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VISCOELASTIC ELLIPTIC MEMBRANE SHELLS ON BILATERAL FRICTIONAL CONTACT: AN ASYMPTOTIC APPROACH

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Abstract. We consider a family of linearly viscoelastic shells, all sharing the same middle surface, with thickness 2ε, clamped along their entire lateral face and on frictional, and bilateral contact with an obstacle along its lower face. Friction is modeled with a Tresca condition and tractions may act on the upper face of the shell. We prove that, if the shell is an elliptic membrane, the solution of the three-dimensional scaled variational contact problem, in curvilinear coordinates, $u(\varepsilon)$, converges to a limit function, u, which is independent of the transverse variable and can be identified with the solution of a limit two-dimensional variational problem, describing tangential deformations of the middle surface, and giving us a two-dimensional model (obstacle problem) for viscoelastic shells with bilateral frictional contact.
Keywords. Asymptotic analysis; Contact; Elliptic membranes; Friction; Viscoelasticity.

1. Introduction

The asymptotic expansion method, whose foundations can be found in [1], has been successfully applied in the mathematical justification of reduced models in solid mechanics, and particularly, for shells. The seminal works [2, 3] were followed by a series of papers which helped to configure a complete theory for the mathematical justification and analysis of models for elastic shells, as it is beautifully compiled in [4]. There, models for elliptic membranes, generalized membranes and flexural shells are presented. The interested reader will be delighted to find a full detailed description of the asymptotic methods that lead to the justification of all the possible kinds of limit two-dimensional equations for these structures, cast into the realm of the theory of mathematical elasticity.

These references above set the fundamentals of the asymptotic analysis applied to shell-type equations, but they are limited to elasticity, while many applications in real life and industrial processes require to take into account effects such as hardening and memory of the material,

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subjects that can be addressed by using constitutive equations in the field of viscoelasticity (see, for example [5, 6]). Particularly remarkable is the increasing number of studies of viscoelastic shells problems in order to reproduce the complex behaviour of tissues in the field of biomedicine. For example, in [7] the difficulties of this kind of problems are detailed and even though an one-dimensional model is derived for modelling a vessel wall, the author comments the possibility of considering two-dimensional models with a shell-type description and a viscoelastic constitutive law. In this direction, to our knowledge, in [8] we gave the first steps towards the justification of existing models of viscoelastic shells and the finding of new ones. By using the asymptotic expansion method, we found a rich variety of cases, depending on the geometry of the middle surface, the boundary conditions and the order of the applied forces. Convergence results for the different kinds of viscoelastic shells followed in [9, 10, 11].

Nevertheless, none of the papers referenced above take contact conditions into consideration, a void which is starting to be dealt with in the recent years, as we shall address below. In solid mechanics, problems with contact conditions have a wide variety of applications in real life, so this subject has drawn the attention of many researchers throughout the years. One can find a growing number of references which deal with the modeling, mathematical analysis and numerical approximation of contact problems, particularly, with the study of the variational inequalities associated to them (see for example [5, 12, 13, 14, 15] and the references therein).

Nowadays, the mathematical theory of contact mechanics is a well established discipline and many authors have contributed to the knowledge of this sort of problems, providing models and results. A considerable share of the research in the field combine viscoelasticity and contact conditions. Take for example some references in the literature as [16, 17, 18, 19, 20, 21, 22] and check references therein.

Nevertheless, regarding the mathematical analysis of contact problems for viscoelastic shells, there exists a very limited amount of results available. We can reference [23], where the authors present a model for a dynamic contact problem where a short memory (Kelvin-Voigt) material is considered.

Our aim in this work is to provide the mathematical justification of a two-dimensional limit problem for viscoelastic membrane shells in frictional contact with a foundation. It follows a series of papers with which we are contributing to fill the gap in the asymptotic analysis of contact problems for shells. In particular, in [24] the case of an elastic shell in frictionless unilateral contact with a rigid foundation was considered, finding a classification of the two dimensional limit problems as obstacle problems for membranes or flexural shells, which is just a natural extension of what we find in [4] for plain elasticity. The convergence result for the elliptic membrane case is given in [25] and error estimates were provided in [26]. Similar results were obtained by other authors (see for example, [27], were a confinement condition is employed), thus confirming the growing attention of the specialty. Besides, in [28], a convergence result for elastic elliptic membrane shells under normal compliance contact conditions is presented and in [29] and [30] coupled problems for thermoviscoelastic shells in frictionless contact were analyzed. From the mathematical point of view, the main new challenges in comparison with our previous works are related with the frictional conditions, due to the weak convergence of sequences on boundary terms (where the frictional conditions are defined).

So, with this paper, we apply, for the first time to our knowledge, the asymptotic method to a frictional contact problem for elliptic membrane shells in viscoelasticity. To be more specific, the shell is clamped on the lateral face, contact is bilateral on the lower face of the shell and friction is modeled with a Tresca's friction bound (see for example [18]). We will justify that, when the thickness of the shell (the small parameter ε) tends to zero, the right limit variational problem is in the form of a two-dimensional variational inequality, describing the tangential displacement of a viscoelastic elliptic membrane shell over an obstacle, keeping contact on the lower face while being clamped on the lateral face. Notice that displacements normal to the middle surface of the shell will be prevented in the limit problem, due to the influence of the three-dimensional bilateral contact condition. Other choices can be made both for the contact and friction modelizations, thus leading to a vast yet unexplored field of mathematical problems, each one of them with its own mathematical challenges and with useful applications in real life situations. The present paper will serve of basis for future research in this direction.

We will follow the notation and style of [4], where the linear elastic shells are studied. For this reason, we shall reference auxiliary results which apply in the same manner to the viscoelastic bilateral, frictional case.

The structure of the paper is the following. In Section 2 we shall recall the three-dimensional viscoelastic contact problem in Cartesian coordinates and then, considering the problem for a family of viscoelastic shells of thickness 2ε, we formulate the so-called scaled variational problem, in curvilinear coordinates. In Section 3, we show the convergence results when the small parameter ε tends to zero, which is the main result of this paper. In Section 4, we present some conclusions and future work.

2. The Three-Dimensional Linearly Viscoelastic Shell Problem

We denote $\mathbb{S}^d$, where $d = 2,3$ in practice, the space of second-order symmetric tensors on $\mathbb{R}^d$, while " $\cdot$ " represents the inner product and $|\cdot|$ the usual norm in $\mathbb{S}^d$ and $\mathbb{R}^d$. In what follows, unless the contrary is explicitly written, we use summation convention on repeated indices. Moreover, Latin indices $i, j, k, l, \ldots$, take their values in the set $\{1,2,3\}$, whereas Greek indices $\alpha, \beta, \sigma, \tau, \ldots$, do it in the set $\{1,2\}$. Also, we use the standard notation for the Lebesgue and Sobolev spaces. For a time dependent function u, we denote $\dot{u}$ the first derivative of u with respect to the time variable. Recall that " $\to$ " denotes strong convergence, while " $\rightharpoonup$ " denotes weak convergence.

Let Ω^* be a domain of $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\Gamma^* = \partial\Omega^*$. Let $x^* = (x_i^*)$ be a generic point of its closure $\bar{\Omega}^*$, and let ∂_i^* denote the partial derivative with respect to x_i^*. Let dx^* denote the volume element in Ω^*, $d\Gamma^*$ denote the area element along Γ^* ,and n^* denote the unit outer normal vector along Γ^*. Finally, let Γ^* be divided into Γ_0^*, Γ_N^*, and Γ_C^*, disjoint subsets, with the condition that $meas(\Gamma_0^*) > 0$.

The set Ω^* is the region occupied by a deformable body in the absence of applied forces. We assume that this body is made of a Kelvin-Voigt viscoelastic material, which is homogeneous and isotropic, so that the material is characterized by its Lamé coefficients $\lambda \geq 0, \mu > 0$, and its viscosity coefficients, $\theta \geq 0, \rho \geq 0$ (see, e.g., [5, 15, 31]).

Let $T > 0$ be the time period of observation. The body is in bilateral, frictional contact with an obstacle on Γ_C^*, modeled with a Tresca condition. Also, the body in clamped on Γ_0^*. Under the effect of applied forces, the body is deformed and we denote by $u_i^* : [0,T] \times \bar{\Omega}^* \to \mathbb{R}^3$ the Cartesian components of the displacements field, defined as $u^* := u_i^* e^i : [0,T] \times \bar{\Omega}^* \to \mathbb{R}^3$, where $\{e^i\}$ denotes the Euclidean canonical basis in $\mathbb{R}^3$. Hence, the displacements field $u^* =$

$(u_i^*) : [0,T] \times \Omega^* \longrightarrow \mathbb{R}^3$ is a solution to the following three-dimensional problem in Cartesian coordinates.

Problem 2.1. Find $u^* = (u_i^*) : [0,T] \times \Omega^* \longrightarrow \mathbb{R}^3$ such that

$$-\partial_j^* \sigma^{ij,*}(u^*) = f^{i,*} \ \text{ in } (0,T) \times \Omega^*, \tag{2.1}$$

$$u_i^* = 0 \ \text{ on } (0,T) \times \Gamma_0^*, \tag{2.2}$$

$$\sigma^{ij,*}(u^*) n_j^* = h^{i,*} \ \text{ on } (0,T) \times \Gamma_N^*, \tag{2.3}$$

$$\left. \begin{array}{l} u_n^* = 0, \\ |\sigma_\tau^*| \leq g_T^*, \left\{ \begin{array}{l} |\sigma_\tau^*| < g_T^* \Rightarrow \dot{u}_\tau^* = 0, \\ |\sigma_\tau^*| = g_T^* \Rightarrow \sigma_\tau^* = -g_T^* \frac{\dot{u}_\tau^*}{|\dot{u}_\tau^*|}, \end{array} \right. \end{array} \right\} \text{ on } (0,T) \times \Gamma_C^*, \tag{2.4}$$

$$u^*(0,\cdot) = u_0^* \ \text{ in } \Omega^*, \tag{2.5}$$

where $\sigma^{ij,*}(u^*) := A^{ijkl,*} e_{kl}^*(u^*) + B^{ijkl,*} e_{kl}^*(\dot{u}^*)$, are the components of the linearized stress tensor field. The functions

$$A^{ijkl,*} := \lambda \delta^{ij} \delta^{kl} + \mu \left(\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right), \quad B^{ijkl,*} := \theta \delta^{ij} \delta^{kl} + \frac{\rho}{2} \left(\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right),$$

are the components of the three-dimensional elasticity and viscosity fourth order tensors, respectively, and $e_{ij}^*(u^*) := \frac{1}{2}(\partial_j^* u_i^* + \partial_i^* u_j^*)$, designate the components of the linearized strain tensor associated with the displacement field u^* of the set $\bar{\Omega}^*$. Moreover, $u_n^* = u^* \cdot n^*$ is the normal displacement and $u_\tau^* = u^* - u_n^* n^*$ is the tangential displacement, while $\sigma_n^* = (\sigma^* n^*) \cdot n^*$ stands for the normal stress and $\sigma_\tau^* = \sigma^* n^* - \sigma_n^* n^*$ for the tangential stress.

We now proceed to describe the equations in Problem 2.1. Expression (2.1) is the equilibrium equation, where $f^{i,*}$ are the components of the body force densities. Equality (2.2) is the Dirichlet condition of place, while (2.3) is the Neumann condition, where $h^{i,*}$ are the components of surface force densities. The expressions in (2.4) are the contact conditions. First, the contact with the obstacle is bilateral, so the normal component of the displacement is zero on that part of the boundary. Second, friction is modeled with a Tresca condition, so the norm of the tangential stress has an upper bound, $g_T^* > 0$. Further, when this bound is reached, the tangential stress is in the opposite direction of the tangential velocity. Finally, (2.5) is the initial condition, where u_0^* denotes the initial displacements.

