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GAP FUNCTIONS AND GLOBAL ERROR BOUNDS FOR HISTORY-DEPENDENT VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

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Abstract. This paper is devoted to a generalized time-dependent variational-hemivariational inequality with history-dependent operators. First, we introduce a new concept of gap functions to the time-dependent variational-hemivariational inequality under consideration. Then, we consider a regularized function, which is proved to be a gap function of the inequality problem, and establish several important properties to the regularized function. Furthermore, an global error bound to the time-dependent variational-hemivariational inequality, which implicitly depends on the regularized gap function, is obtained. Finally, a quasi-static contact problem with the constitutive law involving a convex subdifferential inclusion and long memory effect is studied as an illustrative application.

Keywords. Gap function; Global error bound; History-dependent operator; Quasi-static contact problem; Locking material.

1. INTRODUCTION

Variational and hemivariational inequalities have became useful and powerful mathematical tools in the study of both the qualitative and numerical analysis of nonlinear boundary value problems of PDEs arising in mechanics, physics, and engineering science. Essentially speaking, the theory of variational inequalities is based on arguments of monotonicity and convexity, including the properties of the subdifferential of convex functions, see [1, 2, 3, 4] for the numerical analysis of variational inequalities and their applications in mechanics and engineering. The notion of hemivariational inequalities started by Panagiotopoulos [5, 6, 7] studies the boundary value problems for PDEs governed by nonconvex and nonmonotone potentials. The research of hemivariational inequalities, including existence and uniqueness results, and numerical analysis, can be found in [8, 9, 10, 11]. For a description of various problems arising in mechanics and engineering science which lead to hemivariational inequalities, we refer to [1, 12].

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Variational-hemivariational inequalities represent a special class of variational inequalities and hemivariational inequalities, in which both convex and nonconvex potentials are involved. The motivation of investigation on variational-hemivariational inequalities comes from various mechanical problems, for example, the unilateral contact problems with nonmonotone and multivalued constitutive laws, nonsmooth transportation problems, nonconvex semipermeability problems, and the delamination problems with multilayered composites; see, e.g., [13, 14, 15]. On the other hand, history-dependent operators could model both in the constitutive law of the material and in the interface boundary conditions. Some classical applications of history-dependent operators are the memory terms in the viscoelastic constitutive laws, the total slip, the total slip rate, and the accumulated penetration. So, more and more scholars were attracted to explore variational-hemivariational inequalities involving various history-dependent operators recently. For example, Migórski-Ochal-Sofonea [16] proved the existence and uniqueness to a history-dependent variational-hemivariational inequality by using the arguments of surjectivity for pseudomonotone operators and the fixed point principle, and applied these theoretical results to a new model of viscoelastic frictionless contact, in which both the instantaneous and the memory effects of the foundation were taken into account. For other results on variational-hemivariational inequalities with history-dependent operators, the reader may consult [17, 18, 19, 20, 21] and the references therein.

A popular approach to solve variational inequalities is to transform the considered variational inequality into an optimization-related problem by means of gap functions. In this way, it can apply the descent algorithms with global convergence to obtain the global error estimates. Auslender [22] first introduced the notion of gap function $h\colon R^n \to R$ to the variational inequality

$$\langle H(u), v-u\rangle \geq 0 \text{ for all } v \in K, \tag{1.1}$$

and suggested that variational inequality (1.1) is equivalent to a minimizing problem with the cost function h in K:

$$h(u) := \sup_{v\in K}\langle H(u), u-v\rangle \text{ for all } u \in K,$$

where $u \in K \subset R^n$, $H\colon R^n \to R^n$, and $\langle\cdot,\cdot\rangle$ is the scalar product in R^n. It is obvious that u solves variational inequality (1.1) if and only if u minimizes h on K and $h(u) = 0$. Whereas, the function h in general is nondifferentiable even if H is differentiable. In order to overcome the flaw, Fukushima [23] considered the regularized gap function $h_\xi\colon R^n \to R$ defined by

$$h_\xi(u) := \sup_{v\in K}\left\{\langle H(u), u-v\rangle - \frac{\xi}{2}\|v-u\|^2\right\},$$

where ξ is a positive constant. The major advantage of the regularized gap function h_ξ is that h_ξ is differentiable whenever H is differentiable and this gap function could help us to constitute a constrained differentiable optimization problem, which is reformulated by the considered variational inequality. Based on the idea of the regularized gap function h_ξ, Fukushima [23] developed a descent algorithm for the equivalent optimization problem and proved its global convergence. Thereafter, Yamashita-Fukushima [24] explored the following function called Moreau-Yosida regularized gap function $\theta_{h_{\xi,\eta}}\colon R^n \to R$:

$$\theta_{h_{\xi,\eta}}(u) := \inf_{w\in K}\left\{h_\xi(w) + \eta\|u-w\|^2\right\},$$

where η is a positive parameter. The unconstrained differentiability leads to that the optimization problem driven by Moreau-Yosida regularized gap function is equivalent to the corresponding variational inequality. Moreover, the authors in [24] established several global error bounds to the variational inequality under consideration.

Furthermore, the theory of error bounds not only provides the upper estimates of the distance between an arbitrary feasible point and the solution set of inequality problem, but also delivers the convergence rate of iterative algorithms for solving optimization problems; see, e.g., [25, 26, 27]. Based on this motivation, different gap functions and the corresponding error bounds have been extended to a lot of problems with constraints, for instance, quasi-variational inequalities, vector variational inequalities, hemivariational inequalities, equilibrium problems, and optimization problems. More recently, Fan-Wang [28] and Tang-Huang [29] developed the regularized gap functions for a set-valued variational inequality by using the projection operator method. Aussel-Correa-Marechal [30] defined a new kind of gap functions through an axiomatic approach, and provided several error bounds for quasi-variational inequalities and generalized Nash equilibrium problems, respectively. Li-Ng [31] presented some error bounds to a class of generalized D-gap functions for nonsmooth and nonmonotone variational inequalities and introduced a derivative-free descent method. Hung-Migórski-Tam-Zeng [32] investigated the gap functions and global error bounds for a class of variational-hemivariational inequalities and applied those abstract results to a nonsmooth semipermeability obstacle problem. For more details on this topic, we refer to [33, 34, 35, 36, 37, 38] and the references therein.

