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APPROXIMATION OF SOLUTIONS OF THE SPLIT MINIMIZATION PROBLEM WITH MULTIPLE OUTPUT SETS AND COMMON FIXED POINT PROBLEMS IN REAL BANACH SPACES

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Abstract. In this paper, we introduce and study a split minimization problem with multiple output sets. We propose a new iterative method, which employs the inertial Halpern approximation technique, for a common solution of the split minimization problem and the fixed point problem with a finite family of Bregman relatively nonexpansive mappings in the framework of p-uniformly convex and uniformly smooth Banach spaces. Our iterative method uses the step sizes which do not require prior knowledge of the operators norm, and we prove a strong convergence result under some mild conditions. Moreover, we present some applications of our result and further demonstrate the efficiency and applicability of our algorithm with some numerical examples. The results presented in this paper unify and complement several existing results in the literature.

Keywords. Bregman relatively nonexpansive mapping; Inertial method; Fixed point problem; Split minimization problem; Resolvent operators.

1. INTRODUCTION

Let C and Q be nonempty, closed, and convex subsets of two real Banach spaces E_1 and E_2 with duals E_1^* and E_2^*, respectively. Let $T : E_1 \to E_2$ be a bounded linear operator. The Split Feasibility Problem (in short, SFP) considered by Censor and Elving [1] is defined as follows:

$$\text{Find } x^* \in C \text{ such that } Tx^* \in Q. \tag{1.1}$$

The beauty of SFP is made manifest in many fields such as phase retrieval, medical image reconstruction, radiation therapy treatment planning, signal processing, and so on; see, e.g., [1, 2]. Note that the SFP (1.1) can be formulated as a fixed point equation of the form:

$$P_C\big(I - \tau T^*(I - P_Q)T\big)x^* = x^*; \tag{1.2}$$

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where $\tau \in \left(0, \dfrac{2}{L}\right)$ with L being the spectral radius of the operator T^*T, T^* is the adjoint of T, and P_C and P_Q are metric projections on C and Q, respectively. That is, x^* is a solution of the SFP (1.1) if and only if x^* is a solution of fixed point equation (1.2).

In 2003, Byrne [3], in finite-dimensional Euclidean space $\mathbb{R}^n$, proposed a CQ algorithm , which is found to be a gradient projection method (GPM) in convex minimization for solving (1.1):

$$x_{n+1} = P_C(I - \tau T^*(I - P_Q)T)x_n, \; n \geq 1, \tag{1.3}$$

where $\tau \in \left(0, \frac{2}{L}\right)$ with L being the spectral radius of the operator T^*T, T^* is the adjoint of T, and P_C and P_Q are metric projections on C and Q, respectively. Byrne [3] proved that the sequence generated by (1.3) converges weakly to a solution of the SFP (1.1). Schöpfer et al. [4] in 2008 studied (1.1) in the framework of p-uniformly convex and uniformly smooth real Banach spaces. They proposed the following iterative scheme. For $x_1 \in E_1$, let

$$x_{n+1} = \Pi_C J^q_{E_1^*}\left[J^p_{E_1}(x_n) - \tau_n T^* J^p_{E_2}(Tx_n - P_Q(Tx_n))\right], \; n \geq 1, \tag{1.4}$$

where Π_C denotes the Bregman projection from E_1 onto C, and J^p_E is the duality mapping. Note that algorithm (1.4) generalizes algorithm (1.3).

Several optimization problems, such as Split Variational Inequality Problem (SVIP), Split Variational Inclusion Problem (SVIP), Split Minimization Problem (SMP), Split Equilibrium Problem (SEP), have been defined in terms of SFP (1.1); see, e.g., [5, 6, 7, 8] and the references therein. One of the most important problems in the study of optimization theory and nonlinear analysis is the problem of approximating solutions of Convex Minimization Problem (CMP) ,which is defined as follows: Find a point $x \in H$ such that

$$f(x) = \min_{y \in H} f(y), \tag{1.5}$$

where $f : H \to (-\infty, +\infty]$ is a proper, convex, and lower semicontinuous function, and H is a real Hilbert space. Recall that a mapping f is said to be convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, $\forall x, y \in H$, $\lambda \in (0,1)$. f is said to be proper if $D(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset$, where $D(f)$ denotes the domain of f. The mapping $f : D(f) \to (-\infty, +\infty]$ is said to be lower semicontinuous at a point $x \in D(f)$ if $f(x) \leq \liminf_{n\to\infty} f(x_n)$, for each sequence $\{x_n\} \in D(f)$ such that $\lim_{n\to\infty}(x_n) = x$. f is said to be lower semicontinuous on $D(f)$ if it is lower semicontinuous at each point in $D(f)$. A typical example of a convex and lower semicontinuous function is the indicator function given by $\delta_C : H \to \mathbb{R}$ of a nonempty, closed, and convex subset C of H defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

In 1970, Martinet [9] introduced the concept of Proximal Point Algorithm (for short, PPA) which is a powerful and most popular tool for solving solutions of CMP (1.5). Now, for any $\lambda > 0$, the resolvent (or Moreau-Yosida approximation) of f in H is defined as (see [10])

$$J^f_\lambda(x) = \mathrm{Prox}_\lambda f(x) = \arg\min_{y \in H}\left[f(y) + \frac{1}{2\lambda}\|y - x\|^2\right], \; \forall\, x \in H,$$

where $\arg\min f := \{\bar{x} \in H : f(\bar{x}) \leq f(x) \text{ for all } x \in H\}$. It is well known that J_λ^f is well-defined and firmly nonexpansive for all $\lambda > 0$ (see [10]). Hence, J_λ^f is nonexpansive for all $\lambda > 0$. Furthermore, we denote the solution set of problem (1.5) by $\arg\min f$. It is also known that $Fix(J_\lambda^f)$ coincides with $\arg\min f$.

Recently, Moudafi and Thakur [11] studied the following CMP $\min\{g(x) + f_\lambda(Tx) : x \in H_1\}$, where $g : H_1 \to (-\infty, +\infty]$ is a proper, convex, and lower semicontinuous function, and $f_\lambda(y) := \min_{u \in H_2}\{f(u) + \frac{1}{2\lambda}\|u - y\|^2\}$ is the Moreau-Yosida approximate, and $T : H_1 \to H_2$ is a bounded linear operator.

Let C and Q be nonempty, closed, and convex subsets of real Banach spaces E_1 and E_2, respectively. Let $g : E_1 \to (-\infty, +\infty]$ and $f : E_2 \to (-\infty, +\infty]$ be two proper and lower semicontinuous convex functions. Let $T : H_1 \to H_2$ be a bounded linear operator. The Split Convex Minimization Problem (for short, SCMP) is to find

$$x^* \in C \text{ such that } x^* = \arg\min_{x \in C} g(x), \tag{1.6}$$

and

$$\text{the point } y^* = Tx^* \in Q \text{ solves } y^* = \arg\min_{y \in Q} f(y). \tag{1.7}$$

We denote the solution set of problem (1.6)-(1.7) by Υ.

In 2020, Reich and Tuyen [12] introduced the following generalized split common null point problem (for short, GSCNPP): For $i = 1, 2, \cdots, N$, let H_i be a real Hilbert space, and let $B_i : H_i \to 2^{H_i}$ be a maximal monotone operator. Let $T_i : H_i \to H_{i+1}$ be a bounded linear operator for $i = 1, 2, \cdots, N-1$ such that $T_i \neq 0$. The GSCNPP is defined to find $x^* \in H_1$ such that

$$0 \in B_1(x^*),\ 0 \in B_2(T_1(x^*)), \cdots, 0 \in B_N(T_{N-1}T_{N-2}, \cdots T_1(x^*)). \tag{1.8}$$

Very recently, Reich and Tuyen [13] introduced and studied the Split Common Null Point Problem with Multiple output sets (SCNPPWMOS) in real Hilbert spaces as follows: Let $H, H_1, ..., H_N$ be real Hilbert spaces, and let $T_i : H \to H_i,\ i = 1, 2, ...N$ be bounded linear operators. Let $B : H \to 2^H$, $B_i : H_i \to 2^{H_i}$, $i = 1, 2, ..., N$ be maximal monotone. The SCNPPWMOS is to find an element x^* such that

$$x^* \in B^{-1}(0) \cap \Big(\bigcap_{i=1}^{N} T_i^{-1}(B_i^{-1}(0))\Big) \neq \emptyset. \tag{1.9}$$

Reich and Tuyen [13] proposed two algorithms for the solutions of the SCNPPWMOS. Moreover, they established the relationship between (1.8) and (1.9), and also proved two strong convergence theorems for the proposed algorithms (SCNPPWMOS).

In this study, in the framework of p-uniformly convex and uniformly smooth real Banach spaces, we introduce and study the Split Convex Minimization Problem with Multiple Output Sets (SCMPWMOS) as follows: Let E, E_i, for $i = 1, 2, \cdots, N$, be real Banach spaces, and let $T_i : E \to E_i$ be bounded linear operators. Let $f : E \to (-\infty, +\infty]$, $f_i : E_i \to (-\infty, +\infty]$, for $i = 1, 2, \cdots, N$, be proper convex and lower semicontinuous function. Then the SCMPWMOS is formulated as finding a point x^* such that

$$x^* \in \arg\min f \cap \Big(\bigcap_{i=1}^{N} T_i^{-1}(\arg\min f_i)\Big) \neq \emptyset. \tag{1.10}$$

We denote by Ω the solution set of (1.10). On the other hand, Fixed Point Problem (shortly FPP) finds wide applications in the real world and fixed point methods are powerful in deal with various optimization; see, e.g., [14, 15, 16, 17, 18]. We denote by $Fix(S)$ the fixed points set of S, that is, $Fix(S) := \{x^* \in C : x^* = Sx^*\}$; where $S : C \to C$ is a nonlinear mapping.

In this paper, we consider the problem of finding a common solution of the SCMPWMOS (1.10) and the fixed point problem of a finite family of Bregman relatively nonexpansive mappings $S_j,\ j = 1,2,...m$, in the framework of p-uniformly convex Banach spaces which are also uniformly smooth. That is, find an element x^* such that

$$x^* \in \bigcap_{j=1}^{m} \mathrm{Fix}(S_j) \cap \arg\min f \cap \left(\bigcap_{i=1}^{N} T_i^{-1}\left(\arg\min f_i\right) \right), \tag{1.11}$$

We denote by Γ the solution set of problem (1.11). Inertial methods attracted great interest due to its nice convergence characteristics as well as the performance of algorithms. The inertial methods are based upon a discrete analogue of a second order dissipative dynamical system. They were first considered in [19] for solving the smooth convex minimization problems. They were later made famous by Nesterov's acceleration convex minimization problem [20], and further developed by Beck and Teboulle in the case of structured convex minimization problem [21]. For recent extensions, we refer to [22, 23].

