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ON APPROXIMATE CONTROLLABILITY FOR SYSTEMS OF FRACTIONAL EVOLUTION HEMIVARIATIONAL INEQUALITIES WITH RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

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Abstract. In this paper, we deal with the control systems governed by the systems of fractional evolution hemivariational inequalities involving Riemann-Liouville fractional derivatives. We establish suitable sufficient conditions to guarantee the existence of mild solutions. Under these conditions, the approximate controllability of the associated fractional evolution systems involving Riemann-Liouville fractional derivatives is formulated and proved.

Keywords. Fractional evolution hemivariational inequalities; Mild solutions; Riemann-Liouville fractional derivatives; Partial Clarke's generalized subdifferential.

1. INTRODUCTION

Hemivariational inequalities have important applications in mechanics and engineering, especially in nonsmooth analysis and optimization; see, e.g., [1, 2, 3]. In recent years, under various assumptions, the existence theorems and well-posedness results for hemivariational inequalities have been proven by many authors; see, e.g., [4, 5, 6, 7] and the references therein.

Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$, and let $A : D(A) \subseteq H \to H$ be the infinitesimal generator of a uniformly bounded compact C_0-semigroup $\{T(t)\}_{t\geq 0}$ on H. Let E be a reflexive Banach space, $u : [0,b] \to E$ a control function, and $B : E \to H$ a bounded linear operator. The notation $J^\circ(t,\cdot;\cdot)$ stands for Clarke's generalized directional derivative [1] of a locally Lipschitz $J(t,\cdot) : H \to \mathbb{R}$. Denote by $P(E)$ the collection of all nonempty subsets of E. We use the following notations:

$P_{f(c)}(E) := \{\Omega \subseteq E : \Omega \text{ is nonempty, closed (convex)}\}$;

$P_{(w)k(c)}(E) := \{\Omega \subseteq E : \Omega \text{ is nonempty, (weakly) compact (convex)}\}$.

In 2015, Huang, Liu and Zeng [3] studied the existence of solutions of the following evolution hemivariational inequality:

$$\begin{cases} \langle -x'(t) + Ax(t) + Bu(t), v\rangle_H + J^\circ(t,x(t);v) \geq 0, \quad \text{a.e. } t \in [0,b],\ \forall v \in H, \\ x(0) = x_0 \in H. \end{cases}$$

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Moreover, they were also concerned with the existence of feasible pairs of the following feedback control problem:

$$\begin{cases} \langle -x'(t)+Ax(t)+Bu(t),v\rangle_H + J^\circ(t,x(t);v) \geq 0, \quad \text{a.e. } t\in[0,b],\ \forall v\in H,\\ u(t)\in \mathscr{U}(t,x(t)),\\ x(0)=x_0\in H, \end{cases}$$

where $\mathscr{U}:[0,b]\times H\to P(E)$ is a multimap.

On the other hand, Liu and Li [8] investigated the approximate controllability of the following fractional evolution control problem involving Riemann-Liouville fractional derivative:

$$\begin{cases} D_t^\alpha x(t) = Ax(t)+Bu(t))+f(t,x(t)),\ t\in(0,b],\ 0<\alpha\leq 1,\\ I_t^{1-\alpha}x(t)|_{t=0}=x_0\in X, \end{cases} \tag{1.1}$$

where D_t^α denotes the Riemann-Liouville fractional derivative of order α with the lower limit zero, $A:D(A)\subseteq X\to X$ is the infinitesimal generator of a C_0-semigroup $T(t)$ $(t\geq 0)$ on a Banach space X, and $f:[0,b]\times X\to X$ is a given function. The control function u takes values in $L^p([0,b];Y)$, $p>\frac{1}{\alpha}$, Y is a Banach space, and B is a linear operator from $L^p([0,b];Y)$ into $L^p([0,b];X)$. In addition, fractional differential equations were proved to be valuable tools in the modeling of many phenomena because they are more accurate than integer-order models. Since fractional derivatives provide an excellent instrument for the description of memory and hereditary properties in a model, they found many applications in the mathematical modeling of systems and processes in the fields of physics, aerodynamics, electrodynamics of complex medium, viscoelasticity, heat conduction, electricity mechanics, control theory, and so forth. For more details on this topic, we refer to [9] and the references therein. From the definition of Riemann-Liouville fractional derivatives or integrals, initial conditions play an important role in some practical problems. Heymans and Podlubny [10] demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives or the integrals on the field of the viscoelasticity, and such initial conditions are more appropriate than physically interpretable initial conditions. The concept of the controllability, first introduced by Kalman [11], has become an active area of investigation due to its great applications in the field of physics. There are various results on complete controllability of systems represented by differential equations, integrodifferential equations, differential inclusions, neutral fractional differential equations, and impulsive differential equations in Banach spaces; see, e.g., [12, 13, 14]. The main approach used by Liu an Li [8] is to convert the controllability problem into a fixed point problem. However, it is worth to mention that the complete controllability results of abstract control problems were obtained under the assumption that the controllability operator has an induced inverse on a quotient space and even restricted the finite dimensional spaces when the semigroup $T(t)$ is compact. Therefore, the concept of complete controllability is too strong in infinite dimensional spaces. In order to establish more appropriate conditions about controllability, more and more authors are concerned with approximate controllability; see [15, 16, 17]. In particular, the control problems governed by Caputo fractional evolution equations were extensively studied; see, e.g., [14, 15, 16, 18]. However, there are few results on the approximate controllability of fractional evolution differential equations with Riemann-Liouville fractional derivatives in the literature. In 2015, Liu and Li [8] gave some suitable sufficient conditions for the existence and uniqueness of mild solutions and approximate controllability results for the fractional abstract Cauchy problems with the Riemann-Liouville fractional derivatives.

Let $\alpha \in (0,1]$, $I=[0,b]$ for some $0<b<\infty$, $V=V_1\times V_2$, and $U=U_1\times U_2$, where V_i is a separable Hilbert space, and U_i is a reflexive Banach space for $i=1,2$. Endowed with the norm defined by $\|\mathbf{x}\|_V := \|x_1\|_{V_1}+\|x_2\|_{V_2}$ for all $\mathbf{x}=(x_1,x_2)\in V$, V is a reflexive Banach space; see e.g., [19]. Inspired by the results in [3, 8], the aim of this paper is to provide some suitable sufficient conditions for the existence of mild solutions and approximate controllability of the following fractional evolution control system involving Riemann-Liouville fractional derivatives, which is formulated below:

Find $(x_1,x_2)\in C_{1-\alpha}(I,V_1)\times C_{1-\alpha}(I,V_2)$ such that

$$\begin{cases} \langle -D_t^\alpha x_1(t)+A_1x_1(t)+B_1u_1(t),v_1\rangle_{V_1}+J_1^\circ(t,x_1(t),x_2(t))\geq 0, & \text{a.e. } t\in I, \forall v_1\in V_1,\\ \langle -D_t^\alpha x_2(t)+A_2x_2(t)+B_2u_2(t),v_2\rangle_{V_2}+J_2^\circ(t,x_1(t),x_2(t))\geq 0, & \text{a.e. } t\in I, \forall v_2\in V_2,\\ I_t^{1-\alpha}x_i(t)|_{t=0}=x_{i,0}\in V_i, \quad i=1,2, \end{cases} \tag{1.2}$$

where D_t^α denotes the Riemann-Liouville fractional derivative of order α with the lower limit zero. For $i=1,2$, $A_i : D(A_i)\subseteq V_i\to V_i$ is the infinitesimal generator of a uniformly bounded compact C_0-semigroup $T_i(t)$ $(t\geq 0)$ on V_i, the control function u_i takes values in $L^p(I;U_i)$, $p>\frac{1}{\alpha}$, and B_i is a linear operator from $L^p(I;U_i)$ into $L^p(I;V_i)$. For $l,k=1,2$ and $l\neq k$, the notation $J_l^\circ(t,x_1,x_2;v_l)$ stands for the partial Clarke's generalized directional derivative [7] of a locally Lipschitz $J(t,\cdot,\cdot):V_1\times V_2\to\mathbb{R}$ w.r.t. the lth variable at x_l in the direction v_l for the given x_k. It is worth to point out that the approximate controllability for the systems of fractional evolution hemivariational inequalities with Riemann-Liouville fractional derivatives is still untreated topic in the literature and hence this fact is the motivation of this paper.

The remainder of this paper is assigned below. In Section 2, we present some basic definitions and preliminary facts, which are used throughout the following sections. In Section 3, some sufficient conditions are established for the existence of mild solutions of system (1.2). In Section 4, we study the approximate controllability of the systems of fractional evolution hemivariational inequalities with Riemann-Liouville fractional derivatives. We present an example to demonstrate our main results in Section 5. Finally, we give a concluding remark in Section 6, the last section.

2. Preliminaries

In this section, we introduce some basic definitions and preliminaries, which are used throughout this paper. The norm of a Banach space X is denoted by $\|\cdot\|_X$. $L_b(X,Y)$ denotes the space of bounded linear operators from X to Y. For the uniformly bounded C_0-semigroup $T(t)$ $(t\geq 0)$, we set $M:=\sup_{t\in[0,\infty)}\|T(t)\|_{L_b(X)}<\infty$. Let $C(I,X)$ denote the Banach space of all X-valued continuous functions from $I=[0,b]$ into X with norm $\|x\|_{C(I,X)}=\sup_{t\in I}\|x(t)\|_X$. Let $AC(I,X)$ be the space of functions f, which are absolutely continuous on I and $AC^m(I,X)=\{f: I\to X \text{ and } f^{(m-1)}(x)\in AC(I,X)\}$. To define the mild solutions of system (1.2), we also consider the Banach space $C_{1-\alpha}(I,X)=\{x: t^{1-\alpha}x(t)\in C(I,X), 0<\alpha\leq 1\}$ with norm $\|x\|_{C_{1-\alpha}(I,X)}=\sup\{t^{1-\alpha}\|x(t)\|_X : t\in I, 0<\alpha\leq 1\}$. Obviously, $C_{1-\alpha}(I,X)$ is a Banach space.

