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BERGMAN-MORREY TYPE SPACES AND VOLTERRA INTEGRAL OPERATORS

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Abstract. A family of Bergman-Morrey type spaces in the unit disc are introduced. The boundedness of the embedding from Bergman-Morrey type spaces to a class of tent spaces is studied. The boundedness, compactness, norm and essential norm of Volterra integral operators on Bergman-Morrey type spaces are also investigated in this paper.
Keywords. Bergman space; Carleson measure; Volterra integral operator.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disc in the complex plane, and let $H(\mathbb{D})$ be the set of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, a function $f \in H(\mathbb{D})$ belongs to the Bergman space A^p if $\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < \infty$, where dA denotes the normalized area measure on $\mathbb{D}$. The Bloch space $\mathscr{B}$ consists of all $f \in H(\mathbb{D})$ for which $\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$. The little Bloch space $\mathscr{B}_0$ consists of all $f \in H(\mathbb{D})$ such that $\lim_{|z|\to 0}(1-|z|^2)|f'(z)| = 0$. Let H^∞ denote the bounded analytic function space. It is well known that $H^\infty \subset \mathscr{B}$.

Let $0 < p < \infty$, $-2 < q < \infty$, and $0 \le s < \infty$. A function $f \in H(\mathbb{D})$ belongs to the general function space $F(p,q,s)$, which introduced by Zhao in [1], if

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dA(z) < \infty,$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$. When $q+s > -1$, the space $F(p,q,s)$ is nontrivial. By the classical Littlewood-Paley formula, $F(p,p,0)$ is just the Bergman space A^p, $F(p,p-2,0)$ is the Besov space B_p, $F(p,q,0)$ is the Dirichlet type space $\mathscr{D}_q^p$, and $F(2,0,1)$ is the BMOA space. If $s > 1$, then $F(p,p-2,s) = \mathscr{B}$.

Let $g, f \in H(\mathbb{D})$. The Volterra integral operators T_g and I_g induced by g are defined by

$$T_g f(z) = \int_0^z f(w)g'(w)dw, \; I_g f(z) = \int_0^z f'(w)g(w)dw, \; z \in \mathbb{D},$$

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respectively. The operators T_g and I_g are closely related the multiplication operator M_g as $T_gf + I_gf = M_gf - f(0)g(0)$, where $M_gf(z) = f(z)g(z)$, $f \in H(\mathbb{D})$. Moreover, the operator T_g is the generalization of the Cesàro operator. The operator T_g was introduced by Pommerenke in [2]. In [2], Pommerenke showed that T_g is bounded on the Hardy space H^2 if and only if $g \in BMOA$. In [3], the authors proved that T_g is bounded on $H^p (p \geqslant 1)$ if and only if $g \in BMOA$. Recently, the boundedness, compactness, norm and essential norm of T_g and I_g between various function spaces were investigated; see, e.g., [3]-[19] for more results of T_g and I_g.

Throughout this paper, we always assume that $K : [0,\infty) \to [0,\infty)$ is a nondecreasing and right-continuous function, not identically zero. In [20], Wulan and Zhou defined a Morrey type space H^2_K, which consisting of all $f \in H(\mathbb{D})$ such that

$$\sup_{a\in\mathbb{D}} \frac{1-|a|^2}{K(1-|a|^2)} \|f\circ\sigma_a(z) - f(a)\|^2_{H^2} < \infty.$$

$H^2_K = BMOA$ when $K(t) = t$. When $K(t) = t^{\lambda} (0 < \lambda < 1)$, H^2_K gives the Morrey space $\mathscr{L}^{2,\lambda}$, which was introduced and studied by Wu and Xie in [21]. In [8], the authors studied the boundedness of T_g and I_g on $\mathscr{L}^{2,\lambda} (0 < \lambda < 1)$. In [14], Qian and Li investigated the boundedness of T_g and I_g on the space H^2_K. In [16], Shi and Li investigated the essential norm and compactness of T_g and I_g on H^2_K.

Let $0 \le \lambda \le 2$ and $p > 0$. Recently, Yang and Liu in [18] defined a Bergman-Morrey space $A^{p,\lambda}$, which consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^{p,\lambda}} = |f(0)| + \sup_{a\in\mathbb{D}} (1-|a|^2)^{\frac{2-\lambda}{p}} \|f\circ\sigma_a - f(a)\|_{A^p} < \infty.$$

Moreover, $A^{p,0} = A^p$. They characterized the boundedness of the identity operator $I_d : A^{p,\lambda} \to \mathscr{T}_s^p(\mu)$ and studied the boundedness, compactness, norm and essential norm of operators T_g and I_g on $A^{p,\lambda}$.