Motivated by the results presented in [11, 12, 13] and the ongoing research in this direction, we propose a self-adaptive inertial Halpern iterative method for finding the solution of split minimization problem with multiple output sets, which is also a common fixed point of a finite family of Bregman relatively nonexpansive mappings in the framework of p-uniformly convex and uniformly smooth Banach spaces. Furthermore, we obtain a strong convergence result of the problem and apply our result to equilibrium problems and zero point problems. The result obtained in this paper mainly extends and generalizes the results presented in [4, 11, 12, 13].

Below are some highlights of this paper to our knowledge.

(1) We study the problem of finding the solution of problem (1.11) in a p-uniformly convex Banach spaces, which is more general than the result presented in [11, 12, 13], which were obtained in Hilbert spaces.
(2) Our method uses self-adaptive step sizes. Thus, the implementation of our method does not require the prior knowledge of the norm of the bounded linear operators T_i, $i = 1,2,\ldots,N$.
(3) The sequences generated by our proposed method converges strongly to the solution of problem (1.11) in p-uniformly convex and uniformly smooth Banach spaces, which is more desirable than the weak convergence result obtained in [11]. Moreover, the strong convergence analysis of our method does not rely on the usual "Two-cases Approach" widely used in many papers to guarantee the strong convergence (see, e.g., [13]).

Finally, we emphasize that the problem of finding a common solution of the SMP and the FPP has some possible applications to mathematical models whose constraints can be expressed as the SMP and the FPP. An instance of this can be found in practical problems like signal processing, network resource allocation, and image recovery; see, e.g., [23].

2. PRELIMINARIES

We state some known and useful results, which are needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. Let $K(E) := \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. The modulus of convexity is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\big\{1 - \frac{\|x+y\|}{2} : x,y \in K(E),\ \|x-y\| \geq \varepsilon\big\}.$$

The space E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. Let $p > 1$. Then E is said to be p-uniformly convex (or to have a modulus of convexity of power type p) if there exists $c_p > 0$ such that $\delta_E(\varepsilon) \geq c_p\varepsilon^p$ for all $\varepsilon \in (0,2]$. Note that every p-uniformly convex space is uniformly convex. The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ := [0,\infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \sup\big\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : x,y \in K(E)\big\}.$$

The space E is said to be uniformly smooth if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. Let $q > 1$. Then a Banach space E is said to be q-uniformly smooth if there exists $\kappa_q > 0$ such that $\rho_E(\tau) \leq \kappa_q\tau^q$ for all $\tau > 0$. It is known that a Banach space E is p-uniformly convex if and only if E^* is q-uniformly smooth, where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$; see, e.g., [24]. Let $p > 1$ be a real number. The generalized duality mapping $J_E^p : E \to 2^{E^*}$ is defined by

$$J_E^p(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^p, \|x^*\| = \|x\|^{p-1}\},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between elements of E and E^*. In particular, $J_E^p = J_E^2$ is called the normalized duality mapping. If E is p-uniformly convex and uniformly smooth, then E^* is q-uniformly smooth and uniformly convex. In this case, the generalized duality mapping J_E^p is one-to-one, single-valued, and satisfies $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ is the generalized duality mapping of E^*. Furthermore, if E is uniformly smooth, then J_E^p is norm-to-norm uniformly continuous on bounded subsets of E; see [25] for more details.

Let $g : E \to (-\infty,+\infty]$ be a proper, lower semicontinuous, and convex function. Then the Frenchel conjugate of g denoted as $g^* : E^* \to (-\infty,+\infty]$ is defined as

$$g^*(x^*) = \sup\{\langle x, x^*\rangle - g(x) : x \in E,\ x^* \in E^*\}.$$

Let the domain of g be denoted by $\mathrm{dom}g = \{x \in E : g(x) < +\infty\}$. Then, for any $x \in int(domg)$ and $y \in E$, define the right-hand derivative of g at x in the direction y by

$$g^0(x,y) = \lim_{t\to 0^+} \frac{g(x+ty) - g(x)}{t}.$$

The function g is said to be Gâteaux differentiable at x if $\lim_{t\to 0^+} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case, $g^0(x,y)$ coincides with $\bigtriangledown g(x)$ (the value of the gradient $\bigtriangledown g$ of g at x). The function g is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in int(domg)$. The function g is said to be Fréchet differentiable at x if its limit is attained uniformly in $\|y\| = 1$. In conclusion, g is said to be uniformly Fréchet differentiable on a subset C of E if the above limit is attained uniformly for $x \in C$ and $\|y\| = 1$. A function g is said to be Legendre if it satisfies the following conditions:

(1) The interior of the domain of g, $int(domg)$, is nonempty, g is Gâteaux differentiable on $int(domg)$, and $dom \bigtriangledown g = int(domg)$.
(2) The interior of the domain of g^*, $int(domg^*)$, is nonempty, g^* is Gâteaux differentiable on $int(domg^*)$, and $dom \bigtriangledown g^* = int(domg)$.

Definition 2.1. [26] Let $g : E \to (-\infty,+\infty]$ be a convex and Gâteaux differentiable function. The function $\Delta_g : domg \times int(domg) \to [0,+\infty)$ defined by $\Delta_g(x,y) := g(y) - g(x) - \langle \bigtriangledown g(x), y - x\rangle$, $\forall x \in domg$, $y \in int(domg)$ is called the Bregman distance with respect to g.

It is worth mentioning that the Bregman distance possesses the following interesting properties (see, e.g., [27, 28, 29]):

(i) $\Delta_g(x,x) = 0$, but $\Delta_g(x,y) = 0$ does not necessarily imply that $x = y$;

(ii) for $x \in domg$ and $y,z \in int(domg)$,
$$\Delta_g(x,y) + \Delta_g(y,z) - \Delta_g(x,z) \le \langle \bigtriangledown g(z) - \bigtriangledown g(y), x - y\rangle;$$

(iii) for each $z \in E$, $\{x_i\}_{i=1}^N$ and $\{\alpha_n\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N \alpha_i = 1$,
$$\Delta_g\left(z, \nabla g^*\left(\sum_{i=1}^N \alpha_i \nabla g(x_i)\right)\right) \le \sum_{i=1}^N \alpha_i \Delta_g(z, x_i).$$

However, it is known that the Bregman distance Δ_g does not satisfy the properties of a metric because Δ_g fails to satisfy the symmetric and triangle inequality properties. Moreover, it is also known that the duality mapping J_p^E is the sub-differential of the functional $g_p(.) = \frac{1}{p}\|.\|^p$ for $p > 1$; see [30]. The Bregman distance Δ_p is defined with respect to g_p as follows:
$$\begin{aligned}\Delta_p(x,y) &= \frac{1}{p}\|y\|^p - \frac{1}{p}\|x\|^p - \langle J_E^p x, y - x\rangle \\ &= \frac{1}{q}\|x\|^p - \langle J_E^p x, y\rangle + \frac{1}{p}\|y\|^p \\ &= \frac{1}{q}\|x\|^p - \frac{1}{q}\|y\|^p - \langle J_E^p x - J_E^p y, y\rangle.\end{aligned}$$

Definition 2.2. Let $T : C \to int(domg)$ be a mapping.

(i) A point $x^* \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x_n\}$, which converges weakly to x^* and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. We denote by $\hat{Fix}(T)$ the set of asymptotic fixed points of T;

(ii) T is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for each $x, y, \in C$,

(iii) T is said to be quasi-nonexpansive if $F(T) \ne \emptyset$ and $\|Tx - Ty^*\| \le \|x - y^*\|$ for each $x \in C$ and $y^* \in F(T)$;

(iv) T is said to be Bregman firmly nonexpansive if, for all $x, y \in C$,
$$\Delta_p(Tx,Ty) + \Delta_p(Ty,Tx) + \Delta_p(Tx,x) + \Delta_p(Ty,y) \le \Delta_p(Tx,y) + \Delta_p(Ty,x);$$

(v) T is said to be Bregman nonexpansive if
$$\Delta_p(Tx,Ty) \le \Delta_p(x,y) \ \forall \, x,y \in C;$$

(vi) T is said to be Bregman quasi-nonexpansive if
$$Fix(T) \ne \emptyset \text{ and } \Delta_p(Tx,y^*) \le \Delta_p(x,y^*), \ \forall \, x \in C, \ y^* \in Fix(T);$$

(vii) T is said to be Bregman relatively nonexpansive if

$$\hat{Fix}(T) = Fix(T) \neq \emptyset \text{ and } \Delta_p(Tx, y^*) \leq \Delta_p(x, y^*),\ \forall\, x \in C,\ y^* \in Fix(T).$$

Let E be a p-uniformly convex uniformly smooth Banach space and $f : E \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. The proximal operator $\text{Prox}_{\lambda f} : E \to E$ associated with f with respect to the Bregman distance is defined as

$$\text{Prox}_{\lambda} f(x) := \arg\min_{u\in E} \left[f(u) + \frac{1}{\lambda}\Delta_p(u, x) \right],\ \forall\, u \in E.$$

Proximal operators have very interesting properties which are suitable for solving minimization problems. Take for example, $\text{Prox}_{\lambda} f$ is firmly nonexpansive; that is,

$$\|\text{Prox}_{\lambda} f(x) - \text{Prox}_{\lambda} f(y)\|^2 \leq \|x-y\|^2 - \|(x - \text{Prox}_{\lambda} f(x)) - (y - \text{Prox}_{\lambda} f(x))\|^2,\ \forall\, x, y \in E.$$

On the other hand, the set of fixed points of $\text{Prox}_{\lambda} f(x)$ is the set of minimizers of f; see, e.g., [31]) for more properties of the proximal operators and the references contained therein. Furthermore, we note from [31] that dom $\text{Prox}_{\lambda} f \subset$ int dom$g$ and ran $\text{Prox}_{\lambda} f \subset \text{dom} g \cap \text{dom} f$, where $g(x) = \frac{1}{p}\|x\|^p$ and it is convex and Gâteaux differentiable. Also, if ran $\text{Prox}_{\lambda} f \subset$ int dom$g$, then $\text{Prox}_{\lambda} f$ is Bregman firmly nonexpansive and single-valued on its domain if int dom$g$ is strictly convex.

