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VISCOSITY AND INERTIAL ALGORITHMS FOR THE SPLIT COMMON FIXED POINT PROBLEM WITH APPLICATIONS TO COMPRESSED SENSING

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Abstract. In this paper, several inertial algorithms which combine the established viscosity iteration and cyclic processes for finding a solution of a multi-set split common fixed problem are investigated. The inertial parameters in our algorithms can be positive or negative, which may have fewer steps to meet the requirements than selecting all positive parameters. Under the appropriate conditions, our proposed algorithms are proven to converge strongly. A compressed sensing example is reported to demonstrate the computational implementation of our algorithms and the quality of image restoration.

Keywords. Alternated inertia; Compressed sensing; Multi-step inertia; Split common fixed point problem; Viscosity iteration.

1. Introduction

In 2009, Censor and Segal [1] first proposed the split common fixed point problem (SCFPP for short) in a finite dimensional Hilbert space. Let H_1 and H_2 be two real Hilbert spaces with scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Suppose that $A : H_1 \to H_2$ is a bounded linear operator. Given operators $U : H_1 \to H_1$ and $T : H_2 \to H_2$ with nonempty fixed point sets $Fix(U)$ and $Fix(T)$, respectively, the SCFPP reads as follows:

$$Finding\ x^* \in Fix(U)\ such\ that\ Ax^* \in Fix(T). \tag{1.1}$$

Given mappings $\{U_i\}_{i=1}^t : H_1 \to H_1$ and $\{T_j\}_{j=1}^r : H_2 \to H_2$, the multi-set split common fixed point problem (MSCFPP for short) consists of

$$Finding\ x^* \in \bigcap_{i=1}^{t} Fix(U_i)\ such\ that\ Ax^* \in \bigcap_{j=1}^{r} Fix(T_j), \tag{1.2}$$

where both t and r are fixed positive integers. The solution set of the MSCFPP is denoted by $\Omega = \{x^* | x^* \in \bigcap_{i=1}^t Fix(U_i)\ and\ Ax^* \in \bigcap_{j=1}^r Fix(T_j)\}$.

The SCFPP is a generalization of the known split feasibility problem (SFP for short), which is to find an element

$$x^* \in C\ such\ that\ Ax^* \in Q, \tag{1.3}$$

where C and Q are nonempty, closed, and convex subsets of H_1 and H_2, respectively.

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For the inverse problems arising from phase retrievals, the SFP in finite-dimensional Hilbert spaces was introduced by Censor and Elfving [2]. In particular, if $U := P_C$ and $T := P_Q$, then problem (1.3) is a special case of SCFPP (1.1).

In the last two decades, both SCFPP and SFP were widely investigated from the viewpoint of real applications, such as, compressed sensing, computed tomography, and intensity adjustable radiation therapy. In recent years, the MSCFPP was under the spotlight of research, and many scholars proposed numerous efficient algorithms. For various convergence results, we refer to [2, 3, 4, 5, 6] and the references therein.

The following well-known algorithm starting with any initial x_0 is popular and efficient in deal with the SCFPP $x_{n+1} = U(I - \gamma A^*(I-T)A)x_n$, $\forall n \geq 0$, where A^* is the adjoint operator of A, γ is the step length of iteration that satisfies $\gamma \in (0, \frac{2}{\lambda})$ with the largest eigenvalue of operator A^*A being λ. λ is called the spectral radius of A^*A, denoted by $\rho(A^*A)$. In 2011, Wang and Xu [7] introduced a cyclic iterative process for finding the MSCFPP with directed operators. Given an initial point $x_0 \in H_1$, they define an iterative sequence, $\{x_n\}$, by the following recursion manner: $x_{n+1} = U_{[n]_1}(x_n - \gamma A^*(I - T_{[n]_2})Ax_n)$, $\forall n \geq 0$, where $0 < \gamma < \frac{2}{\rho(A^*A)}$, $[n]_1 := n(mod\ t) + 1$, and $[n]_2 := n(mod\ r) + 1$ are the mod functions which take values in the indexing sets $\{1, 2, ..., t\}$ and $\{1, 2, ..., r\}$, respectively. They gained the weak convergence of this method in Hilbert spaces.

In 2016, Tang and Liu [8] proposed several iterative algorithms for the MSCFPP with directed operators. For any initial $x_0 \in H_1$, their methods generate the following sequences

$$x_{n+1} = \sum_{i=1}^{t} \omega_i U_i(x_n - \gamma_n \sum_{j=1}^{r} \eta_j A^*(I - T_j)Ax_n), \ \ \forall\, n \geq 0,$$

$$x_{n+1} = U_{[n]_1}(x_n - \gamma_n \sum_{j=1}^{r} \eta_j A^*(I - T_j)Ax_n), \ \ \forall\, n \geq 0,$$

and

$$x_{n+1} = \sum_{i=1}^{t} \omega_i U_i(x_n - \gamma_n A^*(I - T_{[n]_2})Ax_n), \ \ \forall\, n \geq 0,$$

where $\{\omega_i\}_{i=1}^t \subset (0,1)$, $\{\eta_j\}_{j=1}^r \subset (0,1)$, $\sum_{i=1}^t \omega_i = 1$, $\sum_{j=1}^r \eta_j = 1$, and $0 < a \leq \gamma_n \leq b < \frac{2}{\lambda}$ with $\lambda = \rho(A^*A)$. The definitions of $[n]_1$ and $[n]_2$ are the same as those in [7]. They investigated the convergence of the iterative sequences.

In 1964, Polyak [9] first considered accelerating the convergence rate of gradient descent methods. In [10], Nesterov further investigated interial gradient descent methods. In 2016, Dang, Sun and Xu [11] constructed an inertial relaxed CQ algorithm for solving SFP (1.3), and they proved the convergence of the following algorithm

$$\begin{cases} y_n = x_n + \delta_n(x_n - x_{n-1}), \\ x_{n+1} = P_{C_n}(I - \gamma A^*(I - P_{Q_n})A)y_n, \end{cases}$$

where P_{C_n} and P_{Q_n} are orthogonal projection operators onto halfspaces C_n and Q_n, respectively (see [11] for more details).

In 2021, Zhao, Zhao and Hou [12] introduced two inertial-type accelerated methods for solving the split common fixed point problem of directed operators. Their iterative algorithms are based on the primal-dual method and the inertial method; see [12] for more details. However, some researchers discovered that these methods with inertial steps lose the monotonicity of the iteration. In order to recover the monotonicity to some extent, many authors proposed alternated inertial acceleration algorithms; see, e.g., [13, 14, 15, 16, 17] and the references therein.

In 2017, Iutzeler and Hendrickx [13] presented a generic online acceleration scheme for solving optimization problem and found that the alternated inertial technique achieved a better numerical effect. The following alternated inertial step was introduced:

$$x_{n+1} = \begin{cases} T(x_n), & n \text{ is even}, \\ T(x_n + \delta_n(x_n - x_{n-1})), & n \text{ is odd}, \end{cases}$$

where T is the affine operator. In [13], they presented the parameter adaptive alternated inertial acceleration algorithm, and the convergence of the algorithm was proved under appropriate conditions.

An important generalization of inertial algorithms is the multi-step inertial iterative method. Ortega and Rheinboldt [18] introduced a d-step inertial iterative method in 1970. It reads $x_{n+1} = \Theta_n(x_n, x_{n-1}, \cdots, x_{n-d})$, $\forall n \geq 1$, where $d \geq 1$ is an integer, and $\Theta_n(\cdot)$ is the operator, which performs "extrapolation" onto the points $x_n, x_{n-1}, \cdots, x_{n-d}$. In 2016, Liang [19] proposed the multi-step inertial operator splitting method. Let $D = \{0, \cdots, d-1\}$, $d \in \mathbb{N}_+$, Liang defined the following multi-step inertial form. $y_n = x_n + \sum_{i \in D} \delta_{i,n}(x_{n-i} - x_{n-i-1})$. Moreover, Liang's method also indicates the superiority in numerical examples.

The MSCFPP of the directed operators has attracted the attention of scholars, however, algorithms for solving the MSCFPP of the averaged operator is relatively raw. In this paper, mainly based on the works mentioned above, we study the MSCFPP of the averaged operators and explore some new approaches for the viewpoint of acceleration. First, we introduce a viscosity algorithm combining cyclic iterative processes to solve the MSCFPP. Second, we extend the viscosity method to an inertial acceleration method and an alternating inertial accelerating method. Third, a multi-step inertial method and an alternated multi-step inertial method are introduced. In particular, we consider that the inertial parameters can be chosen as the positive or the negative parameters to obtain better results. Based on this, we prove that the sequences generated by our methods converge strongly under suitable parameter conditions. Finally, we provide a compressed sensing example to confirm computational implementation of our algorithms and the quality of image restoration.

2. PRELIMINARIES

In this section, we start by collecting some important definitions, propositions, and lemmas which are needed in the proof of the convergence of our algorithms. Below we adopt these notations:

(1) $x_n \to x$ indicates that $\{x_n\}$ converges strongly to x; $x_n \rightharpoonup x$ indicates that $\{x_n\}$ converges weakly to x.

(2) The set of fixed points of a mapping U is denoted by $Fix(U)$, that is, $Fix(U) = \{x \in H : Ux = x\}$.

(3) The set of weak limit points of $\{x_n\}$ is denoted by $\omega_w(x_n)$, that is, $\omega_w(x_n) = \{x \mid \exists \{x_{n_j}\} \subset \{x_n\} \text{ such that } x_{n_j} \rightharpoonup x\}$.

(4) $[n]_1$ and $[n]_2$ are defined by $[n]_1 := n(mod\ t) + 1$ and $[n]_2 := n(mod\ r) + 1$, respectively.

(5) H, H_1, and H_2 always denote real Hilbert spaces.

Let H be a real Hilbert space. Recall that a mapping $U : H \to H$ is said to be

(i) nonexpansive iff $\|Ux - Uy\| \le \|x - y\|$, $\forall x, y \in H$.