First, let us recall the following concepts from fractional calculus (see [9] for more details). For a function $f(t)$ given on the interval $[0,\infty)$, the integral

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0,$$

is called the Riemann-Liouville fractional integral of order α, where Γ is the gamma function. Meanwhile, the expression

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order α. Give $\alpha > 0$ and $m = [\alpha]+1$. For $i = 1,2$, let $x_{i,m-\alpha}(t) = I_t^{m-\alpha} x_i(t)$ be the fractional integral of order $m-\alpha$. Whenever $x_i(t) \in L^1(I,V_i)$ and $x_{i,m-\alpha}(t) \in AC^m(I,V_i)$, one sees from [9] that the following equality holds:

$$I_t^{\alpha} D_t^{\alpha} x_i(t) = x_i(t) - \sum_{k=1}^{m} \frac{x_{i,m-\alpha}^{(m-k)}(0)}{\Gamma(\alpha-k+1)} t^{\alpha-k}.$$

For $i = 1,2$, the Laplace transform formula for the Riemann-Liouville fractional integral is defined by $L\{I_t^{\alpha} x_i(t);\lambda\} = \frac{1}{\lambda^{\alpha}} \widehat{x}_i(\lambda)$, where $\widehat{x}_i(\lambda)$ is the Laplace of x_i defined by

$$\widehat{x}_i(\lambda) = \int_0^{\infty} e^{-\lambda t} x_i(t)dt, \quad \text{Re}\lambda > \omega, \text{ and } \|x_i(t)\|_{V_i} \le c_i e^{\omega t},\ c_i \text{ is a constant.}$$

Lemma 2.1. [8, Lemma 2.4] *For $i = 1,2$, let $\alpha \in (0,1]$ and $h_i \in L^p(I,V_i)$, $p > \frac{1}{\alpha}$. If $x_i(t) \in L^1(I,V_i)$, $x_{i,1-\alpha}(t) \in AC(I,V_i)$, and x_i is a solution to*

$$\begin{cases} D_t^{\alpha} x_i(t) = A_i x_i(t) + h_i(t), & t \in (0,b], \\ I_t^{1-\alpha} x_i(t)|_{t=0} = x_{i,0} \in V_i, \end{cases}$$

then x_i satisfies the following equation

$$x_i(t) = t^{\alpha-1} T_{i,\alpha}(t) x_{i,0} + \int_0^t (t-s)^{\alpha-1} T_{i,\alpha}(t-s) h_i(s)ds, \quad t \in I,$$

where

$$T_{i,\alpha}(t) = \alpha \int_0^{\infty} \theta \xi_{\alpha}(\theta) T_i(t^{\alpha}\theta) d\theta,$$

$$\xi_{\alpha}(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \overline{\omega}_{\alpha}(\theta^{-\frac{1}{\alpha}}),$$

$$\overline{\omega}_{\alpha}(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0,\infty),$$

and ξ_{α} is a probability density function defined on $(0,\infty)$, that is,

$$\xi_{\alpha}(\theta) \ge 0,\ \theta \in (0,\infty), \quad \text{and} \quad \int_0^{\infty} \xi_{\alpha}(\theta) d\theta = 1.$$

Now, let us proceed to the concept of Clarke's generalized subdifferential for a locally Lipschitz $h : X \to \mathbb{R}$, where X is a Banach space with its dual X^* (see [1]) and $\langle \cdot,\cdot \rangle_X$ is the duality pairing between X^* and X. We denote by $h^\circ(x;v)$ the Clarke's generalized directional derivative of h at the point $x \in X$ in the direction $v \in X$, that is,

$$h^\circ(x;v) := \limsup_{\lambda \to 0^+, y \to x} \frac{h(y+\lambda v) - h(y)}{\lambda}.$$

Recall also that Clarke's generalized subdifferential (or Clarke's generalized gradient) of h at $x \in X$, denoted by $\partial h(x)$, is the subset of X^* formulated below

$$\partial h(x) := \{x^* \in X^* : h^\circ(x;v) \geq \langle x^*, v\rangle_X, \ \forall\, v \in X\}.$$

Lemma 2.2. [1, Proposition 3.23] *If $h : X \to \mathbf{R}$ is locally Lipschitz on an open subset Ω of X, then*

(i) $(x,v) \mapsto h^\circ(x;v)$ is u.s.c. from $\Omega \times X$ into $\mathbb{R}$, i.e., for all $x \in \Omega$, $v \in X$, $\{x_n\} \subset \Omega$, $\{v_n\} \subset X$ such that $x_n \to x$ in Ω and $v_n \to v$ in X, one has $\limsup_{n\to\infty} h^\circ(x_n;v_n) \leq h^\circ(x;v)$;

(ii) for every $x \in \Omega$, the gradient $\partial h(x)$ is a nonempty, convex and weakly* compact subset of X^*, and $\|x^*\|_{X^*} \leq \ell_0$ for any $x^* \in \partial h(x)$ (where $\ell_0 > 0$ is the Lipschitz constant of h near x);

(iii) the graph of ∂h is closed in $X \times X^*_{w^*}$;

(iv) for every $v \in X$, $h^\circ(x;v) = \max\{\langle x^*, v\rangle_X : x^* \in \partial h(x)\}$;

(v) the multimap ∂h is u.s.c. from Ω into $X^*_{w^*}$.

Let E_1 and E_2 be nonempty subsets of a Banach space X. The Hausdorff metric $\mathscr{H}(\cdot,\cdot)$ between E_1 and E_2 is formulated by $\mathscr{H}(E_1,E_2) = \max\{e(E_1,E_2), e(E_2,E_1)\}$, where $e(E_1,E_2) = \sup_{x\in E_1} d(x,E_2)$ with $d(x,E_2) = \inf_{y\in E_2} \|x-y\|$. According to [20], one knows that if E_1 and E_2 are compact subsets in X, then $\forall x \in E_1$, $\exists y \in E_2$ s.t. $\|x-y\| \leq \mathscr{H}(E_1,E_2)$.

The key tool in one of our main results is the fixed point theorem below.

Theorem 2.1. [20] *(Nadler's Fixed Point Theorem for Multivalued Contraction Maps.) Let Λ be a nonempty, closed, and convex subset of a Banach space X, and let $CB(\Lambda)$ be the collection of all nonempty, closed, and bounded subsets of Λ. If $\mathscr{F} : \Lambda \to CB(\Lambda)$ is a multivalued contraction map, then $\mathscr{F}$ has a fixed point.*

In order to investigate the existence of solutions of system (1.2), we need the following conditions.

(HT): $T_i(t)$ $(t > 0)$ is a compact operator for $i = 1,2$.

Let $J : I \times V_1 \times V_2 \to \mathbb{R}$ be a functional satisfying the following conditions:

(HJ1) the function $t \mapsto J(t,x_1,x_2)$ is measurable for all $(x_1,x_2) \in V_1 \times V_2$;

(HJ2) the function $(x_1,x_2) \to J(t,x_1,x_2)$ is locally Lipschitz on $V_1 \times V_2$ for a.e. $t \in I$;

(HJ3) for $i = 1,2$, there exist a function $\phi_i(\cdot) \in L^p(I,\mathbb{R}^+)$, $p > \frac{1}{\alpha}$, and a constant $c_i > 0$ such that $\|\partial_i J(t,x_1,x_2)\|_{V_i} = \sup\{\|\zeta_i\|_{V_i} : \zeta_i \in \partial_i J(t,x_1,x_2)\} \leq \phi_i(t) + c_i t^{1-\alpha}\|x_i\|_{V_i}$ for a.e. $t \in I$ and all $(x_1,x_2) \in V_1 \times V_2$;

(HJ4) $J(t,x_1,x_2) + J(t,y_1,y_2) = J(t,x_1,y_2) + J(t,y_1,x_2)$ for a.e. $t \in I$ and all (x_1,x_2), $(y_1,y_2) \in V_1 \times V_2$;

(HJ5) for $i = 1,2$, there exists a constant $L_i > 0$ such that $\|\xi_i - \eta_i\|_{V_i} \leq L_i\|x_i - y_i\|_{V_i}$ for any $(x_1,x_2), (y_1,y_2) \in V_1 \times V_2$, $\xi_i \in \partial_i J(t,x_1,x_2)$ and $\eta_i \in \partial_i J(t,y_1,y_2)$.

Lemma 2.3. [7, Lemma 3.6]. *Suppose that the functional $J : I \times V_1 \times V_2 \to \mathbb{R}$ satisfies the hypotheses (HJ2), and (HJ4). Then, for any sequence $\mathbf{x}^n = (x_1^n, x_2^n) \in V = V_1 \times V_2$ converging strongly to $\mathbf{x} = (x_1,x_2) \in V$ and $y_i^n \in V_i$ converging strongly to $y_i \in V_i$, $\limsup_{n\to\infty} J_i^\circ(t,x_1^n,x_2^n;y_i^n) \leq J_i^\circ(t,x_1,x_2;y_i)$, a.e. $t \in I$, for $i = 1,2$.*

Lemma 2.4. [1, Proposition 3.44] *Let E be a separable reflexive Banach space, $0 < b < \infty$, and $h : (0,b) \times E \to \mathbb{R}$ be a function such that $h(\cdot, x)$ is measurable for all $x \in E$, and $h(t, \cdot)$ is locally Lipschitz on E for all $t \in (0,b)$. Then the multifunction $(t,x) \in (0,b) \times E \mapsto \partial h(t,x) \subset E^*$ is measurable, where ∂h denotes the Clarke's generalized gradient of $h(t, \cdot)$.*

Suppose that both X and Y are topological spaces, and $F : X \to 2^Y$ is a multivalued mapping. According to [21], one knows that there hold the statements below.

(i) If F is u.s.c. and closed-valued, then F is closed.

(ii) If F is compact-valued, then F is u.s.c. at $x \in X$ if and only if, for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \to x$, and for any net $\{y_\alpha\} \subseteq Y$ with $y_\alpha \in F(x_\alpha)$ for all α, there exist $y \in F(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y$.

(iii) F is l.s.c. at $x \in X$ if and only if, for any $y \in F(x)$, and for any net $\{x_\alpha\}$ with $x_\alpha \to x$, there exists a net $\{y_\alpha\}$ with $y_\alpha \in F(x_\alpha)$ for all α such that $y_\alpha \to y$.

The following theorem is useful to the proof of the main results in this paper.

Theorem 2.2. [22, Theorem 2.2.1] *(Kuratowski-Ryll Nardzewski Selection Theorem) If (Ω, Σ) is a measurable space, X is a Polish space (i.e., separable completely metric space), and $F : \Omega \to P_f(X)$ is measurable, then $F(\cdot)$ admits a measurable selection (i.e., $\exists f : \Omega \to X$ (measurable), s.t. $f(x) \in F(x)$, $\forall x \in \Omega$).*

Proposition 2.1. [1, Proposition 3.16] *Let (Ω, Σ, μ) be a σ-finite measure space, E be a Banach space, and $1 \le p < \infty$. If $f_n, f \in L^p(\Omega, E)$, $f_n \to f$ weakly in $L^p(\Omega, E)$ and $f_n(x) \in G(x)$ for μ-a.e. $x \in \Omega$ and all $n \in \mathbb{N}$ where $G(x) \in P_{wk}(E)$ for μ-a.e. $x \in \Omega$, then $f(x) \in \overline{\text{conv}}(w\!-\limsup\{f_n(x)\}_{n \in \mathbb{N}})$ for $\mu-$a.e.* on *$x \in \Omega$, where $\overline{\text{conv}}$ denotes the closed convex hull of a set.*

Next, for $i = 1,2$, we formulate the superposition multimap $\mathcal{N}_i : C(I,V_1) \times C(I,V_2) \to P(L^p(I,V_i))$, $p > \frac{1}{\alpha}$ below

$$\mathcal{N}_i(x_1,x_2) = \{w_i \in L^p(I,V_i) : w_i(t) \in \partial_i J(t, x_1(t), x_2(t)) \text{ a.e. } t \in I\},$$

for all $(x_1,x_2) \in C(I,V_1) \times C(I,V_2)$.