Note that, for the sake of briefness, we omit the explicit dependence on the space and time variables when there is no ambiguity. Let us define the space of admissible unknowns

$$V(\Omega^*) = \{v^* = (v_i^*) \in [H^1(\Omega^*)]^3; v^* = 0 \text{ on } \Gamma_0^*,\ v_n^* = 0 \text{ on } \Gamma_C^*\}.$$

Therefore, assuming enough regularity, the unknown $u^* = (u_i^*)$ satisfies the following variational problem in Cartesian coordinates (see [18]).

Problem 2.2. Find $u^* = (u_i^*) : [0,T] \times \Omega^* \to \mathbb{R}^3$ such that

$$u^*(t,\cdot) \in V(\Omega^*)\ \forall\, t \in [0,T], \quad u^*(0,\cdot) = u_0^*(\cdot),$$

$$\int_{\Omega^*} A^{ijkl,*} e_{kl}^*(u^*) e_{ij}^*(v^* - \dot{u}^*) dx^* + \int_{\Omega^*} B^{ijkl,*} e_{kl}^*(\dot{u}^*) e_{ij}^*(v^* - \dot{u}^*) dx^* + \int_{\Gamma_C^*} g_T^* (|v_\tau^*| - |\dot{u}_\tau^*|) d\Gamma^*$$

$$\geq \int_{\Omega^*} f^{i,*}(v_i^* - \dot{u}_i^*) dx^* + \int_{\Gamma_N^*} h^{i,*}(v_i^* - \dot{u}_i^*) d\Gamma^* \quad \forall v^* \in V(\Omega^*),\ a.e. \text{ in } (0,T).$$

Let us now consider that Ω^* is a viscoelastic shell of thickness 2ε. Therefore, we shall express the equations of the Problem 2.2 in terms of curvilinear coordinates. Let ω be a domain of $\mathbb{R}^2$ with a Lipschitz-continuous boundary $\gamma = \partial\omega$. Let $y = (y_\alpha)$ be a generic point of its closure $\bar{\omega}$, and let ∂_α denote the partial derivative with respect to y_α.

Let $\theta \in \mathscr{C}^2(\bar{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $a_\alpha(y) := \partial_\alpha\theta(y)$ are linearly independent. These vectors form the covariant basis of the tangent plane to the surface $S := \theta(\bar{\omega})$ at the point $\theta(y) = y^*$. We assume that the surface S is uniformly elliptic. Further, consider the two vectors $a^\alpha(y)$ of the same tangent plane defined by the relations $a^\alpha(y)\cdot a_\beta(y) = \delta^\alpha_\beta$, that constitute the contravariant basis. The normalized vector product of the two vectors in that basis, denoted $a_3(y) = a^3(y)$, is the normal vector to S at the point $\theta(y) = y^*$.

We can define the first fundamental form, given as metric tensor, in covariant or contravariant components, respectively, by $a_{\alpha\beta} := a_\alpha\cdot a_\beta$, $a^{\alpha\beta} := a^\alpha\cdot a^\beta$, the second fundamental form, given as curvature tensor, in covariant or mixed components, respectively, by $b_{\alpha\beta} := a^3\cdot\partial_\beta a_\alpha$, $b^\beta_\alpha := a^{\beta\sigma}b_{\sigma\alpha}$, and the Christoffel symbols of the surface S by $\Gamma^\sigma_{\alpha\beta} := a^\sigma\cdot\partial_\beta a_\alpha$. The area element along S is $\sqrt{a}dy = dy^*$, where $a := \det(a_{\alpha\beta})$.

Next, for each $\varepsilon > 0$, we define the three-dimensional domain $\Omega^\varepsilon := \omega\times(-\varepsilon,\varepsilon)$ and its boundary $\Gamma^\varepsilon = \partial\Omega^\varepsilon$. We also define the following parts of the boundary $\Gamma^\varepsilon_N := \omega\times\{\varepsilon\}$, $\Gamma^\varepsilon_C := \omega\times\{-\varepsilon\}$, and $\Gamma^\varepsilon_0 := \gamma\times[-\varepsilon,\varepsilon]$. Let $x^\varepsilon = (x^\varepsilon_i)$ be a generic point of $\bar{\Omega}^\varepsilon$, and let ∂^ε_i denote the partial derivative with respect to x^ε_i. Note that $x^\varepsilon_\alpha = y_\alpha$ and $\partial^\varepsilon_\alpha = \partial_\alpha$. Let $\Theta : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ be the mapping defined by

$$\Theta(x^\varepsilon) := \theta(y) + x^\varepsilon_3 a_3(y),\ \forall x^\varepsilon = (y,x^\varepsilon_3) = (y_1,y_2,x^\varepsilon_3) \in \bar{\Omega}^\varepsilon. \tag{2.6}$$

In [4, Theorem 3.1-1], it was shown that if the injective mapping $\theta : \bar{\omega} \to \mathbb{R}^3$ is smooth enough, then $\Theta : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ is also injective for $\varepsilon > 0$ small enough (say $\varepsilon \le \varepsilon_0$).

For each ε, $0 < \varepsilon \le \varepsilon_0$, the set $\bar{\Omega}^* = \Theta(\bar{\Omega}^\varepsilon)$ is the reference configuration of a viscoelastic shell with middle surface $S = \theta(\bar{\omega})$ and thickness $2\varepsilon > 0$. Furthermore for $\varepsilon > 0$, $g^\varepsilon_i(x^\varepsilon) := \partial^\varepsilon_i\Theta(x^\varepsilon)$ are linearly independent. Hence, the three vectors $g^\varepsilon_i(x^\varepsilon)$ form the covariant basis of the tangent space at the point $x^* = \Theta(x^\varepsilon)$ and $g^{i,\varepsilon}(x^\varepsilon)$ defined by the relations $g^{i,\varepsilon}\cdot g^\varepsilon_j = \delta^i_j$ form the contravariant basis at the same point. We define the metric tensor, in covariant or contravariant components, respectively, by $g^\varepsilon_{ij} := g^\varepsilon_i\cdot g^\varepsilon_j$, $g^{ij,\varepsilon} := g^{i,\varepsilon}\cdot g^{j,\varepsilon}$, and Christoffel symbols by $\Gamma^{p,\varepsilon}_{ij} := g^{p,\varepsilon}\cdot\partial^\varepsilon_i g^\varepsilon_j$. The volume element in the set $\Theta(\bar{\Omega}^\varepsilon) = \bar{\Omega}^*$ is $\sqrt{g^\varepsilon}dx^\varepsilon = dx^*$, and the surface element in $\Theta(\Gamma^\varepsilon) = \Gamma^*$ is $\sqrt{g^\varepsilon}d\Gamma^\varepsilon = d\Gamma^*$, where $g^\varepsilon := \det(g^\varepsilon_{ij})$.

Next, let $n^\varepsilon(x^\varepsilon)$ denote the unit outward normal vector on $x^\varepsilon \in \Gamma^\varepsilon_C$ and $n^*(x^*)$ the unit outward normal vector on $x^* = \Theta(x^\varepsilon) \in \Theta(\Gamma^\varepsilon_C)$.

We can show (see [24]) that

$$n^*(x^*) = -g^\varepsilon_3(x^\varepsilon) = -a_3(y),\ \text{where } x^* = \Theta(x^\varepsilon),\ \text{and } x^\varepsilon = (y,-\varepsilon) \in \Gamma^\varepsilon_C. \tag{2.7}$$

Now, for a vector field v^* defined in $\Theta(\bar{\Omega}^\varepsilon)$, where the cartesian basis is denoted by $\{e^i\}^3_{i=1}$, we define its covariant curvilinear coordinates (v^ε_i) in $\bar{\Omega}^\varepsilon$ as follows:

$$v^*(x^*) = v^*_i(x^*)e^i =: v^\varepsilon_i(x^\varepsilon)g^{i,\varepsilon}(x^\varepsilon),\ \text{with } x^* = \Theta(x^\varepsilon). \tag{2.8}$$

Therefore, by combining (2.7) and (2.8), it can be shown that, on Γ^ε_C,

$$v^*_n := v^*\cdot n^* = (v^*_i n^{i,*}) = (v^*_i e^i)\cdot(-g_3) = (v^\varepsilon_i g^{i,\varepsilon})\cdot(-g^\varepsilon_3) = -v^\varepsilon_3,$$

where here and below it is assumed that each function with * or $^\varepsilon$ superindices are defined on x^* or x^ε, respectively, while having in mind that $x^* = \Theta(x^\varepsilon)$.

We deduce that $v_3^\varepsilon = 0$ on Γ_C^ε is the bilateral condition in curvilinear coordinates. With respect to the friction condition, we note that

$$v_\tau^* = v^* - v_n^* n^* = v_i^* e^i - v_n^* n^* = v_i^\varepsilon g^{i,\varepsilon} - (-v_3^\varepsilon)(-g^{3,\varepsilon}) = v_\alpha^\varepsilon g^{\alpha,\varepsilon}.$$

Thus we define $v_\tau^\varepsilon := v_\alpha^\varepsilon g^{\alpha,\varepsilon}$ the tangential component of v^ε on Γ_C^ε.

Keeping in mind that the true displacements field is $\mathscr{U}^\varepsilon := u_i^\varepsilon g^{i,\varepsilon}$, for convenience, we define the vector of unknowns $u^\varepsilon = (u_i^\varepsilon) : [0,T] \times \Omega^\varepsilon \to \mathbb{R}^3$. Following the ideas in [4], we obtain from Problem 2.2, in curvilinear coordinates, the variational problem of a three-dimensional viscoelastic shell in bilateral, frictional contact on the lower face.