(ii) firmly nonexpansive iff $2U - I$ is nonexpansive, or equivalently, $\langle x - y, Ux - Uy\rangle \ge \|Ux - Uy\|^2$, $\forall x, y \in H$.

(iii) quasi-nonexpansive iff $Fix(U) \ne \emptyset$ and $\|Ux - p\| \le \|x - p\|$, $\forall x \in H,\ p \in Fix(U)$.

(iv) directed (also called firmly quasi-nonexpansive) iff $Fix(U) \ne \emptyset$ and $\|Ux - p\|^2 \le \langle x - p, Ux - p\rangle$, $\forall x \in H,\ p \in Fix(U)$.

(v) contractive iff $\|Ux - Uy\| \le \theta\|x - y\|$, $\forall x, y \in H$, where θ is a constant in $(0, 1)$.

Recall that a mapping $U : H \to H$ is said to be an averaged mapping iff it can be written in the following form: $U = (1 - \alpha)I + \alpha S$, where $\alpha \in (0, 1)$, and $S : H \to H$ is a nonexpansive mapping. More precisely, when this formula holds, we say that U is α-averaged (for short α-*av*) Thus, a firmly nonexpansive mapping is $\frac{1}{2}$-averaged.

Next, we list two examples to show that averaged operators and directed operators do not contain each other.

Example 2.1. Let $H = \mathbb{R}^2$ and $S : H \to H$ be a rotation operator. Let $S = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, where $\theta = \frac{\pi}{2}$. Obviously, S is a nonexpansive mapping. Define $T = \frac{1}{3}I + \frac{2}{3}S$. Thus T is an averaged operator. Taking $p \in Fix\{T\} = \{(0, 0)\}$ yields

$$\|Tx - p\|^2 = \|\frac{1}{3}x + \frac{2}{3}Sx\|^2 = \frac{1}{9}\| \begin{pmatrix} x_1 + 2x_2 \\ x_2 - 2x_1 \end{pmatrix} \|^2 = \frac{5}{9}(x_1^2 + x_2^2).$$

However, $\langle x - p, Tx - p\rangle = \frac{1}{3}(x_1^2 + x_2^2) \le \|Tx - p\|^2$. Thus T is not a directed operator.

Example 2.2. Let $H = \mathbb{R}$, and let the operator $S : H \to H$ with $Fix(S) = \{0\}$ be defined as follows:

$$Sx = \begin{cases} \frac{2}{3}x\sin\frac{1}{x}, & x \ne 0, \\ 0, & x = 0. \end{cases}$$

It is easy to prove that S is a quasi-nonexpansive mapping. Define $T = \frac{1}{2}I + \frac{1}{2}S$. Thus

$$Tx = \begin{cases} \frac{1}{2}x + \frac{x}{3}\sin\frac{1}{x}, & x \ne 0, \\ 0, & x = 0. \end{cases}$$

Taking $p \in Fix(T) = \{0\}$, for $\forall x \ne 0$, we obtain $|Tx - p|^2 = \frac{1}{4}x^2 + \frac{1}{9}x^2(\sin\frac{1}{x})^2 + \frac{x^2}{3}\sin\frac{1}{x} \le \frac{1}{2}x^2 + \frac{x^2}{3}\sin\frac{1}{x} = \langle x - p, Tx - p\rangle$. However, taking $x = \frac{2}{\pi}$ and $y = \frac{2}{3\pi}$, we have $|Tx - Ty| = \frac{14}{9\pi} > |x - y| = \frac{12}{9\pi}$. As a result, T is not an averaged operator, but T is a directed operator.

We remark here that it is meaningful to study the inertial accelerated algorithms for solving the MSCFPP of averaged operators.

Let α be a positive real number. Recall that $U : H \to H$ is said to be α-inverse strongly monotone (for short α-ism) if $\langle x - y, Ux - Uy\rangle \ge \alpha\|Ux - Uy\|^2$, $\forall x, y \in H$. Obviously, U is $\frac{1}{\alpha}$-Lipschitz if U is α-ism.

Proposition 2.1. [20]

(i) *If $U_1, U_2, \cdots, U_{N-1}$, and U_N are averaged, then $U_N U_{N-1} \cdots U_2 U_1$ is also averaged. Especially, if U_1 and U_2 are, respectively, α_1- and α_2-av, then U_2U_1 is $(\alpha_2 + \alpha_1 - \alpha_1\alpha_2)$-av.*

(ii) *If* $\{U_i\}_{i=1}^N$ *are averaged with common fixed points, then* $\bigcap_{i=1}^N Fix(U_i) = Fix(U_1U_2\cdots U_N)$.
(iii) U *is nonexpansive if and only if* $I-U$ *is* $\frac{1}{2}$*-ism.*
(iv) *If* U *is* ν*-ism, then, for* $\tau > 0$, τU *is* $\frac{\nu}{\tau}$*-ism.*
(v) *For* $0 < \alpha < 1$, U *is* α*-averaged if and only if* $I-U$ *is* $\frac{1}{2\alpha}$*-ism.*

We also need the following lemmas.

Lemma 2.1. [5] *Let* $A : H_1 \to H_2$ *be a bounded linear operator,* A^* *be the adjoint operator of* A*, and* $T : H_2 \to H_2$ *be a nonexpansive mapping. If there exists a point* $z \in H_1$ *such that* $Az \in Fix(T)$*, then* $(I-T)Az = 0 \Leftrightarrow A^*(I-T)Az = 0$, $\forall z \in H_1$.

Lemma 2.2. *If* U *is* α_1*-av, then* U *is* α_2*-av for* $\alpha_2 \geq \alpha_1$*, where* $\alpha_i \in (0,1)$, $i = 1,2$.

Proof. Since U is α_1-*av*, we have

$$\begin{aligned} U &= (1-\alpha_1)I + \alpha_1 S_1 = (1-\alpha_2)I + (\alpha_2-\alpha_1)I + \alpha_1 S_1 \\ &= (1-\alpha_2)I + \alpha_2\left(\frac{\alpha_2-\alpha_1}{\alpha_2}I + \frac{\alpha_1}{\alpha_2}S_1\right). \end{aligned}$$

Using the nonexpansivity of S_1, we have

$$\begin{aligned} &\left\|\left(\frac{\alpha_2-\alpha_1}{\alpha_2}I + \frac{\alpha_1}{\alpha_2}S_1\right)x - \left(\frac{\alpha_2-\alpha_1}{\alpha_2}I + \frac{\alpha_1}{\alpha_2}S_1\right)y\right\| \\ &\leq \frac{\alpha_2-\alpha_1}{\alpha_2}\|x-y\| + \frac{\alpha_1}{\alpha_2}\|S_1x - S_1y\| \leq \|x-y\|, \quad \forall x,y \in H. \end{aligned}$$

Thus $(\frac{\alpha_2-\alpha_1}{\alpha_2}I + \frac{\alpha_1}{\alpha_2}S_1)$ is also a nonexpansive mapping, denoted by S_2. It follows that $U = (1-\alpha_2)I + \alpha_2 S_2$. □

Lemma 2.3. [21] *Let* $U : H \to H$ *be a nonexpansive mapping with* $Fix(U) \neq \emptyset$*. If* $\{x_n\}$ *is weakly convergent to* x *and* $\{(I-U)x_n\}$ *is strongly convergent to* y*, then* $(I-U)x = y$*. Particularly, if* $y = 0$*, then* $x \in Fix(U)$.

Lemma 2.4. [22] *Let* $\{b_n\}$ *be a nonnegative real number sequence satisfying*

$$b_{n+1} \leq (1-\sigma_n)b_n + \sigma_n\zeta_n,$$

$$b_{n+1} \leq b_n - \kappa_n + \psi_n,$$

where the real number sequences $\{\sigma_n\} \subset (0,1)$, $\{\kappa_n\} \subset [0,\infty)$, $\{\zeta_n\}$ *and* $\{\psi_n\}$ *are in* $\mathbb{R}$ *satisfying the following conditions:*

(i) $\sum_{n=0}^\infty \sigma_n = \infty$;
(ii) $\lim_{n\to\infty} \psi_n = 0$;
(iii) $\lim_{k\to\infty} \kappa_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \zeta_{n_k} \leq 0$ *for any subsequence* $\{n_k\} \subset \{n\}$.

Then $\lim_{n\to\infty} b_n = 0$.

3. Viscosity Iterative Algorithms for Solving the MSCFPP

In this section, based on the viscosity iteration and cyclic iterative processes, we introduce several algorithms for solving problem (1.2). Some strong convergence results are established under some mild assumptions. The first algorithm is described as follows.

Algorithm 3.1. For any initial x_0, $n \geq 0$, $\{x_n\}$ is generated by the following manner:

$$x_{n+1} = \beta_n g(x_n) + (1-\beta_n) U_{[n]_1}(x_n - \gamma_n A^*(I - T_{[n]_2})Ax_n), \tag{3.1}$$

where $g: H_1 \to H_1$ is a θ-contractive mapping with $\theta \in (0,1)$, and $0 < a \leq \gamma_n \leq b < \frac{2}{\lambda}$ with $\lambda = \rho(A^*A)$.