Let $\mathscr{C} = C(I,V_1) \times C(I,V_2)$. Endowed with the norm formulated by $\|\mathbf{x}\|_{\mathscr{C}} := \|x_1\|_{C(I,V_1)} + \|x_2\|_{C(I,V_2)}$ for all $\mathbf{x} = (x_1,x_2) \in \mathscr{C} = C(I,V_1) \times C(I,V_2)$, $\mathscr{C}$ is a reflexive Banach space; see e.g., [19].

We have the following property for the operator $\mathcal{N}_i$ for $i = 1,2$.

Proposition 2.2. *If conditions (HJ1)-(HJ4) are satisfied, then, for every $(x_1,x_2) \in C(I,V_1) \times C(I,V_2)$, $\mathcal{N}_i(x_1,x_2)$ has nonempty, convex, and weakly compact values for $i = 1,2$.*

Proof. The main idea of the proof comes from [1, Lemma 5.3] and [2, Lemma 2.6]. First of all, for $i = 1,2$, from the reflexivity of V_i and Lemma 2.2 (ii), we know that, for every $(t,x_1,x_2) \in I \times V_1 \times V_2$, $\partial_i(t,x_1,x_2)$ is nonempty, convex, and weakly compact in V_i, and the multifunction $\partial_i J$ is $P_{wkc}(V_i)$-valued. Therefore, it is not difficult to check that, for $i = 1,2$, $\mathcal{N}_i(x_1,x_2)$ has convex and weakly compact values. Next, we show that, for $i = 1,2$, $\mathcal{N}_i(x_1,x_2)$ is nonempty. Indeed, let $(x_1,x_2) \in C(I,V_1) \times C(I,V_2)$. Then, for $i = 1,2$, there exists a sequence $\{\varphi_i^n\} \subseteq C(I,V_i)$ of step functions such that

$$\varphi_i^n(t) \to x_i(t) \text{ in } C(I,V_i), \text{ a.e. } t \in I. \tag{2.1}$$

From hypotheses (HJ1) and (HJ2) and Lemma 2.4, we conclude that, for $i=1,2$, $(t,x_1,x_2)\mapsto \partial_i J(t,x_1,x_2)$ is measurable. Thus, for $i=1,2$, $t\mapsto \partial_i J(t,\varphi_1^n(t),\varphi_2^n(t))$ is measurable from I into $P_{fc}(V_i)$. For $i=1,2$, applying Theorem 2.2, for every $n\geq 1$, there exists a measurable function $\zeta_i^n: I\to V_i$ such that $\zeta_i^n(t)\in \partial_i J(t,\varphi_1^n(t),\varphi_2^n(t))$ a.e. $t\in I$. Next, from hypothesis (HJ3), we obtain that, for $i=1,2$, $\|\zeta_i^n\|_{L^p(I,V_i)}\leq \|\phi_i\|_{L^p(I,\mathbf{R}^+)}+c_i b^{1-\frac{1}{p}}\|\varphi_i^n\|_{L^p(I,V_i)}$. Hence, for $i=1,2$, $\{\zeta_i^n\}$ remains in a bounded subset of $L^p(I,V_i)$. Thus, for $i=1,2$, by passing to a subsequence if necessary, we may suppose that $\zeta_i^n\to\zeta_i$ weakly in $L^p(I,V_i)$ with $\zeta_i\in L^p(I,V_i)$. Then it follows from Proposition 2.1 that, for $i=1,2$,

$$\zeta_i(t)\in\overline{\mathrm{co}}(w-\limsup\{\zeta_i^n(t)\}_{n\geq 1}),\quad \text{a.e. } t\in I. \tag{2.2}$$

We claim that for a.e. $t\in I$, the multifunction $(x_1,x_2)\mapsto \partial_i J(t,x_1,x_2)$ is u.s.c. from $V_1\times V_2$ into $(V_i)_w$, where, for $i=1,2$, $(V_i)_w$ is the space furnished with the w-topology of V_i. Indeed, for any sequence $\{(x_1^n,x_2^n)\}\subseteq V_1\times V_2$ with $(x_1^n,x_2^n)\to(x_1,x_2)$ in $V=V_1\times V_2$, and for any sequence $\{y_i^n\}\subseteq V_i$ with $y_i^n\in\partial_i J(t,x_1^n,x_2^n)$ for all $n\geq 1$, we infer from the definition of partial Clarke's generalized gradient $\partial_i J(t,x_1^n,x_2^n)$ of a locally Lipschitz $J(t,\cdot,\cdot): V_1\times V_2\to\mathbb{R}$ that $\langle y_i^n,v_i\rangle_{V_i}\leq J_i^\circ(t,x_1^n,x_2^n;v_i)$, $\forall v_i\in V_i$. Since $(x_1^n,x_2^n)\to(x_1,x_2)$ in $V_1\times V_2$, it follows from hypotheses (HJ2) and (HJ4), and Lemma 2.3 that, for $i=1,2$,

$$\limsup_{n\to\infty}\langle y_i^n,v_i\rangle_{V_i}\leq \limsup_{n\to\infty} J_i^\circ(t,x_1^n,x_2^n;v_i)\leq J_i^\circ(t,x_1,x_2;v_i),\quad \forall v_i\in V_i. \tag{2.3}$$

Also, for $i=1,2$, from hypothesis (HJ3) and $y_i^n\in\partial_i J(t,x_1^n,x_2^n)$, we obtain $\|y_i^n\|_{V_i}\leq\phi_i(t)+c_i b^{1-\frac{1}{p}}\|x_i^n\|_{V_i}$, which together with the boundedness of $\{x_i^n\}$ implies that $\{y_i^n\}$ is bounded. Taking into account the reflexivity of V_i, we know that there exists a subsequence $\{y_i^{n_k}\}$ of $\{y_i^n\}$ such that $y_i^{n_k}\to y_i$ weakly in V_i. So, it follows from (2.3) that

$$\langle y_i,v_i\rangle=\lim_{k\to\infty}\langle y_i^{n_k},v_i\rangle_{V_i}\leq\limsup_{n\to\infty}\langle y_i^n,v_i\rangle_{V_i}\leq J_i^\circ(t,x_1,x_2;v_i),\quad\forall v_i\in V_i,$$

which yields $y_i\in\partial_i J(t,x_1,x_2)$. Therefore, in terms of Lemma 2.2 (ii), we deduce that, for a.e. $t\in I$, the multifunction $(x_1,x_2)\mapsto\partial_i J(t,x_1,x_2)$ is u.s.c. from $V_1\times V_2$ into $(V_i)_w$, with $i=1,2$. Noticing that the graph of an u.s.c. multifunction with closed values is closed (due to Lemma 2.2 (i)), we obtain that, for $i=1,2$ and a.e. $t\in I$, if $f_i^n\in\partial_i J(t,\zeta_1^n,\zeta_2^n)$, $f_i^n\in V_i$, $f_i^n\to f_i$ weakly in V_i, $\zeta_i^n\in C(I,V_i)$, $\zeta_i^n\to\zeta_i$ in $C(I,V_i)$, then $f_i\in\partial_i J(t,\zeta_1,\zeta_2)$. Therefore, it follows from (2.1) that

$$w-\limsup\partial_i J(t,\zeta_1^n(t),\zeta_2^n(t))\subset\partial_i J(t,x_1(t),x_2(t))\quad \text{a.e. } t\in I, \tag{2.4}$$

where the Kuratowski limit superior (see [1, Definition 3.14]) is formulated below

$$\begin{aligned}&w-\limsup\partial_i J(t,\varphi_1^n(t),\varphi_2^n(t))\\&=\{\zeta_i\in V_i:\zeta_i=w-\lim\zeta_i^{n_k},\ \zeta_i^{n_k}\in\partial_i J(t,\varphi_1^{n_k}(t),\varphi_2^{n_k}(t)),\ n_1<\cdots<n_k<\cdots\}.\end{aligned}$$

So, from (2.2) and (2.4), we obtain, for $i=1,2$,

$$\begin{aligned}\zeta_i(t)\ &\in\overline{\mathrm{co}}(w-\limsup\{\zeta_i^n(t)\}_{n\geq1})\\&\subset\overline{\mathrm{co}}(w-\limsup\partial_i J(t,\varphi_1^n(t),\varphi_2^n(t)))\\&\subset\partial_i J(t,x_1(t),x_2(t)),\quad \text{a.e. } t\in I.\end{aligned}$$

Since, for $i=1,2$, $\zeta_i\in L^p(I,V_i)$ and $\zeta_i(t)\in\partial_i J(t,x_1(t),x_2(t))$ a.e. $t\in I$, it is clear that $\zeta_i\in\mathscr{N}_i(x_1,x_2)$. Therefore, $\mathscr{N}_i(x_1,x_2)$ is nonempty. □

Proposition 2.3. *If conditions (HJ1)-(HJ4) are satisfied, then, for $i = 1,2$, $\mathscr{N}_i$ is closed in $\mathscr{C} \times L^p_w(I,V_i)$, where $L^p_w(I,V_i)$ is the space furnished with the w-topology of $L^p(I,V_i)$.*

Proof. By Proposition 2.2, we know that, for each $\mathbf{x} = (x_1,x_2) \in \mathscr{C} = C(I,V_1) \times C(I,V_2)$, the set $\mathscr{N}_i(x_1,x_2)$ has nonempty, convex, and weakly compact values for $i = 1,2$. Utilizing the similar arguments to those in the proof of Proposition 2.2, we can prove that, for $i = 1,2$, $\mathscr{N}_i : \mathscr{C} = C(I,V_1) \times C(I,V_2) \to P(L^p(I,V_i))$ is u.s.c. from $\mathscr{C} = C(I,V_1) \times C(I,V_2)$ into $L^p_w(I,V_i)$. So, it follows from Proposition 2.1 (ii) that, for $i = 1,2$, the graph of the u.s.c. multifunction $\mathscr{N}_i$ with closed values is closed (due to Lemma 2.2 (i)), which hence implies that, for $i = 1,2$, if $\mathbf{x}^n = (x^n_1,x^n_2) \to \mathbf{x} = (x_1,x_2)$ in $\mathscr{C} = C(I,V_1) \times C(I,V_2)$, $w^n_i \to w_i$ weakly in $L^p(I,V_i)$, and $w^n_i \in \mathscr{N}_i(x^n_1,x^n_2)$, then $w_i \in \mathscr{N}_i(x_1,x_2)$. □

Inspired by [8, Definition 2.5], we now formulate the mild solution of system (1.2).