We omit its explicit formulation here, and refer the reader to [32] for the details, and we pass to describe the so-called scaled version of the problem, posed in a domain independent of ε. Hence, let us define $\Omega := \omega \times (-1,1)$ and its boundary $\Gamma = \partial\Omega$. We also define the following parts of the boundary,

$$\Gamma_N := \omega \times \{1\}, \quad \Gamma_C := \omega \times \{-1\}, \quad \Gamma_0 := \gamma \times [-1,1].$$

Let $x = (x_1, x_2, x_3)$ be a generic point in $\bar\Omega$, and we consider the notation ∂_i for the partial derivative with respect to x_i. We define the following projection map

$$\pi^\varepsilon : x = (x_1,x_2,x_3) \in \bar\Omega \longrightarrow \pi^\varepsilon(x) = x^\varepsilon = (x_i^\varepsilon) = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) = (x_1, x_2, \varepsilon x_3) \in \bar\Omega^\varepsilon.$$

Hence, $\partial_\alpha^\varepsilon = \partial_\alpha$ and $\partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3$. We consider the scaled unknown $u(\varepsilon) = (u_i(\varepsilon)) : [0,T] \times \bar\Omega \longrightarrow \mathbb{R}^3$ and the scaled vector field $v = (v_i) : \bar\Omega \longrightarrow \mathbb{R}^3$ defined as

$$u_i^\varepsilon(t, x^\varepsilon) =: u_i(\varepsilon)(t,x) \text{ and } v_i^\varepsilon(x^\varepsilon) =: v_i(x) \ \forall x^\varepsilon = \pi^\varepsilon(x) \in \bar\Omega^\varepsilon, \ \forall\, t \in [0,T].$$

Also, the scaled functions $\Gamma_{ij}^p(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon), B^{ijkl}(\varepsilon)$ are defined in terms of their corresponding de-scales versions:

$$\Gamma_{ij}^p(\varepsilon)(x) := \Gamma_{ij}^{p,\varepsilon}(x^\varepsilon), \quad g(\varepsilon)(x) := g^\varepsilon(x^\varepsilon),$$

$$A^{ijkl}(\varepsilon)(x) := A^{ijkl,\varepsilon}(x^\varepsilon) = \lambda g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}), \tag{2.9}$$

$$B^{ijkl}(\varepsilon)(x) := B^{ijkl,\varepsilon}(x^\varepsilon) = \theta g^{ij,\varepsilon} g^{kl,\varepsilon} + \frac{\rho}{2}(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon}), \tag{2.10}$$

for all $x^\varepsilon = \pi^\varepsilon(x) \in \bar\Omega^\varepsilon$. In particular, some components are null (see [4] and [9]) since

$$\Gamma_{\alpha 3}^{3,\varepsilon} = \Gamma_{33}^{p,\varepsilon} = 0 \text{ in } \bar\Omega^\varepsilon, \quad A^{\alpha\beta\sigma 3,\varepsilon} = A^{\alpha 333,\varepsilon} = B^{\alpha\beta\sigma 3,\varepsilon} = B^{\alpha 333,\varepsilon} = 0 \text{ in } \bar\Omega^\varepsilon. \tag{2.11}$$

For all $v = (v_i) \in [H^1(\Omega)]^3$, let there be associated the scaled linearized strains components $e_{i||j}(\varepsilon; v) \in L^2(\Omega)$ (see [4] for the details) defined by

$$e_{\alpha||\beta}(\varepsilon; v) := \frac{1}{2}(\partial_\beta v_\alpha + \partial_\alpha v_\beta) - \Gamma_{\alpha\beta}^p(\varepsilon) v_p, \tag{2.12}$$

$$e_{\alpha||3}(\varepsilon; v) := \frac{1}{2}\left(\frac{1}{\varepsilon}\partial_3 v_\alpha + \partial_\alpha v_3\right) - \Gamma_{\alpha 3}^p(\varepsilon) v_p, \tag{2.13}$$

$$e_{3||3}(\varepsilon; v) := \frac{1}{\varepsilon}\partial_3 v_3. \tag{2.14}$$

The functions $\Gamma^p_{ij}(\varepsilon), g(\varepsilon), A^{ijkl}(\varepsilon)$, and $B^{ijkl}(\varepsilon)$ converge in $\mathscr{C}^0(\bar{\Omega})$ when ε tends to zero. In particular, in [9], we extended the results of [4, Theorem 3.3-2] to the viscoelastic case and proved that the three-dimensional elasticity and viscosity tensors are uniformly positive definite, not only with respect to $x \in \bar{\Omega}$, but also with respect to ε, $0 < \varepsilon \leq \varepsilon_0$. Moreover, their limits are functions of $y \in \bar{\omega}$ only, that is, independent of the transversal variable x_3.

Now, we introduce $u_0(\varepsilon) : \Omega \longrightarrow \mathbb{R}^3$, the initial "displacement" for the scaled setting, by

$$u_0(\varepsilon)(x) := u_0^\varepsilon(x^\varepsilon),\ \forall x \in \Omega,\ \text{where } x^\varepsilon = \pi^\varepsilon(x) \in \Omega^\varepsilon,$$

where $\mathscr{U}^0 = (u_0^\varepsilon)_i g^{i,\varepsilon}$ is the true initial displacement, in curvilinear coordinates, and define the space of admissible unknowns

$$V(\Omega) := \{v = (v_i) \in [H^1(\Omega)]^3; v = \mathbf{0} \text{ on } \Gamma_0,\ v_3 = 0 \text{ on } \Gamma_C\},$$

which is a Hilbert space with associated norm denoted by $||\cdot||_{1,\Omega}$. Regarding contact, we have the bilateral condition

$$u_3(\varepsilon)(t,x) = u_3^\varepsilon(t,x^\varepsilon) = 0,\ \forall x \in \Gamma_C,\ \text{where } x^\varepsilon = \pi^\varepsilon(x) \in \Gamma_C^\varepsilon \text{ and } \forall t \in [0,T],$$

and we scale the Tresca friction bound as $g_T(\varepsilon)(x) := g_T^\varepsilon(x^\varepsilon) = g_T^*(x^*)$, where $\pi^\varepsilon(x) = x^\varepsilon$ and $\Theta(x^\varepsilon) = x^*$. Besides, since (see [4, p. 156]), $g^{\alpha,\varepsilon} = a^\alpha + \varepsilon x_3 b^\alpha_\sigma a^\sigma + O(\varepsilon^2)$, we can calculate $|v_\tau^\varepsilon| = |v_\alpha^\varepsilon g^{\alpha,\varepsilon}| = |v_\alpha^\varepsilon(a^\alpha + \varepsilon x_3 b^\alpha_\sigma a^\sigma + O(\varepsilon^2))|$, so that $|v_\tau^\varepsilon| = (v_\alpha^\varepsilon a^{\alpha\beta} v_\beta^\varepsilon)^{\frac{1}{2}} + O(\varepsilon)$. Notice that $(a^{\alpha\beta})$ is a symmetric positive definite matrix. Similarly, $|\dot{u}_\tau^\varepsilon| = (\dot{u}_\alpha^\varepsilon a^{\alpha\beta} \dot{u}_\beta^\varepsilon)^{\frac{1}{2}} + O(\varepsilon)$. In this way,

$$|v_\tau(\varepsilon)(x)| = |v_\tau^\varepsilon(x^\varepsilon)| = (v_\alpha a^{\alpha\beta} v_\beta)^{\frac{1}{2}} + O(\varepsilon),\ \forall x \in \Gamma_C \text{ and } x^\varepsilon = \pi^\varepsilon(x) \in \Gamma_C^\varepsilon, \tag{2.15}$$

for all $v \in V(\Omega)$ and, in particular,

$$\begin{aligned} |\dot{u}_\tau(\varepsilon)(x)| &= |\dot{u}_\tau^\varepsilon(x^\varepsilon)| = (\dot{u}_\alpha(\varepsilon) g^{\alpha\beta}(\varepsilon) \dot{u}_\beta(\varepsilon))^{\frac{1}{2}} \\ &= (\dot{u}_\alpha(\varepsilon) a^{\alpha\beta} \dot{u}_\beta(\varepsilon))^{\frac{1}{2}} + O(\varepsilon),\ \forall x \in \Gamma_C \text{ and } x^\varepsilon = \pi^\varepsilon(x) \in \Gamma_C^\varepsilon. \end{aligned} \tag{2.16}$$

Further, we assume that the scaled friction bound and the scaled applied forces are given by

$$\left.\begin{array}{l} g_T(\varepsilon)(x) = \varepsilon^{p+1} g_T^{p+1}(x), \\ f(\varepsilon)(t,x) = \varepsilon^p f^p(t,x),\ \forall x \in \Omega \text{ and } \forall t \in [0,T], \\ h(\varepsilon)(t,x) = \varepsilon^{p+1} h^{p+1}(t,x),\ \forall x \in \Gamma_N \text{ and } \forall t \in [0,T], \end{array}\right\} \tag{2.17}$$

where g_T^{p+1}, f^p, and h^{p+1} are functions independent of ε, where p is a natural number that specifies the order, in terms of ε, of the Tresca's friction bound, volume densities and surface forces, respectively. In practice, we take $p = 0$ and drop the superindices. This choice is led by a formal asymptotic analysis (see [32]). Then, the scaled variational problem can be written as follows.

Problem 2.3. Find $u(\varepsilon) : [0,T] \times \Omega \longrightarrow \mathbb{R}^3$ such that

$$u(\varepsilon)(t,\cdot) \in V(\Omega)\ \forall\, t \in [0,T], \quad u(\varepsilon)(0,\cdot) = u_0(\varepsilon)(\cdot),$$

$$\int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon, u(\varepsilon)) e_{i||j}(\varepsilon, v - \dot u(\varepsilon)) \sqrt{g(\varepsilon)}dx + \int_\Omega B^{ijkl}(\varepsilon) e_{k||l}(\varepsilon, \dot u(\varepsilon)) e_{i||j}(\varepsilon, v - \dot u(\varepsilon)) \sqrt{g(\varepsilon)}dx$$
$$+ \int_{\Gamma_C} g_T(|v_\tau(\varepsilon)| - |\dot u_\tau(\varepsilon)|) \sqrt{g(\varepsilon)} d\Gamma$$
$$\geq \int_\Omega f^i (v_i - \dot u_i(\varepsilon)) \sqrt{g(\varepsilon)}dx + \int_{\Gamma_N} h^i (v_i - \dot u_i(\varepsilon)) \sqrt{g(\varepsilon)} d\Gamma, \quad \forall v \in V(\Omega),\ a.e. \text{ in } (0,T).$$

From now on, for each $\varepsilon > 0$, we use the shorter notation $e_{i||j}(\varepsilon) \equiv e_{i||j}(\varepsilon; u(\varepsilon))$ and $\dot e_{i||j}(\varepsilon) \equiv e_{i||j}(\varepsilon; \dot u(\varepsilon))$ for its time derivative. Problem 2.3 has a unique solution, and this can be shown by using the arguments of the theory of variational inequalities (see [18]).

Theorem 2.1. *Let Ω be a domain in $\mathbb{R}^3$ defined previously in this section, and let Θ be a $\mathscr{C}^2$-diffeomorphism of $\bar\Omega$ onto its image $\Theta(\bar\Omega)$ such that the three vectors $g_i = \partial_i \Theta(x)$ are linearly independent for all $x \in \bar\Omega$. Let $f^i \in L^2(0,T; L^2(\Omega))$ and $h^i \in L^2(0,T; L^2(\Gamma_N))$. Let $u_0(\varepsilon) \in V(\Omega)$. Then, there exists a unique solution $u(\varepsilon) = (u_i(\varepsilon)) : [0,T] \times \Omega \to \mathbb{R}^3$ satisfying Problem 2.3. Moreover $u(\varepsilon) \in H^1(0,T; V(\Omega))$.*

Under the previous assumptions (see [4]), there exist constants a_0, g_0 and g_1 such that

$$\begin{aligned} &0 < a_0 \leq a(y)\ \forall y \in \bar\omega, \\ &0 < g_0 \leq g(\varepsilon)(x) \leq g_1\ \forall x \in \bar\Omega \text{ and } \forall\, \varepsilon, 0 < \varepsilon \leq \varepsilon_0. \end{aligned} \tag{2.18}$$

We now introduce the average with respect to the transversal variable, which plays a major role in this study. To that end, let v represent scalar or vectorial real functions defined almost everywhere over $\Omega = \omega \times (-1,1)$. We define the transversal average as

$$\bar v(y) = \frac{1}{2} \int_{-1}^{1} v(y, x_3) dx_3, \text{ a.e. } y \in \omega.$$

Given $\eta = (\eta_i) \in [H^1(\omega)]^3$, let

$$\gamma_{\alpha\beta}(\eta) := \frac{1}{2}(\partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta) - \Gamma^\sigma_{\alpha\beta} \eta_\sigma - b_{\alpha\beta} \eta_3 \tag{2.19}$$

denote the covariant components of the linearized change of metric tensor associated with a displacement field $\eta_i a^i$ of the surface S.