We have the following result from the work of Aoyoma et al. [32]:

$$\left\langle \text{Prox}_{\lambda}^{f}(x) - x^*, J\left(x - \text{Prox}_{\lambda}^{f}(x)\right) \right\rangle \geq 0,\ \forall x \in E,\ x^* \in \arg\min f. \tag{2.1}$$

Recall that a metric projection P_C from E onto C satisfies the following property: $\|x - P_C x\| \leq \inf_{y\in C}\|x-y\|, \forall x \in E$. Moreover, $P_C x$ is characterized by the property: $\langle J_E^p(x - P_C x), y - P_C x\rangle \leq 0,\ \forall\, y \in C$. The Bregman projection from E onto C denoted by Π_C also satisfies the property $\Delta_p(x, \Pi_C(x)) = \inf_{y\in C} \Delta_p(x, y),\ \forall\, x \in E$. Also, if C is a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space E and $x \in E$, then the following assertions hold (see [24]):

(i) $z = \Pi_C x$ if and only if

$$\langle y - z,\ J_E^p(x) - J_E^p(z)\rangle \leq 0,\ \forall\, y \in C; \tag{2.2}$$

(ii)

$$\Delta_p(\Pi_C x, y) + \Delta_p(x, \Pi_C x) \leq \Delta_p(x, y),\ \forall\, y \in C.$$

Lemma 2.1. [30] *Let E be a Banach space and $x, y \in E$. If E is q-uniformly smooth, then there exists $C_q > 0$ such that $\|x-y\|^q \leq \|x\|^q - q\langle J_E^q(x), y\rangle + C_q\|y\|^q$.*

Lemma 2.2. [4] *Let E be a p-uniformly convex Banach space. The metric and Bregman distance have the following relation, for all $x, y \in E$, $\tau\|x-y\|^p \leq \Delta_p(x, y) \leq \langle x - y, J_E^p(x) - J_E^p(y)\rangle$, where $\tau > 0$ is a fixed number.*

Lemma 2.3. [33] *Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $z, x_k \in E$ $(k = 1, 2, ..., N)$ and $\alpha_k \in (0, 1)$ with $\sum_{k=1}^N \alpha_k = 1$. Then,*

$$\Delta_p(J_{E^*}^q(\sum_{k=1}^N \alpha_k J_E^p(x_k)), z) \leq \sum_{k=1}^N \alpha_k \Delta_p(x_k, z) - \alpha_i\alpha_j g_r^*(\|J_E^p(x_i) - J_E^p(x_j)\|),$$

for all $i, j \in 1, 2, ..., N$, where $g_r^* : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing function such that $g_r^*(0) = 0$.

Lemma 2.4. [34] *Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $V_p : E^* \times E \to [0,+\infty)$ be defined by $V_p(x,x^*) = \frac{1}{p}\|x\|^p - \langle x,x^*\rangle + \frac{1}{q}\|x^*\|^q$, $\forall x \in E$, $x^* \in E^*$. Then the following assertions hold:*
(i) V_p is nonnegative and convex in the first variable;
(ii) $\Delta_p(x, J^q_{E^*}(x^*)) = V_p(x,x^*)$, $\forall\, x \in E$, $x^* \in E^*$;
(iii) $V_p(x,x^*) + \langle J^q_{E^*}(x^*) - x, y^*\rangle \le V_p(x,x^*+y^*)$, $\forall x \in E$, $x^*, y^* \in E^*$.

Lemma 2.5. [24] *Let E be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E. Then the following assertions are equivalent:*
(i) $\lim_{n\to\infty} \Delta_p(x_n,y_n) = 0$;
(ii) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.6. [35] *Let $q \ge 1$ and $r > 0$ be two fixed real numbers. Then, a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : \mathbb{R}^+ \to \mathbb{R}^*$, $g(0) = 0$ such that, for all $x,y \in B_r$ and $0 \le \alpha < 1$, $\|\alpha x + (1-\alpha)y\|^q \le \alpha\|x\|^q + (1-\alpha)\|y\|^q - W_q(\alpha)g(\|x-y\|)$, where $W_q(\alpha) := \alpha^q(1-\alpha) + \alpha(1-\alpha)^q$ and $B_r := \{x \in E : \|x\| \le r\}$.*

Lemma 2.7. [36] *Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Let $f : E \to (-\infty,+\infty]$ be lower semi-continuous and convex function such that $dom(g) \cap dom(f) \neq \emptyset$ and $ran(Prox^f_\lambda) \subset int(domg)$. For all $x \in E$, $u \in F(Prox^g_\lambda)$ and $\lambda > 0$, $\Delta_g(u, Prox^f_\lambda(x)) + \Delta_g(Prox^f_\lambda(x), x) \le \Delta_g(u,x)$.*

Lemma 2.8. [37] *Let E be a uniformly convex and uniformly smooth Banach space. If $x_0 \in E$ and the sequence $\{\Delta_p(x_n,x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.9. [38] *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ with the condition: $\sum_{n=1}^\infty \alpha_n = \infty$, and $\{d_n\}$ be a sequence of real numbers. Assume that $a_{n+1} \le (1-\alpha_n)a_n + \alpha_n d_n$, $\forall n \ge 0$. If $\limsup_{k\to\infty} d_{n_k} \le 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition: $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \ge 0$, then $\lim_{n\to\infty} a_n = 0$.*

3. The Proposed Algorithm

In this section, we present our proposed method and highlight its features. We start with the following assumptions under which our strong convergence result is obtained.

Assumption 3.1.

(1) E_i, $i = 0,1,2,...,N$ (where $E_0 = E$) are p-uniformly convex real Banach spaces, which are also uniformly smooth;
(2) $T_i : E \to E_i$, $i = 0,1,2,..,N$ (where $T_0 = I^E$) are bounded linear operators;
(3) $f : E \to (-\infty,+\infty]$ and $f_i : E_i \to (-\infty,+\infty]$, $i = 0,1,2,...N$ (where $f_0 = f$) are proper, convex, and lower semi-continuous functions;
(4) $S_j : E \to E$, for $j = 1,2,\ldots,m$ is a Bregman relatively nonexpansive mapping;
(5) $\Gamma := \{x^* \in \bigcap_{j=1}^m Fix(S_j) \cap \arg\min f \cap \bigcap_{i=1}^N T_i^{-1}(\arg\min f_i)\} \neq \emptyset$.
Let $\{\alpha_n\}$ and $\{\beta_{i,n}\}$ be positive sequences satisfying the following conditions:
(i) $\{\alpha_n\} \subset (0,1)$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\beta_{i,n} \subset [a,b] \subset (0,1)$, and $\sum_{i=0}^N \beta_{i,n} = 1$;

(ii) let $\{\varepsilon_n\}$ be a positive sequence such that $\varepsilon_n = \circ(\alpha_n)$, that is, $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$;

(iii) $\theta > 0$ and for $i = 0, 1, \cdots, N$, let λ_n^i be such that $\min_{i=0,1,\cdots,N} \{\inf_n \lambda_n^i\} = \lambda > 0$;

(iv) $\{\phi_{n,j}\} \in (0,1)$, $\sum_{j=0}^m \phi_{n,j} = 1$, and $\liminf_{n\to\infty} \phi_{n,0}\phi_{n,j} > 0$ for each j.

Algorithm 3.1. Initialization: Let $x_0, x_1 \in E$, and choose θ_n such that $\theta_n \in [0, \bar{\theta}_n]$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \dfrac{\varepsilon_n}{\|J_E^P(x_n) - J_E^P(x_{n-1})\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

For $x_0 \in E$, let $E_0 = E, T_0 = I^E, \mathrm{Prox}_{\lambda_n^0}^{f_0} = \mathrm{Prox}^f$, and set $n = 1$.

Iterative steps: Given iterates $x_{n-1},\ x_n$, compute $\{x_n\}$ as follows:

$$\begin{cases} w_n = J_{E^*}^q\Big[J_E^p(x_n) + \theta_n\big(J_E^p(x_{n-1}) - J_E^p(x_n)\big)\Big], \\ y_n = J_{E^*}^q\Big[\sum_{i=0}^N \beta_{i,n}\Big(J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p\big(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i}\big)T_i(w_n)\Big)\Big], \\ z_n = J_{E^*}^q\Big(\phi_{n,0}J_E^p(y_n) + \sum_{j=1}^m \phi_{n,j}J_E^p(S_j y_n)\Big), \\ x_{n+1} = J_{E^*}^q\Big(\alpha_n J_E^p(u) + (1-\alpha_n)J_E^p(z_n)\Big). \end{cases} \tag{3.2}$$

For $\varepsilon > 0$, choose the stepsize $\tau_{i,n}$ in such a way that

$$\tau_{i,n} \in \left(\varepsilon, \left(\frac{q\|T_i(w_n) - \big(\mathrm{Prox}_{\lambda_n^i}^{f_i}\big)T_i(w_n)\|^p}{C_q\|T_i^* J_{E_i}^p\big(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i}\big)T_i(w_n)\|^q} - \varepsilon\right)^{\frac{1}{q-1}}\right),\ \forall\, n \in \Omega, \tag{3.3}$$

for small enough ε, where the index set $\Omega := \{n \in \mathbb{N} : T_i(w_n) - \big(\mathrm{Prox}_{\lambda_n^i}^{f_i} T_i(w_n) \neq 0\big)\}$, otherwise, $\tau_{i,n} = \tau_i$, where τ_i is any nonnegative real number for each $i = 0, 1, \ldots, N$.

We now highlight some of the features of our proposed algorithm.

Remark 3.1.

- The step size $\{\tau_{i,n}\}$ given by (3.3) is generated at each iteration by some simple computation. Thus Algorithm 3.1 is easily implemented without the prior knowledge of the operators norm.
- The inertial technique employed is easily implemented since the value of $\|x_n - x_{n-1}\|$ is a priori known before choosing α_n.

Remark 3.2. By conditions (i) and (ii), it can be verified from (3.1) that

$$\lim_{n\to\infty} \theta_n\|x_n - x_{n-1}\| = 0 \text{ and } \lim_{n\to\infty} \theta_n\|J_E^P(x_n) - J_E^P(x_{n-1})\| = 0.$$

4. Convergence Analysis

We first establish the following lemmas, which are needed to prove the strong convergence theorem for our proposed algorithm.

Lemma 4.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Let conditions (i) and (ii) hold. If $\{x_n\}$ is bounded, then, for all $x^* \in \Gamma$, $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\Big(\triangle_p(x_{n-1},x^*) - \triangle_p(x_n,x^*)\Big) = 0$.*

Proof. Let $x^* \in \Gamma$. Observe that

$$\begin{aligned}
&\Delta_p(x_{n-1},x^*) - \Delta_p(x_n,x^*)\\
&= \frac{1}{q}\|x_{n-1}\|^p - \langle J_E^P(x_{n-1}),x^*\rangle + \frac{1}{p}\|x^*\|^p - \Big(\frac{1}{q}\|x_n\|^p - \langle J_E^P(x_n),x^*\rangle + \frac{1}{p}\|x^*\|^p\Big)\\
&= \frac{1}{q}\Big(\|x_{n-1}\|^p - \|x_n\|^p\Big) + \langle J_E^P(x_n) - J_E^P(x_{n-1}),x^*\rangle\\
&\leq \frac{1}{q}M\|x_{n-1} - x_n\| + \|J_E^P(x_n) - J_E^P(x_{n-1})\|\|x^*\| \qquad (4.1)
\end{aligned}$$

for some constant $M > 0$. Since $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$, then it follows from Remark 3.2 that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$. Similarly, we have that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|J_E^P(x_n) - J_E^P(x_{n-1})\| = 0$. It follows from (4.1) that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\Big(\Delta_p(x_{n-1},x^*) - \Delta_p(x_n,x^*)\Big)\\
&\leq \lim_{n\to\infty}\Big(\frac{M}{q}\cdot\frac{\theta_n}{\alpha_n}\|x_{n-1} - x_n\| + \|x^*\|\frac{\theta_n}{\alpha_n}\|J_E^P(x_n) - J_E^P(x_{n-1})\|\Big) = 0,
\end{aligned}$$

which is the required result. □

Lemma 4.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 and Assumption 3.1 hold. Then, $\{x_n\}$ is bounded.*