Theorem 3.1. *Let $A: H_1 \to H_2$ be a bounded linear operator with $\lambda = \rho(A^*A)$, and let A^* be its adjoint. Assume that $\{U_i\}_{i=1}^t : H_1 \to H_1$ and $\{T_j\}_{j=1}^r : H_2 \to H_2$ are $\{\tau_{1_i}\}_{i=1}^t$-av and $\{\tau_{2_j}\}_{j=1}^r$-av, respectively, where t and r are two positive integer numbers. Suppose that $\Omega \neq \emptyset$. Let $g: H_1 \to H_1$ be a θ-contractive mapping with $\theta \in (0,1)$, and let $\{\beta_n\}$ be a sequence in $(0,1)$. If (i) $\lim\limits_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$ and (ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \frac{1}{max_j \tau_{2_j} \lambda}$, then $\{x_n\}$ converges strongly to $x^* \in \Omega$.*

Proof. Set $V_{\gamma_n} := U_{[n]_1}(I - \gamma_n A^*(I - T_{[n]_2})A)$, $\tau_1 = \max_{1\leq i\leq t} \tau_{1_i}$, and $\tau_2 = \max_{1\leq j\leq r} \tau_{2_j}$, where $0 < \gamma_n < \frac{1}{\tau_2 \lambda}$. For arbitrary $i \in \{1,2,\dots,t\}$ and $j \in \{1,2,\dots,r\}$, since U_i and T_j are averaged, using Proposition 2.1, we obtain U_i and T_j are τ_1- and τ_2-av, respectively. More specifically, we infer that V_{γ_n} is a $\tau_1 + \lambda\gamma_n\tau_2(1-\tau_1)$-*av* operator. Putting $\mu_n := \tau_1 + \lambda\gamma_n\tau_2(1-\tau_1)$, by condition (ii), we conclude that $\tau_1 < \liminf_{n\to\infty} \mu_n \leq \limsup_{n\to\infty} \mu_n < 1$. Then V_{γ_n} is rewritten as $V_{\gamma_n} = (1-\mu_n)I + \mu_n W_n$, where W_n is a nonexpansive operator.

The proof is divided into three steps.

Step 1. Prove that sequence $\{x_n\}$ is bounded.

For any $z \in \Omega$, we find from (3.1) that

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\beta_n(g(x_n) - z) + (1-\beta_n)(V_{\gamma_n}x_n - z)\| \\
&\leq \beta_n\|g(x_n) - g(z)\| + \beta_n\|g(z) - z\| + (1-\beta_n)\|V_{\gamma_n}x_n - z\| \\
&\leq (1 - \beta_n(1-\theta))\|x_n - z\| + \beta_n(1-\theta)\frac{\|g(z) - z\|}{1-\theta}.
\end{aligned}$$

It follows that $\|x_{n+1} - z\| \leq \max\{\|x_n - z\|, \frac{\|g(z)-z\|}{1-\theta}\}$. Using mathematical induction, one has $\|x_n - z\| \leq \max\{\|x_0 - z\|, \frac{\|g(z)-z\|}{1-\theta}\}$. This indicates that $\{x_n\}$ is bounded. Taking $\widetilde{x} \in \omega_\omega(x_n)$, one finds that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \widetilde{x}$. Since g is a θ-contractive operator, and V_{γ_n} is an averaged operator, one concludes that $\{g(x_n)\}$ and $\{V_{\gamma_n}(x_n)\}$ are also bounded.

Step 2. Prove that $\lim_{k\to\infty} \kappa_{n_k} = 0$ implies $\lim_{k\to\infty} \|x_{n_k} - V_{\gamma_{n_k}} x_{n_k}\| = 0$ for any subsequence $\{n_k\} \subset \{n\}$.

It follows from (3.1) that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\beta_n(g(x_n) - z) + (1-\beta_n)(V_{\gamma_n}x_n - z)\|^2 \\
&\leq \beta_n^2(\|g(x_n) - g(z)\| + \|g(z) - z\|)^2 + (1-\beta_n)^2\|V_{\gamma_n}x_n - z\|^2 \\
&\quad + 2\beta_n(1-\beta_n)\langle g(x_n) - g(z) + g(z) - z, V_{\gamma_n}x_n - z\rangle \\
&\leq (1 - \beta_n(2 - \beta_n(1+2\theta^2) - 2\theta(1-\beta_n)))\|x_n - z\|^2 \\
&\quad + 2\beta_n(1-\beta_n)\langle g(z) - z, V_{\gamma_n}x_n - z\rangle + 2\beta_n^2\|g(z) - z\|^2.
\end{aligned}$$

Observe that

$$\|x_{n+1}-z\|^2 = \|V_{\gamma_n}x_n - z\|^2 + \beta_n^2\|g(x_n) - V_{\gamma_n}x_n\|^2 + 2\beta_n\langle V_{\gamma_n}x_n - z, g(x_n) - V_{\gamma_n}x_n\rangle.$$

It follows that

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|(1-\mu_n)(x_n - z) + \mu_n(W_nx_n - z)\|^2 + \beta_n^2\|g(x_n) - V_{\gamma_n}x_n\|^2 \\ &\quad + 2\beta_n\langle V_{\gamma_n}x_n - z, g(x_n) - V_{\gamma_n}x_n\rangle \\ &= (1-\mu_n)\|x_n - z\|^2 + \mu_n\|W_nx_n - z\|^2 - (1-\mu_n)\mu_n\|W_nx_n - x_n\|^2 \\ &\quad + \beta_n^2\|g(x_n) - V_{\gamma_n}x_n\|^2 + 2\beta_n\langle V_{\gamma_n}x_n - z, g(x_n) - V_{\gamma_n}x_n\rangle \\ &\le \|x_n - z\|^2 - (1-\mu_n)\mu_n\|W_nx_n - x_n\|^2 + \beta_n^2\|g(x_n) - V_{\gamma_n}x_n\|^2 \\ &\quad + 2\beta_n\langle V_{\gamma_n}x_n - z, g(x_n) - V_{\gamma_n}x_n\rangle.\end{aligned}$$

Define $b_n = \|x_n - z\|^2$, $\sigma_n = \beta_n(2 - \beta_n(1+2\theta^2) - 2\theta(1-\beta_n))$,

$$\zeta_n = \frac{2}{2-\beta_n(1+2\theta^2) - 2\theta(1-\beta_n)}((1-\beta_n)\langle g(z) - z, V_{\gamma_n}x_n - z\rangle + \beta_n\|g(z) - z\|^2),$$

$$\kappa_n = (1-\mu_n)\mu_n\|W_nx_n - x_n\|^2,$$

and

$$\psi_n = \beta_n^2\|g(x_n) - V_{\gamma_n}x_n\|^2 + 2\beta_n\langle V_{\gamma_n}x_n - z, g(x_n) - V_{\gamma_n}x_n\rangle.$$

Since $\lim_{n\to\infty}\frac{\sigma_n}{\beta_n} = 2(1-\theta)$, we have that $\sum_{n=0}^{\infty}\sigma_n$ is divergent to ∞, and ψ_n tends to 0 from condition (i). Thus $\{\sigma_n\}$ and $\{\psi_n\}$ satisfy the conditions (i) and (ii) of Lemma 2.4. Next, we only need to prove that $\kappa_{n_k} \to 0$ implies that $\limsup_{k\to\infty}\zeta_{n_k} \le 0$ for any subsequence $\{n_k\} \subset \{n\}$. In fact, $\kappa_{n_k} \to 0$ leads to $\|W_{n_k}x_{n_k} - x_{n_k}\| \to 0$. Then, it follows that $\|x_{n_k} - V_{\gamma_{n_k}}x_{n_k}\| = \mu_{n_k}\|x_{n_k} - W_{n_k}x_{n_k}\| \to 0$. By the boundedness of $\{x_{n_k}\}$, we find that $\{x_{n_k}\}$ has a convergent subsequence $\{x_{n_{k_m}}\}$. Without lose of generality, we still use $\{x_{n_k}\}$ instead of $\{x_{n_{k_m}}\}$ and $\{V_{\gamma_{n_k}}\}$ instead of $\{V_{\gamma_{n_{k_m}}}\}$. Assume $x_{n_k} \rightharpoonup \tilde{x}$ and $\gamma_{n_k} \to \gamma$. Since both t and r are finite integers, then tr is also a finite integer. Thus there exists $\{n_{k_{ij}}\} \subset \{n_k\}$ satisfying $[n_{k_{ij}}]_1 = i$, $[n_{k_{ij}}]_2 = j$, where $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant r$ are fixed. Since γ_n is bounded, might as well set that $\gamma_{k_{ij}} \to \gamma$ with $0 < \gamma < \frac{1}{\max_j \tau_{2_j}\lambda}$.

Put $V_{ij} := U_i(I - \gamma A^*(I - T_j)A)$. Thus

$$\begin{aligned}&\|V_{\gamma_{n_{k_{ij}}}}x_{n_{k_{ij}}} - V_{ij}x_{n_{k_{ij}}}\| \\ &= \|U_i(x_{n_{k_{ij}}} - \gamma_{n_{k_{ij}}}A^*(I-T_j)Ax_{n_{k_{ij}}}) - U_i(x_{n_{k_{ij}}} - \gamma A^*(I-T_j)Ax_{n_{k_{ij}}})\| \\ &\le \|(x_{n_{k_{ij}}} - \gamma_{n_{k_{ij}}}A^*(I-T_j)Ax_{n_k}) - (x_{n_{k_{ij}}} - \gamma A^*(I-T_j)Ax_{n_{k_{ij}}})\| \\ &\le |\gamma_{n_{k_{ij}}} - \gamma|\|A^*(I-T_j)Ax_{n_{k_{ij}}}\| \to 0,\end{aligned}$$

which implies that $\|x_{n_{k_{ij}}} - V_{ij}x_{n_{k_{ij}}}\| \le \|x_{n_{k_{ij}}} - V_{\gamma_{n_{k_{ij}}}}x_{n_{k_{ij}}}\| + \|V_{\gamma_{n_{k_{ij}}}}x_{n_{k_{ij}}} - V_{ij}x_{n_{k_{ij}}}\| \to 0$.

Step 3. Prove that $\omega_w(x_{n_k}) \subset \Omega$.