In [4, Theorem 4.3-1], we find a three-dimensional inequality of Korn's type for a family of elliptic membrane shells. Indeed, there exist a constant ε_1 verifying $0 < \varepsilon_1 < \varepsilon_0$ and a constant $C > 0$ such that, for all $\varepsilon, 0 < \varepsilon \leq \varepsilon_1$, the following holds

$$\left(\sum_\alpha ||v_\alpha||^2_{1,\Omega} + |v_3|^2_{0,\Omega} \right)^{1/2} \leq C \left(\sum_{i,j} |e_{i||j}(\varepsilon; v)|^2_{0,\Omega} \right)^{1/2}, \ \forall v = (v_i) \in V(\Omega). \tag{2.20}$$

Further, as mentioned in the introduction, the limit problem will not only be two-dimensional, but the solution will have just two non trivial components, in agreement with the mechanical point of view since the limit displacements must be purely tangential. So let us introduce some particular notation and preliminary results adapted to this situation. Let

$$\underset{\sim}{\gamma}_{\alpha\beta}(\eta) := \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma^\sigma_{\alpha\beta} \eta_\sigma,$$

denote the covariant components of the linearized change of metric tensor associated with a tangential displacement field $\eta_\alpha a^\alpha$ of the surface S. Besides, for convenience, let us define $\underset{\sim}{V}(\omega) = H_0^1(\omega) \times H_0^1(\omega)$ as the set of admissible two-dimensional solutions. Following the proof of [4, Theorem 2.7-3], we may also find a useful Korn inequality in the two-dimensional framework and limited to the tangential components.

Theorem 2.2. *Let ω be a domain in $\mathbb{R}^2$, and let $\theta \in \mathscr{C}^{2,1}(\bar{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $a_\alpha = \partial_\alpha \theta$ are linearly independent at all points of $\bar{\omega}$ and such that the surface $S = \theta(\bar{\omega})$ is elliptic. Then, there exists $C_M > 0$ such that*

$$\left(\sum_\alpha ||\eta_\alpha||_{1,\omega}^2\right)^{1/2} \leq C_M \left(\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\underset{\sim}{\eta})|_{0,\omega}^2\right)^{1/2}, \ \forall \underset{\sim}{\eta} = (\eta_1,\eta_2) \in \underset{\sim}{V}(\omega). \tag{2.21}$$

Note that this implies that $|\cdot|_\omega^M := \left(\sum_{\alpha,\beta} |\gamma_{\alpha\beta}(\underset{\sim}{\eta})|_{0,\omega}^2\right)^{1/2}$ is equivalent to $\|\cdot\|_{1,\omega}$ in $\underset{\sim}{V}(\omega)$. In the following sections, given a vector $v = (v_1, v_2, v_3)$, we use the notation $\underset{\sim}{v}$ to refer to the pair composed of the first two components, (v_1, v_2).

3. Convergence Results as $\varepsilon \to 0$. Two-Dimensional Model

In this section, we present our main result. We prove that, in the particular case of a viscoelastic elliptic membrane shell, if the applied body force density is $O(1)$ with respect to ε, the surface tractions density is $O(\varepsilon)$, and the friction bound is $O(\varepsilon)$, that is, $p = 0$ in (2.17), choices that lead to Problem 2.3, the components of the solution verify that $u_\alpha(\varepsilon) \to u_\alpha$ in $H^1(0,T;H^1(\Omega))$, $u_3(\varepsilon) \to 0$ in $H^1(0,T;L^2(\Omega))$. Further, u_α can be identified with $\xi_\alpha \in H^1(0,T;H^1(\omega))$, where $\underset{\sim}{\xi} = (\xi_1, \xi_2)$ is the unique solution of the two-dimensional variational problem below.

Problem 3.1. Find $\underset{\sim}{\xi} : [0,T] \times \omega \longrightarrow \mathbb{R}^2$ such that

$$\begin{aligned}
&\underset{\sim}{\xi}(t,\cdot) \in \underset{\sim}{V}(\omega) \ \forall\, t \in [0,T], \quad \underset{\sim}{\xi}(0,\cdot) = \underset{\sim}{\xi}_0(\cdot) \\
&\int_\omega \underset{\sim}{a}^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\underset{\sim}{\xi}) \gamma_{\alpha\beta}(\underset{\sim}{\eta} - \dot{\underset{\sim}{\xi}}) \sqrt{a} dy + \int_\omega \underset{\sim}{b}^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\dot{\underset{\sim}{\xi}}) \gamma_{\alpha\beta}(\underset{\sim}{\eta} - \dot{\underset{\sim}{\xi}}) \sqrt{a} dy \\
&\qquad + \int_{\Gamma_C} g_T((\eta_\alpha a^{\alpha\beta} \eta_\beta)^{\frac{1}{2}} - (\dot{\xi}_\alpha a^{\alpha\beta} \dot{\xi}_\beta)^{\frac{1}{2}}) \sqrt{a} d\Gamma \\
&\geq \int_\omega p^\alpha (\eta_\alpha - \dot{\xi}_\alpha) \sqrt{a} dy \quad \forall \underset{\sim}{\eta} = (\eta_\alpha) \in \underset{\sim}{V}(\omega), \ a.e.\ t \in (0,T),
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
&\underset{\sim}{a}^{\alpha\beta\sigma\tau} = 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}), \quad \underset{\sim}{b}^{\alpha\beta\sigma\tau} = \rho(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}), \\
&\textit{and } p^\alpha(t) = \int_{-1}^1 f^\alpha(t) dx_3 + h^\alpha(t) \text{ with } h^\alpha(t) = h^\alpha(t,\cdot,1).
\end{aligned}$$

If we compare the two-dimensional fourth-order tensors $\underset{\sim}{a}^{\alpha\beta\sigma\tau}, \underset{\sim}{b}^{\alpha\beta\sigma\tau}$ with the corresponding tensors in the viscoelastic case without contact conditions (see [9]), we can see that in Problem 3.1 only the "deviatoric" part intervenes. Problem 3.1 is well posed and it has a unique solution. Indeed, we can formulate the following result (see [32]) for the proof).

Theorem 3.1. *Let ω be a domain in $\mathbb{R}^2$, and let $\theta \in \mathscr{C}^2(\bar{\omega};\mathbb{R}^3)$ be an injective mapping such that the two vectors $a_\alpha = \partial_\alpha \theta$ are linearly independent at all points of $\bar{\omega}$. Let $f^\alpha \in L^2(0,T;L^2(\Omega))$ and $h^\alpha \in L^2(0,T;L^2(\Gamma_N))$. Let $\underset{\sim}{\xi}_0 \in \underset{\sim}{V}(\omega)$. Then Problem 3.1 has a unique solution $\underset{\sim}{\xi} \in H^1(0,T;\underset{\sim}{V}(\omega))$.*

For each $\varepsilon > 0$, we assume that the initial condition for the scaled linear strains is

$$e_{i||j}(\varepsilon)(0,\cdot) = 0, \tag{3.2}$$

this is, the domain is on its natural state with no strains on it at the beginning of the period of observation.

Theorem 3.2. *Assume that $\theta \in \mathscr{C}^3(\bar{\omega};\mathbb{R}^3)$. Consider a family of viscoelastic elliptic membrane shells with thickness 2ε approaching zero and with each having the same elliptic middle surface $S = \theta(\bar{\omega})$, and let the assumptions on the data be as in Theorem 3.1. For all ε, $0 < \varepsilon \leq \varepsilon_0$,let $u(\varepsilon)$ be the solution of the associated three-dimensional scaled Problem 2.3. Assume (3.2) and that $(u_0)_3(\varepsilon) \rightharpoonup 0$ in $L^2(\Omega)$. Then, there exist functions $u_\alpha \in H^1(0,T;H^1(\Omega))$ satisfying $u_\alpha = 0$ on $\gamma \times [-1,1]$ such that*

(i) $u_\alpha(\varepsilon) \to u_\alpha$ in $H^1(0,T;H^1(\Omega))$ and $u_3(\varepsilon) \to 0$ in $H^1(0,T;L^2(\Omega))$ when $\varepsilon \to 0$;

(ii) u_α is independent of the transversal variable x_3, thus it can be identified with its average: $u_\alpha \equiv \bar{u}_\alpha = \frac{1}{2}\int_{-1}^{1} u_\alpha dx_3$;

(iii) the average $\bar{\underset{\sim}{u}} = (\bar{u}_1, \bar{u}_2)$ is the unique solution, $\underset{\sim}{\xi} = (\xi_1, \xi_2)$, of Problem 3.1.

Proof. We follow the same structure of the proof in [9, Theorem 11] (itself based on [4, Theorem 4.4-1]). Hence, we shall reference some steps which apply in the same manner. The proof is divided into several parts, numbered from (i) to (xi). We start by considering that the proposed problem is subjected only to volume forces in order to simplify the exposition. We will add the surface forces in latter steps. Therefore, we suppose that the scaled unknown $u(\varepsilon)$ satisfies the following variational problem: Find $u(\varepsilon)(t,\cdot) \in V(\Omega)\ \forall\ t \in [0,T]$ such that $u(\varepsilon)(0,\cdot) = u_0(\varepsilon)(\cdot)$, and

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\varepsilon) e_{k||l}(\varepsilon) e_{i||j}(\varepsilon, v - \dot{u}(\varepsilon))\sqrt{g(\varepsilon)}dx + \int_\Omega B^{ijkl}(\varepsilon)\dot{e}_{k||l}(\varepsilon) e_{i||j}(\varepsilon, v - \dot{u}(\varepsilon))\sqrt{g(\varepsilon)}dx \\
&+ \int_{\Gamma_C} g_T(|v_\tau(\varepsilon)| - |\dot{u}_\tau(\varepsilon)|)\sqrt{g(\varepsilon)}d\Gamma \\
&\quad \geq \int_\Omega f^i(v_i - \dot{u}_i(\varepsilon))\sqrt{g(\varepsilon)}dx, \quad \forall v \in V(\Omega)\ a.e.\ t \in (0,T).
\end{aligned} \tag{3.3}$$

(i) *A priori boundedness and extraction of weak convergent sequences.*

The norms $|e_{i||j}(\varepsilon)|_{H^1(0,T;L^2(\Omega))}, ||u_\alpha(\varepsilon)||_{H^1(0,T;H^1(\Omega))}$, and $|u_3(\varepsilon)|_{H^1(0,T;L^2(\Omega))}$ are bounded independently of $\varepsilon, 0 < \varepsilon \leq \varepsilon_1$. Consequently, there exists a subsequence, also denoted $(u(\varepsilon))_{\varepsilon>0}$, and functions $e_{i||j} \in H^1(0,T;L^2(\Omega)), u_\alpha \in H^1(0,T;H^1(\Omega))$, satisfying $u_\alpha = 0$ on Γ_0, and $u_3 \in H^1(0,T;L^2(\Omega))$, such that

$$e_{i||j}(\varepsilon) \rightharpoonup e_{i||j} \text{ in } H^1(0,T;L^2(\Omega)), \tag{3.4}$$

$$u_\alpha(\varepsilon) \rightharpoonup u_\alpha \text{ in } H^1(0,T;H^1(\Omega)) \tag{3.5}$$

$$\text{and hence } u_\alpha(\varepsilon) \to u_\alpha \text{ in } H^1(0,T;L^2(\Omega)), \tag{3.6}$$

$$u_3(\varepsilon) \rightharpoonup u_3 \text{ in } H^1(0,T;L^2(\Omega)). \tag{3.7}$$

For the proof of this step, we take $v = 2\dot{u}(\varepsilon)$ and $v = 0$ in (3.3) to find