Let $(V, \|\cdot\|_V)$ and $(X, \|\cdot\|_X)$ be reflexive Banach spaces. Also, let $0 < T < \infty$ and K be a nonempty, closed, and convex subset of V. In what follows, we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V^* (the dual of V) and V. In this paper, we are interesting in the study of the following generalized variational-hemivariational inequality with history-dependent operator:

Problem 1.1. Find function $u\colon [0,T] \to K$ such that

$$\langle g(t,u(t)) + (\mathscr{R}u)(t), v - u(t)\rangle + J^0(\gamma u(t); \gamma(v - u(t))) + \varphi(v, u(t)) \geq \langle f(t), v - u(t)\rangle \quad (1.2)$$

for all $v \in K$ and a.e. $t \in [0,T]$.

Here, functions $g\colon [0,T] \times V \to V^*$, $J\colon X \to R$, $\gamma\colon V \to X$, $\varphi\colon V \times V \to R$, $\mathscr{R}\colon C([0,T];V) \to C([0,T];V^*)$, and $f \in C([0,T];V^*)$ will be specialized in Section 3.

The rest of the paper is organized as follows. In Section 2, we survey some preliminary materials and impose the assumptions on the data of Problem 1.1. In the meanwhile, we prove an equivalent result on the solution of Problem 1.1. Section 3 is devoted to the concept of gap functions to Problem 1.1, and to a regularized function, which is shown a gap function to Problem 1.1. In addition, via delivering several important properties to the regularized gap function, we establish an global error bound to Problem 1.1, which implicitly relies on the regularized function. Finally, in Section 4, we employ the theoretical results obtained in Section 3 to study a quasi-static contact problem in which the constitutive law of the locking material with long memory effect is considered.

2. Preliminaries and Hypotheses

In this section, we briefly review some preliminary materials, which are used in the next sections (these can be found in [8, 11, 14, 39]), and we impose the hypotheses on the data for

Problem 1.1. Moreover, an existence and uniqueness theorem to Problem 1.1 is recalled, and an equivalence result to Problem 1.1 is proved.

Let X be a reflexive Banach space with its dual X^*. We denote by $\|\cdot\|_X$ and $\langle\cdot,\cdot\rangle_{X^*\times X}$ the norm of X and the duality pairing between X^* and X, respectively. A function $f\colon X\to R\cup\{+\infty\}$ is said to be

(i) proper if $f(u)<+\infty$ for some $u\in X$;
(ii) convex if $f(\lambda u+(1-\lambda)v)\le\lambda f(u)+(1-\lambda)f(v)$ for all $\lambda\in[0,1]$ and all $u,v\in X$;
(iii) lower semicontinuous (l.s.c. for short) if it holds $f(u)\le\liminf_{n\to\infty}f(u_n)$ whenever the sequence $\{u_n\}\subset X$ is such that $u_n\to u$ in X;
(iv) upper semicontinuous (u.s.c. for short) if it holds $\limsup_{n\to\infty}f(u_n)\le f(u)$ whenever the sequence $\{u_n\}\subset X$ is such that $u_n\to u$ in X.

A function $J\colon X\to R$ is said to be locally Lipschitz continuous at $u\in X$ if there exist a neighborhood $N(u)$ of u and a constant $L_u>0$ such that

$$|J(w)-J(v)|\le L_u\|w-v\|_X \quad \text{for all} \;\; w,v\in N(u).$$

Definition 2.1. Let $J\colon X\to R$ be a locally Lipschitz function. We denote by $J^0(u;v)$ the generalized (Clarke) directional derivative of J at the point $u\in X$ in the direction $v\in X$ defined by

$$J^0(u;v)=\limsup_{\lambda\to0^+,\,w\to u}\frac{J(w+\lambda v)-J(w)}{\lambda}.$$

The generalized gradient in the sense of Clarke of $J\colon X\to R$ at $u\in X$ is given by

$$\partial J(u)=\left\{\,\zeta\in X^*\mid J^0(u;v)\ge\langle\zeta,v\rangle \;\text{ for all } v\in X\,\right\}.$$

The generalized gradient and generalized directional derivative of a locally Lipschitz function enjoy nice properties and rich calculus. Here we collect some basic results; see, e.g., [11, Proposition 3.23].

Proposition 2.1. *Assume that $J\colon X\to R$ is a locally Lipschitz function. Then the following assertions hold:*

(i) *for every $u\in X$, $X\ni v\mapsto J^0(u;v)\in R$ is positively homogeneous and subadditive, i.e., $J^0(u;\lambda v)=\lambda J^0(u;v)$ for all $\lambda\ge0, v\in X$ and $J^0(u;v_1+v_2)\le J^0(u;v_1)+J^0(u,;v_2)$ for all $u,v_1,v_2\in X$.*
(ii) *for every $v\in X$, it holds $J^0(u;v)=\max\{\,\langle\zeta,v\rangle\mid\zeta\in\partial J(u)\,\}$.*
(iii) *$X\times X\ni(u,v)\mapsto J^0(u;v)\in R$ is upper semicontinuous.*
(iv) *for every $u\in X$, $\partial J(u)$ is a nonempty, convex, weakly* compact subset of X^*, which is bounded by the Lipschitz constant $L_u>0$ of J near u.*

The following proposition reveals that a proper and convex function is bounded below by an affine continuous function; see, e.g., [40, Proposition 1.10]).

Proposition 2.2. *Let $(X,\|\cdot\|_X)$ be a reflexive Banach space. Assume that $\varphi\colon X\to R\cup\{+\infty\}$ is a proper, convex, and lower semicontinuous function. Then φ is bounded from below by an affine function, i.e., there exist $l\in X^*$ and a constant $\alpha\in R$ such that*

$$\varphi(v)\ge l(v)+\alpha \quad \textit{for all } v\in X.$$

Furthermore, we recall the concept of history-dependent operators and provide an important result concerning fixed point property to history-dependent operators (see, e.g., [14, Theorem 67]).