Definition 2.1. A pair of functions $\mathbf{x} = (x_1,x_2) \in C_{1-\alpha}(I,V_1) \times C_{1-\alpha}(I,V_2)$ is called a mild solution of system (1.2) if it satisfies the following system of fractional integral equations:

$$\begin{cases} x_1(t) & = t^{\alpha-1}T_{1,\alpha}(t)x_{1,0} + \int_0^t (t-s)^{\alpha-1}T_{1,\alpha}(t-s)B_1u_1(s)ds \\ & \quad + \int_0^t (t-s)^{\alpha-1}T_{1,\alpha}(t-s)f_1(s)ds, \\ x_2(t) & = t^{\alpha-1}T_{2,\alpha}(t)x_{2,0} + \int_0^t (t-s)^{\alpha-1}T_{2,\alpha}(t-s)B_2u_2(s)ds \\ & \quad + \int_0^t (t-s)^{\alpha-1}T_{2,\alpha}(t-s)f_2(s)ds, \end{cases}$$

where $(f_1,f_2) \in \mathscr{N}_1(\mathbf{x}) \times \mathscr{N}_2(\mathbf{x})$ with

$$\mathscr{N}_i(\mathbf{x}) = \{w_i \in L^p(I,V_i) : w_i(t) \in \partial_i J(t,x_1(t),x_2(t)) \text{ a.e. } t \in I\}, \quad i = 1,2.$$

Definition 2.2. Let $\mathbf{x} := \mathbf{x}(t;0,\mathbf{x}_0,\mathbf{u})$ be a solution to system (1.2) from the initial value $\mathbf{x}_0 = (x_{1,0},x_{2,0}) \in V = V_1 \times V_2$ at time t corresponding to the control $\mathbf{u}(\cdot) = (u_1(\cdot),u_2(\cdot)) \in U = U_1 \times U_2$. The set $K_b(\mathbf{f}) = \{\mathbf{x}(b;0,\mathbf{x}_0,\mathbf{u}) : \mathbf{u}(\cdot) \in U\}$, where $\mathbf{f} = (f_1,f_2) \in \mathscr{N}_1(\mathbf{x}) \times \mathscr{N}_2(\mathbf{x})$, is called the reachable set of system (1.2) at terminal time b. If $\overline{K_b(\mathbf{f})} = V$ $\forall \mathbf{x}_0 \in V$, then system (1.2) is said to be approximately controllable on I.

3. The Existence of Mild Solutions

This section is devoted to the study of the existence results for a class of systems of fractional evolution hemivariational inequalities involving Riemann-Liouville fractional derivatives. According to [18], we can obtain the following.

Lemma 3.1. *For $i = 1,2$, the operator $T_{i,\alpha}(t)$ in Lemma 2.1 has the following properties:*

(i) for any fixed $t \geq 0$, $T_{i,\alpha}(t)$ is a linear and bounded operator, i.e., for any $x_i \in V_i$,

$$\|T_{i,\alpha}(t)x_i\|_{V_i} \leq \frac{M_i}{\Gamma(\alpha)}\|x_i\|_{V_i},$$

where $M_i := \sup_{t\in[0,\infty)} \|T_i(t)\|_{L_b(V_i)} < \infty$; for every $t \geq 0$, $T_{i,\alpha}(t)$ is also a compact operator if $T_i(t)$ $(t > 0)$ is compact.

(ii) $T_{i,\alpha}(t)$ $(t \geq 0)$ is strongly continuous.

Lemma 3.2. [23, Lemma 3.5] *If condition (HT) holds, then, for $i = 1,2$, the operator G_i : $L^p(I,V_i) \to C(I,V_i)$ for some $p > 1$, formulated below*

$$(G_i f_i)(\cdot) = \int_0^{\cdot} T_i(\cdot - s) f_i(s) ds,$$

is compact for $f_i \in L^p(I,V_i)$.

Corollary 3.1. *If condition (HT) holds, then, for $i = 1,2$, the operator $\widetilde{G}_i : L^p(I,V_i) \to C(I,V_i)$ for some $p > 1$, formulated below*

$$(\widetilde{G}_i f_i)(\cdot) = \int_0^{\cdot} T_{i,\alpha}(\cdot - s) f_i(s) ds,$$

is compact for $f_i \in L^p(I,V_i)$.

Proof. For $i = 1,2$, since condition (HT) means that $T_i(t)$ $(t > 0)$ is compact, we know by Lemma 3.1 that $T_{i,\alpha}(t)$ $(t > 0)$ is also compact. From Lemma 3.2, we obtain the desired result immediately. □

Next, for $i = 1,2$, we formulate the operator $Q_i : L^p(I,V_i) \to C(I,V_i)$ below

$$(Q_i f_i)(\cdot) = \int_0^{\cdot} (\cdot - s)^{\alpha-1} T_{i,\alpha}(\cdot - s) f_i(s) ds,$$

for $f_i \in L^p(I,V_i)$. Let $\mathscr{C}_{1-\alpha} := C_{1-\alpha}(I,V_1) \times C_{1-\alpha}(I,V_2)$. Endowed with the norm $\|\mathbf{x}\|_{\mathscr{C}_{1-\alpha}} := \|x_1\|_{C_{1-\alpha}(I,V_1)} + \|x_2\|_{C_{1-\alpha}(I,V_2)}$ for all $\mathbf{x} = (x_1,x_2) \in \mathscr{C}_{1-\alpha}$, $\mathscr{C}_{1-\alpha}$ is a Banach space.

Now, we are in a position to present the main result of this section.

Theorem 3.1. *Assume that hypotheses (HT), and (HJ1)-(HJ5) are satisfied. Then, for each control function $\mathbf{u}(\cdot) = (u_1(\cdot),u_2(\cdot)) \in U = U_1 \times U_2$, control system (1.2) has at least one mild solution on $\mathscr{C}_{1-\alpha}$ provided that $\frac{\Gamma(\alpha)\max\{L_1M_1,L_2M_2\}b^{\alpha}}{\Gamma(2\alpha)} < 1$, and for $i = 1,2$ and any $\mathbf{x} = (x_1,x_2) \in \mathscr{C}$, the operator Q_i is compact on $\mathscr{N}_i(\mathbf{x})$ from the $L^p_w(I,V_i)$-topology to the $C_{1-\alpha}(I,V_i)$-topology.*

Proof. Note that $\mathscr{C} \subseteq \mathscr{C}_{1-\alpha}$. Moreover, it is not hard to check that $\mathscr{C}$ is a closed subspace of $\mathscr{C}_{1-\alpha}$. Consider the multimap $\mathscr{F} : \mathscr{C} \to P(\mathscr{C})$ formulated below

$$\begin{aligned}\mathscr{F}(\mathbf{x}) \ &= \{\mathbf{y} = (y_1,y_2) \in \mathscr{C} : \ \exists \mathbf{f} = (f_1,f_2) \in \mathscr{N}_1(\mathbf{x}) \times \mathscr{N}_2(\mathbf{x}) \text{ s.t.} \\ &\quad y_1(t) = t^{\alpha-1}T_{1,\alpha}(t)x_{1,0} + \textstyle\int_0^t (t-s)^{\alpha-1}T_{1,\alpha}(t-s)(B_1u_1(s) + f_1(s))ds, \\ &\quad y_2(t) = t^{\alpha-1}T_{2,\alpha}(t)x_{2,0} + \textstyle\int_0^t (t-s)^{\alpha-1}T_{2,\alpha}(t-s)(B_2u_2(s) + f_2(s))ds \ \},\end{aligned}$$

for all $\mathbf{x} = (x_1,x_2) \in \mathscr{C}$. We divide the proof into four steps to verify that $\mathscr{F}$ has a fixed point in $\mathscr{C}$.

Step 1. $\mathscr{F}(\mathbf{x})$ is convex for each $\mathbf{x} = (x_1,x_2) \in \mathscr{C}$.

Indeed, if $\mathbf{y}^1,\mathbf{y}^2 \in \mathscr{F}(\mathbf{x})$ with $\mathbf{y}^1 = (y_{1,1},y_{2,1})$ and $\mathbf{y}^2 = (y_{1,2},y_{2,2})$, then there exist $\mathbf{f}^1,\mathbf{f}^2 \in \mathscr{N}_1(\mathbf{x}) \times \mathscr{N}_2(\mathbf{x})$ with $\mathbf{f}^1 = (f_{1,1},f_{2,1})$ and $\mathbf{f}^2 = (f_{1,2},f_{2,2})$ such that, for every $t \in I$,

$$\begin{cases} y_{i,1}(t) = t^{\alpha-1}T_{i,\alpha}(t)x_{i,0} + \int_0^t (t-s)^{\alpha-1}T_{i,\alpha}(t-s)(B_iu_i(s) + f_{i,1}(s))ds, & i = 1,2, \\ y_{i,2}(t) = t^{\alpha-1}T_{i,\alpha}(t)x_{i,0} + \int_0^t (t-s)^{\alpha-1}T_{i,\alpha}(t-s)(B_iu_i(s) + f_{i,2}(s))ds, & i = 1,2. \end{cases}$$

Let $0 \le d \le 1$. Then, for $i = 1,2$ and every $t \in I$,

$$\begin{aligned}&(dy_{i,1} + (1-d)y_{i,2})(t) \\ &= t^{\alpha-1}T_{i,\alpha}(t)x_{i,0} + \int_0^t (t-s)^{\alpha-1}T_{i,\alpha}(t-s)(B_iu_i(s) + (df_{i,1} + (1-d)f_{i,2})(s))ds.\end{aligned}$$

For $i=1,2$, it follows from Proposition 2.2 that $\mathscr{N}_i(\mathbf{x})$ is convex. Hence $df_{i,1}+(1-d)f_{i,2}\in\mathscr{N}_i(\mathbf{x})$. Consequently, $d\mathbf{y}^1+(1-d)\mathbf{y}^2\in\mathscr{F}(\mathbf{x})$.

Step 2. For every $\mathbf{x}=(x_1,x_2)\in\mathscr{C}$, $\mathscr{F}(\mathbf{x})\subset\mathscr{C}$ is bounded in $\mathscr{C}_{1-\alpha}$.