Proof. Let $x^* \in \Gamma$. From (3.1), we have

$$\begin{aligned}
\Delta_p(w_n,x^*) &= \Delta_p(J_{E^*}^q(J_E^p x_n + \theta_n(J_E^p x_{n-1} - J_E^p x_n),x^*)\\
&\leq (1-\theta_n)\Delta_p(x_n,x^*) + \theta_n\Delta_p(x_{n-1},x^*). \qquad (4.2)
\end{aligned}$$

In view of (3.2), we obtain

$$\begin{aligned}
\Delta_p(y_n,x^*) &= V_p\left(\sum_{i=0}^N \beta_{i,n}(J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)),x^*\right)\\
&= \frac{1}{p}\|x^*\|^p - \langle\sum_{i=0}^N \beta_{i,n}(J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)),x^*\rangle\\
&\quad + \frac{1}{q}\|\sum_{i=0}^N \beta_{i,n}(J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n))\|^q\\
&= \frac{1}{p}\|x^*\|^p - \langle J_E^p(w_n),x^*\rangle + \sum_{i=0}^N \beta_{i,n}\tau_{i,n}\langle T_i x^*, J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle\\
&\quad + \frac{1}{q}\|\sum_{i=0}^N \beta_{i,n}(J_E^p(w_n) - \tau_{i,n}T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n))\|^q. \qquad (4.3)
\end{aligned}$$

By the convexity property of Δ_p, we have that

$$\begin{aligned}
&\|\sum_{i=0}^{N}\beta_{i,n}(J_E^p(w_n)-\tau_{i,n}T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n))\|^q \\
&\leq \sum_{i=0}^{N}\beta_{i,n}\|J_E^p(w_n)-\tau_{i,n}T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q.
\end{aligned}$$

From Lemma 2.1, we obtain

$$\begin{aligned}
&\|J_E^p(w_n)-\tau_{i,n}T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q \\
&\leq \|w_n\|^q-q\tau_{i,n}\langle T_iw_n,J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle+C_q\tau_{i,n}^q\|T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q.
\end{aligned}\tag{4.4}$$

Substituting (4.4) into (4.3), we obtain

$$\begin{aligned}
&\Delta_p(y_n,x^*) \\
&=\frac{1}{p}\|x^*\|^p-\langle J_E^p(w_n),x^*\rangle+\sum_{i=0}^{N}\beta_{i,n}\tau_{i,n}\langle T_i(x^*),J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle+\frac{1}{q}\|w_n\|^q \\
&\quad-\sum_{i=0}^{N}\beta_{i,n}\tau_{i,n}\langle T_i(w_n),J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle \\
&\quad+\sum_{i=0}^{N}\beta_{i,n}\frac{C_q\tau_{i,n}^q}{q}\|T_i^*(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q \\
&=\Delta_p(w_n,x^*)+\sum_{i=0}^{N}\beta_{i,n}\tau_{i,n}\langle T_i(x^*)-T_i(w_n),J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle \\
&\quad+\sum_{i=0}^{N}\beta_{i,n}\frac{C_q\tau_{i,n}^q}{q}\|T_i^*(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q.
\end{aligned}\tag{4.5}$$

By applying (2.1), we obtain $\langle T_i(x^*)-T_i(w_n),J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle=\langle T_i(x^*)-T_i(w_n)-\mathrm{Prox}_{\lambda_n^i}^{f_i}T_i(w_n)+\mathrm{Prox}_{\lambda_n^i}^{f_i}T_i(w_n),\,J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle=-\|(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^p+\langle T_i(x^*)-\mathrm{Prox}_{\lambda_n^i}^{f_i}T_i(w_n)$, and $J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\rangle\leq-\|(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^p$. From (4.5) and g (3.3), we obtain

$$\begin{aligned}
\Delta_p(y_n,x^*)\leq\Delta_p(w_n,x^*)-\sum_{i=0}^{N}\beta_{i,n}\tau_{i,n}\Bigg(&\|(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^p \\
&-\frac{C_q\tau_{i,n}^{q-1}}{q}\|T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q\Bigg) \\
\leq\Delta_p(w_n,x^*).&
\end{aligned}\tag{4.6}$$

In view of Algorithm 3.1 and Lemma 2.3 together with the Bregman relative nonexpansivity of S_j for each $j = 1, 2, \ldots, m$, we obtain

$$\begin{aligned}
&\Delta_p(z_n, x^*) \\
&= \Delta_p\left(J^q_{E^*}\left(\phi_{n,0}J^p_E(y_n) + \sum_{j=1}^m \phi_{n,j}J^p_E(S_jy_n)\right), x^*\right) \\
&\leq \phi_{n,0}\Delta_p(y_n, x^*) + \sum_{j=1}^m \phi_{n,j}\Delta_p(S_jy_n, x^*) - \phi_{n,0}\sum_{j=1}^m \phi_{n,j}g^*_r(\|J^p_E(y_n) - J^p_E(S_jy_n)\|) \\
&\leq \phi_{n,0}\Delta_p(y_n, x^*) + \sum_{j=1}^m \phi_{n,j}\Delta_p(y_n, x^*) - \phi_{n,0}\sum_{j=1}^m \phi_{n,j}g^*_r(\|J^p_E(y_n) - J^p_E(S_jy_n)\|) \\
&= \Delta_p(y_n, x^*) - \phi_{n,0}\sum_{j=1}^m \phi_{n,j}g^*_r(\|J^p_E(y_n) - J^p_E(S_jy_n)\|) \qquad (4.7) \\
&\leq \Delta_p(y_n, x^*). \qquad (4.8)
\end{aligned}$$

Furthermore, from Algorithm 3.1, (4.2), (4.6), and (4.8), we have

$$\begin{aligned}
&\Delta_p(x_{n+1}, x^*) \\
&= \Delta_p\left(J^q_{E^*}\left(\alpha_nJ^p_E(u) + (1-\alpha_n)J^p_E(z_n)\right), x^*\right) \\
&\leq \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(z_n, x^*) \\
&\leq \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(y_n, x^*) \\
&\leq \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)\Delta_p(w_n, x^*) \\
&\leq \alpha_n\Delta_p(u, x^*) + (1-\alpha_n)[(1-\theta_n)\Delta_p(x_n, x^*) + \theta_n\Delta_p(x_{n-1}, x^*)] \\
&\leq \max\{\Delta_p(u, x^*), \max\{\Delta_p(x_n, x^*), \Delta_p(x_{n-1}, x^*)\}\} \\
&\vdots \\
&\leq \max\{\Delta_p(u, x^*), \max\{\Delta_p(x_1, x^*), \Delta_p(x_0, x^*)\}\} < \infty.
\end{aligned}$$

Thus $\{\Delta_p(x_n, x^*)\}$ is bounded. Consequently, $\{\Delta_p(w_n, x^*)\}$, $\{\Delta_p(y_n, x^*)\}$, and $\{\Delta_p(z_n, y^*)\}$ are bounded. Therefore, it follows from Lemma 2.8 that $\{w_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are all bounded. □

We now prove the strong convergence theorem for the proposed algorithm.

Theorem 4.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 under Assumption 3.1. Then, $\{x_n\}$ converges strongly to $x^* \in \Gamma$, where $x^* = \Pi_\Gamma u$.*

Proof. Let $x^* \in \Gamma$. From Algorithm 3.1, (4.6), (4.7), and Lemma 2.4(iii), we have

$$\begin{aligned}
&\Delta_p(x_{n+1},x^*)\\
&= V_p(\alpha_n J_E^p(u)+(1-\alpha_n)J_E^p(z_n),x^*)\\
&\leq V_p(\alpha_n J_E^p(x^*)+(1-\alpha_n)J_E^p(z_n),x^*)+\alpha_n\langle J_E^p(u)-J_E^p(x^*),x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)V_p(J_E^p(z_n),x^*)+\alpha_n\langle J_E^p(u)-J_E^p(x^*),x_{n+1}-x^*\rangle\\
&= (1-\alpha_n)\Delta_p(z_n,x^*)+\alpha_n\langle J_E^p(u)-J_E^p(x^*),x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)\left[\Delta_p(y_n,x^*)-\phi_{n,0}\sum_{j=1}^m \phi_{n,j}g_r^*(\|J_E^p(y_n)-J_E^p(S_jy_n)\|)\right]\\
&\quad +\alpha_n\langle J_E^p(u)-J_E^p(x^*),x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)\Delta_p(w_n,x^*)-(1-\alpha_n)\sum_{i=0}^N \beta_{i,n}\tau_{i,n}\Bigg(\|(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^p\\
&-\frac{C_q\tau_{i,n}^{q-1}}{q}\|T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q\Bigg)-(1-\alpha_n)\phi_{n,0}\sum_{j=1}^m \phi_{n,j}g_r^*(\|J_E^p(y_n)-J_E^p(S_jy_n)\|)\\
&+\alpha_n\langle J_E^p(u)-J_E^p(x^*),x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)[(1-\theta_n)\Delta_p(x_n,x^*)+\theta_n\Delta_p(x_{n-1},x^*)]+\alpha_n\langle J_E^p(u)-J_E^p(x^*),x_{n+1}-x^*\rangle\\
&\quad -(1-\alpha_n)\sum_{i=0}^N \beta_{i,n}\tau_{i,n}\left(\|(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^p-\frac{C_q\tau_{i,n}^{q-1}}{q}\|T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q\right)\\
&-(1-\alpha_n)\phi_{n,0}\sum_{j=1}^m \phi_{n,j}g_r^*(\|J_E^p(y_n)-J_E^p(S_jy_n)\|)\\
&\leq (1-\alpha_n)\Delta_p(x_n,x^*)-(1-\alpha_n)\sum_{i=0}^N \beta_{i,n}\tau_{i,n}\Bigg(\|(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^p\\
&\quad -\frac{C_q\tau_{i,n}^{q-1}}{q}\|T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q\Bigg)\\
&\quad -(1-\alpha_n)\phi_{n,0}\sum_{j=1}^m \phi_{n,j}g_r^*(\|J_E^p(y_n)-J_E^p(S_jy_n)\|)+\alpha_n\psi_n,
\end{aligned} \tag{4.9}$$

where

$$\psi_n := \left(\frac{\theta_n}{\alpha_n}(\Delta_p(x_{n-1},x^*)-\Delta_p(x_n,x^*))+\langle J_E^p(u)-J_E^p(x^*),x_{n+1}-x^*\rangle\right).$$

Thus it follows from (4.9) that

$$\begin{aligned}
(1-\alpha_n)\sum_{i=0}^N \beta_{i,n}\tau_{i,n}\left(\|(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^p-\frac{C_q\tau_{i,n}^{q-1}}{q}\|T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_n^i}^{f_i})T_i(w_n)\|^q\right)&\\
\leq \Delta_p(x_n,x^*)-\Delta_p(x_{n+1},x^*)+\alpha_n\psi_n.&
\end{aligned} \tag{4.10}$$

