J. Nonlinear Var. Anal. 6 (2022), No. 4, pp. 407-420
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.6.2022.4.07

ORDER-PRESERVATION PROPERTIES OF RESOLVENT OPERATORS AND THEIR APPLICATIONS TO VARIATIONAL INEQUALITIES

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Abstract. In this paper, we aim to investigate a class of new order-theoretic properties for the resolvent operators, which are called order-preservation properties. As applications, we use these order-preservation properties and several order-theoretic fixed point theorems to prove the existence of extremal solutions for several kinds of variational inequalities arising in mechanics and economics. In contrast to many previous studies, the approaches used in this paper are mainly based on the partial order structure of the underlying spaces, and the obtained results weaken the continuity and monotonicity of the associated mappings.

Keywords. Differential variational inequalities; Maximal solution; Minimal solution; Order-preservation property; Order-theoretic fixed point.

1. INTRODUCTION

In economics and management science fields, many practical problems can be modeled as a Nash equilibrium problem. In the classical game theory, the utility functions of the players are often assumed to be real-valued functions, that is, the outcomes in such games have ranges in totally ordered sets. However, in the real world, there are some games in which the utilities for the players on the outcomes are not totally ordered (maybe partially ordered). For this type of game, we refer to [1, 2, 3] for some concrete examples. To study the Nash equilibrium problems with incomplete preferences, the traditional topological approaches usually do not work well. Actually, in addition to the aforesaid Nash equilibrium problems, the traditional topological approaches may also be ineffective for some nonlinear complementarity problems and variational inequalities involved discontinuous mappings; see, e.g., [4, 5, 6, 7] and the references therein. Driven by the nonlinear problems mentioned above, the order-theoretic

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Received March 12, 2022; Accepted June 15, 2022.

methods have drawn much attention in recent years; see, e.g., [8, 9, 10] and the references therein.

To use order-theoretic methods to deal with (generalized) variational inequalities and Nash equilibrium problems, the projection operator and its order-preservation properties played a key role. In 2012, Nishimura and Ok [6] developed a kind of order-theoretic approach for set-valued variational inequalities. By using the order-preservation properties of the projection operator, they investigated the solvability of variational inequalities on Hilbert lattices. Li, Xie, and Wen proved the order-preservation properties for generalized (metric) projection operators and used the order-theoretic fixed point theorems to examine some other nonlinear problems, such as generalized variational inequalities, generalized saddle point problems and extended Nash equilibriums of nonmonetized noncooperative games; see, e.g., [1, 3, 10] for more details. Recently, Wang and Liu further applied these order-theoretic approaches to the equilibrium problems, which include variational inequalities, saddle point problems and Nash equilibrium problems as special cases; see, e.g., [11, 12]. Since the above approaches are order-theoretic, all of the obtained results related to variational inequalities and equilibrium problems are not required the considered mappings to be continuous and monotone.

On the other hand, as an important extension of variational inequalities, mixed variational inequalities (MVIs, for short) also provide a unified framework for us to study various kinds of practical problems arising in mechanics and economics; see, e.g., [13, 14, 15, 16]). It is well known that the MVIs can be converted to a fixed point problem by using the resolvent operator techniques. Thus it is similar to the role of projection operators in variational inequalities and Nash equilibrium problems that the resolvent operators play a crucial role in the existence studies and convergence analysis for MVIs; see, e.g, [17, 18]). In traditional research, in order to guarantee the existence of solutions to MVIs and ensure the iteration algorithms to be convergent, the mappings involved in MVIs are always required to be monotone and Lipschitz continuous. Since these assumptions may not hold in the real world, it is necessary to explore some novel approaches which can weaken the continuity and monotonicity of the considered mappings.

From the above literature, we see that the projection operators and its order-preservation properties can be used to deal with some variational inequalities involved discontinuous or non-monotone mappings. Moreover, we also notice that the projection operators can be regard as the special form of the resolvent operators. These observations motivate us to consider the following question of interest: **Do the resolvent operators have any order-preservation properties under some suitable assumptions**? The main goal of this paper is to give the positive answer to the above question. Furthermore, by virtue of the order-preservation properties of the resolvent operators and some order-theoretic fixed point theorems, we also examine the existence of extremal solutions for (generalized) MVIs and a class of differential quasivariational inequalities.

The reminder of this paper is as follows. Section 2 is devoted to some basic concepts related to poset and several useful lemmas. In Section 3, we investigate the order-preservation properties of the generalized resolvent operators under some suitable conditions, then apply the order-preservation properties of resolvent operators and some order-theoretic fixed point theorems to prove the existence of extremal solutions to a class of generalized mixed variational inequalities. In Section 4, we focus on a class of differential quasivariational inequalities, which couple a differential equation with a time-dependent quasivariational inequality. Section 5 provides a summary and conclusions.

2. Preliminaries

In this section, we review some concepts related to poset (partially ordered set) and several order-theoretic fixed point theorems, which are frequently used in the following sections. One can refer to [6, 19, 20, 21, 22] for more details.

Definition 2.1. Let $(P, \preccurlyeq_P)$ be a poset and D be a nonempty subset of $(P, \preccurlyeq_P)$.

(i) An element p of P is said to be an $\preccurlyeq_P$-*upper bound* of D if $x \preccurlyeq_P p$ for each $x \in D$. In addition, if $p \in D$, then we say that p is the $\preccurlyeq_P$-maximum in D. The $\preccurlyeq_P$-lower bound and $\preccurlyeq_P$-minimum element of D can be similarly understood.

(ii) An element $p \in P$ is said to be the $\preccurlyeq_P$-*supremum* of D if it is the $\preccurlyeq_P$-minimum of the set of all $\preccurlyeq_P$-upper bounds of D, and is denoted by $\bigvee_P D$. If $y \in D$ and $y \preccurlyeq_P z$ does not hold for any $z \in D \setminus \{y\}$, then we say that y is a $\preccurlyeq_P$-*maximal* element of D. The $\preccurlyeq_P$-*infimum* of D, $\bigwedge_P D$, and the $\preccurlyeq_P$-*minimal* element of D are defined similarly. D is said to be a $\preccurlyeq_P$-*chain* of P if either $x \preccurlyeq_P y$ or $y \preccurlyeq_P x$ hold for any $x, y \in D$.

(iii) The poset D is said to be $\preccurlyeq_P$-*bounded from above* if there exists some $x \in P$ such that $D \preccurlyeq_P x$. D is said to be $\preccurlyeq_P$-*bounded from below* if there exists some $x \in P$ such that $x \preccurlyeq_P D$. In turn, D is said to be $\preccurlyeq_P$-*bounded* if it is $\preccurlyeq_P$-bounded from above and below.