Definition 2.2. Let X and Y be Banach spaces. An operator $\mathscr{R}\colon C([0,T];X) \to C([0,T];Y)$ is called history-dependent if there exists a constant $L_{\mathscr{R}} > 0$ such that

$$\|(\mathscr{R}u)(t) - (\mathscr{R}v)(t)\|_Y \le L_{\mathscr{R}} \int_0^t \|u(s) - v(s)\|_X \, ds$$

for all u, $v \in C([0,T];X)$ and $t \in [0,T]$.

Lemma 2.1. *Let X be a Banach space, and let $\mathscr{R}\colon C([0,T];X) \to C([0,T];X)$ be a history-dependent operator. Then there exists a unique function $u^* \in C([0,T];X)$ such that $\mathscr{R}u^* = u^*$.*

We are now in a position to make the following assumptions on the data to Problem 1.1.

$\underline{H(K)}$: K is a nonempty, closed, and convex subset of V.

$\underline{H(K')}$: K is a nonempty, bounded, closed, and convex subset of V.

$\underline{H(J)}$: $J\colon X \to R$ is a locally Lipschitz function such that

(i) there exist constants $\alpha_J \ge 0$ and $b_J > 0$ satisfying

$$\|\partial J(w)\|_{X^*} \le \alpha_J + b_J \|w\|_X \quad \text{for all } w \in X;$$

(ii) there exists a constant $m_J \ge 0$ such that

$$J^0(u; v-u) + J^0(v; u-v) \le m_J \|u - v\|_X^2 \quad \text{for all } u, v \in X.$$

$\underline{H(g)}$: $g\colon [0,T] \times V \to V^*$ is such that

(i) for all $t \in [0,T]$, the mapping $u \mapsto g(t,u)$ is continuous, and is strongly monotone, i.e., there exists a constant $m_g > 0$ such that the following inequality holds

$$\langle g(t,u) - g(t,v), u - v \rangle \ge m_g \|u - v\|_V^2$$

for all $u, v \in V$ and all $t \in [0,T]$;

(ii) there exists a constant $L_g > 0$ such that

$$\|g(t_1,u) - g(t_2,u)\|_{V^*} \le L_g |t_1 - t_2|$$

for all $t_1, t_2 \in [0,T]$ and all $u \in K$.

$\underline{H(\varphi)}$: $\varphi\colon V \times V \to R$ is a bounded function such that

(i) $v \mapsto \varphi(v,u)$ is convex and lower semicontinuous for all $u \in V$;
(ii) $u \mapsto \varphi(v,u)$ is concave and upper semicontinuous for all $v \in V$;
(iii) for all $v \in K$, we have $\varphi(v,v) = 0$.

$\underline{H(\gamma)}$: $\gamma\colon V \to X$ is a linear, bounded, and compact operator.

$\underline{H(\mathscr{R})}$: $\mathscr{R}\colon C([0,T];V) \to C([0,T];V^*)$ is a history-dependent operator.

$\underline{H(f)}$: $f \in C([0,T];V^*)$.

$\underline{H(0)}$: $m_g > m_J \|\gamma\|_{\mathscr{L}(V,X)}^2$.

For the simplicity of calculating, in what follows, we assume that $0_V \in K$. Arguing as in the proof of [19, Theorem 5], [16, Theorem 16] and [41, Theorem 3.10], we have the following existence and uniqueness theorem to Problem 1.1.

Theorem 2.1. *Assume that $H(g)$, $H(J)$, $H(\varphi)$ with $\varphi(u,v)+\varphi(v,u) \le 0$ for all $u,v \in K$, $H(\gamma)$, $H(\mathscr{R})$, $H(f)$, $H(K)$, and $H(0)$ are satisfied. Then, Problem 1.1 has a unique solution $u \in C([0,T];K)$.*

In order to reveal the essential reason why we can assume that the constraint set K is bounded, (namely, in the sequel, we suppose that $H(K')$ is satisfied), we end this section to provide the following theorem, which shows that, for $k \in \mathbb{N}$ large enough, the solution of the following inequality, problem (2.1), coincides with the unique solution of Problem 1.1:
find function $u\colon [0,T] \to K_k$ such that

$$\langle g(t,u(t)) + (\mathscr{R}u)(t), v-u(t)\rangle + J^0(\gamma u(t); \gamma(v-u(t))) + \varphi(v,u(t)) \ge \langle f(t), v-u(t)\rangle \tag{2.1}$$

for all $v \in K_k$ and a.e. $t \in [0,T]$, where $K_k \subset K$ is defined by

$$K_k := \{u \in K \mid \|u\|_V \le k\}.$$

Theorem 2.2. *Assume that $H(g)$, $H(J)$, $H(\varphi)$ with $\varphi(u,v)+\varphi(v,u) \le 0$ for all $u,v \in K$, $H(\gamma)$, $H(\mathscr{R})$, $H(f)$, $H(K)$, and $H(0)$ are satisfied. Then,*

(i) *for each $k \in \mathbb{N}$ such that $K_k \neq \emptyset$, problem (2.1) has a unique solution $u_k \in C([0,T];K_k)$.*
(ii) *there exists a constant $k_0 \in \mathbb{N}$ large enough such that, for each $k \ge k_0$, the solution of problem (2.1) coincides with the unique solution of Problem 1.1.*

Proof. (i) It can be proved directly via using Theorem 2.1.

(ii) Let $k \in \mathbb{N}$ and $u_k \in C([0,T];K_k)$ be the unique solution to problem (2.1). Recall that $\mathscr{R}\colon C([0,T];V) \to C([0,T];V^*)$ is a history-dependent operator, so we can see that it is continuous. Indeed, if $\{u_n\} \subset C([0,T];V)$ is a convergent sequence, namely, there exists a function $u \in C([0,T];V)$ such that $u_n(t) \to u(t)$ in V as $n \to \infty$ for all $t \in [0,T]$. Hence,

$$\|(\mathscr{R}u_n)(t) - (\mathscr{R}u)(t)\|_{V^*} \le L_{\mathscr{R}} \int_0^t \|u_n(s) - u(s)\|_V \, ds$$

for all $t \in [0,T]$. Passing to the limit as $n \to \infty$ in the above inequality and invoking the Lebesgue-dominated convergence theorem, it yields

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|(\mathscr{R}u_n)(t) - (\mathscr{R}u)(t)\|_{V^*} \le \lim_{n\to\infty} L_{\mathscr{R}} \int_0^T \|u_n(s) - u(s)\|_V \, ds$$

$$\le L_{\mathscr{R}} \int_0^T \lim_{n\to\infty} \|u_n(s) - u(s)\|_V \, ds = 0.$$

This implies $(\mathscr{R}u_n)(t) \to (\mathscr{R}u)(t)$ in V^* as $n \to \infty$, i.e., $\mathscr{R}\colon C([0,T];V) \to C([0,T];V^*)$ is continuous.