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\Omega} A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon)e_{i||j}(\varepsilon)\sqrt{g(\varepsilon)}dx + \int_{\Omega} B^{ijkl}(\varepsilon)\dot{e}_{k||l}(\varepsilon)\dot{e}_{i||j}(\varepsilon)\sqrt{g(\varepsilon)}dx$$
$$+\int_{\Gamma_C} g_T|\dot{u}_\tau(\varepsilon)|\sqrt{g(\varepsilon)}d\Gamma = \int_{\Omega} f^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dx,\ a.e.\text{ in } (0,T). \tag{3.8}$$

Now, integrating over $[0,t]$ and using (3.2), we find that

$$\int_{\Omega} A^{ijkl}(\varepsilon)e_{k||l}(t)(\varepsilon)e_{i||j}(\varepsilon)(t)\sqrt{g(\varepsilon)}dx + \int_0^t\left(\int_{\Omega} B^{ijkl}(\varepsilon)\dot{e}_{k||l}(\varepsilon)(s)\dot{e}_{i||j}(\varepsilon)(s)\sqrt{g(\varepsilon)}dx\right)ds$$
$$+\int_0^t\int_{\Gamma_C} g_T|\dot{u}_\tau(\varepsilon)(s)|\sqrt{g(\varepsilon)}d\Gamma ds = \int_0^t\left(\int_{\Omega} f^i(s)\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dx\right)ds\ \ a.e.\text{ in } (0,T). \tag{3.9}$$

Now, using the Cauchy-Schwartz inequality and (2.18) yields

$$\int_0^t\left(\int_{\Omega} f^i(s)\dot{u}_i(\varepsilon)(s)\sqrt{g(\varepsilon)}dx\right)ds \le g_1^{1/2}\left(\int_0^t(\sum_i|f^i(s)|^2_{0,\Omega})ds\right)^{1/2}\left(\int_0^t(\sum_i|\dot{u}_i(\varepsilon)(s)|^2_{0,\Omega})ds\right)^{1/2}. \tag{3.10}$$

On the left hand side of (3.9), since $A^{ijkl}(\varepsilon)$ and $B^{ijkl}(\varepsilon)$ are elliptic, we obtain from (2.18) and (2.20) that, for $t = T$,

$$\int_0^T\left(\int_{\Omega} B^{ijkl}(\varepsilon)\dot{e}_{k||l}(\varepsilon)(s)\dot{e}_{i||j}(\varepsilon)(s)\sqrt{g(\varepsilon)}dx\right)ds$$
$$\ge g_0^{1/2}C_v\int_0^T\sum_{k,l}|\dot{e}_{k||l}(\varepsilon)(s)|^2_{0,\Omega}ds \ge g_0^{1/2}C_vC^{-2}\left(\int_0^T\sum_{\alpha}||\dot{u}_\alpha(\varepsilon)(s)||^2_{1,\Omega}ds + \int_0^T|\dot{u}_3(\varepsilon)(s)|^2_{0,\Omega}ds\right), \tag{3.11}$$

where $C_v > 0$ is the ellipticity constant for $B^{ijkl}(\varepsilon)$. Now, from (3.9)–(3.11),and the ellipticity of $A^{ijkl}(\varepsilon)$ and $g_T > 0$, we have that

$$g_0^{1/2}C_vC^{-2}\int_0^T\left(\sum_{\alpha}||\dot{u}_\alpha(\varepsilon)(t)||^2_{1,\Omega} + |\dot{u}_3(\varepsilon)(t)|^2_{0,\Omega}\right)dt$$
$$\le g_1^{1/2}\left(\int_0^T|f(t)|^2_{0,\Omega}dt\right)^{1/2}\left(\int_0^T|\dot{u}(\varepsilon)(t)|^2_{0,\Omega}dt\right)^{1/2}.$$

In view of

$$|\dot{u}(\varepsilon)(t)|^2_{0,\Omega} \le \sum_{\alpha}||\dot{u}_\alpha(\varepsilon)(t)||^2_{1,\Omega} + |\dot{u}_3(\varepsilon)(t)|^2_{0,\Omega},\ \forall\, t\in[0,T], \tag{3.12}$$

we conclude that there exists a constant $K_1 > 0$ independent of ε such that

$$\int_0^T\left(\sum_{\alpha}||\dot{u}_\alpha(\varepsilon)(t)||^2_{1,\Omega} + |\dot{u}_3(\varepsilon)(t)|^2_{0,\Omega}\right)dt \le K_1. \tag{3.13}$$

Now, if we go back to (3.9) and (3.10), we will also find that there exists a constant $K_2 > 0$ independent of ε such that

$$\int_{\Omega} A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon)(t)e_{i||j}(\varepsilon)(t)\sqrt{g(\varepsilon)}dx \le \int_0^t\left(\int_{\Omega} f^i\dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dx\right)dt \le K_2,\ \ a.e.\text{ in } (0,T).$$

Since $A^{ijkl}(\varepsilon)$ is elliptic, and using (2.20), we conclude that there exists a constant $K_3 > 0$ independent of ε such that

$$\sum_{\alpha} ||u_\alpha(\varepsilon)(t)||^2_{1,\Omega} + |u_3(\varepsilon)(t)|^2_{0,\Omega} \leq C^2 \sum_{k,l} |e_{k||l}(\varepsilon)(t)|^2_{0,\Omega} \leq K_3 \ \ a.e. \text{ in } (0,T).$$

Therefore, there exists $u_\alpha \in H^1(0,T;H^1(\Omega))$, $u_3 \in H^1(0,T;L^2(\Omega))$, and $e_{i||j} \in H^1(0,T;L^2(\Omega))$, such that the convergence considered in (3.4)–(3.7) is verified.

Remark 3.1. Note that from a functional analysis result which can be found, for example, in [33, Lemma 2.55] convergence (3.4)–(3.7) implies that $u_\alpha(\varepsilon)(t,\cdot) \rightharpoonup u_\alpha(t,\cdot)$ in $H^1(\Omega)$, $\forall\, t \in [0,T]$, $u_3(\varepsilon)(t,\cdot) \rightharpoonup u_3(t,\cdot)$ in $L^2(\Omega)$, $\forall\, t \in [0,T]$ and $e_{i||j}(\varepsilon)(t,\cdot) \rightharpoonup e_{i||j}(t,\cdot)$ in $L^2(\Omega)$, $\forall\, t \in [0,T]$. In particular, $(u_0)_\alpha(\varepsilon) = u_\alpha(\varepsilon)(0,\cdot) \rightharpoonup (u_0)_\alpha = u_\alpha(0,\cdot)$ in $H^1(\Omega)$ and $(u_0)_3(\varepsilon) = u_3(\varepsilon)(0,\cdot) \rightharpoonup u_3(0,\cdot) = (u_0)_3$ in $L^2(\Omega)$, thus the need of the hypotheses $(u_0)_3(\varepsilon) \rightharpoonup 0$, as we shall see after the findings in step (v).

(ii) *The sequence of surface integrals of tangential velocities is bounded as follows:*

$$\int_0^T \int_{\Gamma_C} |\dot{u}_\tau(s)| \sqrt{a}\, d\Gamma ds \leq \liminf_{\varepsilon \to 0} \int_0^T \int_{\Gamma_C} |\dot{u}_\tau(\varepsilon)(s)| \sqrt{g(\varepsilon)}\, d\Gamma ds,$$

where $\dot{u}_\tau = (\dot{u}_\alpha a^{\alpha\beta} \dot{u}_\beta)^{\frac{1}{2}}$.

Indeed, notice that the functional $\phi : L^2(0,T;[H^1(\Omega)]^2) \to \mathbb{R}$ such that

$$\phi(v) = \int_0^T \int_{\Gamma_C} (v_\alpha a^{\alpha\beta} v_\beta)^{\frac{1}{2}} d\Gamma ds, \ \forall v = (v_1, v_2) \in L^2(0,T;[H^1(\Omega)]^2),$$

is continuous and convex, and hence weakly lower semicontinuous. Now, since $\dot{u}_\alpha(\varepsilon) \rightharpoonup \dot{u}_\alpha$ in $L^2(0,T;H^1(\Omega))$ and $\sqrt{g(\varepsilon)} \to \sqrt{a}$, when $\varepsilon \to 0$, (see (3.5) and [4, Theorem 3.3-1], respectively), then

$$\int_0^T \int_{\Gamma_C} |\dot{u}_\tau(s)| \sqrt{a}\, d\Gamma ds = \phi(\sqrt{a}\dot{\underset{\sim}{u}}) \leq \liminf_{\varepsilon \to 0} \phi(\sqrt{g(\varepsilon)}\dot{\underset{\sim}{u}}(\varepsilon)) = \int_0^T \int_{\Gamma_C} |\dot{u}_\tau(\varepsilon)(s)| \sqrt{g(\varepsilon)}\, d\Gamma ds,$$

which gives the desired result.

(iii) *The limits of the scaled unknown, u_i, found in step (i) are independent of x_3.* This proof is analogous to the step (ii) in [9, Theorem 11], so we omit it. This proofs statement (ii) in the theorem.

(iv) *The limits $e_{i||j}$ found in (i) are independent of the variable x_3. Moreover, they are related with the limits $u := (u_i)$ by*

$$e_{\alpha||\beta} = \gamma_{\alpha\beta}(u) := \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \Gamma^\sigma_{\alpha\beta} u_\sigma - b_{\alpha\beta} u_3, \quad e_{\alpha||3} = 0, \tag{3.14}$$

$$\lambda a^{\alpha\beta} e_{\alpha||\beta} + (\lambda + 2\mu) e_{3||3} + \theta a^{\alpha\beta} \dot{e}_{\alpha||\beta} + (\theta + \rho) \dot{e}_{3||3} = 0. \tag{3.15}$$

in Ω, $a.e.\ t \in (0,T)$.

Considering $v = u(\varepsilon)$ in (2.12) and $\eta = u$ in (2.19) (*par abus de langage*, since $u \in V(\Omega)$), taking into account step (i) and the convergences $\Gamma^\sigma_{\alpha\beta}(\varepsilon) \to \Gamma^\sigma_{\alpha\beta}$ and $\Gamma^3_{\alpha\beta}(\varepsilon) \to b_{\alpha\beta}$ in $\mathscr{C}^0(\bar{\Omega})$, (see, [4, Theorem 3.3-1]), we have that

$$e_{\alpha||\beta}(\varepsilon) = \frac{1}{2}(\partial_\beta u_\alpha(\varepsilon) + \partial_\alpha u_\beta(\varepsilon)) - \Gamma^p_{\alpha\beta}(\varepsilon) u_p(\varepsilon) \rightharpoonup e_{\alpha||\beta} = \gamma_{\alpha\beta}(u) \text{ in } H^1(0,T;L^2(\Omega)).$$

Moreover, $e_{\alpha||\beta}$ are independent of x_3, as a straightforward consequence of the independence on x_3 of u_i (step (iii)). In addition, let $v \in V(\Omega)$. As a consequence of the definition of the scaled strains in (2.12)–(2.14), we find

$$\varepsilon e_{\alpha||\beta}(\varepsilon;v) \to 0 \text{ in } L^2(\Omega),\ \varepsilon e_{\alpha||3}(\varepsilon;v) \to \frac{1}{2}\partial_3 v_\alpha \text{ in } L^2(\Omega),$$
$$\varepsilon e_{3||3}(\varepsilon;v) = \partial_3 v_3 \text{ for all } \varepsilon > 0.$$

Using the variational formulation (3.3) for $v = \dot{u}(\varepsilon) + \varepsilon v$, we have