(iv) P is said to be a *lattice* if $\bigvee_P\{x,y\}$ and $\bigwedge_P\{x,y\}$ exist for all $x, y \in P$. P is said to be a *complete lattice* if $\bigvee_P S$ and $\bigwedge_P S$ exist for every nonempty $S \subseteq P$. D is said to be a $\preccurlyeq_P$-*sublattice* of P if D contains $\bigvee_P\{x,y\}$ and $\bigwedge_P\{x,y\}$ for every $x, y \in D$. In turn, if $\bigvee_P S \in D$ and $\bigwedge_P S \in D$ for every $S \subseteq D$, then it is said to be a *subcomplete* $\preccurlyeq_P$-*sublattice*. D is said to be chain-complete if for every chain S in D, there holds $\bigvee_P S \in D$.

(v) A sequence $\{z_n\}_{n\in\mathbb{N}}$ of P is said to be *increasing* if $z_n \preccurlyeq_P z_m$ whenever $n \le m$, *decreasing* if $z_m \preccurlyeq_P z_n$ whenever $n \le m$, and *monotone* if it is increasing or decreasing.

Definition 2.2. A closed subset E_+ of a normed space E is called an order cone if $E_+ \neq \{\mathbf{0}\}$, $E_+ + E_+ \subseteq E_+$, $E_+ \cap (-E_+) = \{\mathbf{0}\}$, and $cE_+ \subseteq E_+$ for each $c \geq 0$. The space E, equipped with an order relation "$\preccurlyeq_E$", defined by $x \preccurlyeq_E y$ if and only if $y - x \in E_+$, is called a partially ordered normed space. As usual, $\mathbf{0}$ denotes the origin of the normed spaces. Moreover, we say that the order cone E_+ is normal if there is a constant $\lambda \geq 1$ such that $0 \preccurlyeq_E x \preccurlyeq_E y$ in E implies $\|x\|_E \leq \lambda \|y\|_E$. E_+ is said to be regular if all order-preserving and $\preccurlyeq_E$-bounded sequences of E_+ converge.

Definition 2.3. Let E be a partially ordered Banach space with norm $\|\cdot\|_E$. If $\bigvee_E\{x,y\}$ and $\bigwedge_E\{x,y\}$ exist for all $x, y \in E$, then we say that E is lattice ordered. For each $x \in E$, let $x^+ := \bigvee_E\{x, \mathbf{0}\}$, $x^- := \bigvee_E\{-x, \mathbf{0}\}$ and $|x| := x^+ + x^-$. If for all $x, y \in E$, $|x| \preccurlyeq_P |y|$ imply $\|x\|_E \leq \|y\|_E$, then E is called a *Banach lattice*. If E is a Hilbert space and the norm is induced by an inner product $\langle \cdot, \cdot \rangle$ on E, then we refer to E as a *Hilbert lattice*.

The following lemma reveals the relationship between $\preccurlyeq_P$-sublattice and subcomplete $\preccurlyeq_P$-sublattice.

Lemma 2.1. *(See [6]) Let $(H, \preccurlyeq_H)$ be a separable Hilbert lattice and K a closed and $\preccurlyeq_H$-bounded $\preccurlyeq_H$-sublattice of H. Then K is a subcomplete $\preccurlyeq_H$-sublattice of H.*

Definition 2.4. Let $(X, \preccurlyeq_X)$ and $(P, \preccurlyeq_P)$ be two given posets. Let $\mathscr{F} : X \to 2^P \setminus \{\emptyset\}$ be a set-valued mapping and $\mathscr{G} : X \to P$ a single-valued mapping.

(i) $\mathscr{F}$ is said to be *upper order-preserving* if, for all $x, y \in X$, $x \preccurlyeq_X y$ and $z \in \mathscr{F}(x)$ imply $\mathscr{F}(y) \bigcap [z) \neq \emptyset$. $\mathscr{F}$ is said to *lower order-preserving* if, for all $x, y \in X$, $x \preccurlyeq_X y$ and $w \in \mathscr{F}(y)$ imply $\mathscr{F}(x) \bigcap (w] \neq \emptyset$. If $\mathscr{F}$ is both upper order-preserving and lower order-preserving, then we say that $\mathscr{F}$ is *order-preserving*. Here and below, set $[z) := \{x \in P : z \preccurlyeq_P x\}$, $(w] := \{x \in P : x \preccurlyeq_P w\}$, and $[z, w] := [z) \cap (w]$.

(ii) If, for all $x, y \in X$, $x \preccurlyeq_X y$ implies $\mathscr{G}(x) \preccurlyeq_P \mathscr{G}(y)$, then we say that $\mathscr{G}$ is *order-preserving*. If, for all $x, y \in X$, $x \preccurlyeq_X y$ implies $\mathscr{G}(y) \preccurlyeq_P \mathscr{G}(x)$, then we say that $\mathscr{G}$ is *order-reversing*.

Definition 2.5. A poset $(X, \preccurlyeq_X)$ equipped with a topology is called a partially ordered topological space if the order intervals $[z)$ and $(z]$ are closed for each $z \in X$. If the topology of X is induced by a metric, then we say that X is a partially ordered metric space.

Lemma 2.2. *(See [19]) Let Y be a topological space and Z a partially ordered metric space. If a pointwise monotone and equicontinuous sequence of functions from Y to Z has a pointwise limit, then this limit function is continuous. If Y is a compact metric space, then the convergence is uniform.*

Next, we recall several order-theoretic fixed point theorems, which are related to order-preserving mappings.

Lemma 2.3. *(See [7]) (**Knaster-Taski fixed point theorem**) Let $(X, \preccurlyeq_X)$ be a complete lattice and let $\mathscr{G} : X \to X$ be an order-preserving single-valued mapping. Then the set of all fixed point of $\mathscr{G}$ constitutes a nonempty complete lattice.*

Lemma 2.4. *(See [19]) Let $(X, \preccurlyeq_X)$ be a partially ordered normed space and $[\underline{y}, \bar{y}] = \{x \in X : \underline{y} \preccurlyeq_X x \preccurlyeq_X \bar{y}\}$ a nonempty order interval. Assume that $\mathscr{G} : [\underline{y}, \bar{y}] \to [\underline{y}, \bar{y}]$ is order-preserving, and $\mathscr{G}$ maps monotone sequences to convergent sequences. Then, $\mathscr{G}$ has the maximum fixed point and minimum fixed point in $[\underline{y}, \bar{y}]$, which are the $\preccurlyeq_X$-maximum element and $\preccurlyeq_X$-minimum element of the set of fixed points of $\mathscr{G}$, respectively.*

In what follows, we say that the poset $(P, \preccurlyeq_P)$ has property (C) if each well-ordered chain C of P whose increasing sequences have limits in P contains an increasing sequence that converges to $\bigvee_P C$, and each inversely well-ordered chain C of P whose decreasing sequences have limits in P contains a decreasing sequence that converges to $\bigwedge_P C$.

Lemma 2.5. *(See [19]) Given a partially ordered topological space $(P, \preccurlyeq_P)$ with property (C), assume that $G : P \to P$ is an order-preserving mapping.*

(a) If $S_+ = \{x \in P : x \preccurlyeq_P G(x)\}$ is nonempty, and G maps increasing sequences of S_+ to convergent sequences, then G has a maximal fixed point.