In this paper, inspired by [14, 16, 17, 18, 20], we define a new Bergman-Morrey type space $A^{p,K}$ as follows. Let $f \in H(\mathbb{D})$ and $0 < p < \infty$. We say that f belongs to the Bergman-Morrey type space $A^{p,K}$ if

$$\|f\|^p_{A^{p,K}} = |f(0)|^p + \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \|f\circ\sigma_a(z) - f(a)\|^p_{A^p} < \infty.$$

If $K(t) = t^{\lambda}$, $0 < \lambda < 2$, then $A^{p,K} = A^{p,\lambda}$.

Let $0 < p < \infty$, μ be a positive Borel measure on $\mathbb{D}$, and $|I|$ be the normalized arc length of I. We denote by $\mathscr{T}^p_K(\mu)$ the set of all measure functions f on $\mathbb{D}$ such that

$$\|f\|^p_{\mathscr{T}^p_K(\mu)} = \sup_{I\subseteq\partial\mathbb{D}} \frac{1}{K(|I|)} \int_{\mathbb{D}} |f(z)|^p d\mu(z) < \infty.$$

In this paper, we investigate some basic properties of Bergman-Morrey type spaces $A^{p,K}$ and study the boundedness of the identity operator $I_d : A^{p,K} \to \mathscr{T}^p_K(\mu)$. Moreover, we completely characterize the boundedness, compactness, norm and essential norm of the operators T_g and I_g on $A^{p,K}$.

In this paper, we say that $f \lesssim g$ if there exists a constant C such that $f \le Cg$. The symbol $f \approx g$ means that $f \lesssim g \lesssim f$.

2. Embedding Maps from $A^{p,K}$ to $\mathscr{T}_K^p(\mu)$

In this section, we consider the boundedness of the identity operator $I_d : A^{p,K} \to \mathscr{T}_K^p(\mu)$. First, let us state some notations and some lemmas, which are used in the proof of main results.

Let μ be a positive Borel measure on $\mathbb{D}$. μ is called a K-Carleson measure if (see [17])

$$\|\mu\|_K = \sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{K(|I|)} < \infty,$$

where $S(I) = \{z = re^{i\theta} \in \mathbb{D} : 1-|I| \le r < 1, e^{i\theta} \in I\}$. If $K(t) = t^s (0 < s < \infty)$, then μ is called an $s-$Carleson measure and $\|\mu\|_s = \sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s}$.

For our purpose, in the rest of this paper, we assume that K satisfies (see, e.g., [17, 22])

$$\int_1^\infty \frac{\varphi_K(x)}{x^{1+\delta}} dx < \infty, \ \ \delta > 0, \tag{2.1}$$

where

$$\varphi_K(x) = \sup_{0<s\le 1} \frac{K(sx)}{K(s)}, \ 0 < x < \infty.$$

Lemma 2.1. [17, Theorem 2.1] *Let K satisfy (2.1) for some $\delta \in (0,2)$. μ is a K-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}} \frac{1}{K(1-|a|^2)} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}z|}\right)^t d\mu(z) < \infty, \delta \le t < \infty.$$

Proposition 2.1. *Let $0 < p < \infty$, $f \in H(\mathbb{D})$, and K satisfy (2.1) for some $\delta \in (0,2)$. Then $f \in A^{p,K}$ if and only if*

$$\sup_{I\subset\partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f'(z)|^p (1-|z|^2)^p dA(z) < \infty.$$

Proof. Given any arc $I \subset \partial\mathbb{D}$, let $a = (1-|I|)\xi$, where ξ is the center of I. We have

$$|1-\bar{a}z| \approx 1-|a|^2 \approx |I|, \ z \in S(I).$$

Let $d\mu_f(z) = |f'(z)|^p (1-|z|^2)^p dA(z)$. By Lemma 2.1, we obtain

$$\begin{aligned}
\|f\|_{A^{p,K}}^p &\approx \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \|f\circ\sigma_a(z) - f(a)\|_{A^p}^p \\
&= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p |\sigma_a'(z)|^2 dA(z) \\
&= \sup_{a\in\mathbb{D}} \frac{1}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p \left(\frac{1-|a|^2}{|1-\bar{a}z|}\right)^4 dA(z). \\
&= \sup_{a\in\mathbb{D}} \frac{1}{K(1-|a|^2)} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}z|}\right)^4 d\mu_f(z) \\
&\approx \sup_{I\subset\partial\mathbb{D}} \frac{\mu_f(S(I))}{K(|I|)} \\
&= \sup_{I\subset\partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f'(z)|^p (1-|z|^2)^p dA(z).
\end{aligned}$$

Then the desired result immediately follows. □

Remark 2.1. From the proof of Proposition 2.1, we see that the following statements are equivalent.