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\varepsilon)\left(\varepsilon e_{k||l}(\varepsilon)e_{i||j}(\varepsilon,v)\right)\sqrt{g(\varepsilon)}dx + \int_\Omega B^{ijkl}(\varepsilon)\left(\varepsilon \dot{e}_{k||l}(\varepsilon)e_{i||j}(\varepsilon,v)\right)\sqrt{g(\varepsilon)}dx\\
&+\int_{\Gamma_C} g_T(|\dot{u}_\tau(\varepsilon)+\varepsilon v_\tau| - |\dot{u}_\tau(\varepsilon)|)\sqrt{g(\varepsilon)}d\Gamma\\
&\quad \geq \varepsilon\int_\Omega f^i v_i\sqrt{g(\varepsilon)}dx \quad \forall v \in V(\Omega),\ a.e. \text{ in } (0,T).
\end{aligned}$$

Using (2.11) in the volume integrals and the triangular inequality in the first surface integral, we find

$$\begin{aligned}
&\int_\Omega \left(A^{\alpha\beta\sigma\tau}(\varepsilon)e_{\sigma||\tau}(\varepsilon) + A^{\alpha\beta 33}(\varepsilon)e_{3||3}(\varepsilon)\right)\left(\varepsilon e_{\alpha||\beta}(\varepsilon;v)\right)\sqrt{g(\varepsilon)}dx\\
&\qquad+\int_\Omega 4A^{\alpha 3\sigma 3}(\varepsilon)e_{\sigma||3}(\varepsilon)\left(\varepsilon e_{\alpha||3}(\varepsilon;v)\right)\sqrt{g(\varepsilon)}dx\\
&\qquad+\int_\Omega \left(A^{33\sigma\tau}(\varepsilon)e_{\sigma||\tau}(\varepsilon) + A^{3333}(\varepsilon)e_{3||3}(\varepsilon)\right)\left(\varepsilon e_{3||3}(\varepsilon;v)\right)\sqrt{g(\varepsilon)}dx\\
&\qquad+\int_\Omega \left(B^{\alpha\beta\sigma\tau}(\varepsilon)\dot{e}_{\sigma||\tau}(\varepsilon) + B^{\alpha\beta 33}(\varepsilon)\dot{e}_{3||3}(\varepsilon)\right)\left(\varepsilon e_{\alpha||\beta}(\varepsilon;v)\right)\sqrt{g(\varepsilon)}dx\\
&\qquad+\int_\Omega 4B^{\alpha 3\sigma 3}(\varepsilon)\dot{e}_{3||3}(\varepsilon)\left(\varepsilon e_{\alpha||3}(\varepsilon;v)\right)\sqrt{g(\varepsilon)}dx\\
&\qquad+\int_\Omega \left(B^{33\sigma\tau}(\varepsilon)\dot{e}_{\sigma||\tau}(\varepsilon) + B^{3333}(\varepsilon)\dot{e}_{3||3}(\varepsilon)\right)\left(\varepsilon e_{3||3}(\varepsilon;v)\right)\sqrt{g(\varepsilon)}dx\\
&+\int_{\Gamma_C} g_T|\varepsilon v_\tau|\sqrt{g(\varepsilon)}d\Gamma \geq \varepsilon\int_\Omega f^i v_i\sqrt{g(\varepsilon)}dx \quad \forall v \in V(\Omega),\ a.e. \text{ in } (0,T).
\end{aligned}$$

We pass to the limit as $\varepsilon \to 0$ and by taking into account the asymptotic behavior of the contravariant components of the fourth order tensors $A^{ijkl}(\varepsilon)$, $B^{ijkl}(\varepsilon)$ (see [9, Theorem 3]), $g(\varepsilon)$ (see (2.18)) and the convergence above, we obtain the following integral equation

$$\begin{aligned}
&\int_\Omega \left(2\mu a^{\alpha\sigma}e_{\alpha||3}\partial_3 v_\sigma + (\lambda+2\mu)e_{3||3}\partial_3 v_3\right)\sqrt{a}dx + \int_\Omega \lambda a^{\alpha\beta}e_{\alpha||\beta}\partial_3 v_3\sqrt{a}dx\\
&\quad+\int_\Omega \left(\rho a^{\alpha\sigma}\dot{e}_{\alpha||3}\partial_3 v_\sigma + (\theta+\rho)\dot{e}_{3||3}\partial_3 v_3\right)\sqrt{a}dx + \int_\Omega \theta a^{\alpha\beta}\dot{e}_{\alpha||\beta}\partial_3 v_3\sqrt{a}dx \geq 0, \qquad (3.16)
\end{aligned}$$

in Ω, $a.e.$ in $(0,T)$. From this inequality, we also get the equation, since $v \in V(\Omega)$ implies $-v \in V(\Omega)$. Hence, we proceed as in [9, Theorem 11, step (iii)] to find $e_{\alpha||3} = 0$ and (3.15).

(v) *The third component of the limit unknown, u, is null, i.e.,* $u_3 = 0$ *a.e. in* Ω

We are going to show that, since $u_3(\varepsilon) = 0$ a.e. on Γ_C, $u_3(\varepsilon) \rightharpoonup u_3$ in $L^2(\Omega)$ and $\partial_3 u_3(\varepsilon) \to 0$ in $L^2(\Omega)$, then $u_3 = 0$ a.e. in Ω. To do that, assume that it is not true. Since u_3 is independent of x_3, this assumption can be expressed as follows: there exists a domain $a \subset \omega$ with meas $(a) > 0$

such that $u_3 > 0$ in $A = a \times [-1,1]$ or there exists a domain $b \subset \omega$ with meas $(b) > 0$ such that $u_3 < 0$ in $B = b \times [-1,1]$.

In the case that a exists, take $\varphi \in \mathscr{D}(a)$ such that $\varphi \geq 0$. Let $A_\varphi :=$ support $(\varphi) \times [-1,1]$, thus $A_\varphi \subset A$. Consider $\chi = (1-x_3)\varphi$. We observe the following properties of this function:

$$\chi > 0 \text{ in } A_\varphi \setminus \Gamma_N, \quad \partial_3\chi = -\varphi < 0 \text{ in } A_\varphi, \quad \chi = 0 \text{ on } \Gamma_N, \quad \chi \geq 0 \text{ on } \Gamma_C.$$

With these properties in mind, and using Green's formula (recall that Γ_C is the lower face of the set), we find that

$$\int_\Omega \partial_3(u_3(\varepsilon))\chi\, dx = -\int_\Omega u_3(\varepsilon)\partial_3\chi\, dx - \int_{\Gamma_C} u_3(\varepsilon)\chi\, d\Gamma. \tag{3.17}$$

Since $u_3(\varepsilon) = 0$ on Γ_C, then the surface integral above is null. Passing to the limit as $\varepsilon \to 0$, the left hand side of (3.17) tends to zero, while the right hand side is strictly positive, which is a contradiction:

$$0 = \int_{A_\varphi} u_3\varphi\, dx > 0.$$

In the case that b exists, take $\varphi \in \mathscr{D}(b)$ such that $\varphi \geq 0$. Let $B_\varphi :=$ support $(\varphi) \times [-1,1]$, thus $B_\varphi \subset B$. Consider $\chi = (1-x_3)\varphi$. Also, by using a similar argument (see [32]), we conclude the contradiction

$$0 = \int_{B_\varphi} u_3\varphi\, dx < 0.$$

In any case, the conclusion follows that $u_3 = 0$ a.e. in Ω and, given the independence of u_3 on x_3, we can also postulate that $u_3 = \bar{u}_3 = 0$ on ω.

(vi) *The function $\underset{\sim}{\xi} = \underset{\sim}{\bar{u}} = (\bar{u}_\alpha) \equiv (u_\alpha)$ satisfies the two-dimensional variational Problem 3.1 with $p^\alpha := \int_{-1}^{1} f^\alpha dx_3$. In particular, since the solution of this problem is unique (by the Theorem 2.1), the convergence on (i) is verified for all the family $(u(\varepsilon))_{\varepsilon>0}$. We have that $\underset{\sim}{\xi}(t,\cdot) = \underset{\sim}{\bar{u}}(t,\cdot) = (\bar{u}_\alpha(t,\cdot)) \in \underset{\sim}{V}(\omega)$, $\forall\, t \in [0,T]$.*

Going back to (3.14), since $u_3 = 0$ by the previous step, we find that

$$e_{\alpha||\beta} = \gamma_{\alpha\beta}(\underset{\sim}{u}). \tag{3.18}$$

Let $v = (v_i) \in V(\Omega)$ be independent of x_3 in (3.3), integrate in $[0,t]$, and take the inferior limit when $\varepsilon \to 0$.

$$\begin{aligned}
&\int_0^t \int_\Omega f^i(s)(v_i - \dot{u}_i(s))\sqrt{a}dxds \\
&\leq \int_0^t \int_\Omega A^{ijkl}(0)e_{k||l}(s)e_{i||j}(v)\sqrt{a}dxds + \liminf_{\varepsilon\to 0}\{-\frac{1}{2}\int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon)(t)e_{i||j}(\varepsilon)(t)\sqrt{g(\varepsilon)}dx\} \\
&+ \int_0^t \int_\Omega B^{ijkl}(0)\dot{e}_{k||l}(s)e_{i||j}(v)\sqrt{a}dxds + \int_0^t \liminf_{\varepsilon\to 0}\{-\int_\Omega B^{ijkl}(\varepsilon)\dot{e}_{k||l}(\varepsilon)(s)\dot{e}_{i||j}(\varepsilon)(s)\sqrt{g(\varepsilon)}dx\}ds \\
&+ \int_0^t \int_{\Gamma_C} g_T\left(v_\alpha a^{\alpha\beta}v_\beta\right)^{\frac{1}{2}}\sqrt{a}d\Gamma ds + \int_0^t \liminf_{\varepsilon\to 0}\{-\int_{\Gamma_C} g_T\left(\dot{u}_\alpha(\varepsilon)(s)g^{\alpha\beta}(\varepsilon)\dot{u}_\beta(\varepsilon)(s)\right)^{\frac{1}{2}}\sqrt{g(\varepsilon)}d\Gamma\}ds.
\end{aligned}$$