(b) If $S_- = \{x \in P : G(x) \preccurlyeq_P x\}$ is nonempty, and G maps decreasing sequences of S_- to convergent sequences, then G has a minimal fixed point.

Let H be a Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Now, we recall the concept of resolvent operators for the maximal monotone operator. For the definition of maximal monotone operators, we refer to monograph [19].

Definition 2.6. (See [20, 21]) Let H be a Hilbert space and ψ a maximal monotone operator on H. Then, the resolvent operator associated with ψ is defined as

$$J_{\psi}^{\rho}(u) = (id_H + \rho\psi)^{-1}(u), \ \ \forall u \in H, \tag{2.1}$$

where id_H is the identity operator on H and $\rho > 0$ is a constant.

It is well known that if $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous functional , then the subdifferential $\partial\phi$ of ϕ is a maximal monotone operator. Replacing ψ by $\partial\phi$ in (2.1), we can obtain the resolvent operator related to ϕ, which is denoted by $J_{\phi}^{\rho} := (id_H + \rho\partial\phi)^{-1}$. Regarding J_{ϕ}^{ρ}, we have the following equivalence relation.

Lemma 2.6. *(See [21]) For a given $z \in H$, $u \in H$ satisfies the inequality $\langle u - z, v - u\rangle + \rho\phi(v) - \rho\phi(u) \geq 0$, $\forall v \in H$ if and only if $u = J_{\phi}^{\rho}(z)$, where $\rho > 0$ is a constant.*

By Lemma 2.6, one can explore the existence of solutions to the following mixed variational inequality (MVI, for short): find $u \in H$ such that

$$\langle Tu, v - u\rangle + \phi(v) - \phi(u) \geq 0, \ \ \forall v \in H, \tag{2.2}$$

where $T : H \to H$.

Lemma 2.7. *(See [22]) The function $u \in H$ is a solution of MVI (2.2) if and only if $u \in H$ satisfies the relation $u = J_{\phi}^{\rho}(u - \rho Tu)$, where $\rho > 0$ is a constant.*

3. Order-Preservation Properties of the Generalized Resolvent Operators

In this section, we examine the following **generalized mixed variational inequality** (GMVI, briefly): find an element $u \in H$ such that

$$\langle Tu, v - u\rangle + j(u, v) - j(u, u) \geq 0, \ \forall v \in H, \tag{3.1}$$

where H and T are understood as in Section 2, and $j : H \times H \to \mathbb{R} \cup \{+\infty\}$ denotes a bifunctional.

Obviously, if, for all $u \in H$, we define $\phi : H \to \mathbb{R}$ by $\phi(v) := j(u, v)$, then GMVI (3.1) is reduced to the mixed variational inequality (MVI, for short): find an element $u \in H$ such that

$$\langle Tu, v - u\rangle + \phi(v) - \phi(u) \geq 0, \ \forall v \in H. \tag{3.2}$$

If $T := A - f$, where A is an elliptic operator and $f \in H$, then GMVI (3.1) is reduced to the elliptic quasivariational inequality, which was studied by Sofonea and Benraouda in [13] as a mathematical model of the spring-rod system with unilateral constraints. If, in addition, $\phi(v) := j(u, v)$ for all $u \in H$, then GMVI (3.1) collapses to the elliptic variational inequality of the second kind: find an element $u \in H$ such that $\langle Au, v - u\rangle + \phi(v) - \phi(u) \geq \langle f, v - u\rangle$, $\forall v \in H$.

Next, we establish some order-theoretic approaches for the solvability of GMVI (3.1). To this end, consider a generalized version of the resolvent operator J_{ϕ}^{ρ}, where ϕ is perturbed by a parameter $\eta \in \mathfrak{I}$, that is, $\phi : \mathfrak{I} \times H \to \mathbb{R} \cup \{+\infty\}$. Here and below, let $\mathfrak{I}$ denote a poset. Assume that $\phi(\eta, \cdot)$ is proper, convex and lower semicontinuous for any $\eta \in \mathfrak{I}$. In the same manner as in Definition 2.6, for given $\eta \in \mathfrak{I}$, one can define a resolvent operator $J_{\phi(\eta,\cdot)}^{\rho}$ associated with $\phi(\eta, \cdot)$ by

$$J_{\phi(\eta,\cdot)}^{\rho}(z) = (id_H + \rho\partial\phi(\eta, \cdot))^{-1}(z), \ \ \forall z \in H, \tag{3.3}$$

where $\rho > 0$ is a constant.

Note that the functional $\phi(\eta,\cdot)$ depends on $\eta\in\mathfrak{I}$. Thus $J^{\rho}_{\phi(\eta,\cdot)}(u)$ depends both η and u. In other words, equation (3.3) actually defines a new operator $\mathscr{J}^{\rho}_{\phi}:\mathfrak{I}\times H\to H$ by

$$\mathscr{J}^{\rho}_{\phi}(\eta,z):=J^{\rho}_{\phi(\eta,\cdot)}(z),\ \ \forall(\eta,z)\in\mathfrak{I}\times H.$$

In the sequel, we call $\mathscr{J}^{\rho}_{\phi}$ the **generalized resolvent operator**. From Lemma 2.6, it follows that $(\eta,z)\in\mathfrak{I}\times H$ satisfies the inequality

$$\langle u-z,v-u\rangle+\rho\phi(\eta,v)-\rho\phi(\eta,u)\geq 0,\ \forall v\in H,$$

if and only if $u=\mathscr{J}^{\rho}_{\phi}(\eta,z)$.

Next, we examine the order-preservation properties of $\mathscr{J}^{\rho}_{\phi}:\mathfrak{I}\times H\to H$ with respect to the first argument and the second argument, respectively. To present our result, we need the following lemma.

Lemma 3.1. *(See [6]) Let $(H,\preccurlyeq_H)$ be a Hilbert lattice. Then, for every $x,y\in H_+$, $y\preccurlyeq_H x$* and *$\langle x,y\rangle\leq 0$ imply $y=\mathbf{0}$, and $\langle z-z\wedge w,z\vee w-z\rangle=0$ for every $z,w\in H$, where the positive cone $H_+:=\{h:\mathbf{0}\preccurlyeq_H h\}$.*

Theorem 3.1. *Let $(H,\preccurlyeq_H)$ be a Hilbert lattice and $(\mathfrak{I},\preccurlyeq_{\mathfrak{I}})$ a poset. Assume that $\phi:\mathfrak{I}\times H\to\mathbb{R}\cup\{+\infty\}$ is a bifunctional such that $\phi(\eta,\cdot)$ is proper, convex, and lower semicontinuous for any $\eta\in\mathfrak{I}$; and for any $\eta_1,\eta_2\in\mathfrak{I}$ with $\eta_1\preccurlyeq_{\mathfrak{I}}\eta_2$, there holds*

$$\phi(\eta_1,u_1\wedge u_2)+\phi(\eta_2,u_1\vee u_2)\leq\phi(\eta_1,u_1)+\phi(\eta_2,u_2),\forall u_1,u_2\in H. \tag{3.4}$$

Then, $\mathscr{J}^{\rho}_{\phi}(\cdot,z)$ is order-preserving on $\mathfrak{I}$ for every $z\in H$.