(i) $f \in A^{p,K}$;

(ii)

$$M_1 := \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p |\sigma_a'(z)|^2 dA(z) < \infty;$$

(iii)

$$M_2 := \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\sigma_a(z)|^2)^2 dA(z) < \infty.$$

Moreover,

$$\|f\|^p_{A^{p,K}} \approx M_1 \approx M_2.$$

Lemma 2.2. [23] *Let $s,t>0$, $r>-1$, and $s+t-r>2$. If $t<2+r<s$, then*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^r}{|1-\bar{b}z|^s|1-\bar{a}z|^t} dA(z) \lesssim \frac{1}{(1-|b|^2)^{s-r-2}|1-\bar{b}a|^t},\ a,b\in\mathbb{D}.$$

Lemma 2.3. [17] *Let $0<\alpha\le\beta<\infty$ and K satisfy (2.1) for some $\delta\in(0,2)$. Then, for sufficiently small positive constants $c<\delta$,*

$$\frac{K(\beta)}{K(\alpha)} \le \left(\frac{\beta}{\alpha}\right)^{\delta-c} \le \left(\frac{\beta}{\alpha}\right)^{\delta}.$$

Proposition 2.2. *Let $0<p<\infty$, $b\in\mathbb{D}$ and K satisfy (2.1) for some $\delta\in(0,2)$. Then the function*

$$f_b(z) = \frac{(1-|b|^2)^{\frac{2}{p}} K^{\frac{1}{p}}(1-|b|^2)}{(1-\bar{b}z)^{\frac{4}{p}}},\ z\in\mathbb{D},$$

belongs to $A^{p,K}$. Moreover, $\|f_b\|_{A^{p,K}} \lesssim 1$.

Proof. Using Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
\|f_b\|^p_{A^{p,K}} &\lesssim \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f_b'(z)|^p (1-|z|^2)^{p-2} (1-|\sigma_a(z)|^2)^2 dA(z) \\
&\lesssim \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^4 K(1-|b|^2)(1-|b|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{(1-|z|^2)^p}{|1-\bar{b}z|^{4+p}|1-\bar{a}z|^4} dA(z) \\
&\lesssim \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^4 K(1-|b|^2)(1-|b|^2)^2}{K(1-|a|^2)} \frac{1}{(1-|b|^2)^2|1-\bar{a}b|^4} \\
&\lesssim \sup_{a\in\mathbb{D}} \frac{K(|1-\bar{a}b|)}{K(1-|a|^2)} \left(\frac{1-|a|^2}{|1-\bar{a}b|}\right)^4 \\
&\lesssim \sup_{a\in\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}b|}\right)^{4-\delta} \lesssim 2^{4-\delta} \lesssim 1,
\end{aligned}$$

as desired. □

Proposition 2.3. *Let $0 < p < \infty$ and K satisfy (2.1) for some $\delta \in (0,2)$. Then, for any $f \in A^{p,K}$,*

$$|f(z) - f(0)| \lesssim \frac{K^{\frac{1}{p}}(1-|z|^2)}{(1-|z|^2)^{\frac{2}{p}}}\|f\|_{A^{p,K}},\ z \in \mathbb{D}.$$

Proof. For $z \in \mathbb{D}$ and $r > 0$, set $\mathbb{D}(z,r) = \{w \in \mathbb{D} : \beta(z,w) < r\}$. From [24], we see that

$$\frac{(1-|z|^2)^2}{|1-\bar{z}w|^4} \approx \frac{1}{(1-|w|^2)^2} \approx \frac{1}{(1-|z|^2)^2},$$

when $w \in \mathbb{D}(z,r)$. Hence,

$$\begin{aligned}
|f'(z)|^p &\lesssim \frac{1}{(1-|z|^2)^p}\int_{\mathbb{D}(z,r)} |f'(w)|^p(1-|w|^2)^{p-2}dA(w) \\
&\lesssim \frac{1}{(1-|z|^2)^p}\int_{\mathbb{D}(z,r)} |f'(w)|^p(1-|w|^2)^{p-2}(1-|\sigma_z(w)|^2)^2 dA(w) \\
&\lesssim \frac{K(1-|z|^2)}{(1-|z|^2)^{p+2}}\|f\|^p_{A^{p,K}}.
\end{aligned}$$

Therefore,

$$|f'(z)| \lesssim \frac{K^{\frac{1}{p}}(1-|z|^2)}{(1-|z|^2)^{\frac{2}{p}+1}}\|f\|_{A^{p,K}}.$$