We assert that there exists a constant $k_0 \in \mathbb{N}$ large enough such that the unique solution of problem (2.1) with $k = k_0$ satisfies the following inequality

$$\|u_{k_0}(t)\|_V < k_0 \quad \text{for all } t \in [0,T]. \tag{2.2}$$

Arguing by contradiction, for each $k \in \mathbb{N}$, there exists $t_k \in [0,T]$ such that $\|u_k(t_k)\|_V = k$. This points out $\|u_k(t_k)\|_V \to +\infty$ as $k \to \infty$. Taking $v = 0_V \in K_k$ in (2.1) implies

$$\langle g(t,u_k(t)) + (\mathscr{R}u_k)(t), u_k(t)\rangle - J^0(\gamma u_k(t);\gamma(0_V - u_k(t))) \leq \varphi(0_V, u_k(t)) + \langle f(t), u_k(t)\rangle.$$

Applying hypotheses $H(g)$ and the definition of $\mathscr{R}$, we have

$$\begin{aligned}
&\langle g(t,u_k(t)) + (\mathscr{R}u_k)(t), u_k(t)\rangle \\
=&\langle g(t,u_k(t)) - g(t,0_V), u_k(t) - 0_V\rangle + \langle g(t,0_V) - g(0,0_V), u_k(t)\rangle + \langle g(0,0_V), u_k(t)\rangle \\
&+ \langle (\mathscr{R}u_k)(t) - (\mathscr{R}0_V)(t), u_k(t)\rangle + \langle (\mathscr{R}0_V)(t), u_k(t)\rangle \\
\geq& m_g\|u_k(t)\|_V^2 - \|g(t,0_V) - g(0,0_V)\|_{V^*}\|u_k(t)\|_V - \|g(0,0_V)\|_{V^*}\|u_k(t)\|_V \\
&- L_{\mathscr{R}}\int_0^t \|u_k(s)\|_V\, ds\, \|u_k(t)\|_V - \|(\mathscr{R}0_V)(t)\|_{V^*}\|u_k(t)\|_V \\
\geq& m_g\|u_k(t)\|_V^2 - L_gT\|u_k(t)\|_V - \|g(0,0_V)\|_{V^*}\|u_k(t)\|_V - L_{\mathscr{R}}\int_0^t \|u_k(s)\|_V\, ds\, \|u_k(t)\|_V \\
&- \|\mathscr{R}0_V\|_{C([0,T];V^*)}\|u_k(t)\|_V.
\end{aligned} \tag{2.3}$$

Moreover, assumptions $H(J)$ and $H(\gamma)$ indicate that

$$\begin{aligned}
&-J^0(\gamma u_k(t);\gamma(0_V - u_k(t))) \\
=&-J^0(\gamma u_k(t);\gamma(0_V - u_k(t))) - J^0(\gamma 0_V;\gamma(u_k(t) - 0_V)) + J^0(\gamma 0_V;\gamma u_k(t)) \\
\geq& -m_J\|\gamma\|_{L(V,X)}^2\|u_k(t)\|_V^2 + J^0(\gamma 0_V;\gamma u_k(t)) \\
\geq& -m_J\|\gamma\|_{L(V,X)}^2\|u_k(t)\|_V^2 - \|\zeta_{\gamma 0_V}\|_{X^*}\|\gamma\|_{L(V,X)}\|u_k(t)\|_V \\
\geq& -m_J\|\gamma\|_{L(V,X)}^2\|u_k(t)\|_V^2 - \alpha_J\|\gamma\|_{L(V,X)}\|u_k(t)\|_V
\end{aligned} \tag{2.4}$$

for all $\zeta_{\gamma 0_V} \in \partial J(\gamma 0_V)$. It follows from hypothesis $H(\varphi)$(ii) that $u \mapsto -\varphi(v,u)$ is convex and lower semicontinuous. Using Proposition 2.2, we have

$$-\varphi(0_V, u_k(t)) \geq -\alpha_\varphi\|u_k(t)\|_V - \beta_\varphi \tag{2.5}$$

for some constants $\alpha_\varphi, \beta_\varphi \geq 0$ which are independent of k. Notice that

$$\langle f(t), u_k(t)\rangle \leq \|f(t)\|_{V^*}\|u_k(t)\|_V \leq \|f\|_{C([0,T];V^*)}\|u_k(t)\|_V. \tag{2.6}$$

Using the Young's inequality with $\varepsilon = \frac{m_g - m_J\|\gamma\|_{L(V,X)}^2}{4}$, Hölder inequality, and inequalities (2.3)–(2.6), one has

$$\begin{aligned}
&\left(m_g - m_J\|\gamma\|_{L(V,X)}^2 - 2\varepsilon\right)\|u_k(t)\|_V^2 \\
\leq& \frac{1}{4\varepsilon}\left[L_gT + \|g(0,0_V)\|_{V^*} + \|\mathscr{R}0_V\|_{C([0,T];V^*)} + \alpha_J\|\gamma\|_{L(V,X)} + \alpha_\varphi + \|f\|_{C([0,T];V^*)}\right]^2 \\
&+ \beta_\varphi + \frac{L_{\mathscr{R}}^2 T}{4\varepsilon}\int_0^t \|u_k(s)\|_V^2\, ds
\end{aligned} \tag{2.7}$$

for all $t \in [0,T]$. Let $\alpha_0 > 0$ be such that

$$\alpha_0 := \frac{2}{m_g - m_J\|\gamma\|_{L(V,X)}^2}\max\{\alpha_1, \alpha_2\},$$

where $\alpha_1, \alpha_2 > 0$ are defined by

$$\alpha_1 := \frac{1}{4\varepsilon}\left[L_g T + \|g(0,0_V)\|_{V^*} + \|\mathscr{R}0_V\|_{C([0,T];V^*)} + \alpha_J\|\gamma\|_{L(V,X)} + \alpha_\varphi + \|f\|_{C([0,T];V^*)}\right]^2 + \beta_\varphi,$$
$$\alpha_2 := \frac{L_{\mathscr{R}}^2 T}{2(m_g - m_J\|\gamma\|_{L(V,X)}^2)}.$$

It follows Granwall's inequality that

$$\|u_k(t)\|_V^2 \leq \alpha_0 \cdot exp\{T\alpha_0\}$$

for all $t \in [0,T]$ and all $k \in \mathbb{N}$. This contradicts the assumption that $\|u_k(t_k)\|_V \to +\infty$ as $k \to \infty$. Therefore, the claim (2.2) is verified.