Without lose of generality, we use $\{x_{n_k}\}$ instead of $\{x_{n_{k_{ij}}}\}$. By Lemma 2.3, we have $\widetilde{x} \in Fix(V_{ij})$. From Proposition 2.1 (ii), we conclude that $\widetilde{x} \in Fix(U_i)$ and $\widetilde{x} \in Fix(I - \gamma A^*(I - T_j)A)$. Further we have $A\widetilde{x} \in Fix(T_j)$ by Lemma 2.1. Similarly, for each $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant r$, we can derive the above results. Thus we deduce $\widetilde{x} \in \bigcap_{i=1}^{t} Fix(U_i)$ and $A\widetilde{x} \in \bigcap_{j=1}^{r} Fix(T_j)$. It follows that $\widetilde{x} \in \omega_w(x_{n_k}) \subset \Omega$. Since $P_\Omega g$ is a contractive operator, then there exists a unique fixed point x^* such that $P_\Omega g(x^*) = x^*$. Using the equivalent property of projections, we have $\langle g(x^*) - x^*, \widetilde{x} - x^* \rangle \leq 0$, $\forall \widetilde{x} \in \Omega$. Hence we conclude that

$$\limsup_{n\to\infty} \langle g(x^*) - x^*, x_n - x^* \rangle = \lim_{k\to\infty} \langle g(x^*) - x^*, x_{n_k} - x^* \rangle = \langle g(x^*) - x^*, \widetilde{x} - x^* \rangle \leq 0.$$

This together with the result of Step 2, which indicates that $\limsup_{k\to\infty} \zeta_{n_k} \leq 0$. Lemma 2.4 yields $\{x_n\}$ converges to a point x^* in Ω strongly.

□

Next, based on the work [17], we extend the viscosity method to an inertial acceleration method and an alternated inertial acceleration method.

Algorithm 3.2. For any initial x_{-1}, x_0, for $n \geq 0$, choose $\{\varepsilon_n\}$ satisfying condition (iii) of Theorem 3.2, and take $\beta \geq 3$. Set $n := 0$.
Given the iterates x_{n-1} and x_n, choose δ_n such that $0 \leq |\delta_n| \leq \bar{\delta}_n$ with $\bar{\delta}_n$ defined by

$$\bar{\delta}_n := \begin{cases} \min\{\dfrac{n-1}{n+\beta-1}, \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}\}, \; if \; x_n \neq x_{n-1}, \\ \dfrac{n-1}{n+\beta-1}, \; otherwise. \end{cases}$$

Proceed with the following format:

$$\begin{cases} y_n = x_n + \delta_n(x_n - x_{n-1}), \\ x_{n+1} = \beta_n g(y_n) + (1-\beta_n) U_{[n]_1}(y_n - \gamma_n A^*(I - T_{[n]_2}) A y_n), \end{cases} \tag{3.2}$$

where $g : H_1 \to H_1$ is a θ-contractive mapping with $\theta \in (0,1)$ and $0 < a \leq \gamma_n \leq b < \frac{2}{\lambda}$ with $\lambda = \rho(A^*A)$.

Theorem 3.2. *Let $A : H_1 \to H_2$ be a bounded linear operator with $\lambda = \rho(A^*A)$, and let A^* be its adjoint. Assume that $\{U_i\}_{i=1}^t : H_1 \to H_1$ and $\{T_j\}_{j=1}^r : H_2 \to H_2$ are $\{\tau_{1_i}\}_{i=1}^t$-av and $\{\tau_{2_j}\}_{j=1}^r$-av, respectively, where t and r are two positive integer numbers. Suppose that $\Omega \neq \emptyset$. Let $g : H_1 \to H_1$ be a θ-contractive operator with $\theta \in (0,1)$, and let $\{\beta_n\}$ be a sequence in $(0,1)$. If*

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \min_j \frac{1}{\tau_{2_j} \lambda}$;
(iii) $\varepsilon_n = o(\beta_n)$, *i.e.* $\lim_{n\to\infty} (\varepsilon_n / \beta_n) = 0$ *(e.g.,* $\varepsilon_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{1}{n+1}$*).*

then $\{x_n\}$ converges strongly to $x^* \in \Omega$.

Proof. Set $V_{\gamma_n} := U_{[n]_1}(I - \gamma_n A^*(I - T_{[n]_2})A)$, then V_{γ_n} can be rewritten as $V_{\gamma_n} = (1-\mu_n)I + \mu_n W_n$, where W_n is a nonexpansive operator. The proof is divided into three steps.
Step 1. Prove that $\{x_n\}$ is bounded.

For any $z \in \Omega$, one has $\|y_n - z\| \leq \|x_n - z\| + |\delta_n|\|x_n - x_{n-1}\|$, which together with (3.2) deduces that

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\beta_n(g(y_n) - z) + (1-\beta_n)(V_{\gamma_n} y_n - z)\| \\
&\leq \beta_n\|g(y_n) - g(z)\| + \beta_n\|g(z) - z\| + (1-\beta_n)\|V_{\gamma_n} y_n - z\| \\
&\leq \beta_n\theta\|y_n - z\| + \beta_n\|g(z) - z\| + (1-\beta_n)\|V_{\gamma_n} y_n - z\| \\
&\leq (1-\beta_n(1-\theta))\|y_n - z\| + \beta_n(1-\theta)\frac{\|g(z)-z\|}{1-\theta} \\
&\leq (1-\beta_n(1-\theta))\|x_n - z\| + \beta_n(1-\theta)\frac{\|g(z)-z\| + \frac{|\delta_n|\|x_n-x_{n-1}\|}{\beta_n}}{1-\theta}.
\end{aligned}$$

From condition (iii), we obtain that $\lim_{n\to\infty}|\delta_n|\|x_n - x_{n-1}\| = 0$ and $\lim_{n\to\infty}\frac{|\delta_n|\|x_n-x_{n-1}\|}{\beta_n} = 0$. Thus there exists a positive constant c_1 such that $c_1 \geq \sup_n\{\|g(z)-z\| + \frac{|\delta_n|\|x_n-x_{n-1}\|}{\beta_n}\}$. It follows that $\|x_{n+1} - z\| \leq \max\{\|x_n - z\|, \frac{c_1}{1-\theta}\}$, which further implies $\|x_n - z\| \leq \max\{\|x_0 - z\|, \frac{c_1}{1-\theta}\}$. This indicates that $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{g(y_n)\}$, and $\{V_{\gamma_n}(y_n)\}$.

Step 2. Prove that $\lim_{k\to\infty}\kappa_{n_k} = 0$ implies $\lim_{k\to\infty}\|x_{n_k} - V_{\gamma_{n_k}} x_{n_k}\| = 0$ for any subsequence $\{n_k\} \subset \{n\}$.

For any $z \in \Omega$, we obtain

$$\begin{aligned}
\|y_n - z\|^2 &\leq \|x_n - z\|^2 + 2\langle x_n - z + \delta_n(x_n - x_{n-1}), \delta_n(x_n - x_{n-1})\rangle \\
&\leq \|x_n - z\|^2 + 2|\delta_n|\|x_n - x_{n-1}\|(\|x_n - z\| + |\delta_n|\|x_n - x_{n-1}\|),
\end{aligned}$$

which together with (3.2) deduces that

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&= \beta_n^2\|g(y_n) - z\|^2 + (1-\beta_n)^2\|V_{\gamma_n} y_n - z\|^2 + 2\beta_n(1-\beta_n)\langle g(y_n) - z, V_{\gamma_n} y_n - z\rangle \\
&= \beta_n^2\|g(y_n) - g(z) + g(z) - z\|^2 + (1-\beta_n)^2\|V_{\gamma_n} y_n - z\|^2 \\
&\quad + 2\beta_n(1-\beta_n)\langle g(y_n) - g(z) + g(z) - z, V_{\gamma_n} y_n - z\rangle \\
&\leq 2\beta_n^2\theta^2\|y_n - z\|^2 + 2\beta_n^2\|g(z) - z\|^2 + (1-\beta_n)^2\|y_n - z\|^2 \\
&\quad + 2\beta_n(1-\beta_n)\theta\|y_n - z\|^2 + 2\beta_n(1-\beta_n)\langle g(z) - z, V_{\gamma_n} y_n - z\rangle \\
&\leq (1 - \beta_n(2 - \beta_n(1+2\theta^2) - 2\theta(1-\beta_n)))\|x_n - z\|^2 \\
&\quad + 2|\delta_n|(1 - \beta_n(2 - \beta_n(1+2\theta^2) - 2\theta(1-\beta_n)))\|x_n - x_{n-1}\|(\|x_n - z\| + |\delta_n|\|x_n - x_{n-1}\|) \\
&\quad + 2\beta_n(1-\beta_n)\langle g(z) - z, V_{\gamma_n} y_n - z\rangle + 2\beta_n^2\|g(z) - z\|^2.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|V_{\gamma_n} y_n - z + \beta_n(g(y_n) - V_{\gamma_n} y_n)\|^2 \\
&= \|V_{\gamma_n} y_n - z\|^2 + \beta_n^2\|g(y_n) - V_{\gamma_n} y_n\|^2 + 2\beta_n\langle V_{\gamma_n} y_n - z, g(y_n) - V_{\gamma_n} y_n\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x_{n+1}-z\|^2 \\
&= \|(1-\mu_n)(y_n-z)+\mu_n(W_ny_n-z)\|^2+\beta_n^2\|g(y_n)-V_{\gamma_n}y_n\|^2 \\
&\quad +2\beta_n\langle V_{\gamma_n}y_n-z, g(y_n)-V_{\gamma_n}y_n\rangle \\
&= (1-\mu_n)\|y_n-z\|^2+\mu_n\|W_ny_n-z\|^2-(1-\mu_n)\mu_n\|W_ny_n-y_n\|^2 \\
&\quad +\beta_n^2\|g(y_n)-V_{\gamma_n}y_n\|^2+2\beta_n\langle V_{\gamma_n}y_n-z, g(y_n)-V_{\gamma_n}y_n\rangle \\
&\le \|y_n-z\|^2-(1-\mu_n)\mu_n\|W_ny_n-y_n\|^2+\beta_n^2\|g(y_n)-V_{\gamma_n}y_n\|^2 \\
&\quad +2\beta_n\langle V_{\gamma_n}y_n-z, g(y_n)-V_{\gamma_n}y_n\rangle \\
&\le \|x_n-z\|^2-(1-\mu_n)\mu_n\|W_ny_n-y_n\|^2+\beta_n^2\|g(y_n)-V_{\gamma_n}y_n\|^2 \\
&\quad +2\beta_n\langle V_{\gamma_n}y_n-z, g(y_n)-V_{\gamma_n}y_n\rangle+2|\delta_n|\|x_n-x_{n-1}\|(\|x_n-z\|+|\delta_n|\|x_n-x_{n-1}\|).
\end{aligned}$$