Indeed, for any $\varphi=(\varphi_1,\varphi_2)\in\mathscr{F}(\mathbf{x})$, there exists $\mathbf{f}=(f_1,f_2)\in\mathscr{N}_1(\mathbf{x})\times\mathscr{N}_2(\mathbf{x})$ such that

$$\begin{cases}\varphi_1(t)=t^{\alpha-1}T_{1,\alpha}(t)x_{1,0}+\int_0^t(t-s)^{\alpha-1}T_{1,\alpha}(t-s)(B_1u_1(s)+f_1(s))ds,\\ \varphi_2(t)=t^{\alpha-1}T_{2,\alpha}(t)x_{2,0}+\int_0^t(t-s)^{\alpha-1}T_{2,\alpha}(t-s)(B_2u_2(s)+f_2(s))ds.\end{cases}$$

By (HJ3) and the Holder inequality, we obtain that, for $i=1,2$ and every $t\in I$,

$$\begin{aligned}t^{1-\alpha}\|\varphi_i(t)\|_{V_i} &\le \|T_{i,\alpha}(t)x_{i,0}\|_{V_i}+t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\|T_{i,\alpha}(t-s)B_iu_i(s)\|_{V_i}ds\\ &\quad+t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\|T_{i,\alpha}(t-s)f_i(s)\|_{V_i}ds\\ &\le \frac{M_i}{\Gamma(\alpha)}[\|x_{i,0}\|_{V_i}+t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}\|B_iu_i(s)\|_{V_i}ds\\ &\quad+t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}(\phi_i(s)+c_is^{1-\alpha}\|x_i(s)\|_{V_i})ds]\\ &\le \frac{M_i}{\Gamma(\alpha)}[\|x_{i,0}\|_{V_i}+(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}}b^{1-\frac{1}{p}}(\|B_iu_i\|_{L^p(I,V_i)}+\|\phi_i\|_{L^p(I,\mathbf{R}^+)})\\ &\quad+c_ib^{1-\alpha}\int_0^t(t-s)^{\alpha-1}s^{1-\alpha}\|x_i(s)\|_{V_i}ds]\\ &\le \frac{M_i}{\Gamma(\alpha)}[\|x_{i,0}\|_{V_i}+(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}}b^{1-\frac{1}{p}}(\|B_iu_i\|_{L^p(I,V_i)}+\|\phi_i\|_{L^p(I,\mathbf{R}^+)})\\ &\quad+c_ib^{1-\alpha}\sup_{t\in I}t^{1-\alpha}\|x_i(t)\|_{V_i}\int_0^t(t-s)^{\alpha-1}ds]\\ &\le \frac{M_i}{\Gamma(\alpha)}[\|x_{i,0}\|_{V_i}+(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}}b^{1-\frac{1}{p}}(\|B_iu_i\|_{L^p(I,V_i)}+\|\phi_i\|_{L^p(I,\mathbf{R}^+)})\\ &\quad+\alpha^{-1}c_ib\|x_i\|_{C_{1-\alpha}(I,V_i)}]\\ &:=\ell_i,\end{aligned}$$

which hence yields

$$\begin{aligned}\|\varphi\|_{\mathscr{C}_{1-\alpha}} &=\|\varphi_1\|_{C_{1-\alpha}(I,V_1)}+\|\varphi_2\|_{C_{1-\alpha}(I,V_2)}\\ &=\sup_{t\in I}t^{1-\alpha}\|\varphi_1(t)\|_{V_1}+\sup_{t\in I}t^{1-\alpha}\|\varphi_2(t)\|_{V_2}\\ &\le\ell_1+\ell_2.\end{aligned}$$

Consequently, $\mathscr{F}(\mathbf{x})$ is bounded in $\mathscr{C}_{1-\alpha}$.

Step 3. For every $\mathbf{x}=(x_1,x_2)\in\mathscr{C}$, $\mathscr{F}(\mathbf{x})\subset\mathscr{C}$ is closed in $\mathscr{C}_{1-\alpha}$.

Indeed, assume that $\{\varphi^n\}\subset\mathscr{F}(\mathbf{x})$ with $\varphi^n=(\varphi_{1,n},\varphi_{2,n})$ and $\varphi^n\to\varphi$ in $\mathscr{C}_{1-\alpha}$ with $\varphi=(\varphi_1,\varphi_2)$. Then there exists $\mathbf{f}^n=(f_{1,n},f_{2,n})\subset\mathscr{N}_1(\mathbf{x})\times\mathscr{N}_2(\mathbf{x})$ such that

$$\begin{cases}t^{1-\alpha}\varphi_{1,n}(t)=T_{1,\alpha}(t)x_{1,0}+t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}T_{1,\alpha}(t-s)(B_1u_1(s)+f_{1,n}(s))ds,\\ t^{1-\alpha}\varphi_{2,n}(t)=T_{2,\alpha}(t)x_{2,0}+t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}T_{2,\alpha}(t-s)(B_2u_2(s)+f_{2,n}(s))ds.\end{cases}\tag{3.1}$$

In terms of Proposition 2.2, we know that, for $i=1,2$, the set $\mathscr{N}_i(\mathbf{x})$ is a nonempty, convex, and weakly compact subset in $L^p(I,V_i)$, which together with $\{f_{i,n}\}\subset\mathscr{N}_i(\mathbf{x})$ implies that there exists a subsequence $\{f_{i,n_k}\}$ of $\{f_{i,n}\}$ such that $f_{i,n_k}\to f_i\in\mathscr{N}_i(\mathbf{x})$ weakly in $L^p(I,V_i)$. Meantime, from (3.1), we have

$$\begin{cases}t^{1-\alpha}\varphi_{1,n_k}(t)=T_{1,\alpha}(t)x_{1,0}+t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}T_{1,\alpha}(t-s)(B_1u_1(s)+f_{1,n_k}(s))ds,\\ t^{1-\alpha}\varphi_{2,n_k}(t)=T_{2,\alpha}(t)x_{2,0}+t^{1-\alpha}\int_0^t(t-s)^{\alpha-1}T_{2,\alpha}(t-s)(B_2u_2(s)+f_{2,n_k}(s))ds.\end{cases}\tag{3.2}$$

Set $\mathbf{f} := (f_1, f_2)$. Then $\mathbf{f} \in \mathcal{N}_1(\mathbf{x}) \times \in \mathcal{N}_2(\mathbf{x})$. For $i = 1,2$, since the operator Q_i is compact on $\mathcal{N}_i(\mathbf{x})$ from the $L^p_w(I,V_i)$-topology to the $C_{1-\alpha}(I,V_i)$-topology, we deduce that, as $k \to \infty$,

$$\begin{aligned}&\|Q_i f_{i,n_k} - Q_i f_i\|_{C_{1-\alpha}(I,V_i)}\\ &= \sup_{t\in I} t^{1-\alpha} \|\int_0^t (t-s)^{\alpha-1} T_{2,\alpha}(t-s) f_{i,n_k}(s)ds - \int_0^t (t-s)^{\alpha-1} T_{2,\alpha}(t-s) f_i(s)ds\|_{V_i} \to 0.\end{aligned}$$

In view of $\varphi^n \to \varphi$ in $\mathscr{C}_{1-\alpha}$, we can see easily that, for $i = 1,2$,

$$\|\varphi_{i,n_k} - \varphi_i\|_{C_{1-\alpha}(I,V_i)} = \sup_{t\in I} t^{1-\alpha} \|\varphi_{i,n_k}(t) - \varphi_i(t)\|_{V_i} \to 0.$$

Therefore, it is clear from (3.2) that, as $k \to \infty$,

$$\begin{cases} t^{1-\alpha}\varphi_1(t) = T_{1,\alpha}(t)x_{1,0} + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1} T_{1,\alpha}(t-s)(B_1u_1(s) + f_1(s))ds,\\ t^{1-\alpha}\varphi_2(t) = T_{2,\alpha}(t)x_{2,0} + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1} T_{2,\alpha}(t-s)(B_2u_2(s) + f_2(s))ds,\end{cases}$$

that is,

$$\begin{cases} \varphi_1(t) = t^{\alpha-1} T_{1,\alpha}(t)x_{1,0} + \int_0^t (t-s)^{\alpha-1} T_{1,\alpha}(t-s)(B_1u_1(s) + f_1(s))ds,\\ \varphi_2(t) = t^{\alpha-1} T_{2,\alpha}(t)x_{2,0} + \int_0^t (t-s)^{\alpha-1} T_{2,\alpha}(t-s)(B_2u_2(s) + f_2(s))ds.\end{cases}$$

This means that $\varphi = (\varphi_1, \varphi_2) \in \mathscr{F}(\mathbf{x})$. Thus $\mathscr{F}(\mathbf{x})$ is closed in $\mathscr{C}_{1-\alpha}$.

Step 4. $\mathscr{F} : \mathscr{C} \to CB(\mathscr{C})$ is a multivalued contraction map in $\mathscr{C}_{1-\alpha}$.

Indeed, take any $\mathbf{x}, \mathbf{y} \in \mathscr{C}$ with $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, $\varphi \in \mathscr{F}(\mathbf{x})$, and $\psi \in \mathscr{F}(\mathbf{y})$ with $\varphi = (\varphi_1, \varphi_2)$ and $\psi = (\psi_1, \psi_2)$. Then there exist $\xi = (\xi_1, \xi_2) \in \mathcal{N}_1(\mathbf{x}) \times \mathcal{N}_2(\mathbf{x})$ and $\eta = (\eta_1, \eta_2) \in \mathcal{N}_1(\mathbf{y}) \times \mathcal{N}_2(\mathbf{y})$ such that, for $i = 1,2$,

$$\begin{cases} \varphi_i(t) = t^{\alpha-1} T_{i,\alpha}(t)x_{i,0} + \int_0^t (t-s)^{\alpha-1} T_{i,\alpha}(t-s)(B_iu_i(s) + \xi_i(s))ds,\\ \psi_i(t) = t^{\alpha-1} T_{i,\alpha}(t)x_{i,0} + \int_0^t (t-s)^{\alpha-1} T_{i,\alpha}(t-s)(B_iu_i(s) + \eta_i(s))ds.\end{cases}$$