Let $w \in K$ and $t \in [0,T]$ be arbitrary fixed. Then, from (2.2), we can pick a sufficiently small $\alpha > 0$ to satisfy

$$(1-\alpha)u_{k_0}(t) + \alpha w \in K_{k_0}.$$

Putting $v = (1-\alpha)u_{k_0}(t) + \alpha w$ into inequality (2.1) with $k = k_0$, we have

$$\begin{aligned}&\langle g(t,u_{k_0}(t)) + (\mathscr{R}u_{k_0})(t), w - u_{k_0}(t)\rangle + J^0(\gamma u_{k_0}(t); \gamma(w - u_{k_0}(t))) + \varphi(w, u_{k_0}(t))\\ \geq &\langle f(t), w - u_{k_0}(t)\rangle,\end{aligned}$$

where we have used the positive homogeneity of $v \mapsto J^0(u;v)$, the convexity of $v \mapsto \varphi(v,u)$, and the fact $\varphi(v,v) = 0$ for all $v \in K$. Because $w \in K$ and $t \in [0,T]$ are arbitrary, u_{k_0} is the unique solution to Problem 1.1 as well. The proof of this theorem is complete. □

Remark 2.1. Note the constraint set K of Problem 1.1 is unbounded in general. However, Theorem 2.2 reveals that the unique solution to Problem 1.1 with the suitable bounded set K_{k_0} coincides with the unique solution of the original problem with the constraint set K. Therefore, it is reasonable to assume that the constraint set K is a bounded in the sequel.

3. Gap Functions and Global Error Bounds

This section is devoted to the concept of gap functions to Problem 1.1. Using this concept, we introduce a regularized function (see (3.1) below) and prove that this regularized function (3.1) is a gap function to Problem 1.1 which is l.s.c. and uniformly bounded. Finally, we derive a global error bound of Problem 1.1 described by the regularized gap function (3.1).

We first give the exact definition of gap functions for Problem 1.1 as follows.

Definition 3.1. A real-valued function $h\colon [0,T] \times C([0,T];K) \to R$ is said to be a gap function for Problem 1.1 if it satisfies the following properties:

(a) $h(t,u) \geq 0$ for all $u \in C([0,T];K)$ and all $t \in [0,T]$.
(b) $u^* \in C([0,T];K)$ is such that $h(t,u^*) = 0$ for all $t \in [0,T]$ if and only if $u^* \in C([0,T];K)$ is a solution of Problem 1.1.

Let $\xi > 0$ be a fixed parameter. Following Fukushima [23] and Yamashita-Fukushima [24], we construct the function $\Xi_\xi : [0,T] \times C([0,T];K) \to R$ defined by

$$\Xi_\xi(t,u) = \sup_{v\in K}\Big(\langle g(t,u(t)) + (\mathscr{R}u)(t), u(t)-v\rangle - J^0(\gamma u(t);\gamma(v-u(t))) \\ -\varphi(v,u(t)) + \langle f(t), v-u(t)\rangle - \frac{1}{2\xi}\|u(t)-v\|_V^2\Big) \tag{3.1}$$

for all $u \in C([0,T];K)$ and all $t \in [0,T]$.

The following theorem shows that Ξ_ξ is a gap function for Problem 1.1.

Theorem 3.1. *Assume that $H(J)$, $H(\varphi)$, $H(\gamma)$, $H(f)$, and $H(K)$ are fulfilled. Then, the function Ξ_ξ defined by (3.1) for any parameter $\xi > 0$ is a gap function to Problem 1.1.*

Proof. For any fixed parameter $\xi > 0$, we are going to show that Ξ_ξ satisfies the conditions of Definition 3.1.

(a) Let $u \in C([0,T];K)$ be arbitrary. By the definition of Ξ_ξ, we have

$$\begin{aligned}\Xi_\xi(t,u) \geq &\langle g(t,u(t)) + (\mathscr{R}u)(t), u(t)-u(t)\rangle - J^0(\gamma u(t);\gamma(u(t)-u(t))) - \varphi(u(t),u(t)) \\ &+ \langle f(t), u(t)-u(t)\rangle - \frac{1}{2\xi}\|u(t)-u(t)\|_V^2 \\ =&0\end{aligned}$$

for all $t \in [0,T]$. This means that $\Xi_\xi(t,u) \geq 0$ for all $t \in [0,T]$ and all $u \in C([0,T];K)$.

(b) Assume that $u^* \in C([0,T];K)$ is such that $\Xi_\xi(t,u^*) = 0$ for all $t \in [0,T]$, namely,

$$\sup_{v\in K}\Big(\langle g(t,u^*(t)) + (\mathscr{R}u^*)(t), u^*(t)-v\rangle - J^0(\gamma u^*(t);\gamma(v-u^*(t))) - \varphi(v,u^*(t)) \\ + \langle f(t), v-u^*(t)\rangle - \frac{1}{2\xi}\|u^*(t)-v\|_V^2\Big) = 0.$$