Proof. For any given $z\in H$, take any $\eta_1,\eta_2\in\mathfrak{I}$ with $\eta_1\preccurlyeq_{\mathfrak{I}}\eta_2$, and set $u_1:=J_{\phi}(\eta_1,z)$ and $u_2:=J_{\phi}(\eta_2,z)$. By contradiction, suppose that $u_2\prec_H u_1$ or $u_1\bowtie_H u_2$, where the notation "$\bowtie_H$" means that u_1 and u_2 are not $\preccurlyeq_H$-comparable. From the definition of the resolvent operator, we have

$$\langle u_1-z,v-u_1\rangle+\rho\phi(\eta_1,v)-\rho\phi(\eta_1,u_1)\geq 0 \tag{3.5}$$

and

$$\langle u_2-z,v-u_2\rangle+\rho\phi(\eta_2,v)-\rho\phi(\eta_2,u_2)\geq 0,\ \forall v\in H. \tag{3.6}$$

Since H is a Hilbert lattice, both $u_1\wedge u_2$ and $u_1\vee u_2$ exist in H. Setting $v=u_1\wedge u_2$ in (3.5) and $v=u_1\vee u_2$ in (3.6), we have

$$\langle u_1-z,u_1\wedge u_2-u_1\rangle+\rho\phi(\eta_1,u_1\wedge u_2)-\rho\phi(\eta_1,u_1)\geq 0, \tag{3.7}$$

and

$$\langle u_2-z,u_1\vee u_2-u_2\rangle+\rho\phi(\eta_2,u_1\vee u_2)-\rho\phi(\eta_2,u_2)\geq 0. \tag{3.8}$$

Adding the two sides of (3.8) to those of (3.7) and using condition (3.4), we obtain

$$\langle u_2-u_1,u_1-u_1\wedge u_2\rangle\geq 0, \tag{3.9}$$

where the equality $u_1\vee u_2-u_2=u_1-u_1\wedge u_2$ is used.

The rest of this proof is divided into two cases.

Case I. $u_2\prec_H u_1$.

In this case, we have $u_2=u_1\wedge u_2$. Thus (3.9) is reduced to $\|u_1-u_2\|^2\leq 0$, which implies that $u_1=u_2$. This contradicts $u_2\prec_H u_1$.

Case II. $u_2\bowtie_H u_1$.

In this case, setting $x := u_1 - u_2$ and $y := u_1 - u_1 \wedge u_2$, it is easy to see $x \preccurlyeq_H y$ and $\langle x, y\rangle \leq 0$. Invoking Lemma 3.1, we obtain $x = \mathbf{0}$, that is, $u_1 = u_2$. This contradicts $u_2 \bowtie_H u_1$.

From Case I and Case II, we conclude that $u_1 \preccurlyeq_H u_2$. Therefore, $J_\phi(\cdot, z)$ is order-preserving for every $z \in H$. $\square$

Using the similar approach as in Theorem 3.1, it is readily to prove the following result.

Proposition 3.1. *Under the assumptions of Theorem 3.1, $\mathscr{J}_\phi^\rho(\eta,\cdot)$ is order-preserving for every $\eta \in \Im$.*

In order to study to the existence of solutions to GMVI (3.1), we consider the following auxiliary problem (AP, for short): for any given $\eta \in \Im$, find $u_\eta \in H$ such that

$$\langle Tu_\eta, v - u_\eta\rangle + \phi(\eta, v) - \phi(\eta, u_\eta) \geq 0, \ \forall v \in H. \tag{3.10}$$

If the solution set of AP (3.10) is nonempty, then we can define a solution mapping $\Lambda : \Im \to H \setminus \{\emptyset\}$ for AP (3.10) by

$$\Lambda\eta := \{u_\eta : u_\eta \text{ is a solution of AP(3.10)}\}, \ \forall \eta \in \Im.$$

We are now in a position to apply Theorem 3.1 to prove the order-preservation property of Λ.

Theorem 3.2. *Let $(H, \preccurlyeq_H)$ be a weakly sequentially complete Hilbert lattice and $(\Im, \preccurlyeq_\Im)$ a poset. Let $T : H \to H$ be a function such that there exists a real number $\rho > 0$ such that $id_H - \rho T$ is order-preserving and the range of $id_H - \rho T$ is $\preccurlyeq_H$-bounded. Assume that $\phi : \Im \times H \to \mathbb{R} \cup \{+\infty\}$ satisfies the conditions of Theorem 3.1. Then $\Lambda(\eta)$ is nonempty for any $\eta \in \Im$, and Λ is order-preserving on $\Im$.*

Proof. Construct a mapping $\digamma : \Im \times H \to H$ by $\digamma(\eta, u) := \mathscr{J}_\phi^\rho(\eta, u - \rho Tu)$. By the assumptions and Proposition 3.1, both $id_H - \rho T$ and $\mathscr{J}_\phi^\rho(\eta,\cdot)$ are order-preserving for any $\eta \in \Im$. Thus $\digamma(\eta,\cdot)$ is order-preserving on H. As the range of $id_H - \rho T$ is $\preccurlyeq_H$-bounded, both $S_+ := \{x \in P : x \preccurlyeq_P \digamma(\eta, x)\}$ and $S_- := \{x \in P : \digamma(\eta, x) \preccurlyeq_P x\}$ are nonempty. Let $\{y_n\}_{n\in\mathbb{N}}$ be an increasing sequence in S_+. Since $\digamma(\eta,\cdot)$ is order-preserving, the sequence $\{\digamma(\eta, y_n)\}_{n\in\mathbb{N}}$ is also increasing. Moreover, the $\preccurlyeq_H$-boundedness of $id_H - \rho T$ implies that $\{\digamma(\eta, y_n)\}_{n\in\mathbb{N}}$ is $\preccurlyeq_H$-bounded. Notice that H is a weakly sequentially complete Hilbert lattice. Thus the positive cone of H is normal, and hence regular (see [19, Lemma 9.3]). Therefore, $\{\digamma(\eta, y_n)\}_{n\in\mathbb{N}}$ converges. Invoking Lemma 2.5, the set of all fixed points of $\digamma(\eta,\cdot)$ is nonempty. Thus, by Lemma 2.7, $\Lambda(\eta) \neq \emptyset$ for any $\eta \in \Im$.