So, by (HJ5), we conclude that, for $i = 1,2$ and every $t \in I$,

$$\begin{aligned} t^{1-\alpha}\|\varphi_i(t) - \psi_i(t)\|_{V_i} &\le t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\|T_{i,\alpha}(t-s)(\xi_i(s) - \eta_i(s))\|_{V_i} ds\\ &\le \frac{L_iM_i}{\Gamma(\alpha)} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1} s^{\alpha-1} s^{1-\alpha}\|x_i(s) - y_i(s)\|_{V_i} ds\\ &\le \frac{\Gamma(\alpha)L_iM_i t^{\alpha}}{\Gamma(2\alpha)}\|x_i - y_i\|_{C_{1-\alpha}(I,V_i)},\end{aligned}$$

which immediately yields

$$\|\varphi_i - \psi_i\|_{C_{1-\alpha}(I,V_i)} \le \frac{\Gamma(\alpha)L_iM_i b^{\alpha}}{\Gamma(2\alpha)}\|x_i - y_i\|_{C_{1-\alpha}(I,V_i)}.$$

So, it follows that

$$\begin{aligned} \|\varphi - \psi\|_{\mathscr{C}_{1-\alpha}} &= \sup_{t\in I} t^{1-\alpha}\|\varphi_1(t) - \psi_1(t)\|_{V_1} + \sup_{t\in I} t^{1-\alpha}\|\varphi_2 - \psi_2\|_{V_2}\\ &\le \frac{\Gamma(\alpha)L_1M_1 b^{\alpha}}{\Gamma(2\alpha)}\|x_1 - y_1\|_{C_{1-\alpha}(I,V_1)} + \frac{\Gamma(\alpha)L_2M_2 b^{\alpha}}{\Gamma(2\alpha)}\|x_2 - y_2\|_{C_{1-\alpha}(I,V_2)}\\ &\le \frac{\Gamma(\alpha)\max\{L_1M_1, L_2M_2\} b^{\alpha}}{\Gamma(2\alpha)}[\|x_1 - y_1\|_{C_{1-\alpha}(I,V_1)} + \|x_2 - y_2\|_{C_{1-\alpha}(I,V_2)}]\\ &= \frac{\Gamma(\alpha)\max\{L_1M_1, L_2M_2\} b^{\alpha}}{\Gamma(2\alpha)}\|\mathbf{x} - \mathbf{y}\|_{\mathscr{C}_{1-\alpha}}\\ &= \kappa\|\mathbf{x} - \mathbf{y}\|_{\mathscr{C}_{1-\alpha}}.\end{aligned}$$

where $\kappa := \frac{\Gamma(\alpha)\max\{L_1M_1, L_2M_2\} b^{\alpha}}{\Gamma(2\alpha)} < 1$. Thus $\mathscr{H}(\mathscr{F}(\mathbf{x}), \mathscr{F}(\mathbf{y})) \le \kappa\|\mathbf{x} - \mathbf{y}\|_{\mathscr{C}_{1-\alpha}}$, $\forall \mathbf{x}, \mathbf{y} \in \mathscr{C}$. This means that $\mathscr{F}$ is a multivalued contraction map in $\mathscr{C}_{1-\alpha}$. In terms of Theorem 2.1, $\mathscr{F}$ has a fixed point in $\mathscr{C}$. Therefore, system (1.2) has at least one mild solution on $\mathscr{C}_{1-\alpha}$. □

Remark 3.1. From [24, Lemma 2.3.2 and Lemma 2.4.2], one knows that if the C_0-semigroup $T(t)$ is compact or differentiable for $t > t_0 \geq 0$, then $T(t)$ is continuous in the uniform operator topology for $t > t_0$. Therefore, compared with hypothesis H(1) in [8], our hypothesis (HT) is stronger than H(1). By using the modified version of Banach's contraction mapping principle, Liu and Li [8] proved the existence of mild solutions of frictional evolution control problem (1.1) involving Riemann-Liouville fractional derivative. Actually, Nadler's fixed point theorem for multivalued contraction maps plays a key role in the demonstration of the existence of mild solutions of system (1.2) because, for the treatment of the existence of fixed points of multimaps, such a theorem is more advantageous than Banach's contraction mapping principle and its modified versions.

4. Approximate Controllability Results

In this section, we are concerned with the approximate controllability results of the fractional evolution control system with Riemann-Liouville fractional derivatives.

For $i = 1,2$, we define the bounded and linear operator $\mathscr{G}_i : L^p(I,V_i) \to V_i$ by

$$\mathscr{G}_i h_i = \int_0^b (b-s)^{\alpha-1} T_{i,\alpha}(b-s) h_i(s) ds, \quad h_i(\cdot) \in L^p(I,V_i).$$

From Definition 2.2, we know that if, for any $\mathbf{x}_0 = (x_{1,0}, x_{2,0}) \in V = V_1 \times V_2$ and $\mathbf{u}(\cdot) = (u_1(\cdot), u_2(\cdot)) \in U = U_1 \times U_2$, $\overline{K_b(\mathbf{f})} = V$, where $\mathbf{f} = (f_1, f_2) \in \mathscr{N}_1(\mathbf{x}) \times \mathscr{N}_2(\mathbf{x})$, then system (1.2) is approximately controllable on I. Equivalently, if, for every desired final state $\zeta = (\zeta_1, \zeta_2) \in V$ and any $\varepsilon > 0$, there exists a control function $\mathbf{u}_\varepsilon(\cdot) = (u_{1,\varepsilon}(\cdot), u_{2,\varepsilon}(\cdot)) \in U = U_1 \times U_2$ such that the mild solution of system (1.2) satisfies $\|\zeta_i - b^{\alpha-1} T_{i,\alpha}(b) x_{i,0} - \mathscr{G}_i f_{i,\varepsilon} - \mathscr{G}_i B_i u_{i,\varepsilon}\|_{V_i} < \varepsilon$, where $\mathbf{x}_\varepsilon = (x_{1,\varepsilon}, x_{2,\varepsilon})$ with $\mathbf{x}_\varepsilon(t) =: \mathbf{x}(t; 0, \mathbf{x}_0, \mathbf{u}_\varepsilon), t \in [0,b]$ and $\mathbf{f}_\varepsilon = (f_{1,\varepsilon}, f_{2,\varepsilon}) \in \mathscr{N}_1(\mathbf{x}_\varepsilon) \times \mathscr{N}_2(\mathbf{x}_\varepsilon)$, then system (1.2) is approximately controllable on I.

In what follows, to discuss the approximate controllability of system (1.2), we suppose the following assumptions.

(HJ5$'$) for $i = 1,2$, there exists a constant $L_i' > 0$ such that $\|\xi_i - \eta_i\|_{V_i} \leq L_i' t^{1-\alpha} \|x_i - y_i\|_{V_i}$, for any $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$, $\xi_i \in \partial_i J(t, x_1, x_2)$ and $\eta_i \in \partial_i J(t, y_1, y_2)$;

(HA) for $i = 1,2$, any $\varepsilon > 0$ and $\varphi_i(\cdot) \in L^p(I,V_i)$, there exists a $u_i(\cdot) \in L^p(I,U_i)$ such that

$$\|\mathscr{G}_i \varphi_i - \mathscr{G}_i B_i u_i\|_{V_i} < \varepsilon, \tag{4.1}$$

and $\|B_i u_i(\cdot)\|_{L^p(I,V_i)} < N_i \|\varphi_i(\cdot)\|_{L^p(I,V_i)}$, where N_i is a constant, which is independent of $\varphi_i(\cdot) \in L^p(I,V_i)$, and

$$\frac{M_i L_i' N_i}{\Gamma(\alpha)} \left(\frac{p-1}{p\alpha - 1}\right)^{\frac{p-1}{p}} b^{1-\frac{1}{p}} E_\alpha(M_i L_i' b) < 1. \tag{4.2}$$

Since condition (HJ5$'$) is stronger than (HJ5) and by Theorem 3.1, it is obvious that, for each control function $\mathbf{u}(\cdot) = (u_1(\cdot), u_2(\cdot)) \in U = U_1 \times U_2$, control system (1.2) has at least one mild solution on $\mathscr{C}_{1-\alpha}$ if all the hypotheses in Theorem 3.1 hold except that (HJ5) is replaced by (HJ5$'$).

In order to discuss the approximate controllability of system (1.2), we need the following lemma.

Lemma 4.1. *Assume that the hypotheses (HJ1)-(HJ4) and (HJ5$'$) hold. Then, any mild solutions of system (1.2) satisfy the following inequalities: for $i = 1,2$ and any $\mathbf{u}(\cdot) = (u_1(\cdot), u_2(\cdot))$,*

$\mathbf{u}_1(\cdot) = (u_{1,1}(\cdot), u_{2,1}(\cdot))$ and $\mathbf{u}_2(\cdot) = (u_{1,2}(\cdot), u_{2,2}(\cdot))$ in $U = U_1 \times U_2$,

$$\|x_i(\cdot;0,\mathbf{x}_0,\mathbf{u})\|_{C_{1-\alpha}(I,V_i)} \le \kappa_i E_\alpha(M_i c_i b),$$

$$\|x_{i,1}(\cdot) - x_{i,2}(\cdot)\|_{C_{1-\alpha}(I,V_i)} \le \rho_i E_\alpha(M_i L_i' b)\|B_i u_{i,1}(\cdot) - B_i u_{i,2}(\cdot)\|_{L^p(I,V_i)},$$

where

$$\kappa_i = \frac{M_i}{\Gamma(\alpha)}[\|x_{i,0}\|_{V_i} + (\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}}(\|B_i u_i\|_{L^p(I,V_i)} + \|\phi_i\|_{L^p(I,V_i)})b^{1-\frac{1}{p}}],\ \rho_i = \frac{M_i}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}} b^{1-\frac{1}{p}},$$

and E_α is the Mittag-Leffler function defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}.$$

Proof. If $\mathbf{x} = (x_1, x_2)$ is a mild solution to system (1.2) w.r.t. $\mathbf{u}(\cdot) = (u_1(\cdot), u_2(\cdot)) \in U$ on $\mathscr{C}_{1-\alpha} = C_{1-\alpha}(I,V_1) \times C_{1-\alpha}(I,V_2)$, then

$$\begin{cases} x_1(t) &= t^{\alpha-1}T_{1,\alpha}(t)x_{1,0} + \int_0^t (t-s)^{\alpha-1}T_{1,\alpha}(t-s)B_1u_1(s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}T_{1,\alpha}(t-s)f_1(s)ds, \\ x_2(t) &= t^{\alpha-1}T_{2,\alpha}(t)x_{2,0} + \int_0^t (t-s)^{\alpha-1}T_{2,\alpha}(t-s)B_2u_2(s)ds \\ &\quad + \int_0^t (t-s)^{\alpha-1}T_{2,\alpha}(t-s)f_2(s)ds, \end{cases}$$

where $(f_1, f_2) \in \mathscr{N}_1(\mathbf{x}) \times \mathscr{N}_2(\mathbf{x})$ with

$$\mathscr{N}_i(\mathbf{x}) = \{w_i \in L^p(I,V_i) : w_i(t) \in \partial_i J(t, x_1(t), x_2(t)) \text{ a.e. } t \in I\}, \quad i = 1,2.$$