By Lemma 2.3, there exists a constant $c \in (0, 2-\delta)$ such that

$$\begin{aligned}
|f(a) - f(0)| &= \left| a\int_0^1 f'(az)dz \right| \lesssim \|f\|_{A^{p,K}} \int_0^1 \frac{|a|K^{\frac{1}{p}}(1-|az|^2)}{(1-|az|^2)^{\frac{2}{p}+1}}dz \\
&\lesssim \|f\|_{A^{p,K}} \frac{K^{\frac{1}{p}}(1-|a|^2)}{(1-|a|^2)^{\frac{\delta-c}{p}}} \int_0^1 (1-|az|)^{\frac{\delta-c-2}{p}-1}|a|dz \\
&\lesssim \frac{K^{\frac{1}{p}}(1-|a|^2)}{(1-|a|^2)^{\frac{2}{p}}}\|f\|_{A^{p,K}}.
\end{aligned}$$

□

Proposition 2.4. *Let $0 < p < \infty$. Then $A^{p,K} \subseteq A^p$. Moreover, $A^{p,K} = A^p$ if and only if $K(0) > 0$.*

Proof. Let $f \in A^{p,K}$. By Remark 2.1, we have

$$\begin{aligned}
\infty &> \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)}\int_{\mathbb{D}} |f'(w)|^p(1-|w|^2)^{p-2}(1-|\sigma_a(w)|^2)^2 dA(w) \\
&= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)}\int_{\mathbb{D}} |f'(w)|^p(1-|w|^2)^p \frac{(1-|a|^2)^2}{|1-\bar{a}w|^4}dA(w) \\
&\geq \frac{1}{K(1)}\int_{\mathbb{D}} |f'(w)|^p(1-|w|^2)^p dA(w).
\end{aligned}$$

So, $f \in A^p$, that is, $A^{p,K} \subseteq A^p$.

Next, we prove that $A^{p,K} = A^p$ if and only if $K(0) > 0$. First, we assume $A^{p,K} = A^p$. For any $\gamma \in \mathbb{D}$, set

$$f_\gamma(z) = \int_0^z \frac{(1-|\gamma|^2)dw}{(1-\bar{\gamma}w)^{2+\frac{2}{p}}},\ z \in \mathbb{D}.$$

Applying Lemma 3.10 in [24], one has

$$\|f_\gamma\|_{A^p}^p = \int_{\mathbb{D}} |f_\gamma'(z)|^p (1-|z|^2)^p dA(z) = \int_{\mathbb{D}} \frac{(1-|\gamma|^2)^p}{|1-\bar{\gamma}z|^{2p+2}} (1-|z|^2)^p dA(z) \lesssim 1.$$

Thus $f_\gamma \in A^p$. It follows that

$$\begin{aligned}
\infty > \|f_\gamma\|_{A^p}^p = \|f_\gamma\|_{A^{p,K}}^p &= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f_\gamma'(z)|^p (1-|z|^2)^{p-2} (1-|\sigma_a(z)|^2)^2 dA(z) \\
&\gtrsim \frac{(1-|\gamma|^2)^2}{K(1-|\gamma|^2)} \int_{\mathbb{D}} |f_\gamma'(z)|^p (1-|z|^2)^{p-2} (1-|\sigma_\gamma(z)|^2)^2 dA(z) \\
&\gtrsim \frac{(1-|\gamma|^2)^{p+4}}{K(1-|\gamma|^2)} \int_{\mathbb{D}(z,r)} \frac{(1-|z|^2)^p}{|1-\bar{\gamma}z|^{2p+6}} dA(z) \\
&\approx \frac{1}{K(1-|\gamma|^2)},
\end{aligned}$$

which implies that $K(0) > 0$.

Conversely, assume that $f \in A^p$ and $K(0) > 0$. Using the monotonicity of K, one has

$$\begin{aligned}
\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \|f\circ\sigma_a(z) - f(a)\|_{A^p}^p &\lesssim \sup_{a\in\mathbb{D}} \frac{1}{K(0)} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p \frac{(1-|a|^2)^4}{|1-\bar{a}z|^4} dA(z) \\
&\lesssim \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p dA(z) < \infty.
\end{aligned}$$

Therefore, $f \in A^{p,K}$. Furthermore, $A^{p,K} = A^p$. This completes the proof. □

Lemma 2.4. [25] *Let* $1 < p < \infty, s > -1, t \geq 0$ *such that* $t < 2+s$. *If* $f \in H(\mathbb{D})$, *then*

$$\int_{\mathbb{D}} |f(z)-f(0)|^p \frac{(1-|z|^2)^s}{|1-\bar{w}z|^t} dA(z) \lesssim \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|^2)^{s+p}}{|1-\bar{w}z|^t} dA(z),\ w \in \mathbb{D}.$$

Now we are in a position to state and prove the main result in this section.