Next, we prove that $\Lambda(\cdot)$ is order-preserving on $\Im$. By using Theorem 3.1 and Proposition 3.1, we know that $\digamma(\cdot, u)$ is order-preserving for any $u \in H$ and $\digamma(\eta,\cdot)$ is order-preserving for any $\eta \in \Im$. In what follows, we prove that $\Lambda(\cdot)$ is lower order-preserving. To this end, take any $\eta_1, \eta_2 \in \Im$ with $\eta_1 \preccurlyeq_\Im \eta_2$, and pick an arbitrary $x_{\eta_2} \in \Lambda(\eta_2)$. We aim to find an $x_{\eta_1} \in \Lambda(\eta_1)$ such that $x_{\eta_1} \preccurlyeq_H x_{\eta_2}$. Set $K_{\eta_2} := H \bigcap (x_{\eta_2}]$ and $g := \digamma(\eta_1,\cdot)|K_{\eta_2}$. Then g is order-preserving on K_{η_2}, where $\digamma(\eta_1,\cdot)|K_{\eta_2}$ stands for the restriction of $\digamma(\eta_1,\cdot)$ on H to K_{η_2}. Next, we prove that g is a self-map on K_{η_2}. Actually, for any $x \in K_{\eta_2}$, it is easy to see that

$$g(x) = \digamma(\eta_1, x) \preccurlyeq \digamma(\eta_2, x) \preccurlyeq \digamma(\eta_2, x_{\eta_2}) = x_{\eta_2}.$$

Using the same approach as in the first part of this proof, we claim that g admits a fixed point in K_{η_2}. That is, there exists $\bar{x} \in K_{\eta_2}$ such that $\bar{x} = g(\bar{x}) = \digamma(\eta_1, \bar{x})$. Denoting this $\bar{x}$ by x_{η_1}, we have found an $x_{\eta_1} \in H$ such that $x_{\eta_1} \preccurlyeq_H x_{\eta_2}$ and $x_{\eta_1} \in \Lambda(\eta_1)$. Thus Λ is lower order-preserving.

In a similar way, it is readily to verify that Λ is also upper order-preserving. Thus Λ is order-preserving on $\Im$. $\square$

In Theorem 3.2, the range of $id_H - \rho T$ is assumed to be $\preccurlyeq_H$-bounded. More precisely, let $x_\circ$ and x° denote the $\preccurlyeq_H$-upper bounded and $\preccurlyeq_H$-lower bounded, respectively. That is, $x_\circ \preccurlyeq_H (id_H - \rho T)(y) \preccurlyeq_H x^\circ, \forall y \in H$. If we restrict our consideration to the order interval $[x_\circ, x^\circ]$, then the following result can be obtained immediately.

Theorem 3.3. *Assume that all the conditions of Theorem 3.2 are fulfilled and, in addition, H is separable. Then $\Lambda(\eta)$ constitutes a nonempty complete lattice for any $\eta \in \Im$.*

Proof. For any given $\eta \in \Im$, it follows from the proof of Theorem 3.2 that $\mathcal{F}(\eta,\cdot)$ is order-preserving on H for any $\eta \in \Im$. Since the range of $id_H - \rho T$ is bounded, there exists $x_\circ$ and x° such that $x_\circ \preccurlyeq_H (id_H - \rho T)(y) \preccurlyeq_H x^\circ, \forall y \in H$. By the construction of $\mathcal{F}(\eta,\cdot)$, we have $\mathscr{J}_\phi^\rho(\eta, x_\circ) \preccurlyeq_H \mathcal{F}(\eta, y) \preccurlyeq_H \mathscr{J}_\phi^\rho(\eta, x^\circ)$. Setting $x_{\circ\circ} := \mathscr{J}_\phi^\rho(\eta, x_\circ)$ and $x^{\circ\circ} := \mathscr{J}_\phi^\rho(\eta, x^\circ)$, we have that $\mathcal{F}(\eta,\cdot)$ maps the order interval $[x_{\circ\circ}, x^{\circ\circ}]$ into $[x_{\circ\circ}, x^{\circ\circ}]$. Notice that $[x_{\circ\circ}, x^{\circ\circ}]$ is a closed and $\preccurlyeq_H$-bounded $\preccurlyeq_H$-sublattice of H. Thus $[x_{\circ\circ}, x^{\circ\circ}]$ is a subcomplete $\preccurlyeq_H$-sublattice of H (see Lemma 2.1). By the Knaster-Taski fixed point theorem, the set of all fixed points of $\mathcal{F}(\eta,\cdot)$ constitutes a nonempty complete lattice. This implies that $\Lambda(\eta)$ constitutes a nonempty complete lattice for any $\eta \in \Im$. $\square$

By virtue of Theorem 3.2 and the following fixed point theorem given by Li [1], we can establish the solvability for GMVI (3.1).

Lemma 3.2. *(See [1]) Let $(P, \preccurlyeq)$ be a chain-complete poset, and let $\mathscr{G} : P \to 2^P \setminus \{\emptyset\}$ be a set-valued mapping. Assume that F satisfies the following three conditions:*
(i) $\mathscr{G}$ is upper order-preserving;
(ii) $\mathscr{G}(x)$ has a $\preccurlyeq$-maximum element for every $x \in P$;
(iii) There exists a $y \in P$ with $y \preccurlyeq u$ for some $u \in \mathscr{G}(y)$.

Then, $\mathscr{G}$ has a maximal fixed point, which is the $\preccurlyeq$-maximal element of the set of fixed points of $\mathscr{G}$.

Theorem 3.4. *Let $\Im := H$ and all the conditions of Theorem 3.3 hold. Then GMVI (3.1) admits a maximal solution, which is the $\preccurlyeq_H$-maximal element of the solution set of GMVI (3.1).*

Proof. Set $P := [x_{\circ\circ}, x^{\circ\circ}]$ and $\mathscr{G} := \Lambda|_P$, where $x_{\circ\circ}, x^{\circ\circ}$ are understood as in Theorem 3.3, and $\Lambda|_P$ stands for the restriction of Λ on H to $[x_{\circ\circ}, x^{\circ\circ}]$. Obviously, the assumption (iii) of Lemma 3.2 is valid. By Theorem 3.2 and Theorem 3.3, $\mathscr{G}$ is upper order-preserving and complete lattice-valued, which implies that the assumptions (i) and (ii) of Lemma 3.2 are also fulfilled. Invoking Lemma 3.2, $\mathscr{G}$ has a $\preccurlyeq_H$-maximal fixed point, which is equivalent to the existence of maximal solution to GMVI (3.1). $\square$

Remark 3.1. In Theorem 3.1 and Proposition 3.1, the functional ϕ is required to satisfy inequality (3.4). In fact, there are numerous examples of such ϕ. For instance, let $\mathscr{W} : H \to \mathbb{R}$ be a linear functional and $F : \Im \to \mathbb{R}$ a any functional. Define the functional $\phi : \Im \times H \to \mathbb{R}$ by $\phi(\eta, u) = F(\eta) + \mathscr{W}(u)$. Then it is readily to verify that this ϕ satisfies the inequality (3.4). Another example of ϕ that satisfies (3.4) is to define $\phi : \Im \times H \to \mathbb{R}$ by $\phi(\eta, u) = F(\eta)\mathscr{W}(u)$.

It should be noted that if $\phi : H \to \mathbb{R} \cup \{+\infty\}$, then one can deduce a corollary from Theorem 3.1.