Define $b_n=\|x_n-z\|^2$, $\sigma_n=\beta_n(2-\beta_n(1+2\theta^2)-2\theta(1-\beta_n))$, $\kappa_n=(1-\mu_n)\mu_n\|W_ny_n-y_n\|^2$,

$$\begin{aligned}
\zeta_n=&\frac{2}{2-\beta_n(1+2\theta^2)-2\theta(1-\beta_n)}((1-\beta_n)\langle g(z)-z, V_{\gamma_n}y_n-z\rangle \\
&+\frac{|\delta_n|\|x_n-x_{n-1}\|(\|x_n-z\|+|\delta_n|\|x_n-x_{n-1}\|)}{\beta_n} \\
&+\beta_n\|g(z)-z\|^2)-2|\delta_n|\|x_n-x_{n-1}\|(\|x_n-z\|+|\delta_n|\|x_n-x_{n-1}\|),
\end{aligned}$$

and

$$\begin{aligned}
\psi_n=&\beta_n^2\|g(y_n)-V_{\gamma_n}y_n\|^2+2\beta_n\langle V_{\gamma_n}y_n-z, g(y_n)-V_{\gamma_n}y_n\rangle \\
&+2|\delta_n|\|x_n-x_{n-1}\|(\|x_n-z\|+|\delta_n|\|x_n-x_{n-1}\|).
\end{aligned}$$

Since $\lim_{n\to\infty}\frac{\sigma_n}{\beta_n}=2(1-\theta)$, we have $\sum_{n=0}^{\infty}\sigma_n$ is divergent to ∞ and ψ_n tends to 0. Thus the parameters $\{\sigma_n\}$ and $\{\psi_n\}$ satisfy the conditions (i) and (ii) of Lemma 2.4. Now, we only need to prove that $\kappa_{n_k}\to 0$ implies that $\limsup_{k\to\infty}\zeta_{n_k}\le 0$ for any subsequence $\{n_k\}\subset\{n\}$. In fact, $\kappa_{n_k}\to 0$ leads to $\|W_{n_k}y_{n_k}-y_{n_k}\|\to 0$. Thus $\|y_{n_k}-V_{\gamma_{n_k}}y_{n_k}\|=\mu_{n_k}\|y_{n_k}-W_{n_k}y_{n_k}\|\to 0$. According to condition (iii), we find $\|y_{n_k}-x_{n_k}\|=|\delta_{n_k}|\|x_{n_k}-x_{n_k-1}\|\to 0$. Hence,

$$\begin{aligned}
&\lim_{k\to\infty}\|x_{n_k}-V_{\gamma_{n_k}}x_{n_k}\| \\
&=\lim_{k\to\infty}\|x_{n_k}-y_{n_k}+y_{n_k}-V_{\gamma_{n_k}}y_{n_k}+V_{\gamma_{n_k}}y_{n_k}-V_{\gamma_{n_k}}x_{n_k}\| \\
&\le\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\|+\lim_{k\to\infty}\|y_{n_k}-V_{\gamma_{n_k}}y_{n_k}\|+\lim_{k\to\infty}\|V_{\gamma_{n_k}}y_{n_k}-V_{\gamma_{n_k}}x_{n_k}\| \\
&\le\lim_{k\to\infty}2\|x_{n_k}-y_{n_k}\|+\lim_{k\to\infty}\|y_{n_k}-V_{\gamma_{n_k}}y_{n_k}\|.
\end{aligned}$$

which implies that $\|x_{n_k}-V_{\gamma_{n_k}}x_{n_k}\|=0$. As Theorem 3.1, setting $V_{ij}=U_i(I-\gamma A*(I-T_j)A)$, we derive $\|x_{n_{k_{ij}}}-V_{ij}x_{n_{k_{ij}}}\|\le\|x_{n_{k_{ij}}}-V_{\gamma_{n_{k_{ij}}}}x_{n_{k_{ij}}}\|+\|V_{\gamma_{n_{k_{ij}}}}x_{n_{k_{ij}}}-V_{ij}x_{n_{k_{ij}}}\|\to 0$.

Step 3. Prove that $\omega_w(x_{n_k})\subset\Omega$.

The proof of Step 3 can be derived immediately by following Theorem 3.1. Applying Lemma 2.4, one concludes that $\{x_n\}$ converges to a point x^* in Ω strongly. □

Inspired by the ideas of [13, 14, 16, 17] and related parameter conditions, we have the following algorithm.

Algorithm 3.3. For any initial x_{-1}, x_0, for $n \geq 0$, choose sequences $\{\varepsilon_n\}$ satisfying the condition (iii) of Theorem 3.3, and take $\beta \geq 3$. Set $n := 0$.
Given the iterates x_{n-1} and x_n, choose δ_n such that $0 \leq |\delta_n| \leq \bar{\delta}_n$ with $\bar{\delta}_n$ defined by

$$\bar{\delta}_n := \begin{cases} \min\{\frac{n-1}{n+\beta-1}, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\}, \; if \; x_n \neq x_{n-1}, \\ \frac{n-1}{n+\beta-1}, \; otherwise. \end{cases}$$

Proceed with the following format:

$$\begin{cases} y_n = \begin{cases} x_n, \quad if \; n \; is \; even, \\ x_n + \delta_n(x_n - x_{n-1}), \;\; if \; n \; is \; odd. \end{cases} \\ x_{n+1} = \beta_n g(y_n) + (1-\beta_n) U_{[n]_1}(y_n - \gamma_n A^*(I - T_{[n]_2})Ay_n), \end{cases} \tag{3.3}$$

where $g : H_1 \to H_1$ is a θ-contractive mapping with $\theta \in (0,1)$ and $0 < a \leq \gamma_n \leq b < \frac{2}{\lambda}$ with $\lambda = \rho(A^*A)$.

Theorem 3.3. *Let $A : H_1 \to H_2$ be a bounded linear operator with $\lambda = \|A^*A\|$, and let A^* be the adjoint of A. Assume that $\{U_i\}_{i=1}^t : H_1 \to H_1$ and $\{T_j\}_{j=1}^r : H_2 \to H_2$ are $\{\tau_{1_i}\}_{i=1}^t$-av and $\{\tau_{2_j}\}_{j=1}^r$-av, respectively, where t and r are two positive integer numbers. Suppose that $\Omega \neq \emptyset$. Let $g : H_1 \to H_1$ be a θ-contractive operator with $\theta \in (0,1)$, and let $\{\beta_n\}$ be a sequence in $(0,1)$. If*

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \min_j \frac{1}{\tau_{2_j}\lambda}$;
(iii) $\varepsilon_n = o(\beta_n)$, *i.e.* $\lim_{n\to\infty}(\varepsilon_n/\beta_n) = 0$ (*e.g.*, $\varepsilon_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{1}{n+1}$),

then $\{x_n\}$ converges strongly to $x^* \in \Omega$.

Proof. Setting $V_{\gamma_n} := U_{[n]_1}(I - \gamma_n A^*(I - T_{[n]_2})A)$, V_{γ_n} can be rewritten as $V_{\gamma_n} = (1-\mu_n)I + \mu_n W_n$, where W_n is a nonexpansive operator.

The proof is divided into three steps.

Step 1. Prove that $\{x_n\}$ is bounded.

For any $z \in \Omega$, we find from (3.3) that $\|y_{2n+1} - z\| \leq \|x_{2n+1} - z\| + |\delta_{2n+1}|\|x_{2n+1} - x_{2n}\|$. It follows that

$$\begin{aligned} &\|x_{2n+2} - z\| \\ &= \|\beta_{2n+1}(g(y_{2n+1}) - z) + (1-\beta_{2n+1})(V_{\gamma_{2n+1}} y_{2n+1} - z)\| \\ &\leq \beta_{2n+1}\|g(y_{2n+1}) - h(z)\| + \beta_{2n+1}\|g(z) - z\| + (1-\beta_{2n+1})\|V_{\gamma_{2n+1}} y_{2n+1} - z\| \\ &\leq \beta_{2n+1}\theta\|y_{2n+1} - z\| + \beta_{2n+1}\|g(z) - z\| + (1-\beta_{2n+1})\|y_{2n+1} - z\| \\ &\leq (1 - \beta_{2n+1}(1-\theta))\|x_{2n+1} - z\| + \beta_{2n+1}(1-\theta)\frac{\|g(z) - z\| + \frac{|\delta_{2n+1}|\|x_{2n+1} - x_{2n}\|}{\beta_{2n+1}}}{1-\theta}. \end{aligned}$$

Similarly, we deduce

$$\begin{aligned}\|x_{2n+1}-z\| &= \|\beta_{2n}(g(x_{2n})-z)+(1-\beta_{2n})(V_{\gamma_{2n}}x_{2n}-z)\| \\ &\leq \beta_{2n}\|g(x_{2n})-g(z)\|+\beta_{2n}\|g(z)-z\|+(1-\beta_{2n})\|V_{\gamma_{2n}}x_{2n}-z\| \\ &\leq (1-\beta_{2n}(1-\theta))\|x_{2n}-z\|+\beta_{2n}(1-\theta)\frac{\|g(z)-z\|}{1-\theta}.\end{aligned}$$

From condition (iii), we have that $\lim_{n\to\infty}|\delta_{2n+1}|\|x_{2n+1}-x_{2n}\|=0$ and $\lim_{n\to\infty}\frac{|\delta_{2n+1}|\|x_{2n+1}-x_{2n}\|}{\beta_{2n+1}}$ $=0$. Thus we find that there exists some positive constant c_2 such that

$$c_2 \geq \sup_n\{\|g(z)-z\|+\frac{|\delta_{2n+1}|\|x_{2n+1}-x_{2n}\|}{\beta_{2n+1}}\}.$$

It follows that $\|x_{2n+2}-z\| \leq \max\{\|x_{2n+1}-z\|,\frac{c_2}{1-\theta}\}$ and

$$\|x_{2n+1}-z\| \leq \max\{\|x_{2n}-z\|,\frac{\|g(z)-z\|}{1-\theta}\} \leq \max\{\|x_{2n}-z\|,\frac{c_2}{1-\theta}\}.$$

Similarly, using mathematical induction, we have $\|x_n-z\| \leq \max\{\|x_0-z\|,\frac{c_2}{1-\theta}\}$. This immediately obtains that $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{g(y_n)\}$, and $\{V_{\gamma_n}(y_n)\}$.