Theorem 2.1. *Let* $1 < p < \infty$, μ *be a positive Borel measure on* $\mathbb{D}$, *and* K *satisfy* (2.1) *for some* $\delta \in (0,2)$. *Then* $I_d : A^{p,K} \to \mathcal{T}_K^p(\mu)$ *is bounded if and only if* μ *is a 2-Carleson measure.*

Proof. Assume that μ is a 2-Carleson measure. For any $I \subset \partial\mathbb{D}$, let ξ be the center of I and $a = (1-|I|)\xi$. For any $f \in A^{p,K}$,

$$\begin{aligned}
\frac{1}{K(|I|)} \int_{S(I)} |f(z)|^p d\mu(z) &\lesssim \frac{1}{K(|I|)} \int_{S(I)} |f(a)|^p d\mu(z) + \frac{1}{K(|I|)} \int_{S(I)} |f(z)-f(a)|^p d\mu(z) \\
&= G_1 + G_2.
\end{aligned}$$

By Proposition 2.3, one has

$$G_1 = \frac{1}{K(|I|)} \int_{S(I)} |f(a)|^p d\mu(z) \leq \frac{1}{K(|I|)} \int_{S(I)} \frac{K(|I|)}{|I|^2} \|f\|_{A^{p,K}}^p d\mu(z) \lesssim \|f\|_{A^{p,K}}^p.$$

Since μ is a 2-Carleson measure, we see from [24] that $I_d : A^p \to L^p(\mu)$ is bounded. By the fact that $\int_{\mathbb{D}} |f(z)|^p dA(z) \approx \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p dA(z)$, we have

$$G_2 = \frac{1}{K(|I|)} \int_{S(I)} |f(z)-f(a)|^p d\mu(z) \lesssim \frac{(1-|a|^2)^4}{K(1-|a|^2)} \int_{S(I)} \left| \frac{f(z)-f(a)}{(1-\bar{a}z)^{\frac{4}{p}}} \right|^p d\mu(z)$$
$$\lesssim \frac{(1-|a|^2)^4}{K(1-|a|^2)} \int_{\mathbb{D}} \left| \frac{d}{dz} \frac{f(z)-f(a)}{(1-\bar{a}z)^{\frac{4}{p}}} \right|^p (1-|z|^2)^p dA(z).$$

Since

$$\frac{d}{dz} \frac{f(z)-f(a)}{(1-\bar{a}z)^{\frac{4}{p}}} = \frac{f'(z)}{(1-\bar{a}z)^{\frac{4}{p}}} + \frac{4\bar{a}}{p} \frac{f(z)-f(a)}{(1-\bar{a}z)^{\frac{4}{p}+1}},$$

we deduce that $G_2 \lesssim Q+J$, where

$$Q = \frac{(1-|a|^2)^4}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|f'(z)|^p}{|1-\bar{a}z|^4} (1-|z|^2)^p dA(z)$$

and

$$J = \frac{(1-|a|^2)^4}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^p}{|1-\bar{a}z|^{4+p}} (1-|z|^2)^p dA(z).$$

Clearly,

$$Q = \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\sigma_a(z)|^2)^2 dA(z) \lesssim \|f\|^p_{A^{p,K}}.$$

Making the change of variable $w = \sigma_a(z)$, by Lemma 2.4, we obtain

$$\begin{aligned}
J &= \frac{(1-|a|^2)^4}{K(1-|a|^2)} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^p}{|1-\bar{a}z|^{4+p}} (1-|z|^2)^p dA(z) \\
&= \frac{1}{K(1-|a|^2)} \int_{\mathbb{D}} |f\circ\sigma_a(w) - f\circ\sigma_a(0)|^p \frac{(1-|w|^2)^p (1-|a|^2)^2}{|1-\bar{a}w|^p} dA(w) \\
&\lesssim \frac{1}{K(1-|a|^2)} \int_{\mathbb{D}} |(f\circ\sigma_a)'(w)|^p \frac{(1-|w|^2)^{2p} (1-|a|^2)^2}{|1-\bar{a}w|^p} dA(w) \\
&= \frac{1}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(\sigma_a(w))|^p (1-|\sigma_a(w)|^2)^p \frac{(1-|w|^2)^p (1-|a|^2)^2}{|1-\bar{a}w|^p} dA(w) \\
&= \frac{1}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p \frac{(1-|\sigma_a(z)|^2)^p (1-|a|^2)^2}{|1-\bar{a}\sigma_a(z)|^p} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z) \\
&= \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|^2)^{2p} (1-|a|^2)^2}{|1-\bar{a}z|^{p+4}} dA(z) \\
&= \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|^2)^{2p} (1-|a|^2)^2}{|1-\bar{a}z|^{p+4}} dA(z) \\
&\lesssim \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\sigma_a(z)|^2)^2 dA(z) \lesssim \|f\|^p_{A^{p,K}}.
\end{aligned}$$

Thus $G_2 \lesssim \|f\|^p_{A^{p,K}}$. Therefore, for all $f \in A^{p,K}$, $\|f\|_{\mathcal{T}^p_K(\mu)} \lesssim \|f\|_{A^{p,K}}$, as desired.