Corollary 3.1. *Let $(H, \preccurlyeq_H)$ be a Hilbert lattice. Assume that $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous functional such that $\phi(z) + \phi(w) \geq \phi(z \vee w) + \phi(z \wedge w)$ for any $z, w \in H$. Then, the resolvent operator J_ϕ^ρ defined by Definition 2.6 is order-preserving on H.*

Furthermore, invoking Corollary 3.1, one can obtain an existence result for MVI (3.2), which is a special case of Theorem 3.4.

Corollary 3.2. *Let $(H, \preccurlyeq_H)$ be a weakly sequentially complete Hilbert lattice. Assume that $T : H \to H$ is an operator such that there exists a real number $\rho > 0$ such that $id_H - \rho T$ is order-preserving and the range of $id_H - \rho T$ is $\preccurlyeq_H$-bounded. Assume that ϕ satisfies the conditions of Corollary 3.1. Then the solution set of MVI (3.2) is nonempty.*

4. Order-Theoretic Approaches to Differential Quasivariational Inequalities

Throughout this section, let H_1 and H_2 be two Hilbert spaces, whose inner product are denoted by $\langle\cdot,\cdot\rangle_{H_1}$ and $\langle\cdot,\cdot\rangle_{H_2}$, respectively. Let $\|\cdot\|_{H_1}$ and $\|\cdot\|_{H_2}$ represent the norms of H_1 and H_2, which are induced by $\langle\cdot,\cdot\rangle_{H_1}$ and $\langle\cdot,\cdot\rangle_{H_2}$, respectively. $t_f > 0$ is a given constant. For simplicity, set $I := [0, t_f]$, $\mathscr{V} := C(I; H_1)$, $\mathscr{H} := C(I; H_2)$, and $\mathscr{X} := C^1(I; H_2)$. Let $F : I \times H_2 \times H_1 \to H_2$, $x_0 \in H_2$, $A : H_2 \times H_1 \to H_1$, $j : H_2 \times H_1 \times H_1 \to \mathbb{R} \cup \{+\infty\}$, and $f \in C(I; H_1)$ be given.

The aim of this section is to apply the order-preservation properties of resolvent operators to examine the solvability of the following differential quasivariational inequality (for short, DQVI): find a pair of functions $x \in \mathscr{X}$ and $u \in \mathscr{V}$ such that

$$\begin{cases} \dot{x}(t) = F(t, x(t), u(t)), \quad \forall t \in I, \\ u(t) \in S(K, A(x(t), \cdot), f(t), j), \quad t \in I, \\ x(0) = x_0, \end{cases} \tag{4.1}$$

where $S(K, A(x(t), \cdot), f(t), j)$ stands for the solution set of the following quasivariational inequality (QVI, for short): find $u \in \mathscr{V}$ such that

$$\langle A(x(t), u(t)) - f(t), v - u(t)\rangle + j(x(t), u(t), v) - j(x(t), u(t), u(t)) \geq 0, \;\; \forall v \in H_1, t \in I. \tag{4.2}$$

Definition 4.1. A pair of functions (x, u) with $x \in \mathscr{X}$ and $u \in \mathscr{V}$ is said to be a mild solution to DQVI(4.1) if $x(t) = x_0 + \int_0^t F(s, x(s), u(s))ds$, $t \in I$, and $u(s) \in S(K, A(x(t), \cdot), f(t), j)$ for any $s \in I$.

For the pioneer works on differential variational inequalities, one can consult [23, 24]. In the past decade, differential variational inequalities attracted much attention of many scholars; see, e.g., [25, 26, 27, 28, 29] and the references therein. Recently, Liu and Sofonea [30] used this kind differential quasivariational inequality to investigate a class of elastic contact problems with wear. In the sequel, we establish a new existence result by using the order-theoretic techniques. We begin with a list of the hypotheses on the data of the problem.

(H_H) $(H_2, \preccurlyeq_{H_2})$ and $(H_1, \preccurlyeq_{H_1})$ are two separable Hilbert lattices respectively induced by their regular order cones. Moreover, $\mathscr{X}$, $\mathscr{V}$, and $\mathscr{H}$ are pointwise ordered, and the partial orders are denoted by $\preccurlyeq_{\mathscr{X}}$, $\preccurlyeq_{\mathscr{V}}$, and $\preccurlyeq_{\mathscr{H}}$, respectively.

(H_F) $F : I \times H_2 \times H_1 \to H_2$ is a mapping such that
(i) $F(\cdot, x(\cdot), u(\cdot))$ is strongly measurable for all $x \in \mathscr{X}$ and $u \in \mathscr{V}$;

(ii) $F(t,\cdot,\cdot)$ is order-preserving in x and in y for a.e. $t \in I$, and there exist Bochner integrable functions $\vartheta_{\pm} : I \to H_2$ such that $\vartheta_{-}(t) \preccurlyeq_{H_2} F(t,x,u) \preccurlyeq_{H_2} \vartheta_{+}(t)$ for a.e. $t \in I$ and for all $x \in H_2$ and $u \in H_1$.

(H_A) $A : H_2 \times H_1 \to H_1$ is such that
(i) There exists $L' > 0$ such that

$$\|A(x_1,u) - A(x_2,u)\|_{H_1} \le L' \|x_1 - x_2\|_{H_2}, \ \ \forall x_1, x_2 \in H_2, u \in H_1;$$

(ii) There exists $m > 0$ such that

$$\langle A(x,u_1) - A(x,u_2), u_1 - u_2 \rangle \ge m \|u_1 - u_2\|^2_{H_1}, \ \ \forall u_1, u_2 \in H_1, x \in H_2;$$

(iii) $A(\cdot,u)$ is order-preserving for any $u \in H_1$, and $id_{H_1} - A(x,\cdot)$ is order-preserving for any $x \in H_2$; the range of $id_{H_1} - A(x,\cdot)$ is $\preccurlyeq_{H_1}$-bounded.

(H_j) $j : H_2 \times H_1 \times H_1 \to \mathbb{R}$ is such that
(i) for all $x \in H_2$ and $u \in H_1$, $j(x,u,\cdot)$ is proper, convex and lower semicontinuous on H_1;
(ii) there exist $\alpha > 0$ and $\beta > 0$ such that

$$\begin{aligned} & j(x_1,u_1,v_2) - j(x_1,u_1,v_1) + j(x_2,u_2,v_1) - j(x_2,u_2,v_2) \\ & \le \alpha \|x_1 - x_2\|_{H_2} \|v_1 - v_2\|_{H_1} + \beta \|u_1 - u_2\|_{H_1} \|v_1 - v_2\|_{H_1}, \\ & \forall x_1, x_2 \in H_2, u_1, u_2 \in H_1, v_1, v_2 \in H_1; \end{aligned}$$

(iii) for any $(x_1,u_1),(x_2,u_2) \in H_2 \times H_1$ with $(x_1,u_1) \preccurlyeq_{H_2 \times H_1} (x_2,u_2)$, there holds

$$j(x_1,u_1,v_1 \wedge v_2) + j(x_2,u_2,v_1 \vee v_2) \le j(x_1,u_1,v_1) + j(x_2,u_2,v_2), \forall v_1, v_2 \in H_1,$$

where $(x_1,u_1) \preccurlyeq_{H_2 \times H_1} (x_2,u_2)$ means that $x_1 \preccurlyeq_{H_2} x_2$ and $u_1 \preccurlyeq_{H_1} u_2$.

