9}-1 \le 0\right\},$$

and

$$A=\begin{pmatrix}2 & -1 & 3\\ 4 & 2 & 5\\ 2 & 0 & 2\end{pmatrix}.$$

The underlying MSSFP is to find $x^* \in C_1 \bigcap C_2$ such that $Ax^* \in Q_1 \bigcap Q_2$. We use $E_k = \|x^{k+1}-x^k\|/\|x^k\|$ to measure the error of the k-th step iteration. If $E_k < 10^{-5}$, then the iteration process stops. Let $x^1 = x^2$.

We study the effect of the sequence $\{\theta_k\}$ on the iterative scheme by choosing different θ_k such that $0 \le \theta_k \le \theta < \frac{1-\mu}{1+\mu}$. We take $\theta_k = \frac{1}{4}$, $\theta_k = \frac{1}{5}$, $\theta_k = \frac{1}{4k}$, and $\theta_k = \frac{1}{4^k}$, and compare the CPU time and numbers of iterates; see Table 2. The asymptotic behavior of the error E_k for each choice of x^1 is shown in Figure 3.

The choice of initial value is as follows

Choice 1: $x^1 = (1,2,3)^T$;

Choice 2: $x^1 = (5,1,9)^T$;

TABLE 2. Comparison of Different Choices of θ_k.

		$\theta_k=\frac{1}{4}$	$\theta_k=\frac{1}{5}$	$\theta_k=\frac{1}{4k}$	$\theta_k=\frac{1}{4^k}$
Choice 1	No. of Iter	107	113	141	142
	cpu(time)	0.0636	0.0544	0.0641	0.0625
Choice 2	No. of Iter	107	116	150	151
	cpu(time)	0.0563	0.0565	0.0615	0.0633
Choice 3	No. of Iter	17	18	20	20
	cpu(time)	0.0372	0.0352	0.0358	0.0384
Choice 4	No. of Iter	110	17	148	157
	cpu(time)	0.0560	0.0357	0.0618	0.0676
Choice 5	No. of Iter	115	121	149	147
	cpu(time)	0.0598	0.0551	0.0593	0.0608
Choice 6	No. of Iter	14	14	15	16
	cpu(time)	0.0353	0.0283	0.0330	0.0354

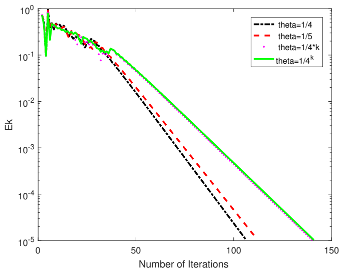
Choice 3: $x^1=(0.1,-2,-1)^T$;
Choice 4: $x^1=(-1,-1,3)^T$;
Choice 5: $x^1=(0.2785,0.547,0.9575)^T$;
Choice 6: $x^1=(0,0.06,1.005)^T$.

From Table 2, we see that for most cases the iterates need less CPU time if $\theta_k\to 0$, but the rate of the convergence may not be too fast.

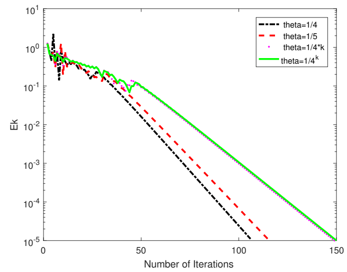
We also compare the iteration numbers and the CPU time of Algorithm 3.1 and the algorithm without alternated inertial term of Chen et al. [21], where $\theta_k=\frac{1}{4}$; see Table 3. The convergence behavior of the error E_k for each choice of x^1 is shown in Figure 4. It is convinced that the algorithm with alternated inertial term is more efficient in that the iteration number and CPU time can both be improved.

TABLE 3. Comparison of Algorithm 3.1 and Chen et al.'s algorithm.