This implies

$$\begin{aligned}&\langle g(t,u^*(t)) + (\mathscr{R}u^*)(t), v-u^*(t)\rangle + J^0(\gamma u^*(t);\gamma(v-u^*(t))) + \varphi(v,u^*(t)) \\ \geq&\langle f(t), v-u^*(t)\rangle - \frac{1}{2\xi}\|u^*(t)-v\|_V^2\end{aligned}$$

for all $v \in K$ and all $t \in [0,T]$. Let $w \in K$ and $t \in [0,T]$ be arbitrary. For any $\lambda \in (0,1)$, set $v_\lambda := (1-\lambda)u^*(t) + \lambda w$. Because K is convex, $v_\lambda \in K$. Note that $v \mapsto J^0(u,v)$ is positive homogeneity, and $v \mapsto \varphi(v,u)$ is convex and $\varphi(v,v) = 0$ for all $v \in K$. Putting $v = v_\lambda$ into the above inequality gives

$$\begin{aligned}&\lambda\langle g(t,u^*(t)) + (\mathscr{R}u^*)(t), w-u^*(t)\rangle + \lambda J^0(\gamma u^*(t);\gamma(w-u^*(t))) + \lambda\varphi(w,u^*(t)) \\ \geq&\lambda\langle f(t), w-u^*(t)\rangle - \frac{\lambda^2}{2\xi}\|u^*(t)-w\|_V^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\langle g(t,u^*(t)) + (\mathscr{R}u^*)(t), w-u^*(t)\rangle + J^0(\gamma u^*(t);\gamma(w-u^*(t))) + \varphi(w,u^*(t)) \\ \geq&\langle f(t), w-u^*(t)\rangle - \frac{\lambda}{2\xi}\|u^*(t)-w\|_V^2.\end{aligned}$$

Passing to the limit as $\lambda \to 0^+$ for the above inequality yields

$$\begin{aligned}&\langle g(t,u^*(t))+(\mathscr{R}u^*)(t),w-u^*(t)\rangle+J^0(\gamma u^*(t);\gamma(w-u^*(t)))+\varphi(w,u^*(t))\\ \geq&\langle f(t),w-u^*(t)\rangle.\end{aligned} \tag{3.2}$$

Because $w\in K$ and $t\in[0,T]$ are arbitrary, $u^*\in C([0,T];K)$ is a solution to Problem 1.1.

Conversely, if $u^*\in C([0,T];K)$ is a solution to Problem 1.1, namely, (3.2) holds for all $w\in K$ and all $t\in[0,T]$. Then,

$$\begin{aligned}\Xi_\xi(t,u^*)=&\sup_{v\in K}\Big(\langle g(t,u^*(t))+(\mathscr{R}u^*)(t),u^*(t)-v\rangle-J^0(\gamma u^*(t);\gamma(v-u^*(t)))\\ &-\varphi(v,u^*(t))+\langle f(t),v-u^*(t)\rangle-\frac{1}{2\xi}\|u^*(t)-v\|_V^2\Big)\\ \leq&\,0.\end{aligned}$$

The latter combined with the fact $\Xi_\xi(t,u)\geq 0$ for all $u\in C([0,T];K)$ and all $t\in[0,T]$ implies that $\Xi_\xi(t,u^*)=0$. Consequently, Ξ_ξ is a gap function to Problem 1.1. □

The following theorem states some important properties to gap function Ξ_ξ defined in (3.1).

Theorem 3.2. *Assume that $H(g)$, $H(J)$, $H(\varphi)$, $H(\gamma)$, $H(\mathscr{R})$, $H(K)$, $H(f)$, and $H(0)$ are fulfilled. Then, for any parameter $\xi>0$ fixed, the following statements hold:*

(i) *for each $t\in[0,T]$, the function $u\mapsto\Xi_\xi(t,u)$ is lower semicontinuous.*

(ii) *if, in addition, K is bounded, then, for each fixed $u\in C([0,T];K)$, the function $t\mapsto \Xi_\xi(t,u(t))$ belongs to $L_+^\infty(0,T)$.*

Proof. Let any parameter $\xi>0$ and any $t\in[0,T]$ be fixed. We consider the function $\hat{\Xi}_\xi\colon [0,T]\times C([0,T];K)\times K\to R$ defined by

$$\begin{aligned}\hat{\Xi}_\xi(t,u,v)=&\langle g(t,u(t))+(\mathscr{R}u)(t),u(t)-v\rangle-J^0(\gamma u(t);\gamma(v-u(t)))-\varphi(v,u(t))\\ &+\langle f(t),v-u(t)\rangle-\frac{1}{2\xi}\|u(t)-v\|_V^2\end{aligned}$$

for all $u\in C([0,T];K)$, $v\in K$, and $t\in[0,T]$. Let sequence $\{u_n\}\subset C([0,T];K)$ be such that $u_n\to u$ in $C([0,T];K)$ for some $u\in C([0,T];K)$. Since $\mathscr{R}\colon C([0,T];V)\to C([0,T];V^*)$ and $u\mapsto g(t,u)$ are continuous, $u\mapsto-\varphi(v,u)$, and $(u,v)\mapsto -J^0(u;v)$ are lower semicontinuous, we have

$$\begin{aligned}&\liminf_{n\to\infty}\hat{\Xi}_\xi(t,u_n,v)\\ \geq&\liminf_{n\to\infty}\langle g(t,u_n(t))+(\mathscr{R}u_n)(t),u_n(t)-v\rangle-\limsup_{n\to\infty}J^0(\gamma u_n(t);\gamma(v-u_n(t)))\\ &-\limsup_{n\to\infty}\varphi(v,u_n(t))+\liminf_{n\to\infty}\langle f(t),v-u_n(t)\rangle-\limsup_{n\to\infty}\frac{1}{2\xi}\|u_n(t)-v\|_V^2\\ \geq&\langle g(t,u(t))+(\mathscr{R}u)(t),u(t)-v\rangle-J^0(\gamma u(t);\gamma(v-u(t)))-\varphi(v,u(t))+\langle f(t),v-u(t)\rangle\\ &-\frac{1}{2\xi}\|u(t)-v\|_V^2.\end{aligned}$$

This points out that $u \mapsto \hat{\Xi}_\xi(t,u,v)$ is lower semicontinuous. Using the equality

$$\Xi_\xi(t,u) = \sup_{v\in K} \hat{\Xi}_\xi(t,u,v) \quad \text{for all } u \in C([0,T];K) \text{ and } t \in [0,T]$$

and the lower semicontinuity of $u \mapsto \hat{\Xi}_\xi(t,u,v)$, we have

$$\begin{aligned}\liminf_{n\to\infty} \Xi_\xi(t,u_n) &= \liminf_{n\to\infty} \sup_{v\in K} \hat{\Xi}_\xi(t,u_n,v)\\ &\geq \liminf_{n\to\infty} \hat{\Xi}_\xi(t,u_n,w) \geq \hat{\Xi}_\xi(t,u,w) \text{ for all } w \in K.\end{aligned}$$

Passing to the supremum with $w \in K$ for the above estimates, we have

$$\liminf_{n\to\infty} \Xi_\xi(t,u_n) \geq \sup_{w\in K} \hat{\Xi}_\xi(t,u,w) = \Xi_\xi(t,u).$$

Therefore, for any parameter $\xi > 0$ and any $t \in [0,T]$ fixed, the function $u \mapsto \Xi_\xi(t,u)$ is lower semicontinuous.