Then, the asymptotic behaviour of the functions $e_{i||j}(\varepsilon; v), A^{ijkl}(\varepsilon), B^{ijkl}(\varepsilon)$ (see [9, Theorem 3]) and $g(\varepsilon)$, expressions (2.15)–(2.16), the weak convergence $e_{i||j}(\varepsilon) \rightharpoonup e_{i||j}$ in $H^1(0,T;L^2(\Omega))$

from step (i) and the result in (ii), lead to

$$\int_0^t \int_\Omega f^i(s)(v_i - \dot{u}_i(s))\sqrt{a}dxds \le \int_0^t \int_\Omega A^{ijkl}(0)e_{k||l}(s)e_{i||j}(v)\sqrt{a}dxds - \int_0^t \int_\Omega A^{ijkl}(0)e_{k||l}(s)\dot{e}_{i||j}(s)\sqrt{a}dxds$$
$$+ \int_0^t \int_\Omega B^{ijkl}(0)\dot{e}_{k||l}(s)e_{i||j}(v)\sqrt{a}dxds - \int_0^t \int_\Omega B^{ijkl}(0)\dot{e}_{k||l}(s)\dot{e}_{i||j}(s)\sqrt{a}dxds$$
$$+ \int_0^t \int_{\Gamma_C} g_T \left(v_\alpha a^{\alpha\beta} v_\beta\right)^{\frac{1}{2}} \sqrt{a}d\Gamma ds - \int_0^t \int_{\Gamma_C} g_T \left(\dot{u}_\alpha(s) a^{\alpha\beta} \dot{u}_\beta(s)\right)^{\frac{1}{2}} \sqrt{a}d\Gamma ds,$$

where we have used the weak lower semicontinuity of the various terms (see, for example, [34, Corollary III.8]). Therefore, since $e_{\alpha||\beta}(\varepsilon; v) \to \gamma_{\alpha\beta}(v)$, $e_{3||3}(\varepsilon; v) = 0$ (see, [4, Theorem 3.3-1]) and because of (3.14), we find

$$\begin{aligned}
&\int_0^t \int_\Omega f^i(s)(v_i - \dot{u}_i(s))\sqrt{a}dxds \\
&\le \int_0^t \Big\{ \int_\Omega \left(\lambda a^{\alpha\beta} a^{\sigma\tau} + \mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma})\right) e_{\sigma||\tau}(s)\gamma_{\alpha\beta}(v - \dot{u}(s))\sqrt{a}dx \\
&\qquad + \int_\Omega \lambda a^{\alpha\beta} e_{3||3}(s)\gamma_{\alpha\beta}(v - \dot{u}(s))\sqrt{a}dx - \int_\Omega (\lambda + 2\mu) e_{3||3}(s)\dot{e}_{3||3}(s)\sqrt{a}dx \\
&\qquad + \int_\Omega \left(\theta a^{\alpha\beta} a^{\sigma\tau} + \frac{\rho}{2}(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma})\right) \dot{e}_{\sigma||\tau}(s)\gamma_{\alpha\beta}(v - \dot{u}(s))\sqrt{a}dx \\
&\qquad + \int_\Omega \theta a^{\alpha\beta} \dot{e}_{3||3}(s)\gamma_{\alpha\beta}(v - \dot{u}(s))\sqrt{a}dx - \int_\Omega (\theta + \rho)\dot{e}_{3||3}(s)\dot{e}_{3||3}(s)\sqrt{a}dx \\
&+ \int_{\Gamma_C} g_T \left(v_\alpha a^{\alpha\beta} v_\beta\right)^{\frac{1}{2}} \sqrt{a}d\Gamma - \int_{\Gamma_C} g_T \left(\dot{u}_\alpha(s) a^{\alpha\beta} \dot{u}_\beta(s)\right)^{\frac{1}{2}} \sqrt{a}d\Gamma \Big\} ds,
\end{aligned} \tag{3.19}$$

$a.e.\ t \in (0,T)$. Now, by taking $v = \dot{u}$, we find

$$\int_0^t \{ \int_\Omega (\lambda + 2\mu) e_{3||3}(s)\dot{e}_{3||3}(s)\sqrt{a}dx + \int_\Omega (\theta + \rho)\dot{e}_{3||3}(s)\dot{e}_{3||3}(s)\sqrt{a}dx \} ds \le 0.$$

Next, because of (3.2), Remark 3.1, and since the viscosity coefficients θ, ρ are non-negative, we find

$$\int_\Omega (\lambda + 2\mu) e_{3||3} e_{3||3} \sqrt{a}dx \le 0 \;\; a.e. \text{ in } (0,T),$$

from which we deduce that

$$e_{3||3} = 0 \;\; a.e. \text{ in } (0,T), \tag{3.20}$$

and as a consequence, $\dot{e}_{3||3} = 0$ as well. Going back to the differential equation (3.15) and because of (3.2), we also find that

$$a^{\alpha\beta} e_{\alpha||\beta} = 0 \;\; a.e. \text{ in } (0,T). \tag{3.21}$$

Besides, recall that $u_3 = 0$, so take $v_3 = 0$ as well. Therefore, we can replace each term $\gamma_{\alpha\beta}(v - \dot{u})$ above by $\underset{\sim}{\gamma}_{\alpha\beta}(\underset{\sim}{v} - \underset{\sim}{\dot{u}})$. Now, given $\underset{\sim}{\eta} = (\eta_\alpha) \in \underset{\sim}{V}(\omega)$,we can define $\underset{\sim}{v} = (v_\alpha)$ such that $v_\alpha(y, x_3) = \eta_\alpha(y)$ and $v_3 = \eta_3 = 0$ for all $(y, x_3) \in \Omega$. Then $v \in V(\Omega)$ and it is independent of x_3. Hence, as a consequence of [9, Theorem 7], the variational inequality in (3.19) is satisfied for $\bar{v} = \eta$, and recall that we identify $\underset{\sim}{\xi} = \underset{\sim}{\bar{u}} \in \underset{\sim}{V}(\omega)$ and $e_{\alpha||\beta} = \underset{\sim}{\gamma}_{\alpha\beta}(\underset{\sim}{\xi})$ (see (iv)), so we find

$$\int_0^t \Big\{ \int_\omega 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})\underset{\sim}{\gamma}_{\sigma\tau}(\underset{\sim}{\xi}(s))\underset{\sim}{\gamma}_{\alpha\beta}(\underset{\sim}{\eta} - \dot{\underset{\sim}{\xi}}(s))\sqrt{a}dx$$
$$+ \int_\omega \rho(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})\underset{\sim}{\gamma}_{\sigma\tau}(\dot{\underset{\sim}{\xi}}(s))\underset{\sim}{\gamma}_{\alpha\beta}(\underset{\sim}{\eta} - \dot{\underset{\sim}{\xi}}(s))\sqrt{a}dx$$
$$+ \int_{\Gamma_C} g_T \left(\eta_\alpha a^{\alpha\beta}\eta_\beta\right)^{\frac{1}{2}} \sqrt{a}d\Gamma - \int_{\Gamma_C} g_T \left(\dot{\xi}_\alpha(s)a^{\alpha\beta}\dot{\xi}_\beta(s)\right)^{\frac{1}{2}} \sqrt{a}d\Gamma\Big\}ds,$$
$$\geq \int_0^t \int_\omega p^\alpha(s)(\eta_\alpha - \dot{\xi}_\alpha(s))\sqrt{a}dxds \;\; a.e. \text{ in } (0,T)\forall \underset{\sim}{\eta} = (\eta_\alpha) \in \underset{\sim}{V}(\omega).$$

Now, by using an argument like in [5, p. 57-58], we remove the time integral from above and obtain a pointwise inequality $a.e.\ t \in (0,T)$, which matches the variational inequality in Problem 3.1. This proofs statement (iii) in the theorem.

(vii) *The weak convergence $e_{i||j}(\varepsilon)(t,\cdot) \rightharpoonup e_{i||j}(t,\cdot)$ in $H^1(0,T;L^2(\Omega))$ is, in fact, strong.*
Indeed, we define

$$\Psi(\varepsilon) := \int_\Omega A^{ijkl}(\varepsilon)(e_{k||l}(\varepsilon) - e_{k||l})(\dot{e}_{i||j}(\varepsilon) - \dot{e}_{i||j})\sqrt{g(\varepsilon)}dx$$
$$+ \int_\Omega B^{ijkl}(\varepsilon)(\dot{e}_{k||l}(\varepsilon) - \dot{e}_{k||l})(\dot{e}_{i||j}(\varepsilon) - \dot{e}_{i||j})\sqrt{g(\varepsilon)}dx.$$

By taking (3.8) into account, we find that

$$\Psi(\varepsilon) = \int_\Omega f^i \dot{u}_i(\varepsilon)\sqrt{g(\varepsilon)}dx - \int_{\Gamma_C} g_T|\dot{u}_\tau(\varepsilon)|\sqrt{g(\varepsilon)}d\Gamma$$
$$- \int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\dot{e}_{i||j}(\varepsilon) - \dot{e}_{i||j})\sqrt{g(\varepsilon)}dx - \int_\Omega A^{ijkl}(\varepsilon)e_{k||l}(\varepsilon)\dot{e}_{i||j}\sqrt{g(\varepsilon)}dx$$
$$- \int_\Omega B^{ijkl}(\varepsilon)\dot{e}_{k||l}(\dot{e}_{i||j}(\varepsilon) - \dot{e}_{i||j})\sqrt{g(\varepsilon)}dx - \int_\Omega B^{ijkl}(\varepsilon)\dot{e}_{k||l}(\varepsilon)\dot{e}_{i||j}\sqrt{g(\varepsilon)}dx. \tag{3.22}$$

Integrating over the interval $[0,t]$, using the ellipticiy of the elasticity and viscosity tensors, and having in mind (2.18) and (3.2), we find that

$$\int_0^t \Psi(\varepsilon)dt = \frac{1}{2}\int_\Omega A^{ijkl}(\varepsilon)(e_{k||l}(\varepsilon) - e_{k||l})(e_{i||j}(\varepsilon) - e_{i||j})\sqrt{g(\varepsilon)}dx$$
$$+ \int_0^t \int_\Omega B^{ijkl}(\varepsilon)(\dot{e}_{k||l}(\varepsilon) - \dot{e}_{k||l})(\dot{e}_{i||j}(\varepsilon) - \dot{e}_{i||j})\sqrt{g(\varepsilon)}dxdt$$
$$\geq C_e^{-1}g_0^{1/2}\sum_{i,j}|e_{i||j}(\varepsilon) - e_{i||j}|^2_{0,\Omega} + C_v^{-1}g_0^{1/2}\int_0^T \sum_{i,j}|\dot{e}_{i||j}(\varepsilon) - \dot{e}_{i||j}|^2_{0,\Omega}dt, \tag{3.23}$$

where $C_e > 0$ is the ellipticity constant for $A^{ijkl}(\varepsilon)$. Let $\varepsilon \to 0$ in (3.22). Taking into account the weak convergence studied in (i), the upper bound in (ii) and the asymptotic behaviour of the functions $A^{ijkl}(\varepsilon), B^{ijkl}(\varepsilon)$, and $g(\varepsilon)$, we find that

$$\liminf_{\varepsilon\to 0}\{\int_0^T \Psi(\varepsilon)(s)ds\} \leq \int_0^T \int_\Omega f^\alpha(s)\dot{u}_\alpha(s)\sqrt{a}dxds - \int_0^T \int_{\Gamma_C} g_T \left(\dot{u}_\alpha a^{\alpha\beta}\dot{u}_\beta(s)\right)^{\frac{1}{2}} \sqrt{a}d\Gamma ds$$
$$- \int_0^T \int_\Omega A^{ijkl}(0)e_{k||l}(s)\dot{e}_{i||j}(s)\sqrt{a}dxds - \int_0^T \int_\Omega B^{ijkl}(0)\dot{e}_{k||l}(s)\dot{e}_{i||j}(s)\sqrt{a}dxds.$$

By the expressions of $A^{ijkl}(0)$ and $B^{ijkl}(0)$ and (3.21), we find that

$$\liminf_{\varepsilon\to 0}\{\int_0^T \Psi(\varepsilon)(s)ds\} \leq \int_0^T\int_\Omega f^\alpha(s)\dot{u}_\alpha(s)\sqrt{a}dxds - \int_0^T\int_{\Gamma_C} g_T\left(\dot{u}_\alpha(s)a^{\alpha\beta}\dot{u}_\beta(s)\right)^{\frac{1}{2}}\sqrt{a}d\Gamma ds$$
$$-\int_0^T\int_\Omega \mu(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma})e_{\sigma||\tau}(s)\dot{e}_{\alpha||\beta}(s)\sqrt{a}dxds$$
$$-\int_0^T\int_\Omega \frac{\rho}{2}(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma})\dot{e}_{\sigma||\tau}(s)\dot{e}_{\alpha||\beta}(s)\sqrt{a}dxds. \tag{3.24}$$

Then, taking $\xi_\alpha = \bar{u}_\alpha$, and both first $\eta_\alpha = 2\dot{u}_\alpha$ and then $\eta_\alpha = 0$ in the inequality (3.1), and using that $e_{\alpha||\beta} = \underset{\sim}{\gamma}_{\alpha\beta}(\underset{\sim}{u})$ (see step (iv)), we conclude that

$$\int_\Omega f^\alpha\dot{u}_\alpha\sqrt{a}dx - \int_{\Gamma_C} g_T\dot{u}_\alpha a^{\alpha\beta}\dot{u}_\beta\sqrt{a}dx - \int_\Omega \mu(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma})e_{\sigma||\tau}\dot{e}_{\alpha||\beta}\sqrt{a}dx$$
$$-\int_\Omega \frac{\rho}{2}(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma})\dot{e}_{\sigma||\tau}\dot{e}_{\alpha||\beta}\sqrt{a}dx = 0 \;\; a.e. \text{ in } (0,T),$$

which upon substitution in (3.24) gives

$$\lim_{\varepsilon\to 0}\{\int_0^T \Psi(\varepsilon)(s)ds\} \leq 0.$$

As a consequence, the strong convergence $e_{i||j}(\varepsilon) \to e_{i||j}$ in $L^2(0,T;L^2(\Omega))$ and $\dot{e}_{i||j}(\varepsilon) \to \dot{e}_{i||j}$ in $L^2(0,T;L^2(\Omega))$ is satisfied. Therefore, we conclude that $e_{i||j}(\varepsilon) \to e_{i||j}$ in $H^1(0,T;L^2(\Omega))$.