Conversely, assume that $I_d : A^{p,K} \to \mathscr{T}_K^p(\mu)$ is bounded. For $I \subset \partial\mathbb{D}$, let ξ be the center of I and $b = (1-|I|)\xi$. It is known that

$$|1-\bar{b}z| \approx 1-|b|^2 \approx |I|,\ z \in S(I).$$

Using the function f_b, given in Proposition 2.2, we find

$$\frac{\mu(S(I))}{|I|^2} \approx \frac{1}{K(|I|)}\int_{S(I)} |f_b(z)|^p d\mu(z) \lesssim \|f_b\|^p_{\mathscr{T}_K^p(\mu)} \lesssim \|f_b\|^p_{A^{p,K}} < \infty,$$

which implies that μ is a 2-Carleson measure. □

3. Integral Operators T_g and I_g

In this section, we characterize the boundedness, compactness and essential norm of the operators T_g and I_g on $A^{p,K}$.

Theorem 3.1. *Let $g \in H(\mathbb{D}), 1 < p < \infty$ and K satisfy (2.1) for some $\delta \in (0,2)$. Then $T_g : A^{p,K} \to A^{p,K}$ is bounded if and only if $g \in \mathscr{B}$. Moreover, $\|T_g\| \approx \|g\|_{\mathscr{B}}$.*

Proof. Assume that $g \in \mathscr{B}$. From [1], we have

$$\begin{aligned}\|g\|^p_{\mathscr{B}} &\approx \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^2 dA(z)\\ &\approx \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^2}\int_{S(I)} |g'(z)|^p(1-|z|^2)^p dA(z) \approx \sup_{I\subset\partial\mathbb{D}} \frac{\mu_g(S(I))}{|I|^2},\end{aligned}$$

where $\mu_g = |g'(z)|^p(1-|z|^2)^p dA(z)$. This means that μ_g is a 2-Carleson measure. By Theorem 2.1, $I_d : A^{p,K} \to \mathscr{T}_K^p(\mu_g)$ is bounded. Letting $f \in A^{p,K}$, we have

$$\begin{aligned}\|T_g f\|^p_{A^{p,K}} &\approx \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |(T_g f)'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^2 dA(z)\\ &= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f(z)|^p|g'(z)|^p(1-|z|^2)^p \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z)\\ &\approx \sup_{I\subset\partial\mathbb{D}} \frac{1}{K(|I|)}\int_{S(I)} |f(z)|^p|g'(z)|^p(1-|z|^2)^p dA(z)\\ &\approx \sup_{I\subset\partial\mathbb{D}} \frac{1}{K(|I|)}\int_{S(I)} |f(z)|^p d\mu_g(z)\\ &= \|f\|^p_{\mathscr{T}_K^p(\mu_g)} \lesssim \|f\|^p_{A^{p,K}} \frac{\mu_g(S(I))}{|I|^2}\\ &\lesssim \|f\|^p_{A^{p,K}}\|g\|^p_{\mathscr{B}} < \infty.\end{aligned}$$

Therefore $T_g : A^{p,K} \to A^{p,K}$ is bounded.

Conversely, suppose that $T_g : A^{p,K} \to A^{p,K}$ is bounded. For any $b \in \mathbb{D}$, let

$$f_b(z) = \frac{(1-|b|^2)^{\frac{2}{p}} K^{\frac{1}{p}}(1-|b|^2)}{(1-\bar{b}z)^{\frac{4}{p}}},\ z \in \mathbb{D}.$$

By Proposition 2.2, $f_b \in A^{p,K}$ and $\|f_b\|_{A^{p,K}} \lesssim 1$. Thus $\|T_g f_b\|_{A^{p,K}} \lesssim \|T_g\| \|f_b\|_{A^{p,K}} \lesssim \|T_g\|$. For any $r > 0$,

$$\begin{aligned}\infty &> \|T_g f_b\|_{A^{p,K}}^p \\ &\gtrsim \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |(T_g f_b)'(z)|^p (1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^2 dA(z) \\ &\geqslant \frac{(1-|b|^2)^2}{K(1-|b|^2)} \int_{\mathbb{D}} |f_b(z)|^p |g'(z)|^p (1-|z|^2)^{p-2}(1-|\sigma_b(z)|^2)^2 dA(z) \\ &\geqslant \int_{\mathbb{D}(b,r)} |g'(z)|^p (1-|z|^2)^{p-2}(1-|\sigma_b(z)|^2)^2 dA(z) \\ &\gtrsim |g'(b)|^p (1-|b|^2)^p,\end{aligned}$$

which implies that $g \in \mathscr{B}$. From the above proof, we see that $\|T_g\| \approx \|g\|_{\mathscr{B}}$. □