Step 2. Prove that $\lim_{k\to\infty}\kappa_{n_k}=0$ implies $\lim_{k\to\infty}\|x_{n_k}-V_{\gamma_{n_k}}x_{n_k}\|=0$ for any subsequence $\{n_k\}\subset\{n\}$.

We select the subsequence $\{x_{2n+1}\}$ and $\{x_{2n+2}\}$ from the sequence $\{x_n\}$. From 3.2, it is easy to see that $\|x_{n_{k_{ij}}}-V_{ij}x_{n_{k_{ij}}}\| \leq \|x_{n_{k_{ij}}}-V_{\gamma_{n_{k_{ij}}}}x_{n_{k_{ij}}}\|+\|V_{\gamma_{n_{k_{ij}}}}x_{n_{k_{ij}}}-V_{ij}x_{n_{k_{ij}}}\| \to 0$.

Step 3. Prove that $\omega_w(x_{n_k})\subset\Omega$.

Following Theorem 3.1, the proof of Step 3 can be completed immediately. In view of Lemma 2.4, one obtains that $\{x_n\}$ converges to a point x^* in Ω strongly. □

In recent years, multi-step inertial algorithms were extensively investigated; see, e.g., [19, 23, 24] and references therein. Next, we purpose a multi-step inertial method.

Algorithm 3.4. Let $d\in\mathbb{N}_+$ and $D:=\{0,\cdots,d-1\}$. For any initial $x_0,x_{-1},\cdots,x_{-d}$, choose a sequence $\{\varepsilon_n\}$ satisfying the condition (iii) of Theorem 3.4, and take $\beta\geq 3$. Set $n:=0$.
Given $x_n,x_{n-1},\cdots,x_{n-d}$, compute $y_n=x_n+\sum_{i\in D}\delta_{i,n}(x_{n-i}-x_{n-1-i})$, and choose $\delta_{i,n}$ such that $0\leq|\delta_{i,n}|\leq\bar{\delta}_n$ with $\bar{\delta}_n$ defined by

$$\bar{\delta}_n=\begin{cases}\min\{\frac{n-1}{n+\beta-1},\frac{\varepsilon_n}{\sum_{i\in D}\|x_{n-i}-x_{n-i-1}\|}\}, & \sum_{i\in D}\|x_{n-i}-x_{n-i-1}\|\neq 0,\\ \frac{n-1}{n+\beta-1}, & otherwise.\end{cases}$$

Proceed with the following format:

$$x_{n+1}=\beta_n g(y_n)+(1-\beta_n)U_{[n]_1}(y_n-\gamma_n A^*(I-T_{[n]_2})Ay_n), \tag{3.4}$$

where $g:H_1\to H_1$ is a θ-contractive mapping with $\theta\in(0,1)$ and $0<a\leq\gamma_n\leq b<\frac{2}{\lambda}$ with $\lambda=\rho(A^*A)$.

Theorem 3.4. *Let $A:H_1\to H_2$ be a bounded linear operator with $\lambda=\rho(A^*A)$, and let A^* be its adjoint. Assume that $\{U_i\}_{i=1}^t:H_1\to H_1$ and $\{T_j\}_{j=1}^r:H_2\to H_2$ are $\{\tau_{1_i}\}_{i=1}^t$-av and $\{\tau_{2_j}\}_{j=1}^r$-av, respectively, where t and r are two positive integer numbers. Suppose that $\Omega\neq\emptyset$.*

Let $g : H_1 \to H_1$ be a θ-contractive operator with $\theta \in (0,1)$, and let $\{\beta_n\}$ be a sequence in $(0,1)$. If

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < \min_j \frac{1}{\tau_{2_j}\lambda}$;
(iii) $\varepsilon_n = o(\beta_n)$, *i.e.*, $\lim_{n\to\infty}(\varepsilon_n/\beta_n) = 0$ ($e.g.$, $\varepsilon_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{1}{n+1}$),

then $\{x_n\}$ converges strongly to $x^* \in \Omega$

Proof. Setting $V_{\gamma_n} := U_{[n]_1}(I - \gamma_n A^*(I - T_{[n]_2})A)$, V_{γ_n} can be rewritten as $V_{\gamma_n} = (1-\mu_n)I + \mu_n W_n$, where W_n is a nonexpansive operator. The proof is divided into three steps.
Step 1. Prove that $\{x_n\}$ is bounded.

For any $z \in \Omega$, we obtain $\|y_n - z\| \le \|x_n - z\| + \|\sum_{i\in D} \delta_{i,n}(x_{n-i} - x_{n-1-i})\|$. From (3.4), we deduce

$$\begin{aligned}\|x_{n+1} - z\| &= \|\beta_n(g(y_n) - z) + (1-\beta_n)(V_{\gamma_n} y_n - z)\| \\ &\le \beta_n\|g(y_n) - g(z)\| + \beta_n\|g(z) - z\| + (1-\beta_n)\|V_{\gamma_n} y_n - z\| \\ &\le \beta_n\theta\|y_n - z\| + \beta_n\|g(z) - z\| + (1-\beta_n)\|y_n - z\| \\ &\le (1 - \beta_n(1-\theta))\|x_n - z\| + \beta_n(1-\theta)\frac{\|g(z) - z\| + \frac{\bar{\delta}_n \sum_{i\in D}\|x_{n-i} - x_{n-1-i}\|}{\beta_n}}{1-\theta}.\end{aligned}$$

From condition (iii), we see that $\lim_{n\to\infty} \bar{\delta}_n \sum_{i\in D}\|x_{n-i} - x_{n-1-i}\| = 0$ and

$$\lim_{n\to\infty} \frac{\bar{\delta}_n \sum_{i\in D}\|x_{n-i} - x_{n-1-i}\|}{\beta_n} = 0.$$

Thus there exists some $c_3 > 0$ such that $c_3 \ge \sup_n\{\|g(z) - z\| + \frac{\bar{\delta}_n \sum_{i\in D}\|x_{n-i} - x_{n-1-i}\|}{\beta_n}\}$. Hence, $\|x_{n+1} - z\| \le \max\{\|x_n - z\|, \frac{c_3}{1-\theta}\}$. Further, we have $\|x_n - z\| \le \max\{\|x_0 - z\|, \frac{c_3}{1-\theta}\}$. We immediately obtain $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{g(y_n)\}$, and $\{V_{\gamma_n}(y_n)\}$.
Step 2. Prove that $\lim\limits_{k\to\infty} \kappa_{n_k} = 0$ implies $\lim_{k\to\infty}\|x_{n_k} - V_{\gamma_{n_k}} x_{n_k}\| = 0$ for any subsequence $\{n_k\} \subset \{n\}$.

For any $z \in \Omega$, we obtain

$$\begin{aligned}&\|y_n - z\|^2 \\ &= \|x_n + \sum_{i\in D} \delta_{i,n}(x_{n-i} - x_{n-1-i}) - z\|^2 \\ &\le \|x_n - z\|^2 + 2\langle x_n - z + \sum_{i\in D} \delta_{i,n}(x_{n-i} - x_{n-1-i}), \sum_{i\in D} \delta_{i,n}(x_{n-i} - x_{n-1-i})\rangle \\ &= \|x_n - z\|^2 + 2\sum_{i\in D} |\delta_{i,n}|\|x_{n-i} - x_{n-1-i}\|(\|x_n - z\| + \sum_{i\in D} |\delta_{i,n}|\|x_{n-i} - x_{n-1-i}\|),\end{aligned}$$

which together with (3.4) deduces that

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\leq (1-\beta_n(2-\beta_n(1+2\theta^2)-2\theta(1-\beta_n)))\|y_n-z\|^2\\
&\quad +2\beta_n(1-\beta_n)\langle g(z)-z, V_{\gamma_n}y_n-z\rangle+2\beta_n^2\|g(z)-z\|^2\\
&\leq (1-\beta_n(2-\beta_n(1+2\theta^2)-2\theta(1-\beta_n)))\|x_n-z\|^2\\
&\quad +2\sum_{i\in D}|\delta_{i,n}|(1-\beta_n(2-\beta_n(1+2\theta^2)-2\theta(1-\beta_n)))\|x_{n-i}-x_{n-1-i}\|(\|x_n-z\|\\
&\quad +\sum_{i\in D}|\delta_{i,n}|\|x_{n-i}-x_{n-1-i}\|)+2\beta_n(1-\beta_n)\langle g(z)-z, V_{\gamma_n}y_n-z\rangle+2\beta_n^2\|g(z)-z\|^2.
\end{aligned}$$

Observe that

$$\begin{aligned}
\|x_{n+1}-z\|^2&=\|V_{\gamma_n}y_n-z+\beta_n(g(y_n)-V_{\gamma_n}y_n)\|^2\\
&=\|V_{\gamma_n}y_n-z\|^2+\beta_n^2\|g(y_n)-V_{\gamma_n}y_n\|^2+2\beta_n\langle V_{\gamma_n}y_n-z, g(y_n)-V_{\gamma_n}y_n\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-z\|^2&\leq \|y_n-z\|^2-(1-\mu_n)\mu_n\|W_ny_n-y_n\|^2+\beta_n^2\|g(y_n)-V_{\gamma_n}y_n\|^2\\
&\quad +2\beta_n\langle V_{\gamma_n}y_n-z, g(y_n)-V_{\gamma_n}y_n\rangle\\
&\leq \|x_n-z\|^2-(1-\mu_n)\mu_n\|W_ny_n-y_n\|^2+\beta_n^2\|g(y_n)-V_{\gamma_n}y_n\|^2\\
&\quad +2\beta_n\langle V_{\gamma_n}y_n-z, g(y_n)-V_{\gamma_n}y_n\rangle\\
&\quad +2\sum_{i\in D}|\delta_{i,n}|\|x_{n-i}-x_{n-1-i}\|(\|x_n-z\|+\sum_{i\in D}|\delta_{i,n}|\|x_{n-i}-x_{n-1-i}\|).
\end{aligned}$$