By (HJ3) and the Holder inequality, we obtain that, for $i = 1,2$ and each $t \in I$,

$$\begin{aligned} t^{1-\alpha}\|x_i(t)\|_{V_i} &\le \|T_{i,\alpha}(t)x_{i,0}\|_{V_i} + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\|T_{i,\alpha}(t-s)B_iu_i(s)\|_{V_i}ds \\ &\quad + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\|T_{i,\alpha}(t-s)f_i(s)\|_{V_i}ds \\ &\le \frac{M_i}{\Gamma(\alpha)}[\|x_{i,0}\|_{V_i} + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}\|B_iu_i(s)\|_{V_i}ds \\ &\quad + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}(\phi_i(s) + c_i s^{1-\alpha}\|x_i(s)\|_{V_i})ds] \\ &\le \frac{M_i}{\Gamma(\alpha)}[\|x_{i,0}\|_{V_i} + (\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}} b^{1-\frac{1}{p}}(\|B_iu_i\|_{L^p(I,V_i)} + \|\phi_i\|_{L^p(I,\mathbf{R}^+)}) \\ &\quad + c_i b^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{1-\alpha}\|x_i(s)\|_{V_i}ds]. \end{aligned} \tag{4.3}$$

For $i = 1,2$, let $W_i(t) = t^{1-\alpha}\|x_i(t)\|_{V_i}$. In view of (4.3), we have

$$W_i(t) \le \kappa_i + \frac{M_i c_i b^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}W_i(s)ds.$$

It follows from [25, Corollary 2] that $W_i(t) \le \kappa_i E_\alpha(M_i c_i b)$. Therefore

$$\|x_i\|_{C_{1-\alpha}(I,V_i)} = \sup_{t\in I} t^{1-\alpha}\|x_i(t)\|_{V_i} \le \kappa_i E_\alpha(M_i c_i b).$$

Similarly, we obtain that, for $i = 1,2$,

$$\|x_{i,1}(\cdot) - x_{i,2}(\cdot)\|_{C_{1-\alpha}(I,V_i)} \le \rho_i E_\alpha(M_i L_i' b)\|B_i u_{i,1}(\cdot) - B_i u_{i,2}(\cdot)\|_{L^p(I,V_i)}.$$

□

By the definition of the operators $T_{i,\alpha}, i = 1,2$, it is easy to show the following lemma.

Lemma 4.2. [8, Lemma 4.2] *For $i=1,2$, let $T_i(t)$ be a differentiable semigroup generated by A_i. Then, for $x_i \in V_i$,*

$$T_{i,\alpha}(t)x_i \in D(A_i)\ \forall t>0,\ T_{i,\alpha}(t)T_{i,\alpha}(s)=T_{i,\alpha}(s)T_{i,\alpha}(t)\ \forall t,s\geq 0,$$

and

$$\frac{d\,T_{i,\alpha}^2(t)x_i}{dt}=2T_{i,\alpha}(t)\frac{d\,T_{i,\alpha}(t)x_i}{dt},\ t>0.$$

Theorem 4.1. *Suppose that hypotheses (HJ1)-(HJ4), (HJ5′), and (HA) are satisfied. Then system (1.2) is approximately controllable on I if, for $i=1,2$, A_i generates a differentiable semigroup $T_i(t)$ on a Hilbert space V_i.*

Proof. Since, for $i=1,2$, the domain $D(A_i)$ of the operator A_i is dense in V_i, it is sufficient to show that $D(A_1)\times D(A_2)\subset K_b(\mathbf{f})$, that is, for any $\varepsilon>0$ and $\eta=(\eta_1,\eta_2)\in D(A_1)\times D(A_2)$, there exists a $\mathbf{u}_\varepsilon(\cdot)=(u_{1,\varepsilon}(\cdot),u_{2,\varepsilon}(\cdot))\in U=U_1\times U_2$ such that, for $i=1,2$,

$$\|\eta_i-b^{\alpha-1}T_{i,\alpha}(b)x_{i,0}-\mathscr{G}_i f_{i,\varepsilon}-\mathscr{G}_i B_i u_{i,\varepsilon}\|_{V_i}<\varepsilon, \tag{4.4}$$

where $\mathbf{x}_\varepsilon=(x_{1,\varepsilon},x_{2,\varepsilon})$ with $\mathbf{x}_\varepsilon(t):=\mathbf{x}(t;0,\mathbf{x}_0,\mathbf{u}_\varepsilon),t\in[0,b]$ and $\mathbf{f}_\varepsilon=(f_{1,\varepsilon},f_{2,\varepsilon})\in\mathscr{N}_1(\mathbf{x}_\varepsilon)\times\mathscr{N}_2(\mathbf{x}_\varepsilon)$. First, for any $\mathbf{x}_0=(x_{1,0},x_{2,0})\in V=V_1\times V_2$, we know that $b^{\alpha-1}T_{i,\alpha}(b)x_{i,0}\in D(A_i)$ because $T_i(t)$ is differentiable for $i=1,2$. Therefore, for any given $\eta=(\eta_1,\eta_2)\in D(A_1)\times D(A_2)$, it can be seen that there exists a function $\varphi(\cdot)=(\varphi_1(\cdot),\varphi_2(\cdot))\in L^p(I,V_1)\times L^p(I,V_2)$ such that, for $i=1,2$, $\mathscr{G}_i\varphi_i=\eta_i-b^{\alpha-1}T_{i,\alpha}(b)x_{i,0}$, for example,

$$\varphi_i(t)=\frac{[\Gamma(\alpha)]^2(b-t)^{1-\alpha}}{b}[T_{i,\alpha}(b-t)-2t\frac{d\,T_{i,\alpha}(b-t)}{dt}][\eta_i-b^{\alpha-1}T_{i,\alpha}(b)x_{i,0}],\ t\in(0,b).$$

Next, we prove that we can obtain a control function $\mathbf{u}_\varepsilon(\cdot)=(u_{1,\varepsilon}(\cdot),u_{2,\varepsilon}(\cdot))\in U=U_1\times U_2$ such that inequality (4.4) holds. In fact, for any given $\varepsilon>0$ and $\mathbf{u}_1(\cdot)=(u_{1,1}(\cdot),u_{2,1}(\cdot))\in U$, we see that from hypothesis (HA) that there exists a $\mathbf{u}_2(\cdot)=(u_{1,2}(\cdot),u_{2,2}(\cdot))\in U$ such that, for $i=1,2$,

$$\|\eta_i-b^{\alpha-1}T_{i,\alpha}(b)x_{i,0}-\mathscr{G}_i f_{i,1}-\mathscr{G}_i B_i u_{i,2}\|_{V_i}<\frac{\varepsilon}{2^2},$$

where $\mathbf{x}_1(t):=\mathbf{x}(t;0,\mathbf{x}_0,\mathbf{u}_1),t\in[0,b]$ and $\mathbf{f}_1=(f_{1,1},f_{2,1})\in\mathscr{N}_1(\mathbf{x}_1)\times\mathscr{N}_2(\mathbf{x}_1)$. Denote $\mathbf{x}_2(t):=\mathbf{x}(t;0,\mathbf{x}_0,\mathbf{u}_2),t\in[0,b]$. By hypothesis (HA) again, one sees that there exists $\mathbf{w}_2(\cdot)=(w_{1,2}(\cdot),w_{2,2}(\cdot))\in U$ such that

$$\|\mathscr{G}_i[f_{i,2}-f_{i,1}]-\mathscr{G}_i B_i w_{i,2}\|_{V_i}<\frac{\varepsilon}{2^3},$$

where $\mathbf{f}_2=(f_{1,2},f_{2,2})\in\mathscr{N}_1(\mathbf{x}_2)\times\mathscr{N}_2(\mathbf{x}_2)$, and

$$\begin{aligned}\|B_i w_{i,2}(\cdot)\|_{L^p(I,V_i)} &\leq N_i\|f_{i,2}(\cdot)-f_{i,1}(\cdot)\|_{V_i}\\ &\leq N_iL_i't^{1-\alpha}\|x_{i,2}(\cdot)-x_{i,1}(\cdot)\|_{V_i}\\ &\leq \tfrac{M_iL_i'N_i}{\Gamma(\alpha)}(\tfrac{p-1}{p\alpha-1})^{\frac{p-1}{p}}b^{1-\frac{1}{p}}E_\alpha(M_iL_i'b)\|B_iu_{i,1}(\cdot)-B_iu_{i,2}(\cdot)\|_{L^p(I,V_i)}.\end{aligned}$$

Now, we define $\mathbf{u}_3(t)=\mathbf{u}_2(t)-\mathbf{w}_2(t)$, $\mathbf{u}_3(\cdot)\in U$. It follows that, for $i=1,2$,

$$\begin{aligned}&\|\eta_i-b^{\alpha-1}T_{i,\alpha}(b)x_{i,0}-\mathscr{G}_i f_{i,2}-\mathscr{G}_i B_i u_{i,3}\|_{V_i}\\ &\leq\|\eta_i-b^{\alpha-1}T_{i,\alpha}(b)x_{i,0}-\mathscr{G}_i f_{i,1}-\mathscr{G}_i B_i u_{i,2}\|_{V_i}+\|\mathscr{G}_i B_i w_{i,2}-[\mathscr{G}_i f_{i,2}-\mathscr{G}_i f_{i,1}]\|_{V_i}\\ &\leq(\tfrac{1}{2^2}+\tfrac{1}{2^3})\varepsilon.\end{aligned}$$

By induction, we can obtain the sequence $\{\mathbf{u}_n(\cdot)\} \subset U$, which follows that, for $i = 1,2$,

$$\|\eta_i - b^{\alpha-1}T_{i,\alpha}(b)x_{i,0} - \mathscr{G}_i f_{i,n} - \mathscr{G}_i B_i u_{i,n+1}\|_{V_i} < (\frac{1}{2^2} + \cdots + \frac{1}{2^n})\varepsilon,$$

where $\mathbf{x}_n(\cdot) := \mathbf{x}(\cdot;0,\mathbf{x}_0,\mathbf{u}_n), t \in [0,b]$ and $\mathbf{f}_n = (f_{1,n}, f_{2,n}) \in \mathscr{N}_1(\mathbf{x}_n) \times \mathscr{N}_2(\mathbf{x}_n)$, and

$$\|B_i u_{i,n+1} - B_i u_{i,n}\|_{L^p(I,V_i)} < \frac{M_i L_i' N_i}{\Gamma(\alpha)}(\frac{p-1}{p\alpha-1})^{\frac{p-1}{p}} b^{1-\frac{1}{p}} E_\alpha(M_i L_i' b)\|B_i u_{i,n}(\cdot) - B_i u_{i,n-1}(\cdot)\|_{L^p(I,V_i)}.$$