Similarly, (4.9) yields that

$$(1-\alpha_n)\phi_{n,0}\sum_{j=1}^{m}\phi_{n,j}g_r^*(\|J_E^p(y_n)-J_E^p(S_jy_n)\|)\le \Delta_p(x_n,x^*)-\Delta_p(x_{n+1},x^*)+\alpha_n\psi_n \tag{4.11}$$

and

$$\Delta_p(x_{n+1},x^*)\le(1-\alpha_n)\Delta_p(x_n,x^*)+\alpha_n\psi_n. \tag{4.12}$$

We now prove that $\{x_n\}$ converges strongly to x^*. Let $a_n:=\Delta_p(x_n,x^*)$. It is easy to see that (4.12) satisfies the inequality $a_{n+1}\le(1-\alpha_n)a_n+\alpha_n\psi_n$. Using Lemma 2.9, it suffices to show that $\limsup_{k\to\infty}\psi_{n_k}\le 0$ (where $\{\psi_{n_k}\}$ is a subsequence of $\{\psi_n\}$), for every subsequence $\{\Delta_p(x_{n_k},x^*)\}$ of $\{\Delta_p(x_n,x^*)\}$ satisfying the condition

$$\liminf_{k\to\infty}\left(\Delta_p(x_{n_k+1},x^*)-\Delta_p(x_{n_k},x^*)\right)\ge 0. \tag{4.13}$$

Now, suppose that $\{\Delta_p(x_{n_k},x^*)\}$ is a subsequence of $\{\Delta_p(x_n,x^*)\}$ such that (4.13) holds. Using (4.10), (4.13), and condition (i), we have that

$$\begin{aligned}
&\limsup_{k\to\infty}(1-\alpha_{n_k})\sum_{i=0}^{N}\beta_{i,n_k}\tau_{i,n_k}\left(\|(I^{E_i}-\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^p-\frac{C_q\tau_{i,n_k}^{q-1}}{q}\|T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^q\right)\\
&\le\limsup_{k\to\infty}\left(\Delta_p(x_{n_k},x^*)-\Delta_p(x_{n_k+1},x^*)+\alpha_{n_k}\psi_{n_k}\right)\\
&=\limsup_{k\to\infty}\left(\Delta_p(x_{n_k},x^*)-\Delta_p(x_{n_k+1},x^*)\right)\\
&\le-\liminf_{k\to\infty}\left(\Delta_p(x_{n_k+1},x^*)-\Delta_p(x_{n_k},x^*)\right)\\
&\le 0.
\end{aligned} \tag{4.14}$$

From the choice of our stepsize τ_{i,n_k}, we see that

$$\tau_{i,n_k}^{q-1}<\frac{q\|T_i(w_{n_k})-(\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^p}{C_q\|T_i^*J_{E_i}^p(I^{E_i}-(\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k}))\|^q}-\varepsilon, \tag{4.15}$$

which implies that

$$\begin{aligned}
\frac{\varepsilon C_q}{q}\|T_i^*J_{E_i}^p(I^{E_i}-(\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k}))\|^q<\Big(&\|T_i(w_{n_k})-\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^p\\
&-\frac{C_q\tau_{i,n_k}^{q-1}}{q}\|T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^q\Big).
\end{aligned} \tag{4.16}$$

Passing the limit as $k\to\infty$, we conclude from (4.14) and (4.16) that

$$\lim_{k\to\infty}\|T_i^*J_{E_i}^p(I^{E_i}-\mathrm{Prox}_{\lambda_{n_k}^i}^{f_i})T_i(w_{n_k})\|^q=0,\quad\forall\, i=0,1,2,\ldots,N. \tag{4.17}$$

Similarly, from (4.14), (4.15), and condition (i) of Assumption 3.1, we obtain that

$$\lim_{k\to\infty} \|T_i(w_{n_k}) - (\mathrm{Prox}^{f_i}_{\lambda^i_{n_k}})T_i(w_{n_k})\|^p = 0, \quad \forall\, i = 0,1,2,...,N. \tag{4.18}$$

Also, we obtain from (4.11) and (4.13) that

$$\begin{aligned}
&(1-\alpha_{n_k})\phi_{n_k,0} \limsup_{k\to\infty} \sum_{j=1}^{m} \phi_{n_k,j} g_r^*(\|J_E^p(y_{n_k}) - J_E^p(S_j y_{n_k})\|)\\
&\le \limsup_{k\to\infty} \Big(\Delta_p(x_{n_k},x^*) - \Delta_p(x_{n_k+1},x^*) + \alpha_{n_k}\psi_{n_k} \Big)\\
&= \limsup_{k\to\infty} \Big(\Delta_p(x_{n_k},x^*) - \Delta_p(x_{n_k+1},x^*) \Big)\\
&= -\liminf_{k\to\infty} \Big(\Delta_p(x_{n_k+1},x^*) - \Delta_p(x_{n_k},x^*) \Big)\\
&\le 0.
\end{aligned}$$

By conditions (i) and (iv), we have $(1-\alpha_{n_k})\phi_{n_k,0}\limsup_{k\to\infty}\sum_{j=1}^m \phi_{n_k,j} g_r^*(\|J_E^p(y_{n_k}) - J_E^p(S_j y_{n_k})\|)$ $= 0$. Consequently, $\lim_{k\to\infty} g_r^*(\|J_E^p(y_{n_k}) - J_E^p(S_j y_{n_k})\|) = 0$, $j = 1,2,\dots,m$. By the properties of g_r^*, we obtain $\lim_{k\to\infty}\|J_E^p(y_{n_k}) - J_E^p(S_j y_{n_k})\| = 0$, $j = 1,2,\dots,m$. Since J_E^p is norm-to-norm uniformly continuous on bounded subsets of E, we obtain

$$\lim_{k\to\infty} \|y_{n_k} - S_j y_{n_k}\| = 0, \quad j = 1,2,\dots,m. \tag{4.19}$$

Recall that $w_{n_k} = J_{E^*}^q\Big[J_E^p(x_{n_k}) + \theta_{n_k}\big(J_E^p(x_{n_k-1}) - J_E^p(x_{n_k})\big)\Big]$. It then follows from Remark 3.2 that $\lim\limits_{k\to\infty}\|J_E^p(w_{n_k}) - J_E^p(x_{n_k})\| = \lim\limits_{k\to\infty}\theta_{n_k}\|J_E^p(x_{n_k-1}) - J_E^p(x_{n_k})\| = 0$. By the uniform continuity of $J_{E^*}^q$ on bounded subsets of E^*, we obtain that

$$\lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = 0. \tag{4.20}$$

Combining (3.2) and (4.17), we have

$$\|J_E^p(y_{n_k}) - J_E^p(w_{n_k})\| \le \sum_{i=0}^{N} \beta_{i,n_k}\tau_{i,n_k}\|T_i^* J_{E_i}^p(I^{E_i} - \mathrm{Prox}^{f_i}_{\lambda^i_{n_k}})T_i(w_{n_k})\| \to 0$$

as $k\to\infty$ for all $i = 0,1,2,...,N$. By the uniform continuity of $J_{E^*}^q$ on bounded subsets of E^*, we obtain that

$$\lim_{k\to\infty} \|y_{n_k} - w_{n_k}\| = 0. \tag{4.21}$$

Thus, it is easy to see from (4.20) and (4.21) that

$$\lim_{k\to\infty} \|y_{n_k} - x_{n_k}\| = 0. \tag{4.22}$$

Moreover, we obtain from condition (iv) and (4.19) that

$$\lim_{k\to\infty}\|J_E^p z_{n_k} - J_E^p y_{n_k}\| \le \lim_{k\to\infty}\Big(\phi_{n_k,0}\|J_E^p(y_{n_k}) - J_E^p(y_{n_k})\| + \sum_{j=1}^{m}\phi_{n_k,j}\|J_E^p(S_j y_{n_k}) - J_E^p(y_{n_k})\|\Big) = 0.$$

It follows from the uniform continuity of $J_{E^*}^q$ on bounded subsets of E^* that $\lim_{k\to\infty}\|z_{n_k}-y_{n_k}\|=0$. It is not difficult to see from (4.21) that $\lim_{k\to\infty}\|z_{n_k}-w_{n_k}\|\le\lim_{k\to\infty}(\|z_{n_k}-y_{n_k}\|+\|y_{n_k}-w_{n_k}\|)=0$. Similarly, from (4.22), we have

$$\lim_{k\to\infty}\|z_{n_k}-x_{n_k}\|\le\lim_{k\to\infty}(\|z_{n_k}-y_{n_k}\|+\|y_{n_k}-x_{n_k}\|)=0. \tag{4.23}$$

In view of (3.1), we obtain

$$\begin{aligned}\lim_{k\to\infty}\|J_E^p(x_{n_k+1})-J_E^p(z_{n_k})\| &= \lim_{k\to\infty}\|\alpha_{n_k}J_E^p(u)+(1-\alpha_{n_k})J_E^p(z_{n_k})-J_E^p(z_{n_k})\|\\ &\le \alpha_{n_k}\|J_E^p(u)-J_E^p(z_{n_k})\|+(1-\alpha_{n_k})\|J_E^p(z_{n_k})-J_E^p(z_{n_k})\|\to 0\end{aligned}$$

as $k\to\infty$. Since $J_{E^*}^q$ is uniformly continuous on bounded subsets of E^*, we obtain $\lim_{k\to\infty}\|x_{n_k+1}-z_{n_k}\|=0$, which together with (4.23) yields that

$$\lim_{k\to\infty}\|x_{n_k+1}-x_{n_k}\|=0. \tag{4.24}$$

We now show that $\lim_{k\to\infty}\psi_{n_k}\le 0$. To do this, we need to show that $\limsup_{k\to\infty}\langle J_E^p(u)-J_E^p(x^*),x_{n_k+1}-x^*\rangle\le 0$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$, which converges weakly to $\bar{x}\in E$ such that $\limsup_{k\to\infty}\langle J_E^p(u)-J_E^p(x^*),x_{n_k+1}-x^*\rangle=\lim_{j\to\infty}\langle J_E^p(u)-J_E^p(x^*),x_{n_{k_j}+1}-x^*\rangle$. To complete the proof, we need to show that $w_\omega(x_n)\subset\Gamma$. Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $\bar{x}\in w_\omega(x_n)$ be an arbitrarily chosen element. Then, from (4.20) and (4.22), there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ which converges weakly to $\bar{x}\in E$ and subsequence $\{y_{n_{k_j}}\}$ of $\{y_{n_k}\}$ which converges weakly to $\bar{x}\in E$. Also, $w_{n_{k_j}}\rightharpoonup\bar{x}\in E$. Since, for every $i=0,1,2,\cdots,N$, T_i is a bounded linear operator, we obtain $T_iw_{n_{j_k}}\rightharpoonup T_i\bar{x}\in E_i$ as $k\to\infty$. From (4.18), we obtain $T_i\bar{x}\in Fix(\text{Prox}_{\lambda_{n_k}^i}^{f_i})$ for $i=0,1,2,\cdots,N$, which implies that $\bar{x}\in arg\min f\cap\bigcap_{i=1}^N T_i^{-1}(\arg\min f_i)$. Also, from (4.19) and the fact that $F\hat{i}x(S_j)=Fix(S_j)$ for all $j=1,2,...,m$, we have that $\bar{x}\in Fix(S_j)$ for all $j=1,2,\ldots,m$. This implies that $\bar{x}\in\bigcap_{j=1}^m Fix(S_j)$. Thus $\bar{x}\in\Gamma$. Since $\bar{x}\in w_\omega(x_n)$ is an arbitrary element, then $w_\omega(x_n)\subset\Gamma$. Since $x^*=\Pi_\Gamma u$, then we obtain from (2.2) and (4.24) that

$$\begin{aligned}\limsup_{k\to\infty}\langle J_E^p(u)-J_E^p(x^*),x_{n_k+1}-x^*\rangle &\le \limsup_{k\to\infty}\langle J_E^p(u)-J_E^p(x^*),x_{n_k}-x^*\rangle\\ &\quad+\limsup_{k\to\infty}\langle J_E^p(u)-J_E^p(x^*),x_{n_k+1}-x_{n_k}\rangle\\ &=\lim_{j\to\infty}\langle J_E^p(u)-J_E^p(x^*),x_{n_{k_j}}-x^*\rangle\\ &=\langle J_E^p(u)-J_E^p(x^*),\bar{x}-x^*\rangle\le 0.\end{aligned}$$