Theorem 3.2. *Let $g \in H(\mathbb{D}), 1 < p < \infty$ and K satisfy (2.1) for some $\delta \in (0,2)$. Then $I_g : A^{p,K} \to A^{p,K}$ is bounded if and only if $g \in H^\infty$. Moreover, $\|I_g\| \approx \|g\|_\infty$.*

Proof. Let $g \in H^\infty$. For any $f \in A^{p,K}$, we have

$$\begin{aligned}\|I_g f\|_{A^{p,K}}^p &\approx \sup_{I\subset\partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |(I_g f)'(z)|^p (1-|z|^2)^p dA(z) \\ &\approx \sup_{I\subset\partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f'(z)|^p |g(z)|^p (1-|z|^2)^p dA(z) \\ &\lesssim \|f\|_{A^{p,K}}^p \|g\|_\infty^p,\end{aligned}$$

which implies that $I_g : A^{p,K} \to A^{p,K}$ is bounded and $\|I_g\| \lesssim \|g\|_\infty$.

Conversely, assume that $I_g : A^{p,K} \to A^{p,K}$ is bounded. For any $0 \neq b \in \mathbb{D}$ and $r > 0$, take f_b in Proposition 2.2. By the Proposition 4.13 in [24], we arrive at

$$\begin{aligned}\|I_g\|^p &\gtrsim \|I_g f_b\|_{A^{p,K}}^p \approx \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f_b'(z)|^p |g(z)|^p (1-|z|^2)^p |\sigma_a'(z)|^2 dA(z) \\ &\geq \int_{\mathbb{D}} \frac{(1-|b|^2)^4}{|1-\bar{b}z|^{4+p}} |g(z)|^p (1-|z|^2)^p |\sigma_b'(z)|^2 dA(z) \\ &= \int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^{-2} (1-|\sigma_b|^2)^2 dA(z) \\ &\gtrsim \frac{1}{|\mathbb{D}(b,r)|} \int_{\mathbb{D}(b,r)} |g(z)|^p dA(z) \gtrsim |g(b)|^p.\end{aligned}$$

Since b is arbitrary, we obtain $\|I_g\| \gtrsim \|g\|_\infty$. The proof is complete. □

Finally, we study the essential norm of T_g and I_g.

Lemma 3.1. [26, Theorem 3.9] *For $g \in \mathscr{B}$, $\limsup_{r\to 1^-} \|g - g_r\|_{\mathscr{B}} \approx \limsup_{|z|\to 1} |g'(z)|(1-|z|^2)$ where $g_r(z) = g(rz), 0 < r < 1, z \in \mathbb{D}$.*

Similar to the proof of [19, Lemma 5], we have the following result.

Lemma 3.2. *Let $g \in H(\mathbb{D})$, $1 < p < \infty$, and K satisfy (2.1) for some $\delta \in (0,2)$. If $0 < r < 1$ and $g \in \mathscr{B}$, then $T_{g_r} : A^{p,K} \to A^{p,K}$ is compact.*

Theorem 3.3. *Let $g \in H(\mathbb{D})$, $1 < p < \infty$ and K satisfy (2.1) for some $\delta \in (0,2)$ such that $T_g : A^{p,K} \to A^{p,K}$ is bounded. Then $\|T_g\|_{e,A^{p,K}\to A^{p,K}} \approx \limsup_{|z|\to 1} |g'(z)|(1-|z|^2)$.*

Proof. For $0 < r < 1$, by Lemma 3.2, we see that $T_{g_r} : A^{p,K} \to A^{p,K}$ is compact. Then, by Theorem 3.1,

$$\|T_g\|_{e,A^{p,K}\to A^{p,K}} \leqslant \|T_g - T_{g_r}\| = \|T_{g-g_r}\| \approx \|g - g_r\|_{\mathscr{B}}.$$

Using Lemma 3.1, we have

$$\|T_g\|_{e,A^{p,K}\to A^{p,K}} \lesssim \limsup_{r\to 1^-} \|g - g_r\|_{\mathscr{B}} \approx \limsup_{|z|\to 1} |g'(z)|(1-|z|^2).$$

On the other hand, let $\{c_j\} \subset \mathbb{D}$ such that $\lim_{j\to\infty} |c_j| = 1$. For each j, let

$$f_j(z) = \frac{(1-|c_j|^2)^{\frac{2}{p}} K^{\frac{1}{p}}(1-|c_j|^2)}{(1-\overline{c_j}z)^{\frac{4}{p}}}.$$