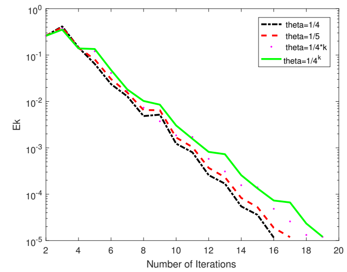
		Algorithm 3.1	Chen et al.'s alg.
Choice 1 $x^1=(0.05,0.01,0.02)^T$	No. of Iter	42	47
	cpu(time)	0.0456	0.0507
Choice 2 $x^1=(-7,-1,0)^T$	No. of Iter	128	167
	cpu(time)	0.0672	0.0743
Choice 3 $x^1=(-0.4,0.555,0.888)^T$	No. of Iter	103	139
	cpu(time)	0.0545	0.0647
Choice 4 $x^1=(-5,-10,6)^T$	No. of Iter	151	172
	cpu(time)	0.0619	0.0744
Choice 5 $x^1=(-24,-42,-10)^T$	No. of Iter	78	196
	cpu(time)	0.0527	0.0755
Choice 6 $x^1=(0.1,0.1,0.1)^T$	No. of Iter	54	60
	cpu(time)	0.0486	0.0510



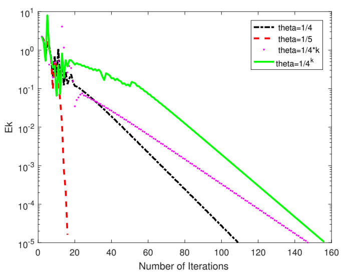
(a) Comparison of Choice 1



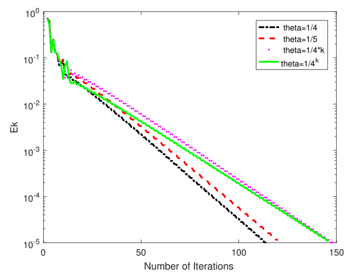
(b) Comparison of Choice 2



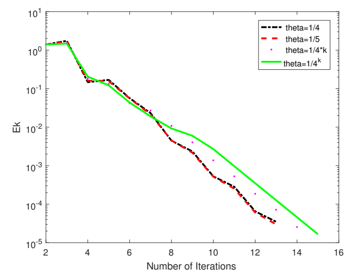
(c) Comparison of Choice 3



(d) Comparison of Choice 4

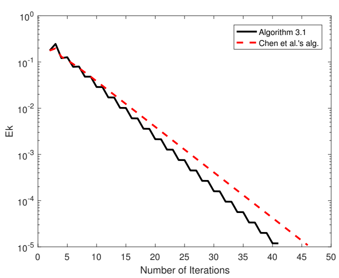


(e) Comparison of Choice 5

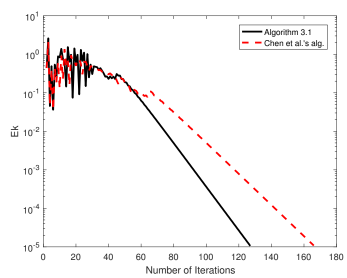


(f) Comparison of Choice 6

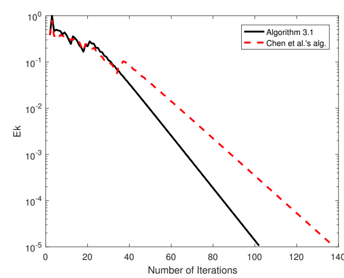
FIGURE 3. Comparison of different choices of θ_k.



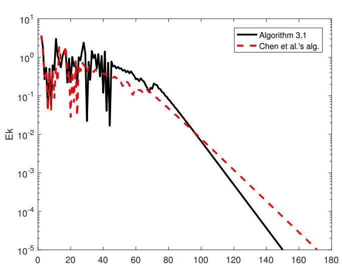
(a) Comparison of Choice 1



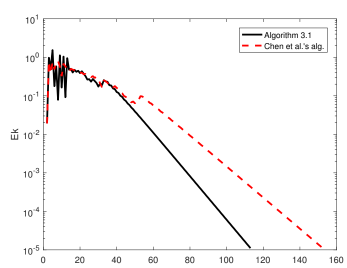
(b) Comparison of Choice 2



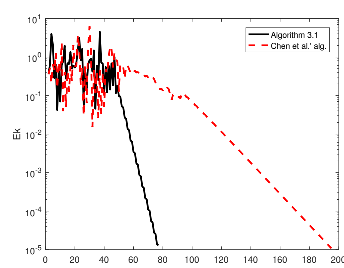
(c) Comparison of Choice 3



(d) Comparison of Choice 4



(e) Comparison of Choice 5



(f) Comparison of Choice 6

FIGURE 4. Comparison of Algorithm 3.1 and Chen et al.'s algorithm.