(ii) For any fixed $u \in C([0,T];K)$, we prove that the function $t \mapsto \Xi_\xi(t,u)$ is measurable and essentially bounded. In fact, if we can prove that, for each $c \in R$, the set

$$\Gamma_c := \big\{t \in [0,T] \mid \Xi_\xi(t,u(t)) \leq c\big\} \neq \emptyset$$

is closed, then $t \mapsto \Xi_\xi(t,u)$ is measurable. Let sequence $\{t_n\} \subseteq \Gamma_c$ be such that $t_n \to t$ in $[0,T]$ as $n \to \infty$ for some $t \in [0,T]$. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned}c \geq &\Xi_\xi(t_n,u)\\ \geq &\langle g(t_n,u(t_n)) + (\mathscr{R}u)(t_n), u(t_n) - v\rangle - J^0(\gamma u(t_n);\gamma(v-u(t_n))) - \varphi(v,u(t_n))\\ &+ \langle f(t_n), v-u(t_n)\rangle - \frac{1}{2\xi}\|u(t_n)-v\|_V^2\end{aligned}$$

for all $v \in K$. Passing to the lower limit as $n \to \infty$ for the inequality above and employing the continuity of $u\colon [0,T] \to K$ and $t \mapsto \mathscr{R}u(t)$, we have

$$\begin{aligned}c \geq &\liminf_{n\to\infty}\bigg(\langle g(t_n,u(t_n)) + (\mathscr{R}u)(t_n), u(t_n) - v\rangle - J^0(\gamma u(t_n);\gamma(v-u(t_n)))\\ &- \varphi(v,u(t_n)) + \langle f(t_n), v-u(t_n)\rangle - \frac{1}{2\xi}\|u(t_n)-v\|_V^2\bigg)\\ \geq &\langle g(t,u(t)) + (\mathscr{R}u)(t), u(t) - v\rangle - J^0(\gamma u(t);\gamma(v-u(t))) - \varphi(v,u(t))\\ &+ \langle f(t), v-u(t)\rangle - \frac{1}{2\xi}\|u(t)-v\|_V^2\end{aligned}$$

for all $v \in K$. Taking the supremum in the above inequality with $v \in K$ gives

$$\begin{aligned}c \geq &\sup_{v\in K}\bigg(\langle g(t,u(t)) + (\mathscr{R}u)(t), u(t) - v\rangle - J^0(\gamma u(t);\gamma(v-u(t))) - \varphi(v,u(t))\\ &+ \langle f(t), v-u(t)\rangle - \frac{1}{2\xi}\|u(t)-v\|_V^2\bigg)\\ = &\Xi_\xi(t,u).\end{aligned}$$

This implies that $t \in \Gamma_c$, i.e., Γ_c is closed. Therefore, the function $t \mapsto \Xi_\xi(t,u)$ is measurable on $[0,T]$.

Let $u \in C([0,T];V)$ be fixed. Next, we prove that the function $t \mapsto \Xi_\xi(t,u)$ is uniformly bounded. By virtue of hypotheses $H(g)$ and $H(\mathscr{R})$, we have

$$\begin{aligned}
&\langle g(t,u(t))+(\mathscr{R}u)(t),u(t)-v\rangle \\
=&\langle g(t,u(t))-g(0,u(t)),u(t)-v\rangle+\langle g(0,u(t)),u(t)-v\rangle \\
&+\langle(\mathscr{R}u)(t)-(\mathscr{R}0)(t),u(t)-v\rangle+\langle(\mathscr{R}0)(t),u(t)-v\rangle \\
\le&\|g(t,u(t))-g(0,u(t))\|_{V^*}\|u(t)-v\|_V+\|g(0,u(t))\|_{V^*}\|u(t)-v\|_V \\
&+\|(\mathscr{R}0)(t)\|_{V^*}\|u(t)-v\|_V+\|(\mathscr{R}u)(t)-(\mathscr{R}0)(t)\|_{V^*}\|u(t)-v\|_V \\
\le&L_gT\|u(t)-v\|_V+\|g(0,u(t))\|_{V^*}\|u(t)-v\|_V+\|(\mathscr{R}0)(t)\|_{V^*}\|u(t)-v\|_V \\
&+L_{\mathscr{R}}\int_0^t\|u(s)\|_V\,ds\,\|u(t)-v\|_V \\
\le&\big(L_gT+\|g(0,u(t))\|_{V^*}+L_{\mathscr{R}}T\|u\|_{C([0,T];V)}+\|(\mathscr{R}0)(t)\|_{V^*}\big)\big(\|u(t)\|_V+\|v\|_V\big).
\end{aligned} \tag{3.3}$$