From (4.2), we know that, for $i = 1,2$, $\{B_i u_{i,n}\}_{n=1}^{\infty}$ is a Cauchy sequence in the Banach space $L^p(I,V_i)$. Therefore, for $i = 1,2$, there exists a $\psi_i(\cdot) \in L^p(I,V_i)$ such that $\lim_{n\to\infty} B_i u_{i,n}(\cdot) = \psi_i(\cdot)$ in $L^p(I,V_i)$. Then, for any $\varepsilon > 0$, there exists a positive integer $N \geq 1$ such that, for $i = 1,2$, $\|\mathscr{G}_i B_i u_{i,N+1} - \mathscr{G}_i B_i u_{i,N}\|_{V_i} < \frac{\varepsilon}{2}$. For $i = 1,2$,

$$\begin{aligned}
&\|\eta_i - b^{\alpha-1}T_{i,\alpha}(b)x_{i,0} - \mathscr{G}_i f_{i,N} - \mathscr{G}_i B_i u_{i,N}\|_{V_i} \\
&\leq \|\eta_i - b^{\alpha-1}T_{i,\alpha}(b)x_{i,0} - \mathscr{G}_i f_{i,N} - \mathscr{G}_i B_i u_{i,N+1}\|_{V_i} + \|\mathscr{G}_i B_i u_{i,N+1} - \mathscr{G}_i B_i u_{i,N}\|_{V_i} \\
&\leq (\tfrac{1}{2^2} + \cdots + \tfrac{1}{2^n})\varepsilon + \tfrac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}$$

This proves the approximate controllability of system (1.2). □

Remark 4.1. Liu-Li's result [8, Theorem 4.3] for the approximate controllability of the fractional evolution differential equation (1.1) with Riemann-Liouville fractional derivative is extended to develop our Theorem 4.1 for the approximate controllability of the system of fractional evolution hemivariational inequalities with Riemann-Liouville fractional derivatives under some similar conditions to those in [8, Theorem 4.3], for example, the conditions (HJ3), (HJ5′), and (HA) are similar to the ones H(2), H(3′) and H(4) in [8, Theorem 4.3], respectively.

5. An Example

As an application of our main results, we consider the following control system described by the system of fractional parabolic hemivariational inequalities with Riemann-Liouville fractional derivatives:

$$\begin{cases}
\langle -D_t^{\frac{2}{3}} x_1(t,y) + \frac{\partial^2}{\partial y^2} x_1(t,y) + B_1 u_1(t), v_1\rangle_{V_1} + J_1^\circ(t, x_1(t,y), x_2(t,y); v_1) \geq 0, \\
\quad t \in I = [0,1],\ y \in [0,\pi],\ \forall v_1 \in V_1 = L^2[0,\pi], \\
\langle -D_t^{\frac{2}{3}} x_2(t,y) + \frac{\partial^2}{\partial y^2} x_2(t,y) + B_2 u_2(t), v_2\rangle_{V_2} + J_2^\circ(t, x_1(t,y), x_2(t,y); v_2) \geq 0, \\
\quad t \in I = [0,1],\ y \in [0,\pi],\ \forall v_2 \in V_2 = L^2[0,\pi], \\
x_i(t,0) = x_i(t,\pi) = 0, \quad t \in I = [0,1],\ i = 1,2, \\
I_{0^+}^{1-\alpha} x_i(t,y)|_{t=0} = x_{i,0}(y), \quad y \in [0,\pi],\ i = 1,2,
\end{cases} \tag{5.1}$$

where, for $i,j = 1,2$ and $i \neq j$, $J_i^\circ(t,\theta_1,\theta_2;v_i)$ denotes the partial Clark's generalized directional derivative (see [7]) of a locally Lipschitz $J(t,\cdot,\cdot) : V_1 \times V_2 \to \mathbf{R}$ w.r.t. the i-th variable at the point $\theta_i \in V_i$ in the direction $v_i \in V_i$ for the given $\theta_j \in V_j$.

For $i = 1,2$, take $V_i = H_i = L^2[0,\pi]$ and the operator $A_i : D(A_i) \subset V_i \to V_i$ is defined by $A_i x_i = x_i''$, where the domain $D(A_i)$ is given by

$$D(A_i) := \{x_i \in V_i : x_i, x_i' \text{ are absolutely continuous},\ x_i'' \in V_i,\ x_i(0) = x_i(\pi) = 0\}.$$

Then, for $i = 1,2$, A_i can be written as $A_i x_i = -\sum_{n=1}^{\infty} n^2 \langle x_i, x_n\rangle x_n$, $x_i \in D(A_i)$, where $x_n(y) = \sqrt{2/\pi}\sin(ny)$ $(n = 1,2,\ldots)$ is an orthonormal basis of V_i. It is well known that, for $i = 1,2$,

A_i is the infinitesimal generator of a differentiable semigroup $T_i(t)$ $(t>0)$ in V_i given by $T_i(t)x_i=\sum_{n=1}^{\infty}e^{-n^2t}\langle x_i,x_n\rangle x_n$, $x_i\in V_i$, and $\|T_i(t)\|\le e^{-1}<1=M_i$. Moreover, for $i=1,2$ and any $u_i(\cdot)\in U_i=L^2(I,H_i)$, we have $u_i(t)=\sum_{n=1}^{\infty}u_{i,n}(t)x_n$, $u_{i,n}(t)=\langle u_i(t),x_n\rangle$. For $i=1,2$, define the operator B_i as $B_iu_i(t)=\sum_{n=1}^{\infty}\overline{u}_{i,n}(t)x_n$, where

$$\overline{u}_{i,n}(t)=\begin{cases}0, & 0\le t<1-\frac{1}{n^2},\\ u_{i,n}(t), & 1-\frac{1}{n^2}\le t\le 1,\end{cases}\qquad n=1,2,\ldots$$

Then, one can easily obtain that $\|B_iu_i(\cdot)\|_{L^2(I,V_i)}\le\|u_i(\cdot)\|_{U_i}$, which implies that $B_i\in L(U_i,L^2(I,V_i))$. First, by the definition of the operators $B_i, i=1,2$, the corresponding linear system of (5.1) is as follows

$$\begin{cases}D_t^{\frac{2}{3}}x_{1,n}(t)+n^2x_{1,n}(t)=\widehat{u}_{1,n}(t),\ 1-\frac{1}{n^2}<t<1,\\ D_t^{\frac{2}{3}}x_{2,n}(t)+n^2x_{2,n}(t)=\widehat{u}_{2,n}(t),\ 1-\frac{1}{n^2}<t<1,\\ I_{0^+}^{1-\alpha}x_{i,n}(t)|_{t=0}=x_{i,0}\in V_i,\quad i=1,2.\end{cases}$$

Next, we check that hypothesis (HA) is satisfied. To check these, let us denote, for $i=1,2$,

$$h_i=\int_0^1(1-s)^{-\frac{1}{3}}T_{i,\frac{2}{3}}(1-s)g_i(s)ds=\sum_{n=1}^{\infty}h_{i,n}x_n,\ h_{i,n}=\langle h_i,x_n\rangle,\ \text{for every } g_i(\cdot)\in L^2(I,V_i).$$

In fact, for $i=1,2$, we can choose $\widetilde{u}_{i,n}(t)$, which follows from

$$\widetilde{u}_{i,n}(t)=\frac{2n^2}{1-e^{-2}}h_{i,n}e^{-n^2(1-t)},\ 1-\frac{1}{n^2}\le t\le 1,$$

and

$$h_{i,n}=\int_{1-\frac{1}{n^2}}^1\int_0^{\infty}(1-t)^{-\frac{1}{3}}\theta\xi_{\frac{2}{3}}(\theta)e^{-n^2\theta(1-t)^{\frac{2}{3}}}\widetilde{u}_{i,n}(t)d\theta dt.$$

Based on this, we define, for $i=1,2$, $u_i(t)=\sum_{n=1}^{\infty}u_{i,n}(t)x_n$, where

$$u_{i,n}(t)=\begin{cases}0, & 0\le t<1-\frac{1}{n^2},\\ \widetilde{u}_{i,n}(t), & 1-\frac{1}{n^2}\le t\le 1,\end{cases}\qquad n=1,2,\ldots.$$

Therefore, for $i=1,2$ and any given function $g_i(\cdot)\in L^2([0,1],V_i)$, there exists $u_i(\cdot)\in U_i$ such that

$$\int_0^1(1-s)^{-\frac{1}{3}}T_{i,\frac{2}{3}}(1-s)B_iu_i(s)ds=\int_0^1(1-s)^{-\frac{1}{3}}T_{i,\frac{2}{3}}(1-s)g_i(s)ds,$$

which implies that condition (4.1) of (HA) is satisfied. Moreover, for $i=1,2$, we obtain

$$\begin{aligned}\|B_iu_i(\cdot)\|^2_{L^2(I,V_i)}&=\sum_{n=1}^{\infty}\int_{1-\frac{1}{n^2}}^1|\widetilde{u}_{i,n}(t)|^2dt\\ &=(1-e^{-2})^{-1}\sum_{n=1}^{\infty}2n^2h_{i,n}^2\\ &=\tfrac{3}{2}(1-e^{-2})^{-1}\sum_{n=1}^{\infty}(1-e^{-2n^2})\int_0^1|g_{i,n}(t)|^2dt\\ &\le\tfrac{3}{2}(1-e^{-2})^{-1}|g_i(\cdot)|^2.\end{aligned}$$

Hence, it can be seen that conditions (HA) are satisfied, and then system (5.1) is approximately controllable on I, if

$$\frac{3\sqrt{3}}{\Gamma(2\frac{2}{3})}(1-e^{-2})^{-1}L_i'E_{\frac{2}{3}}(L_i') < 1$$

holds for $i = 1,2$.

6. Concluding Remark

This paper is concerned with the existence of mild solutions and complete controllability for control systems described by a class of systems of fractional stochastic evolution hemivariational inequalities in Hilbert spaces. Firstly, we introduced the concept of mild solutions for the systems of hemivariational inequalities. Then the existence results were obtained, and the controllability was formulated and proved by using stochastic analysis techniques, fractional calculation, the fixed point theorem of multivalued maps, and the properties of partial Clarke's generalized subdifferential. It is known that the approximate controllability problem is one of the most fundamental issues in the field of control engineering; see e.g., [2, 3, 6, 8, 14, 15, 16, 18] for more details. In the forthcoming paper, we shall consider approximate controllability results for systems of fractional stochastic evolution hemivariational inequalities with nonlocal initial conditions in the Hilbert spaces.

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