By applying Lemma 2.9 and Lemma 4.1 to (4.12), we conclude that $\Delta_p(x_n,x^*)\to 0$ as $n\to\infty$. Therefore, we conclude by Lemma 2.5 that $\{x_n\}$ converges strongly to x^*. □

We have the following corollary as a consequent result of our main result.

Corollary 4.1. *Let $E_i=H_i$, $i=0,1,\cdots,N$, with $H_0=H$, be real Hilbert spaces, and let T_i be bounded linear operators such that $T_i\ne 0$ with $T_0=I^H$. Let $f:H\to(-\infty,+\infty]$, $f_i:H_i\to(-\infty,+\infty]$ be proper, convex, and lower semi-continuous functions. Let $S_j:H\to H$, for $j=1,2,\ldots,m$ be quasi-nonexpansive mappings, which are demiclosed at zero and such that $\Gamma:=$*

$\{x^* \in \bigcap_{j=1}^{m} Fix(S_j) \cap \arg\min f \cap \bigcap_{i=1}^{N} T_i^{-1}(\arg\min f_i)\} \neq \emptyset$. Suppose that the other conditions of Assumption 3.1 are satisfied. Let $x_0,\ x_1 \in H$ and $\{x_n\}$ be a sequence generated as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = \sum_{i=0}^{N} \beta_{i,n}(w_n - \tau_{i,n}T_i^*(I^{H_i} - Prox_{\lambda_n^i}^{f_i})T_i(w_n)) \\ z_n = \phi_{n,0}y_n + \sum_{j=1}^{m} \phi_{n,j}S_j y_n \\ x_{n+1} = \alpha_n u + (1-\alpha_n)z_n,\ n \geq 1. \end{cases}$$

Choose θ_n such that $\theta_n \in [0, \bar{\theta}_n]$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise}. \end{cases}$$

Suppose that, for $\varepsilon > 0$, the step size $\tau_{i,n}$ is chosen such that

$$\tau_{i,n} \in \left(\varepsilon, \frac{2\|T_i(w_n) - \left(Prox_{\lambda_n^i}^{f_i}\right)T_i(w_n)\|^2}{\|T_i^*\left(I^{H_i} - Prox_{\lambda_n^i}^{f_i}\right)T_i(w_n)\|^2} - \varepsilon\right),\ \forall\, n \in \Omega,$$

for small enough ε, where the index set $\Omega := \{n \in \mathbb{N} : T_i(w_n) - \left(Prox_{\lambda_n^i}^{f_i} T_i(w_n) \neq 0\right)\}$, otherwise, $\tau_{i,n} = \tau_i$, where τ_i is any nonnegative real number for each $i = 0, 1, \ldots, N$. Then, $\{x_n\}$ converges strongly to $x^* = P_\Gamma u$, where $P_\Gamma : H \to \Gamma$ is the metric projection of H onto Γ.

5. Applications

5.1. **Equilibrium problems.** Let C be a nonempty, closed, and convex subset of the Banach space E, and let $g : C \times C \to \mathbb{R}$ be a bifunction. We recall that the equilibrium problem (EP) consists of finding the point $x \in C$ such that

$$g(x,y) \geq 0\ \forall\, y \in C. \tag{5.1}$$

Let $\bar{x} \in C$. Setting $g(\bar{x}, \bar{y}) := \phi(\bar{y}) - \phi(\bar{x})$, $\forall \bar{y} \in C$, the equilibrium problem (5.1) coincides with the convex minimization problem (1.5), and the function g satisfies the following conditions:

(i) $g(x,x) = 0\ \forall\, x \in C$,
(ii) g is monotone, i.e., $g(x,y) + g(y,x) \leq 0,\ \forall\, x,y \in C$,
(iii) $\limsup_{t\to 0} g(tz + (1-t)x, y) \leq g(x,y)\ \forall\, x,y,z \in C$.
(iv) for each $x \in C$, the function $\rightarrowtail g(x,y)$ is convex lower semi-continuous.

The solution set of (5.1) is denoted by $\mathrm{EP}(g)$. The resolvent of the bifunction g is the function $\mathrm{Res}_g^\phi : E \to 2^C$ defined by (see, e.g., [37, 39])

$$\mathrm{Res}_g^\phi(x) = \{z \in C : g(z,y) + \langle \Delta\phi(z) - \Delta\phi(x), y - z\rangle \geq 0,\ \forall\, y \in C\}.$$

Proposition 5.1. [37] *Let $\phi : E \to (-\infty, +\infty]$ be a coercive and Legendre function. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies the conditions (i)-(iv), then*

- *Res_g^ϕ is single-valued,*
- *Res_g^ϕ is Bregman firmly nonexpansive,*
- *$F(Res_g^\phi) = EP(g)$,*
- *$EP(g)$ is a closed and convex subset of C,*

- *for all* $x \in E$ *and* $q \in F(Res_g^{\phi})$ $\Delta_{\phi}(q, Res_g^{\phi}(x)) + \Delta_{\phi}(Res_g^{\phi}(x), x) \leq \Delta_{\phi}(q, x)$.

Observe that every Bregman firmly nonexpansive mapping is Bregman relatively nonexpansive mapping. Hence, by setting $S_j = \text{Res}_{g_j}^{\phi_j}, \ j = 1, 2, ..., m$ in Theorem 4.1, then we have a strong convergence theorem for approximating a common solution of split minimization problem with multiple output sets which is also a common solution of a finite family of equilibrium problems in p-uniformly convex Banach spaces which are also uniformly smooth.

5.2. **Zeroes of Bregman inverse-strongly monotone operators.** Let the Legendre function g be such that $\text{ran}(\Delta g - B) \subseteq \text{ran}(\Delta g)$, where $B : E \to 2^{E^*}$ is a Bregman inverse-strongly monotone (BISM) operator, this is, $(\text{dom}B) \cap (\text{int}(\text{dom}g)) \neq \emptyset$ and, for any $x, y \in (\text{int}(\text{dom}g))$ and each $\psi \in Bx$, $\eta \in By$, $\langle \psi - \eta, (\Delta g(x) - \psi) - \Delta g^*(\Delta g(y) - \eta) \rangle \geq 0$. This class of operators was introduced by Butnariu and Kassay (see [6]). For any operator $B : E \to 2^{E^*}$, the anti-resolvent $B^g : E \to 2^E$ of B is defined by $B^g := \Delta g^* \circ (\Delta g - B)$. Observe that $\text{dom}B^g \subseteq (\text{dom}B) \cap (\text{int}(\text{dom}g))$ and $\text{ran}B^g \subseteq \text{int}(\text{dom}g)$. The operator B is BISM if and only if the anti-resolvent B^g is a single-valued Bregman Firmly Nonexpansive Mapping (BFNM) (see [6]). Some examples of BISM operators can be found in [6]. From the definition of anti-resolvent and [6, Lemma 3.5], we obtain the following proposition.

Proposition 5.2. *Let* $g : E \to (-\infty, +\infty]$ *be a Legendre function, and let* $B : E \to 2^{E^*}$ *be a BISM operator such that* $B^{-1}(0)^* \neq \emptyset$. *Then*

(i) $B^{-1}(0)^* = F(B^g)$

(ii) *for any* $u \in B^{-1}(0)^*$ *and* $x \in \text{dom}B^g$, $D_g(u, B_x^g) + D_g(B_x^g, x) \leq D_g(u, x)$.

If g is uniformly Fr*é*chet differentiable and bounded on the bounded subsets of E, then the anti-resolvent B^g of B is a single-valued Bregman Firmly Nonexpansive Mapping (BFNM) which satisfies $F(B^g) = \hat{F}(B^g)$; see [40, Lemma 1.3.2]. In Theorem 4.1, if we let $S_j = B_j^g$ and let g be the Legendre function such that $\text{ran}(\Delta g - B) \subseteq \text{ran}(\Delta g)$ is satisfied, then we obtain a strong convergence theorem for approximating a common zero of a countable family of Bregman inverse strongly monotone operators, which is also a solution of the split minimization problem with multiple output sets in p-uniformly convex and uniformly smooth Banach spaces.

6. Numerical Examples

This section provides some numerical experiments to illustrate Algorithm 3.1 and verify the effects of the key parameters on our method.

Example 6.1. Let $E = \mathbb{R}^2 = E_i, \ i = 0, 1, 2, 3$, and let $f(x) = \|x\|_2$, for all $x \in \mathbb{R}^2$, be the Euclidean norm. For a unit ball B, the projection onto B is given by

$$P_B(x) = \begin{cases} \frac{x}{\|x\|_2}, & \text{if } \|x\|_2 > 1 \\ x, & \text{if } \|x\|_2 \leq 1. \end{cases}$$

Then, the proximal operator $\text{Prox}^{f_i}(x)$ is given by

$$\text{Prox}^{\ f_i}(x) := \begin{cases} \left(1 - \frac{i+1}{\|x\|_2}\right) x, & \text{if } \|x\|_2 \geq 1 \\ 0, & \text{if } \|x\|_2 < 1. \end{cases}$$

The proximal operator is called the block soft thresholding (see [41]).