(H_f) $f \in C(I;H_1)$ and $f(\cdot)$ is order-preserving.

(H_o) $m > \beta$ and $x_0 \in H_2$.

In what follows, we first investigate the existence and order-preservation properties of the solutions to QVI (4.2). Regarding the existence results of QVI (4.2), there have been numerous results; see, e.g., [30, 31, 32].

Lemma 4.1. *(See [30]) Assume that (H_H), (H_A)(i), (H_A)(ii), (H_j)(i), (H_j)(ii), (H_f), and (H_o) hold. Then, for any $x \in \mathscr{X}$, QVI (4.2) has a unique solution $u \in \mathscr{V}$.*

According to Lemma 4.1, we can define a solution mapping $\mathscr{U} : \mathscr{X} \to \mathscr{V}$ by $\mathscr{U}x(t) = u(t), \forall t \in I$. Next, we study the order-preservation properties of $\mathscr{U}$. To this end, we focus the following auxiliary problem (AP, for short): for each $(t,x) \in I \times H_2$, find $u \in H_1$ such that

$$\langle A(x,u), v - u \rangle + j(x,u,v) - j(x,u,u) \ge \langle f(t), v - u \rangle, \ \ \forall v \in H_1. \tag{4.3}$$

Similarly, the solution mapping of the above auxiliary problem $U : I \times H_2 \to H_1$ can be defined by

$$U(t,x) := \{u : u \text{ is a solution of } AP \ (4.3)\} \tag{4.4}$$

Theorem 4.1. *Assume that (H_H), (H_A), (H_j), (H_f), and (H_o) hold. Then the solution mapping U defined by (4.4) is order-preserving.*

Proof. Since (H_j)(i) holds, then the resolvent operator related to $j(x,u,\cdot)$ exists. Now define a mapping $G : I\times H_2\times H_1 \to H_1$ by

$$G(t,x,u) := J_{j(x,u,\cdot)}(u-A(x,u)+f(t)),\ \ \forall (t,x,u)\in I\times H_2\times H_1.$$

Next, we prove that both $G(t,x,\cdot)$ and $G(\cdot,\cdot,u)$ are order-preserving. For any given (t,x), take any $u_1,u_2\in H_1$ with $u_1 \preccurlyeq_{H_1} u_2$. We aim to prove $G(t,x,u_1)\preccurlyeq_{H_1} G(t,x,u_2)$. To this end, set $\eta_1 := (x,u_1)$, $\eta_2 := (x,u_2)$, $z_1 := u_1-A(x,u_1)+f(t)$ and $z_2 := u_2-A(x,u_2)+f(t)$. Then $G(t,x,u_1) = J_{j(\eta_1,\cdot)}(z_1)$, and $G(t,x,u_2) = J_{j(\eta_2,\cdot)}(z_2)$. By Assumption (H_A)(iii), we have $\eta_1 \preccurlyeq_{H_2\times H_1} \eta_2$ and $z_1 \preccurlyeq_{H_1} z_2$. From Theorem 3.1, we obtain $J_{j(\eta_1,\cdot)}(z_1)\preccurlyeq_{H_1} J_{j(\eta_2,\cdot)}(z_2)$, that is, $G(t,x,u_1)\preccurlyeq_{H_1} G(t,x,u_2)$, which implies that $G(t,x,\cdot)$ is order-preserving. For any given u, take any $(t_1,x_1),(t_2,x_2)\in I\times H_2$ with $t_1\le t_2$ and $x_1\preccurlyeq_{H_2} x_2$. Next we show that $G(t_1,x_1,u)\preccurlyeq_{H_1} G(t_2,x_2,u)$. Set $\tilde{\eta}_1 := (x_1,u)$, $\tilde{\eta}_2 := (x_2,u)$, $\tilde{z}_1 := u-A(x_1,u)+f(t_1)$, and $\tilde{z}_2 := u-A(x_2,u)+f(t_2)$. Then $G(t_1,x_1,u) = J_{j(\tilde{\eta}_1,\cdot)}(\tilde{z}_1)$, and $G(t_2,x_2,u) = J_{j(\tilde{\eta}_2,\cdot)}(\tilde{z}_2)$. It follows from Assumption (H_A)(iii) that $\tilde{\eta}_1 \preccurlyeq_{H_2\times H_1} \tilde{\eta}_2$ and $\tilde{z}_1\preccurlyeq_{H_1}\tilde{z}_2$. Invoking Theorem 3.1, we have $G(t_1,x_1,u)\preccurlyeq_{H_1} G(t_2,x_2,u)$, which implies that $G(\cdot,\cdot,u)$ is order-preserving. Obviously, for each $(t,x)\in I\times H_2$, u^* is a solution of AP (4.3) if and only if u^* is a fixed point of $G(t,x,\cdot)$, that is, $u^* = G(t,x,u^*)$. By Lemma 4.1, $G(t,x,\cdot)$ has a unique fixed point for any $(t,x)\in I\times H_2$. Now we prove that $U(\cdot,\cdot)$ is order-preserving. Take any $(t_1,x_1),(t_2,x_2)\in I\times H_2$ with $t_1\le t_2$ and $x_1\preccurlyeq_{H_2} x_2$. Let $u_1 := G(t_1,x_1,u_1)$ and $u_2 := G(t_2,x_2,u_2)$. It only need to prove $u_1\preccurlyeq_{H_1} u_2$. Setting $K_1 := H_1\cap[u_1)$, we claim that $G(t_2,x_2,\cdot)$ maps K_1 into K_1. In fact, for any $u\in K_1$, by virtue of the order-preservation properties of $G(t,x,\cdot)$ and $G(\cdot,\cdot,u)$, we have

$$u_1 = G(t_1,x_1,u_1)\preccurlyeq_{H_1} G(t_1,x_1,u)\preccurlyeq_{H_1} G(t_2,x_2,u).$$

Noticing that the cone of H_1 is regular, the reasoning similar to that used in the proof of Theorem 3.2 shows that there exists $u_*\in K_1$ such that $u^* = G(t_2,x_2,u^*)$. Since u_2 is the unique fixed point of $G(t_2,x_2,\cdot)$, we obtain $u_2 = u^*\in K_1$, i.e. $u_1\preccurlyeq_{H_1} u_2$. Thus $U(\cdot,\cdot)$ is order-preserving. □

Theorem 4.2. *Assume that (H_H), (H_A), (H_j), (H_f), and (H_o) hold. Then the solution mapping $\mathscr{U}$ is order-preserving.*