Define $b_n=\|x_n-z\|^2$, $\sigma_n=\beta_n(2-\beta_n(1+2\theta^2)-2\theta(1-\beta_n))$, $\kappa_n=(1-\mu_n)\mu_n\|W_ny_n-y_n\|^2$,

$$\begin{aligned}
\zeta_n=&\frac{2}{2-\beta_n(1+2\theta^2)-2\theta(1-\beta_n)}((1-\beta_n)\langle g(z)-z, V_{\gamma_n}y_n-z\rangle\\
&+\frac{\sum_{i\in D}|\delta_{i,n}|\|x_{n-i}-x_{n-1-i}\|(\|x_n-z\|+\sum_{i\in D}|\delta_{i,n}|\|x_{n-i}-x_{n-1-i}\|)}{\beta_n}+\beta_n\|g(z)-z\|^2)\\
&-2\sum_{i\in D}|\delta_{i,n}|\|x_{n-i}-x_{n-1-i}\|(\|x_n-z\|+\sum_{i\in D}|\delta_{i,n}|\|x_{n-i}-x_{n-1-i}\|),
\end{aligned}$$

and

$$\psi_n=\beta_n^2\|g(y_n)-V_{\gamma_n}y_n\|^2+2\beta_n\langle V_{\gamma_n}y_n-z, g(y_n)-V_{\gamma_n}y_n\rangle.$$

Since $\lim_{n\to\infty}\frac{\sigma_n}{\beta_n}=2(1-\theta)$, we have that $\sum_{n=0}^{\infty}\sigma_n$ is divergent to ∞ and ψ_n tends to 0. Then $\{\sigma_n\}$ and $\{\psi_n\}$ satisfy the conditions (i) and (ii) of Lemma 2.4. Now, we only need to prove that $\kappa_{n_k}\to 0$ implies that $\limsup_{k\to\infty}\zeta_{n_k}\leq 0$ for any subsequence $\{n_k\}\subset\{n\}$. In fact, $\kappa_{n_k}\to 0$ leads to $\|W_{n_k}y_{n_k}-y_{n_k}\|\to 0$. Then, it follows that $\|y_{n_k}-V_{\gamma_{n_k}}y_{n_k}\|=\mu_{n_k}\|y_{n_k}-W_{n_k}y_{n_k}\|\to 0$. According to condition (iii), we find $\|y_{n_k}-x_{n_k}\|=\sum_{i\in D}|\delta_{i,n_k}|\|x_{n_k-i}-x_{n_k-1-i}\|\to 0$. Hence,

$$\begin{aligned}
&\lim_{k\to\infty}\|x_{n_k}-V_{\gamma_{n_k}}x_{n_k}\|\\
&\leq \lim_{k\to\infty}\|x_{n_k}-y_{n_k}\|+\lim_{k\to\infty}\|y_{n_k}-V_{\gamma_{n_k}}y_{n_k}\|+\lim_{k\to\infty}\|V_{\gamma_{n_k}}y_{n_k}-V_{\gamma_{n_k}}x_{n_k}\|\\
&\leq \lim_{k\to\infty}2\|x_{n_k}-y_{n_k}\|+\lim_{k\to\infty}\|y_{n_k}-V_{\gamma_{n_k}}y_{n_k}\|.
\end{aligned}$$

which implies that $\|x_{n_k} - V_{\gamma_{n_k}} x_{n_k}\| = 0$. It is similar to Theorem 3.1, we deduce that $\|x_{n_{k_{ij}}} - V_{ij} x_{n_{k_{ij}}}\| \le \|x_{n_{k_{ij}}} - V_{\gamma_{n_{k_{ij}}}} x_{n_{k_{ij}}}\| + \|V_{\gamma_{n_{k_{ij}}}} x_{n_{k_{ij}}} - V_{ij} x_{n_{k_{ij}}}\| \to 0$.

Step 3. Prove that $\omega_w(x_{n_k}) \subset \Omega$.

Following Theorem 3.1, the proof of Step 3 is completed easily. Applying Lemma 2.4, we obtain that $\{x_n\}$ converges strongly to a point x^* in Ω. □

Finally, we consider the following alternated multi-step inertial method.

Algorithm 3.5. Let $d \in \mathbb{N}_+$ and $D := \{0, \cdots, d-1\}$. For any initial $x_0, x_{-1}, \cdots, x_{-d}$, choose a sequence $\{\varepsilon_n\}$ satisfying the condition (iii) of Theorem 3.5, and take $\beta \ge 3$. Set $n := 0$.
Given $x_n, x_{n-1}, \cdots, x_{n-d}$, compute

$$y_n = \begin{cases} x_n + \sum_{i \in D} \delta_{i,n}(x_{n-i} - x_{n-1-i}), & n \text{ is odd}, \\ x_n, & n \text{ is even}. \end{cases}$$

When n is odd, choose $\delta_{i,n}$ such that $0 \le |\delta_{i,n}| \le \bar{\delta}_n$ with $\bar{\delta}_n$ defined by

$$\bar{\delta}_n = \begin{cases} \min\{\frac{n-1}{n+\beta-1}, \frac{\varepsilon_n}{\sum_{i \in D} \|x_{n-i} - x_{n-i-1}\|}\}, & \sum_{i \in D} \|x_{n-i} - x_{n-i-1}\| \ne 0, \\ \frac{n-1}{n+\beta-1}, & \text{otherwise}. \end{cases}$$

Proceed with the following format:

$$x_{n+1} = \beta_n g(y_n) + (1-\beta_n) U_{[n]_1}(y_n - \gamma_n A^*(I - T_{[n]_2})Ay_n), \tag{3.5}$$

where $g : H_1 \to H_1$ is a θ-contractive mapping with $\theta \in (0,1)$ and $0 < a \le \gamma_n \le b < \frac{2}{\lambda}$ with $\lambda = \rho(A^*A)$.

Theorem 3.5. *Let $A : H_1 \to H_2$ be a bounded linear operator with $\lambda = \rho(A^*A)$, and let A^* be its adjoint. Assume that $\{U_i\}_{i=1}^t : H_1 \to H_1$ and $\{T_j\}_{j=1}^r : H_2 \to H_2$ are $\{\tau_{1_i}\}_{i=1}^t$-av and $\{\tau_{2_j}\}_{j=1}^r$-av, respectively, where t and r are two positive integer numbers. Suppose that $\Omega \ne \emptyset$. Let $g : H_1 \to H_1$ be a θ-contractive operator with $\theta \in (0,1)$, and let $\{\beta_n\}$ be a sequence in $(0,1)$. If*

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < \min_j \frac{1}{\tau_{2_j}\lambda}$;
(iii) $\varepsilon_n = o(\beta_n)$, *i.e.* $\lim\limits_{n\to\infty} (\varepsilon_n/\beta_n) = 0$ (*e.g.*, $\varepsilon_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{1}{n+1}$).

then the sequence $\{x_n\}$ g converges strongly to $x^* \in \Omega$.

Proof. Following Theorem 3.3 and Theorem 3.4, we can conclude the desired conclusion immediately. □

4. Applications in Compressed Sensing

With the rapid growth of information technology, the demand for information and signal bandwidth is getting higher and higher. In 2006, Donoho [25] and Candes, Romberg, and Tao [26, 27] proposed the compressed sensing theory on signal sampling, compression, and reconstruction. Tt has attracted wide attention in many areas in science and technology. The importance of sparse optimization techniques was recognized on compressed sensing in recent years; see, e.g., [28, 29, 30] and the references therein. Our aim is to find a reconstructed sparse vector $f \in \mathbb{R}^n$ satisfying the following equation $\widetilde{f} = Af$, where $\widetilde{f} \in \mathbb{R}^m$ is a given observed (noised or blurred) image or signal, A is a measurement matrix, and f is the original image or signal. The problem can be modeled as

$$\min_{f\in\mathbb{R}^n}\{\frac{1}{2}\|Af-\widetilde{f}\|_2^2+\lambda\|f\|_0\}, \tag{4.1}$$

where λ is a nonnegative tuning parameter, and $\|f\|_0$ represents the number of nonzero components of f. Actually, it is difficult to solve (4.1) due to it is NP-hard. In order to overcome this disadvantage, some relaxation methods were given, such as l_1-norm instead of l_0-norm. When sensing matrix satisfies restrict isometric property (for short RIP), the accurate image or signal can be restored through the convex relaxation. In [31], the following model was proposed.

$$\min_{f\in\mathbb{R}^n}\{\frac{1}{2}\|Af-\widetilde{f}\|_2^2+\lambda\|f\|_1\}, \tag{4.2}$$

where $\|f\|_1 = \sum_{i=1}^n |f_i|$ denotes l_1-norm of f. It is known that (4.2) is a convex optimization problem. Thus it is easier than (4.1) to solve. Following [31], we deduce that f^* is a solution to (4.2) if and only if f^* solves the fixed point equation $f^* = prox_{\lambda\|\cdot\|_1}(f^* - \lambda A^*(Af^* - \widetilde{f}))$. Measurement matrix and reconstruction algorithm are two main factors affecting the reconstruction quality. On the one hand, for the same reconstruction algorithm, different measurement matrices have different signal reconstruction capabilities, therefore, the construction of measurement matrix is a key problem in compressed sensing. On the other hand, for the same measurement matrix, different reconstruction algorithms have different signal reconstruction capabilities, therefore, the design of reconstruction algorithm is also an important problem in compressed sensing.