Let $T_i x := (i+1)x$, where $x : (x_1, x_2) \in \mathbb{R}^2$. We now consider the following problem:

$$x^* \in \arg\min f \cap \left(\bigcap_{i=1}^{N} T_i^{-1}(\arg\min f_i)\right) \neq \emptyset.$$

Also, let $S_j : C \times C \to \mathbb{R}$ be defined by

$$S_j x := \frac{2}{3j} x, \quad \forall j = 1, 2, \ldots, 7.$$

It is easy to see that S_j is relatively nonexpansive for each $j = 1, 2, \ldots, 7$. Take $\alpha_n = \frac{1}{n+1}$ for all $n \geq 1$, $u = (1,1)$, $\beta_{i,n} = \frac{1}{4}$, $\phi_{n,0} = \frac{1}{2}$, and $\phi_{n,j} = \frac{1}{2^{j+1}}$, $j = 1, 2, \cdots, 7$. Then Algorithm 3.1 becomes

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \sum_{i=0}^{3} \frac{1}{4}\left(w_n - \tau_{i,n} T_i^*\left(T_i(w_n) - \mathrm{Prox}_{\lambda_n^i}^{f_i} T_i(w_n)\right)\right), \\ z_n = \left(\frac{1}{2}(y_n) + \sum_{j=1}^{7} \frac{1}{2^{j+1}}(S_j y_n)\right), \\ x_{n+1} = \frac{1}{n+1} u + \frac{n}{n+1} z_n, \end{cases}$$

for $n \geq 1$. Suppose that, for $\varepsilon > 0$, the step size $\tau_{i,n}$ is chosen such that

$$\tau_{i,n} \in \left(\varepsilon, \frac{2\|T_i(w_n) - \left(\mathrm{Prox}_{\lambda_n^i}^{f_i}\right) T_i(w_n)\|^2}{\|T_i^*\left(I^{H_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i}\right) T_i(w_n)\|^2} - \varepsilon\right), \ \forall\, n \in \Omega, \tag{6.1}$$

for small enough ε, where the index set $\Omega := \{n \in \mathbb{N} : T_i(w_n) - \left(\mathrm{Prox}_{\lambda_n^i}^{f_i} T_i(w_n) \neq 0\right)\}$, otherwise, $\tau_{i,n} = \tau_i$, where τ_i is any nonnegative real number for each $i = 0, 1, 2, 3$.

Using $\|x_{n+1} - x_n\| < 10^{-3}$ as the stopping criterion, we plot the graphs of $\|x_{n+1} - x_n\|$ against the number of iterations in each case. The numerical results can be found in Fig. 1, Fig. 2, Table 1, and Table 2.

Table 1. Numerical Results for Example 6.1 (Experiment 1).

Cases		$\theta = 1.5$	$\theta = 2.0$	$\theta = 2.5$	$\theta = 3.0$	$\theta = 3.5$
1	CPU time (sec.)	0.0125	0.0114	0.0115	0.0108	0.0154
	No. of Iter.	40	40	40	40	40
2	CPU time (sec.)	0.0207	0.0181	0.0131	0.0121	0.0181
	No. of Iter.	40	40	40	40	40
3	CPU time (sec.)	0.0121	0.0141	0.0137	0.0118	0.0157
	No. of Iter.	40	40	40	40	40
4	CPU time (sec.)	0.0128	0.0116	0.0122	0.0107	0.0152
	No. of Iter.	40	40	40	40	40

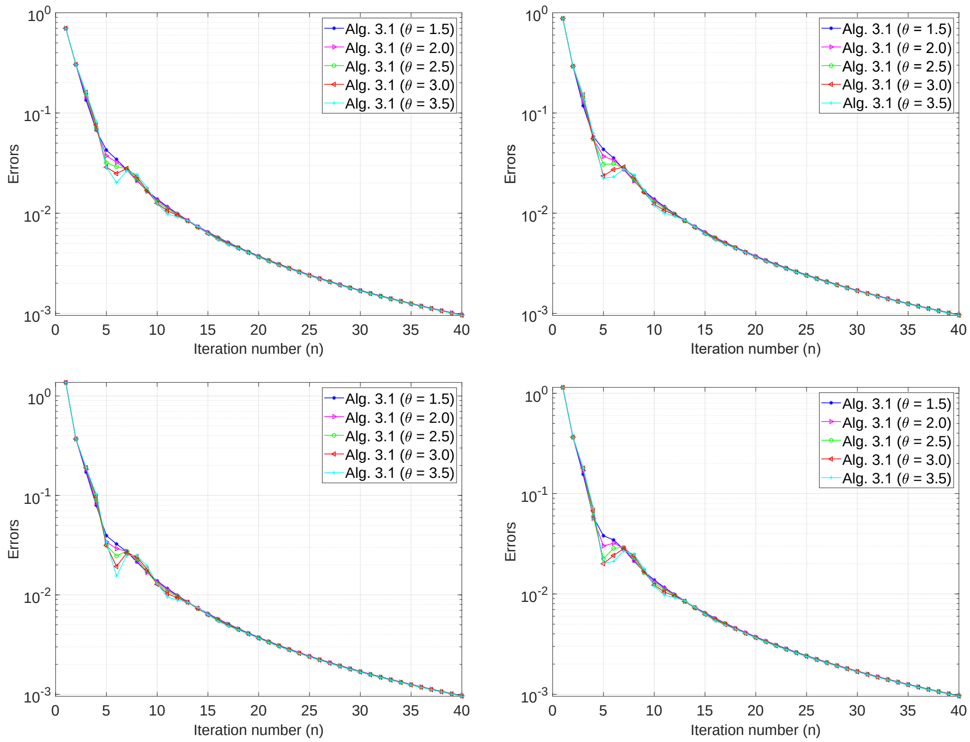

FIGURE 1. Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

Table 2. Numerical Results for Example 6.1 (Experiment 1).

Cases		$\varepsilon=\frac{1}{(n+2)^2}$	$\varepsilon=\frac{2}{(n+5)^2}$	$\varepsilon=\frac{1}{(n+3)^3}$	$\varepsilon=\frac{1}{(n+1)^4}$	$\varepsilon=\frac{2}{(n+4)^4}$
1	CPU time (sec.)	0.0166	0.0127	0.0157	0.0164	0.0157
	No. of Iter.	40	40	40	40	40
2	CPU time (sec.)	0.0122	0.0121	0.0125	0.0114	0.0159
	No. of Iter.	40	40	40	40	40
3	CPU time (sec.)	0.0128	0.0125	0.0120	0.0112	0.0160
	No. of Iter.	40	40	40	40	40
4	CPU time (sec.)	0.0134	0.0119	0.0123	0.0113	0.0159
	No. of Iter.	40	40	40	40	40

The next example is in infinite dimensional Hilbert spaces.

Example 6.2. Let $E=L^2([0,2\pi])=E_i,\ i=0,1,2,3$ with norm $\|x\|=(\int_0^{2\pi}|x(t)|^2dt)^{\frac{1}{2}}$ and the corresponding inner product $\langle x,y\rangle=\int_0^{2\pi}x(t)y(t)dt,\ \forall\, x,y\in L^2([0,2\pi])$. Suppose that $C:=\{x\in L^2([0,2\pi]):\int_0^{2\pi}(t^2+1)x(t)dt\leq 1\}$ and $C_i:=\{x\in L^2([0,2\pi]):\int_0^{2\pi}|x(t)-sin(t)|^2\leq 16\}$ are subsets of E and E_i, respectively. Define $T:L^2([0,2\pi])\to L^2([0,2\pi])$ by $T(x)(t)=\int_0^{2\pi}e^{-st}x(t)dt$ for all $x\in L^2([0,2\pi])$ and $T_ix(t)=\int_0^{2\pi}\frac{1}{10}(x(t))dt$. It is easy to see that T and T_i for $i=0,1,2,3$ are bounded linear operators. Also, for $j=1,2,\ldots,7$, let $S_j:L^2([0,2\pi])\to$

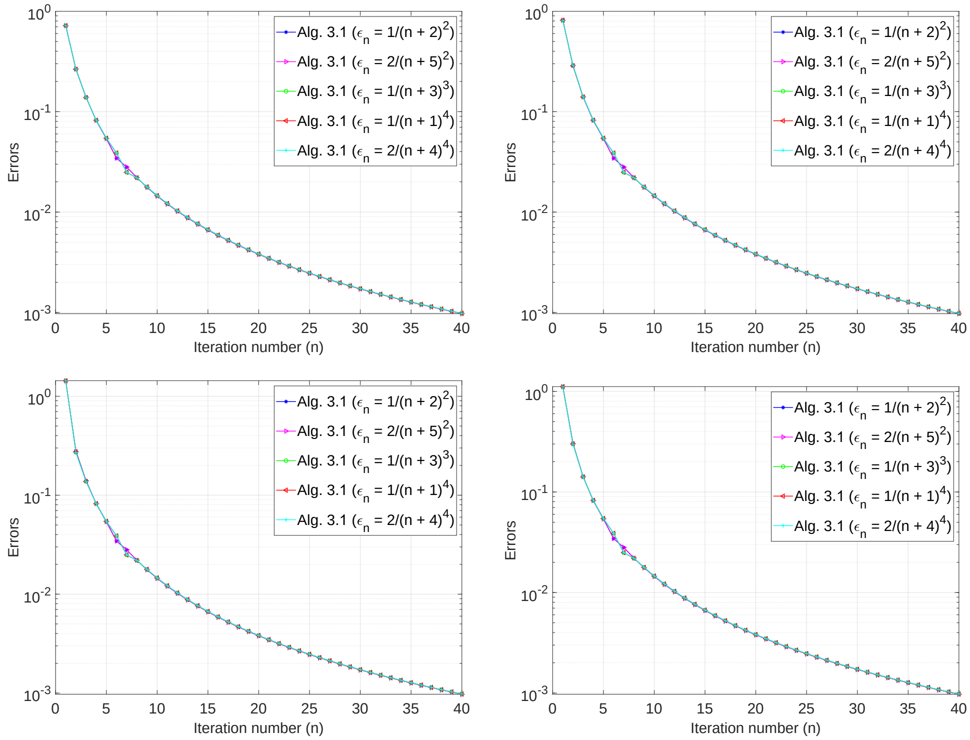

FIGURE 2. Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

$L^2([0,2\pi])$ be defined by

$$S_j x = \frac{1}{2j}x, \ \forall j = 1,2,\dots,7.$$

It is easy to see that S_j is relatively nonexpansive for each $j = 1,2,\dots,7$. Let $f = i_C$ $f_i = i_{C_i}$ be the indicator functions on C and C_i, respectively. Thus $\mathrm{Prox}_\lambda f = \Pi_C$ and $\mathrm{Prox}_\lambda f_i = \Pi_{C_i}$. Also, we choose $u = t+1$, $\alpha_n = \frac{1}{(n+1)}$, $\beta_{i,n} = \frac{1}{4}$, $\phi_{n,0} = \frac{1}{2}$, and $\phi_{n,j} = \frac{1}{2^{j+1}}$. Then Algorithm 3.1 becomes

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \sum_{i=0}^{3} \frac{1}{4}\Big(w_n - \tau_{i,n}T_i^*\big(T_i(w_n) - \mathrm{Prox}_{\lambda_n^i}^{f_i} T_i(w_n)\big)\Big), \\ z_n = \Big(\frac{1}{2}(y_n) + \sum_{j=1}^{7} \frac{1}{2^{j+1}}(S_j y_n)\Big), \\ x_{n+1} = \frac{1}{(n+2)}(t+1) + \frac{n+1}{(n+2)} z_n, \end{cases}$$

for $n \geq 1$. Suppose that, for $\varepsilon > 0$, the step size $\tau_{i,n}$ is chosen such that

$$\tau_{i,n} \in \left(\varepsilon, \frac{2\|T_i(w_n) - \big(\mathrm{Prox}_{\lambda_n^i}^{f_i}\big)T_i(w_n)\|^2}{\|T_i^*\big(I^{H_i} - \mathrm{Prox}_{\lambda_n^i}^{f_i}\big)T_i(w_n)\|^2} - \varepsilon\right), \ \forall\, n \in \Omega,$$

for small enough ε, where the index set $\Omega := \{n \in \mathbb{N} : T_i(w_n) - \big(\mathrm{Prox}_{\lambda_n^i}^{f_i} T_i(w_n) \neq 0\big)\}$, otherwise, $\tau_{i,n} = \tau_i$, where τ_i is any nonnegative real number for each $i = 0,1,2,3$. Using $\|x_{n+1} - x_n\| <$

10^{-3} as the stopping criterion, we plot the graphs of $\|x_{n+1}-x_n\|$ against the number of iterations in each cases. The numerical results are reported in Fig. 3, Fig. 4, Table 3, and Table 4.