(viii) *The family* $(\bar{u}(\varepsilon))_{\varepsilon>0}$ *converges strongly to* $\bar{u}$ *(as* $\varepsilon \to 0$*) in* $H^1(0,T;\underset{\sim}{V}(\omega)\times L^2(\omega))$, *that is,*

$$\bar{u}_\alpha(\varepsilon) \to \bar{u}_\alpha \textit{ in } H^1(0,T;H^1(\omega)), \bar{u}_3(\varepsilon) \to 0 \textit{ in } H^1(0,T;L^2(\omega)).$$

This proof is a corollary of the step (vi) in [4, Th.4.4-1]. In order to do that, we follow the same arguments made there to prove that $\bar{u}_\alpha(\varepsilon) \to \bar{u}_\alpha$ in $L^2(0,T;H^1(\omega))$, $\bar{u}_3(\varepsilon) \to 0$ in $L^2(0,T;L^2(\omega))$ and the corresponding convergence of the time derivatives in the same spaces. Then the conclusion follows.

(ix) *The convergence* $u_3(\varepsilon) \rightharpoonup 0$ *in* $H^1(0,T;L^2(\Omega))$ *is, in fact, strong.*

Indeed, by (2.14) and step (i), we have $\partial_3 u_3(\varepsilon) = \varepsilon e_{3||3}(\varepsilon) \to 0$ in $H^1(0,T;L^2(\Omega))$. On the other hand, we have $\bar{u}_3(\varepsilon) \to 0$ in $H^1(0,T;L^2(\omega))$. Hence by [9, Theorem 7 (c)], the conclusion follows.

(x) *The convergence* $u_\alpha(\varepsilon) \to u_\alpha$ *is strong in* $H^1(0,T;H^1(\Omega))$. This proof is a corollary of the step $(viii)$ in [4, Theorem 4.4-1]. In order to do that, we follow the same arguments made there to prove that $u_\alpha(\varepsilon) \to u_\alpha$ in $L^2(0,T;H^1(\omega))$ and the corresponding convergence of the time derivatives in the same spaces. Then the conclusion follows.

(xi) *We add in this step the surface forces.*

The addition of surface forces essentially involves technicalities, but the same results found in steps (i) to (x) can be obtained with some minor changes. We basically have to make a reiterative use of a time dependent version of the trace result in step (ix) of [4, Theorem 4.4-1], similarly of what was done in [9, Theorem 11].

This proves the statement (i) in the theorem. Thus the proof of the theorem is complete. □

It remains to be proved an analogous result to the previous theorem but in terms of de-scaled unknowns. We consider the de-scalings $\xi_i^\varepsilon := \xi_i$ for each $\varepsilon > 0$ and formulate the de-scaled version of the limit Problem 3.1:

Problem 3.2. Find $\underset{\sim}{\xi}^\varepsilon : [0,T] \times \omega \longrightarrow \mathbb{R}^2$ such that

$$\underset{\sim}{\xi}^\varepsilon(t,\cdot) \in \underset{\sim}{V}(\omega) := H_0^1(\omega) \times H_0^1(\omega),\ \forall\, t \in [0,T],\ \xi_\alpha^\varepsilon(0,\cdot) = (\xi_0^\varepsilon)_\alpha(\cdot),$$

$$\varepsilon \int_\omega \underset{\sim}{a}^{\alpha\beta\sigma\tau,\varepsilon} \underset{\sim}{\gamma}_{\sigma\tau}(\xi^\varepsilon(t)) \underset{\sim}{\gamma}_{\alpha\beta}(\underset{\sim}{\eta} - \underset{\sim}{\dot{\xi}}^\varepsilon(t))\sqrt{a}dy + \varepsilon \int_\omega \underset{\sim}{b}^{\alpha\beta\sigma\tau,\varepsilon} \underset{\sim}{\gamma}_{\sigma\tau}(\underset{\sim}{\dot{\xi}}^\varepsilon(t)) \underset{\sim}{\gamma}_{\alpha\beta}(\underset{\sim}{\eta} - \underset{\sim}{\dot{\xi}}^\varepsilon(t))\sqrt{a}dy$$

$$+ \int_{\Gamma_C^\varepsilon} g_T^\varepsilon (\eta_\alpha a^{\alpha\beta}\eta_\beta)^{\frac{1}{2}}\sqrt{a}dy - \int_{\Gamma_C^\varepsilon} g_T^\varepsilon (\dot{\xi}_\alpha^\varepsilon a^{\alpha\beta}\dot{\xi}_\beta^\varepsilon)^{\frac{1}{2}}\sqrt{a}dy$$

$$\geq \int_\omega p^{\alpha,\varepsilon}(t)(\eta_\alpha - \dot{\xi}_\alpha^\varepsilon)\sqrt{a}dy\ \forall \underset{\sim}{\eta} = (\eta_\alpha) \in \underset{\sim}{V}(\omega),\ a.e. \text{ in } (0,T),$$

where

$$\underset{\sim}{\gamma}_{\alpha\beta}^\varepsilon(\eta) := \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) - \Gamma_{\alpha\beta}^\sigma \eta_\sigma,$$

$$p^{\alpha,\varepsilon}(t) := \int_{-\varepsilon}^{\varepsilon} f^{\alpha,\varepsilon}(t) dx_3^\varepsilon + h^{\alpha,\varepsilon}(t) \text{ and } h^{\alpha,\varepsilon}(t) = h^{\alpha,\varepsilon}(t,\cdot,\varepsilon),$$

and the contravariant components of the fourth order two-dimensional tensors $\underset{\sim}{a}^{\alpha\beta\sigma\tau,\varepsilon}$ and $\underset{\sim}{b}^{\alpha\beta\sigma\tau,\varepsilon}$ are defined as

$$\underset{\sim}{a}^{\alpha\beta\sigma\tau,\varepsilon} := 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}), \quad \underset{\sim}{b}^{\alpha\beta\sigma\tau,\varepsilon} := \rho(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}).$$

The convergence $u_\alpha(\varepsilon) \to u_\alpha$ in $H^1(0,T;H^1(\Omega))$ and $u_3(\varepsilon) \to u_3$ in $H^1(0,T;L^2(\Omega))$ from Theorem 3.2, the scaling proposed in Section 2, and [9, Theorem 7] together lead to the following convergences:

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon dx_3^\varepsilon \to \xi_\alpha \text{ in } H^1(0,T;H^1(\omega)), \quad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon dx_3^\varepsilon \to 0 \text{ in } H^1(0,T;L^2(\omega)).$$

Furthermore, we can prove the following theorem regarding the convergence of the averages of the tangential and normal components of the three-dimensional displacement vector field.

Theorem 3.3. *Assume that $\theta \in \mathscr{C}^3(\bar{\omega};\mathbb{R}^3)$. Consider a family of viscoelastic elliptic membrane shells with thickness 2ε approaching zero and with each having the same elliptic middle surface $S = \theta(\bar{\omega})$, and let the assumptions on the data be as in Theorem 3.1. Let $u^\varepsilon = (u_i^\varepsilon) \in H^1(0,T;V(\Omega^\varepsilon))$ denote the solution to the de-scaled version of Problem 2.3 (see [32]), and let $\underset{\sim}{\xi}^\varepsilon = (\xi_\alpha^\varepsilon) \in H^1(0,T;\underset{\sim}{V}(\omega))$ be the solution two-dimensional Problem 3.2 for each $\varepsilon > 0$. Moreover, let $\underset{\sim}{\xi} = (\xi_\alpha) \in H^1(0,T;\underset{\sim}{V}(\omega))$ denote the solution to Problem 3.1. Then*

$$\xi_\alpha^\varepsilon = \xi_\alpha \textit{ and thus } \xi_\alpha^\varepsilon a^\alpha = \xi_\alpha a^\alpha \textit{ in } H^1(0,T;H^1(\omega)), \forall \varepsilon > 0,$$

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_\alpha^\varepsilon g^{\alpha,\varepsilon} dx_3^\varepsilon \to \xi_\alpha a^\alpha \textit{ in } H^1(0,T;H^1(\omega)) \textit{ as } \varepsilon \to 0,$$

and $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u_3^\varepsilon g^{3,\varepsilon} dx_3^\varepsilon \to 0$ in $H^1(0,T;L^2(\omega))$ as $\varepsilon \to 0$.

The proof is similar to [9, Theorem 12] and can be found in [32].

Remark 3.2. The fields $\xi_T^\varepsilon, \xi_N^\varepsilon : [0,T] \times \bar{\omega} \longrightarrow \mathbb{R}^3$ defined by $\xi_T^\varepsilon := \xi_\alpha^\varepsilon a^\alpha$ and $\xi_N^\varepsilon := \xi_3^\varepsilon a^3$ are known as the limit tangential and normal displacement fields, respectively, of the middle surface S of the shell. If we denote the limit displacement field of S by $\xi^\varepsilon := \xi_i a^i$, then $\xi^\varepsilon = \xi_T^\varepsilon + \xi_N^\varepsilon$. In this case, $\xi_N^\varepsilon = 0$, and any admissible displacement is purely tangential.

4. Conclusions

We found and mathematically justified a model for viscoelastic shells in bilateral, frictional contact, for the particular case of the elliptic membranes, given by Problem 3.2. To this end, we used the insight provided by the asymptotic expansion method (not detailed here for the sake of briefness), and we justified this approach by obtaining convergence results. We consider that the most noticeable result of our study is that in the limit problem only contribute the terms derived from the deviatoric part of the three-dimensional problem, and it can be seen as an obstacle problem with bilateral and frictional contact conditions, in whole domain ω. Besides, the normal component of the solution is null, so the displacement is purely tangential.

To our knowledge this is the first time that a limit problem for a shell in frictional contact was justified with a convergence theorem. In forthcoming papers, we will continue to investigate limit problems for other kinds of shells in frictional contact. Particularly interesting from the physical point of view is the case of a unilateral contact condition instead of the bilateral case studied here. Notice that the mathematical difficulty is associated to the weaker regularity of the normal component of the solutions, $u_3(\varepsilon)$, in curvilinear coordinates, which makes it harder to operate with the corresponding traces on the contact surface.

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