It is easy to check that $f_j \in A^{p,K}$ and $\{f_j\}$ converges to zero uniformly on every compact subsets of $\mathbb{D}$. Let $K : A^{p,K} \to A^{p,K}$ be a compact operator. Using [27, Lemma 2.10], we have $\lim_{j\to\infty} \|Kf_j\|_{A^{p,K}} = 0$. So

$$\begin{aligned}
&\|T_g - K\| \gtrsim \limsup_{j\to\infty} \|(T_g - K)f_j\|_{A^{p,K}} \\
&\gtrsim \limsup_{j\to\infty} \left(\|T_g f_j\|_{A^{p,K}} - \|Kf_j\|_{A^{p,K}}\right) = \limsup_{j\to\infty} \|T_g f_j\|_{A^{p,K}} \\
&\gtrsim \limsup_{j\to\infty} \left(\frac{(1-|c_j|^2)^2}{K(1-|c_j|^2)} \int_{\mathbb{D}} |f_j(z)|^p |g'(z)|^p (1-|z|^2)^{p-2} (1-|\sigma_{c_j}(z)|^2)^2 dA(z)\right)^{\frac{1}{p}} \\
&\gtrsim \limsup_{j\to\infty} |g'(c_j)|(1-|c_j|^2),
\end{aligned}$$

which implies that

$$\|T_g\|_{e,A^{p,K}\to A^{p,K}} \gtrsim \limsup_{|z|\to 1} |g'(z)|(1-|z|^2).$$

The proof is complete. □

It is clear that $T : X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$. The following result can be directly obtained by Theorem 3.3.

Corollary 3.1. *Let $g \in H(\mathbb{D})$, $1 < p < \infty$ and K satisfy (2.1) for some $\delta \in (0,2)$. Then $T_g : A^{p,K} \to A^{p,K}$ is compact if and only if $g \in \mathscr{B}_0$.*

Theorem 3.4. *Let $g \in H(\mathbb{D})$, $1 < p < \infty$ and K satisfy (2.1) for some $\delta \in (0,2)$. If $I_g : A^{p,K} \to A^{p,K}$ is bounded, then $\|I_g\|_e \approx \|g\|_\infty$.*

Proof. By Theorem 3.2, we have $\|I_g\|_e = \inf_K \|I_g - K\| \leq \|I_g\| \lesssim \|g\|_\infty$.

Next we prove that $\|I_g\|_e \gtrsim \|g\|_\infty$. Let $\{c_j\} \subset \mathbb{D}$ such that $|c_j| \to 1$ as $j \to \infty$. Set

$$f_j(z) = \frac{(1-|c_j|^2)^{\frac{2}{p}} K^{\frac{1}{p}}(1-|c_j|^2)}{(1-\overline{c_j}z)^{\frac{4}{p}}}.$$

From the proof of Theorem 3.3 we see that $f_j \in A^{p,K}$ and $\{f_j\}$ converges to zero uniformly on every compact subsets of $\mathbb{D}$. By the proof of Theorem 3.2 , we have

$$\|I_g f_j\|_{A^{p,K}}^p = \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^2}{K(1-|a|^2)} \int_{\mathbb{D}} |f_j'(z)|^p |g(z)|^p (1-|z|^2)^p |\sigma_a'(z)|^2 dA(z) \gtrsim |g(c_j)|^p.$$

Let $K : A^{p,K} \to A^{p,K}$ be a compact operator. By [27, Lemma 2.10], we have

$$\begin{aligned}\|I_g - K\| &\gtrsim \limsup_{j\to\infty} \|(I_g - K) f_j\|_{A^{p,K}} \geq \limsup_{j\to\infty} (\|I_g f_j\|_{A^{p,K}} - \|K f_j\|_{A^{p,K}}) \\ &\geq \limsup_{j\to\infty} \|I_g f_j\|_{A^{p,K}}.\end{aligned}$$

Therefore, $\|I_g\|_e \gtrsim \limsup_{j\to\infty} |g(c_j)|$. Since $\{c_j\}$ is arbitrary, we obtain that $\|I_g\|_e \gtrsim \|g\|_\infty$. The proof is complete. □

The following result can be directly obtained by Theorem 3.4.

Corollary 3.2. *Let $g \in H(\mathbb{D})$, $1 < p < \infty$ and K satisfy (2.1) for some $\delta \in (0,2)$. Then $I_g : A^{p,K} \to A^{p,K}$ is compact if and only if $g = 0$.*

Remark 3.1. From Corollaries 3.1 and 3.2, we see that $M_g : A^{p,K} \to A^{p,K}$ is bounded if and only if $g \in H^\infty$. $M_g : A^{p,K} \to A^{p,K}$ is compact if and only if $g = 0$.

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