The compressed sensing matrix is divided into the randomly compressed sensing matrix (see [26, 32]) and the deterministic compressed sensing matrix (see [33]). At present, although the extensive Gaussian random matrix can recover the original data with high probability, it has some disadvantages, such as high uncertainty, complex algorithms, and difficult to implement in hardware. Recently, some results were obtained in deterministic compressed sensing matrix construction; see, e.g., [33, 34, 35, 36] and the references therein.

In our numerical experiment, the compressed sensing matrix A is a random sparse matrix with 40% distributed as $N(\bar{x}, \sigma^2)$.

Next, we consider an example to show the realization and efficiency of our algorithms.

Example 4.1. Choose the original "cameraman" image, shown in Fig. 1 (a) and the blurred image, shown in Fig. 1 (b). Let $H_1 = \mathbb{R}^n$ $(n = 256^2)$, $H_2 = \mathbb{R}^m$ $(m = 128^2)$, and A represent an $m \times n$ matrix. Assume that $t = r = 1$, $U = prox_{\lambda\|\cdot\|_1}$, and $T = P_Q$ with $Q = \{\widetilde{f}\}$ Let $\{f_k : k \in N\}$ be an image sequence (We transform the image matrix into a vector by column before writing codes). For a pre-given tolerance tol, the iterative process is terminated if the following requirement is satisfied: $\|f_{k+1} - f_k\| \leqslant tol$. The quality of restored images can be determined by value of the peak signal to noise ratio (PSNR) measured in decibel (dB) as follows $PSNR = 10\log_{10}\frac{255n}{\|f-\bar{f}\|}$, where f and $\bar{f}$ represent the original image and the restored image respectively. Generally, the larger PSNR value is, the better image quality is. To better access the differences among the images reconstructed via Algorithms 3.1-3.5, we set up two other stopping criteria.

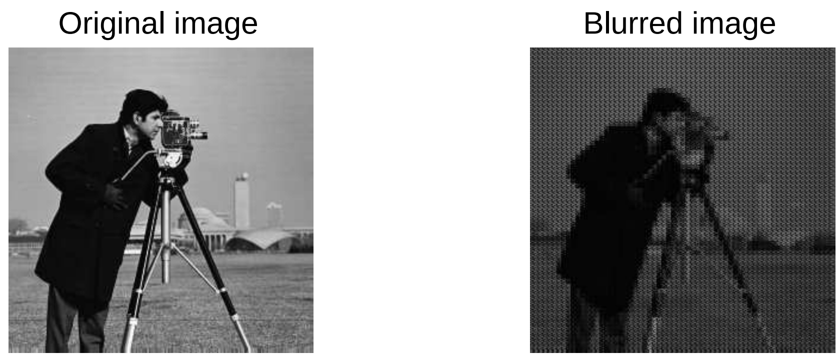


FIGURE 1. (a) The original image (b) The blurred image

The iterative process is terminated if one of the following requirements is satisfied: $PSNR \geqslant p, iterations = t$.

FIGURE 2. The blurred and reconstructed images via Algorithms 3.1-3.5 presented in 200th iterations, respectively.

The blurred and restored image are given in Figure 2. The relative error, CPU time, PSNR, and the iterative numbers are shown in Tab. 1. Obviously, Algorithm 3.1 uses the viscosity approximate to the strong convergence. Note that strong and weak convergence are equivalent in a finite dimensional space. For convenience and simpleness, we take $g = 0$.

In Algorithm 3.2, we adopt a one-step inertial technique. In Algorithm 3.3, we use the alternated inertia technique. In Algorithm 3.4 and Algorithm 3.5, we change the one-step inertia in Algorithm 3.2 and Algorithm 3.3 to multi-step inertia, respectively.

In Table 1, we list the iterations, CPU time, and PSNR of the deblurred images used by Algorithms 3.1-3.5. After examining the table, we obtain that under the same stopping criteria the result of Algorithm 3.5 is better than Algorithms 3.1-3.4. Compared with Algorithm 3.2, Algorithm 3.4 obviously performs better. Compared with Algorithm 3.1, Algorithm 3.3 is more efficient. From (a) of Figure 3, we can see the above elaborate information clearly

TABLE 1. The summary of the restoration results of Algorithms 3.1-3.5 in three err cases

	$tol = 10^{-2}$			$tol = 10^{-3}$			$tol = 10^{-4}$		
Alg	$iterations$	T	$PSNR$	$iterations$	T	$PSNR$	$iterations$	T	$PSNR$
Alg 3.1	108	4.0553	32.3373	702	15.5716	43.8598	1835	41.3321	63.1756
Alg 3.2	123	5.8128	37.8915	323	15.0409	62.3418	551	25.2839	70.4978
Alg 3.3	128	5.8043	34.7612	448	20.0748	45.0333	1110	48.0580	64.5015
Alg 3.4	132	7.8858	38.0378	321	19.0874	71.7508	415	23.7932	85.4383
Alg 3.5	72	4.1802	33.7065	128	7.2258	50.9704	262	14.4096	70.3860

In order to compare the effectiveness of image restoration, we use the second stop criterion. It is apparent from the Table 2 that Algorithms 3.4 and 3.5 are faster to recover the quality of the images. We can obtain that Algorithms 3.2, 3.4, and 3.5 require only 10-20s to reach the ratio of 60 dB, while Algorithms 3.1 and 3.3 need almost 38s. Furthermore, the number of iterations used in Algorithms 3.2, 3.4, and 3.5 is significantly reduced through comparing with Algorithms 3.1 and 3.3

TABLE 2. The summary of the restoration results of Algorithms 3.1-3.5 in three cases

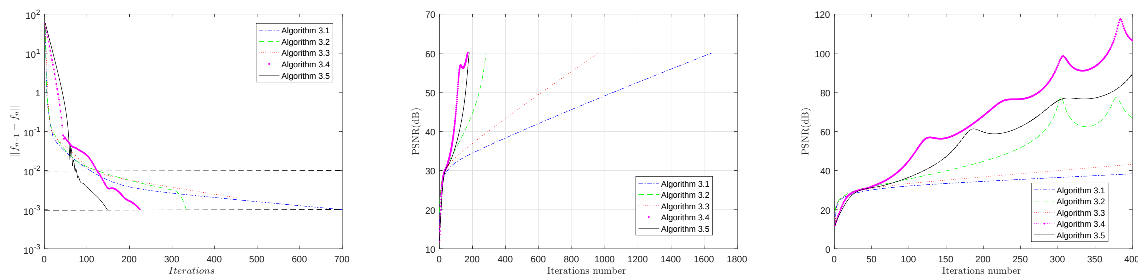
	$p = 40dB$		$p = 50dB$		$p = 60dB$	
Alg	T	$iterations$	T	$iterations$	T	$iterations$
Alg 3.1	11.3644	491	24.9596	1053	38.1014	1645
Alg 3.2	7.8837	153	12.4081	242	14.0667	283
Alg 3.3	14.5798	296	30.5599	621	45.5962	962
Alg 3.4	6.0120	88	7.5407	110	11.7460	171
Alg 3.5	7.9677	120	10.2963	158	11.6840	180

In Table 3, we report on more results for comparing all methods for the same iteration. We observe that the PSNR of the restoration by Algorithms 3.4 and 3.5 are bigger when the number of iteration steps reached 400. The PSNR of the recovered image is actually above 80dB. That means it is very close to the original image.

Finally, we compare the relevant parameters (iterations, error, and PSNR) of Algorithms 3.1-3.5 by graphs in Figure 3. The first is for comparing the number of iterative steps of Algorithms 3.1-3.5 while the error decrease to 10^{-3}, although Algorithm 3.5 produces a shock phenomenon in the late period, the number of iterative steps is still the least. The second is for comparing

TABLE 3. Numerical results of the restorated images by Algorithms 3.1-3.5

	$n=100$		$n=200$		$n=400$	
Alg	*T*	*PSNR*	*T*	*PSNR*	*T*	*PSNR*
Alg 3.1	2.3391	32.1643	4.5880	34.4888	9.0858	38.3696
Alg 3.2	4.7947	36.0338	9.5061	44.6929	18.8627	65.4907
Alg 3.3	4.6913	33.7673	9.2039	37.1858	18.2634	43.5585
Alg 3.4	5.9419	45.1045	11.9636	67.9901	23.3267	106.5321
Alg 3.5	5.6846	36.6459	11.0507	59.6337	21.2697	89.5558



(a) (b) (c)

FIGURE 3. (a) Comparison of iterations of Algorithms 3.1-3.5 when $\|f_{n+1} - f_n\|$ is 10^{-3},
(b) Comparison of iterations of Algorithms 3.1-3.5 when the PSNR reaches 60dB,
(c) PSNR values of Algorithms 3.1-3.5 versus the number of iterations.

the number of iterations of Algorithms 3.1-3.5 when the PSNR reaches 60dB. Algorithms 3.4 and 3.5 are more efficient than other algorithms. The last is for comparing the PSNR of the restored images by Algorithms 3.1-3.5 when the number of iterative steps are the same. It is clearly evident from (c) of Figure 3 that Algorithm 3.4 is the best algorithm and Algorithm 3.5 is second only to Algorithm 3.4 under the conditions of Example 4.1. All in all, through this numerical experiment, the multi-step inertial method and the alternated multi-step inertial acceleration method show better computational efficiency.

5. CONCLUSION

In this paper, we considered the multi-set split common fixed point problem (1.2). Inspired by several cyclic iterative algorithms for solving the MSCFPP of directed operators, proposed by Wang and Xu [7] in 2011, we proposed some new inertial algorithms which combine the viscosity iteration and the cyclic iterative process for solving the MSCFPP. To make full use of the advantage of cyclic iterative processes and different types of inertial acceleration methods, we presented a mixed viscosity iterative and cyclic algorithm and several general inertial acceleration algorithms. A numerical example is presented to verify the image restoration preference. The comparisons between the several iterative algorithms, the multi-step inertial method, and the alternated multi-step inertial acceleration method show better efficiency of calculation and the quality of image recovery.

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