Table 3. Numerical Results for Example 6.2 (Experiment 2).

Cases		$\theta = 0.2$	$\theta = 0.4$	$\theta = 0.6$	$\theta = 0.8$	$\theta = 1.0$
1	CPU time (sec.)	18.2522	9.5605	9.6427	9.9466	6.4868
	No. of Iter.	42	42	42	42	42
2	CPU time (sec.)	76.3470	78.2310	71.6249	73.9699	22.4244
	No. of Iter.	42	42	42	42	42
3	CPU time (sec.)	55.4661	54.4661	50.8791	51.8060	0.0161
	No. of Iter.	42	42	42	42	14
4	CPU time (sec.)	17.936	18.2225	10.8839	6.2995	7.1470
	No. of Iter.	42	42	42	42	42

Table 4. Numerical Results for Example 6.2 (Experiment 2).

Cases		$\varepsilon = \frac{1}{(n+1)^2}$	$\varepsilon = \frac{1}{(n+3)^2}$	$\varepsilon = \frac{2}{(n+1)^3}$	$\varepsilon = \frac{2}{(n+3)^3}$	$\varepsilon = \frac{2}{(n+1)^4}$
1	CPU time (sec.)	7.1705	7.0260	7.0490	7.1383	7.6806
	No. of Iter.	42	42	42	42	42
2	CPU time (sec.)	5.6297	5.5916	5.6409	6.2183	27.9355
	No. of Iter.	42	42	42	42	42
3	CPU time (sec.)	6.1491	5.6183	5.5601	6.2239	27.7076
	No. of Iter.	42	42	42	42	42
4	CPU time (sec.)	6.9287	6.7615	6.7168	6.7157	7.7159
	No. of Iter.	42	42	42	142	42

We test these examples under the following experiments.

In the first experiment, we show the behavior of our method by fixing the other parameters and varying θ and ε in Example 6.1. We do this to verify the effects of these parameters on our method.

Experiment 1. Consider the following cases for the initial values of x_0, x_1 :

Case 1 : $x_0 = (1.00, 1.25)$; $x_1 = (0.34, 1.30)$;
Case 2 : $x_0 = (1.50, 2.60)$; $x_1 = (0.75, 1.40)$;
Case 3 : $x_0 = (-2.50, 1.30)$; $x_1 = (1.75, -0.45)$;
Case 4 : $x_0 = (2.50, -1.30)$ $x_1 = (1.75, 0.45)$.

Also, we consider $\theta \in \{1.5, 2.0, 2.5, 3.0, 3.5\}$ and $\varepsilon_n \in \{\frac{1}{(n+2)^2}, \frac{2}{(n+5)^2}, \frac{1}{(n+3)^3}, \frac{1}{(n+1)^4}, \frac{2}{(n+4)^4}\}$ which satisfies Assumption 3.1 (5)(ii-iii). We use Algorithm 3.1 for the experiment and report the numerical results in Fig. 1, Fig. 2, Table 1, and Table 2.

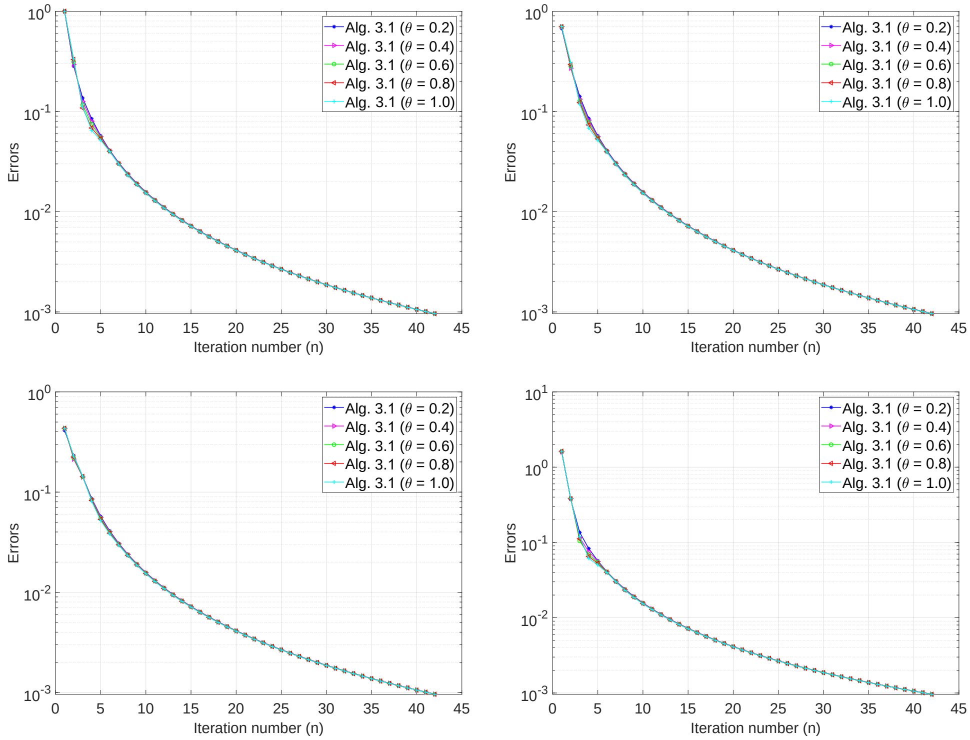

FIGURE 3. Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

In the second experiment, we show the behavior of our method by fixing the other parameters and varying θ and ε in Example 6.1. We do this to verify the effects of these parameters on our method.

Experiment 2.

Consider the following cases for the initial values of x_0, x_1 :

Case 1 : $x_0 = t+5;\ x_1 = t^3+t+1;$
Case 2 : $x_0 = e^{2t};\ x_1 = \frac{3}{10}e^{2t};$
Case 3 : $x_0 = e^{2t};\ x_1 = t+1;$
Case 4 : $x_0 = t^2+t+3;\ x_1 = t+2.$

Consider $\theta \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$ and $\varepsilon_n \in \{\frac{1}{(n+1)^2}, \frac{1}{(n+3)^2}, \frac{2}{(n+1)^3}, \frac{2}{(n+3)^3}, \frac{1}{(n+1)^4}\}$, which satisfies Assumption 3.1 (5)(ii-iii). We use Algorithm 3.1 for the experiment and report the numerical results in Fig. 3, Fig. 4, Table 3, and Table 2.

7. CONCLUSION

We introduced and studied the Split Minimization Problem with Multiple Output Sets. We further proposed an efficient algorithm for finding a common element in the solution set of the split minimization problem with multiple output sets and the fixed point problem for a finite family of Bregman relatively nonexpansive mappings in the framework of p-uniformly convex Banach spaces, which are also uniformly smooth. The proposed algorithm was proved to be strongly convergent under mild conditions on the control parameters. The proof of the strong

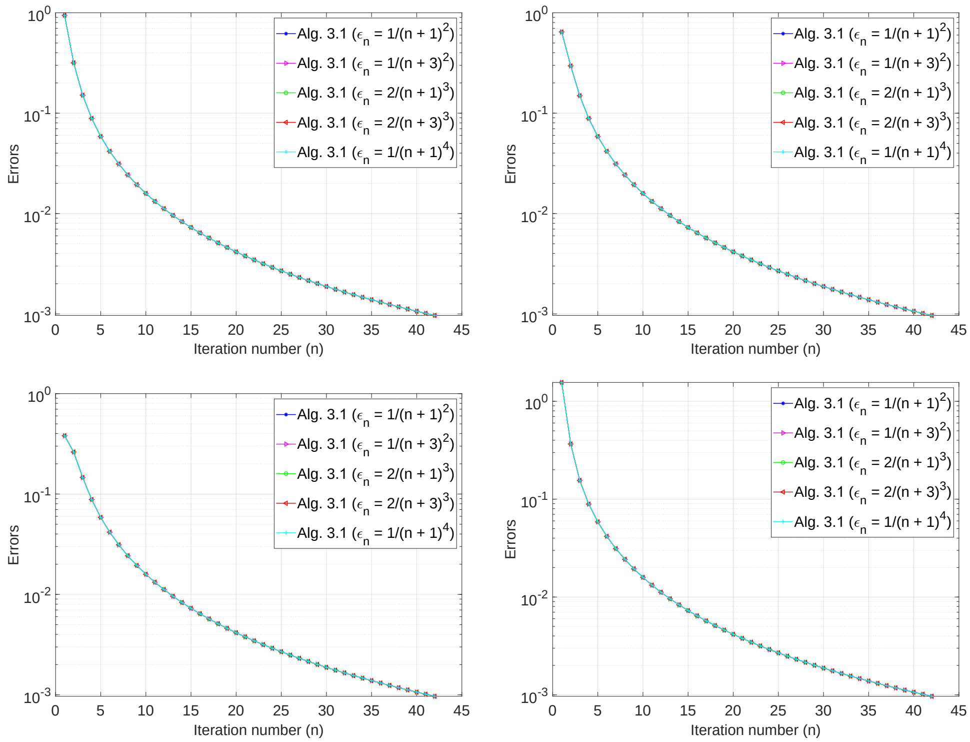

FIGURE 4. Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

convergence theorem is concise as it does not follow the conventional "Two-case Approach" often used in the literature to guarantee strong convergence. Furthermore, we presented some applications of the proposed method and demonstrate with numerical experiments the usefulness of the proposed method. In our numerical experiments, we checked the sensitivity of key parameters for each starting points in order to know if their choices affect the performance of our methods. We can see from the tables and figures that the number of iterations and CPU times for our proposed method remain consistent and well-behaved for different choices of these key parameters. Our result unifies and extends numerous results in the literature in this direction of research.

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