Proof. Taking any $x_1,x_2\in\mathscr{X}$ with $x_1\preccurlyeq_{\mathscr{X}} x_2$, we find from Lemma 4.1 that there exist $u_1,u_2\in\mathscr{V}$ such that $u_1,u_2\in\mathscr{V}$ are the solutions to QVI(4.2) corresponding to x_1 and x_2, respectively. We claim that $u_1\preccurlyeq_{\mathscr{V}} u_2$. Observe that $u_1\preccurlyeq_{\mathscr{V}} u_2$ is equivalent to $u_1(t)\preccurlyeq_{H_1} u_2(t)$ for any $t\in I$. For any given $t\in I$, $x_1\preccurlyeq_{\mathscr{X}} x_2$ implies $x_1(t)\preccurlyeq_{H_2} x_2(t)$. By Theorem 4.1, the solution mapping $U(\cdot,\cdot)$ of AP (4.3) is order-preserving. Thus $u_1(t)\preccurlyeq_{H_1} u_2(t)$ at the given moment t. From the arbitrary of t, we have $u_1\preccurlyeq_{\mathscr{V}} u_2$. This completes the proof. □

Let the partial ordering $\preccurlyeq_{\mathscr{X}\times\mathscr{V}}$ be defined by the similar manner as $\preccurlyeq_{H_2\times H_1}$. Our main result in this section is given by the following theorem.

Theorem 4.3. *Assume that (H_H), (H_F), (H_A), (H_j), (H_f), and (H_o) hold. Then DQVI(4.1) admits extremal solutions, which are the $\preccurlyeq_{\mathscr{X}\times\mathscr{V}}$-maximum element and $\preccurlyeq_{\mathscr{X}\times\mathscr{V}}$-minimum element of its solution set, respectively.*

Proof. For any $x\in\mathscr{X}$, let $u\in\mathscr{V}$ be the solution of QVI (4.2) corresponding to this x. By (H_F), we can use the above x and u to define a operator $G:\mathscr{X}\to\mathscr{X}$ by

$$Gx(t) = x_0+\int_a^t F(s,x(s),u(s))ds,\ \ t\in I.$$

In addition, by using $\vartheta_{\pm} : I \to H_2$, we can construct two functions $\underline{y}$ and $\bar{y}$ in $\mathscr{X}$ by

$$\underline{y}(t) = x_0 + \int_a^t \vartheta_-(s)ds \tag{4.5}$$

and

$$\bar{y}(t) = x_0 + \int_a^t \vartheta_+(s)ds. \tag{4.6}$$

From (H_F)(ii), we conclude that if $x, y \in \mathscr{X}$ and $x \preccurlyeq_{\mathscr{X}} y$, and $a \leq \underline{t} \leq t \leq b$, then

$$\int_{\underline{t}}^t \vartheta_-(s)ds \preccurlyeq_{H_2} \int_{\underline{t}}^t F(s,x(s),u(s))ds \preccurlyeq_{H_2} \int_{\underline{t}}^t F(s,y(s),u(s))ds \preccurlyeq_{H_2} \int_{\underline{t}}^t \vartheta_+(s)ds. \tag{4.7}$$

From (4.5), (4.6) and (4.7), it is readily to see that the operator G maps $\mathscr{X}$ into the order interval $[\underline{y}, \bar{y}]$. Furthermore, the following inequalities hold

$$\mathbf{0} \preccurlyeq_{H_2} Gx(t) - Gx(\underline{t}) - (\underline{y}(t) - \underline{y}(\underline{t})) \preccurlyeq_{H_2} \bar{y}(t) - \bar{y}(\underline{t}) - (\underline{y}(t) - \underline{y}(\underline{t})),$$

whenever $a \leq \underline{t} \leq t \leq b$. Since the order cone of H_2 is regular and hence normal, the following inequality holds

$$\|Gx(t) - Gx(\underline{t})\|_{H_2} \leq (1+\lambda)\|\underline{y}(t) - \underline{y}(\underline{t})\|_{H_2} + \lambda\|\bar{y}(t) - \bar{y}(\underline{t})\|_{H_2}, \;\; a \leq \underline{t} \leq t \leq b, \tag{4.8}$$

where $\lambda > 1$ is the constant with respect to the normal cone of H_2. Let $\{x_n\}_{n\in\mathbb{N}^+}$ be a monotone sequence in $[\underline{y}, \bar{y}]$. Then it follows from (4.7) that, for every $t \in I$, $\{Gx_n(t)\}_{n\in\mathbb{N}^+}$ is also a monotone sequence in the order interval $[\underline{y}, \bar{y}]$. Since the order cone is regular, $\{Gx_n(t)\}_{n\in\mathbb{N}^+}$ converges in H_2 for each $t \in I$. In addition, from (4.8), we see that the sequence $\{Gx_n(t)\}_{n\in\mathbb{N}^+}$ is equicontinuous, which implies from Lemma 2.2 that $\{Gx_n(t)\}_{n\in\mathbb{N}^+}$ converges uniformly on I, and hence with respect to the uniform norm of $\mathscr{X}$.

The above proof indicates that the assumptions of Lemma 2.4 are valid for the restriction to $[\underline{y}, \bar{y}]$ of mapping G. Thus G has the $\preccurlyeq_{\mathscr{X}}$-minimum fixed point x_* and the $\preccurlyeq_{\mathscr{X}}$-maximum fixed point x^* in $[\underline{y}, \bar{y}]$. Denote by u_* the solution of QVI(4.2) that corresponds to x_*, and denote u^* the solution of QVI(4.2) that corresponds to x^*. Then, the pair of functions (x_*, u_*) and (x^*, u^*) are the minimum solution and maximum solution of DQVI(4.1), respectively. This completes the proof. □

5. Conclusion

In this paper, we established some sufficient conditions under which the resolvent operators were order-preserving on Hilbert lattices. As applications, some solvability results for (generalized) mixed variational inequalities were obtained. Furthermore, we also used the order-preservation properties of resolvent operators and order-theoretic fixed point theorems to investigate a class of differential quasivariational inequalities which attracted a lot of attention in recent years. Different from the traditional topology methods, the order-theoretic methods developed in this paper are expected to open up some new research paths for differential quasi-variational inequalities and the relevant nonlinear problems. In future research, we aim to focus on using the developed order-theoretic approaches to investigate some variational inequalities driven by fractional nonlinear evolutionary system, such as fractional differential elliptic hemi-variational inequality, fractional differential parabolic hemi-variational inequality and so on. In addition to the above theoretical research, it is also a research direction to apply these order-theoretic methods and variational inequalities to some practical problems in the real world.

Acknowledgments

The first author was supported by the National Natural Science Foundation of China (Grant No. 72001101), Qing Lan Project in Jiangsu Province, Natural Science Foundation in Jiangsu Province (Grant No. BK20171041), Project of Philosophy and Social Science Research in Colleges and Universities in Jiangsu Province (Grant No. 2017SJB0238), and Natural Science Foundation in colleges and universities in Jiangsu Province (Grant No. 16KJB110009). The second author was supported by the Ministry of Science and Technology, Taiwan under Grant No. 110-2115-M-037-001. The third author was supported by the Ministry of Science and Technology, Taiwan under Grant No. 108-2115-M-039-001.

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