Example 4.3. ([24]) Take $\mathscr{H}_1 = \mathbf{R}^n$, $\mathscr{H}_2 = \mathbf{R}^m$, $A = (a_{ij})_{m\times n}$ with $a_{ij} \in (0,1)$ generated randomly, $C_i = \{x \in \mathbf{R}^n \mid \|x - d_i\|_2^2 \le r_i^2\}$, $i = 1,2,\cdots,t$, $Q_j = \{y \in \mathbf{R}^m \mid \frac{1}{2}y^T B_j y + b_j y + c_j \le 0\}$, $j = 1,2,\cdots,r$, where $d_i \in (6\mathbf{e}_0, 16\mathbf{e}_1)$, $r_i \in (100,120)$, $b_j \in (-30\mathbf{e}_1, -20\mathbf{e}_1)$, $c_j \in (-60,-50)$, and all elements of the matrix B_j are all generated randomly in the interval (2,10). Set $\beta_1 = \beta_2 = \cdots = \beta_r = \frac{1}{r}$ and $\alpha_k = \gamma l^{m_k}$ for all $k \ge 1$, $\gamma = 1$, $l = \frac{1}{2}$, $\mu = \frac{1}{2}$.

We compare Algorithm 3.1 and the algorithm without alternated inertial term of Chen et al. [21], where $\theta_k = \frac{1}{4}$. Let $x^1 = x^2$. The stopping criterion is defined by

$$E_k = \frac{1}{2}\sum_{i=1}^{t} \|x^k - P_{C_i^k}x^k\|^2 + \frac{1}{2}\sum_{j=1}^{r} \|Ax^k - P_{Q_j^k}Ax^k\|^2 < 10^{-4}.$$

We arbitrarily choose three different initial points, and consider iterative steps of the two algorithms with m, n, r, and t being different values. The details are shown in Table 4. From Table 4, we can see that the convergence rate is also improved by the alternated inertial terms.

TABLE 4. Comparison of Algorithm 3.1 and Chen et al.'s algorithm.

		Algorithm 3.1	Chen et al.'s alg.
$r = t = 10, m = n = 20$			
Choice 1 $x^1 = e_1$	No. of Iter	423	477
	cpu(time)	1.1629	2.1611
Choice 1 $x^1 = 0.1e_1$	No. of Iter	212	239
	cpu(time)	0.5338	0.5845
Choice 3 $x^1 = 10rand$	No. of Iter	567	638
	cpu(time)	1.2325	1.3924
$r = t = 10, m = n = 80$			
Choice 1 $x^1 = e_1$	No. of Iter	1950	2194
	cpu(time)	17.7406	20.5005
Choice 2 $x^1 = 0.1e_1$	No. of Iter	1271	1429
	cpu(time)	10.0991	10.7911
Choice 3 $x^1 = 10rand$	No. of Iter	2120	2386
	cpu(time)	15.8109	18.3324
$r = t = 40, m = n = 60$			
Choice 1 $x^1 = e_1$	No. of Iter	1550	1744
	cpu(time)	40.5580	47.8809
Choice 2 $x^1 = 0.1e_1$	No. of Iter	1012	1139
	cpu(time)	26.8444	29.0546
Choice 3 $x^1 = 10rand$	No. of Iter	1978	2226
	cpu(time)	45.6827	53.2812

5. CONCLUSION

In this paper, we proposed an Armijo-line search relaxed CQ algorithm with alternated inertial terms, and proved the weak convergence of the algorithm in Hilbert spaces. In numerical examples, the convergence rate and CPU time of the Armijo-line search relaxed CQ algorithm with and without alternated inertial were compared, and Algorithm 3.1 was compared with classical relaxed CQ algorithm in the LASSO problem. Numerical results show that the relaxed CQ algorithm with the fixed step size outperforms the one with Armijo-line search in CPU time, while imposing alternated inertial terms can accelerate the convergence rate of the iterative sequence.

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