It follows from hypotheses $H(J)$ that

$$\begin{aligned}
&-J^0(\gamma u(t);\gamma(v-u(t))) \\
\le&-\langle\zeta_{\gamma u(t)},\gamma(v-u(t))\rangle_X \\
\le&\|\zeta_{\gamma u(t)}\|_{X^*}\|\gamma\|_{L(V,X)}\|v-u(t)\|_V \\
\le&(\alpha_J+b_J\|\gamma\|_{L(V,X)}\|u(t)\|_V)\|\gamma\|_{L(V,X)}(\|u(t)\|_V+\|v\|_V)
\end{aligned} \tag{3.4}$$

for all $\zeta_{\gamma u(t)} \in \partial J(\gamma u(t))$. Because φ is a bounded map, K is bounded, and $u \in C([0,T];V)$, there exists a constant $M_0 > 0$ such that $\varphi(v,u(t)) \le M_0$ for all $v \in K$ and $t \in [0,T]$. Combining (3.3)–(3.4), we have

$$\begin{aligned}
&\langle g(t,u(t))+(\mathscr{R}u)(t),u(t)-v\rangle-J^0(\gamma u(t);\gamma(v-u(t)))-\varphi(v,u(t)) \\
&+\langle f(t),v-u(t)\rangle-\frac{1}{2\xi}\|u(t)-v\|_V^2 \\
\le&\big(L_gT+\|g(0,u(t))\|_{V^*}+L_{\mathscr{R}}T\|u\|_{C([0,T];V)}+\|(\mathscr{R}0)(t)\|_{V^*}+\alpha_J\|\gamma\|_{L(V,X)} \\
&+b_J\|\gamma\|_{L(V,X)}^2\|u(t)\|_V+\|f(t)\|_{V^*}\big)\big(\|u(t)\|_V+\|v\|_V\big)+M_0 \\
\le&\big(L_gT+\|g(0,u(\cdot))\|_{C([0,T];V^*)}+L_{\mathscr{R}}T\|u\|_{C([0,T];V)}+\|(\mathscr{R}0)\|_{C([0,T];V^*)}+\alpha_J\|\gamma\|_{L(V,X)} \\
&+b_J\|\gamma\|_{L(V,X)}^2\|u\|_{C([0,T];V)}+\|f\|_{C([0,T];V^*)}\big)\big(\|u\|_{C([0,T];V)}+\|v\|_V\big)+M_0
\end{aligned} \tag{3.5}$$

$$\le M_1 \tag{3.6}$$

for all $v \in K$, where $M_1 > 0$ is independent of $t \in [0,T]$ and $v \in K$. Passing to the supremum with $v \in K$ for the above estimates, it gives

$$\Xi_\xi(t,u) \le M_1 \quad \text{for all } t \in [0,T].$$

The latter together with the fact $\Xi_\xi(t,u) \ge 0$ for all $t \in [0,T]$ implies that $t \mapsto \Xi_\xi(t,u)$ is essentially bounded. Therefore, for each fixed $u \in C([0,T];K)$, the function $t \mapsto \Xi_\xi(t,u)$ belongs to $L_+^\infty(0,T)$. The proof of this theorem is complete. □

Under the analysis above, we are now in a position to deliver the following theorem, which gives a global error bound to Problem 1.1 controlled by the gap function (3.1).

Theorem 3.3. *Let $u^* \in C([0,T];K)$ be the unique solution to Problem 1.1. Let ξ be positive parameter such that $m_g - m_J\|\gamma\|^2_{L(V,X)} > \frac{1}{2\xi}$. Assume that $H(g)$, $H(J)$, $H(\varphi)$ with $\varphi(u,v) + \varphi(v,u) \le 0$ for all $u,v \in K$, $H(\gamma)$, $H(\mathscr{R})$, $H(K')$, $H(f)$, and $H(0)$ are satisfied. Then, for each function $u \in C([0,T];K)$, there exists a function $\Pi_u \in L^\infty_+(0,T)$ such that*

$$\|u(t) - u^*(t)\|_V \le \Pi_u(t) \ \ \textit{for all } t \in [0,T]. \tag{3.7}$$

Proof. For any $u \in C([0,T];K)$ fixed, we have

$$\begin{aligned}&\langle g(t,u^*(t)) + (\mathscr{R}u^*)(t), u(t) - u^*(t)\rangle + J^0(\gamma u^*(t); \gamma(u(t) - u^*(t))) + \varphi(u(t), u^*(t))\\ \ge&\langle f(t), u(t) - u^*(t)\rangle\end{aligned} \tag{3.8}$$

for all $t \in [0,T]$. Since $u^* \in C([0,T];K)$ is the unique solution to Problem 1.1, we have

$$\begin{aligned}\Xi_\xi(t,u) =&\sup_{v\in K}\Big(\langle g(t,u(t)) + (\mathscr{R}u)(t), u(t) - v\rangle - J^0(\gamma u(t); \gamma(v - u(t)))\\ &- \varphi(v, u(t)) + \langle f(t), v - u(t)\rangle - \frac{1}{2\xi}\|u(t) - v\|^2_V\Big)\\ \ge&\langle g(t,u(t)) + (\mathscr{R}u)(t), u(t) - u^*(t)\rangle - J^0(\gamma u(t); \gamma(u^*(t) - u(t)))\\ &- \varphi(u^*(t), u(t)) + \langle f(t), u^*(t) - u(t)\rangle - \frac{1}{2\xi}\|u(t) - u^*(t)\|^2_V.\end{aligned} \tag{3.9}$$

Applying hypotheses $H(g)$ and $H(\mathscr{R})$, one has

$$\begin{aligned}&\langle g(t,u(t)) + (\mathscr{R}u)(t), u(t) - u^*(t)\rangle\\ =&\langle g(t,u(t)) - g(t,u^*(t)), u(t) - u^*(t)\rangle + \langle g(t,u^*(t)), u(t) - u^*(t)\rangle\\ &+ \langle (\mathscr{R}u)(t) - (\mathscr{R}u^*)(t), u(t) - u^*(t)\rangle + \langle (\mathscr{R}u^*)(t), u(t) - u^*(t)\rangle\\ \ge& m_g\|u(t) - u^*(t)\|^2_V - \|(\mathscr{R}u)(t) - (\mathscr{R}u^*)(t)\|_{V^*}\|u(t) - u^*(t)\|_V\\ &+ \langle g(t,u^*(t)) + (\mathscr{R}u^*)(t), u(t) - u^*(t)\rangle\\ \ge& m_g\|u(t) - u^*(t)\|^2_V - L_{\mathscr{R}}\int_0^t \|u(s) - u^*(s)\|_V\,ds\,\|u(t) - u^*(t)\|_V\\ &+ \langle g(t,u^*(t)) + (\mathscr{R}u^*)(t), u(t) - u^*(t)\rangle.\end{aligned} \tag{3.10}$$

Moreover, hypothesis $H(J)$(ii) deduces

$$-J^0(\gamma u(t); \gamma(u^*(t) - u(t))) \ge -m_J\|\gamma\|^2_{L(V,X)}\|u(t) - u^*(t)\|^2_V + J^0(\gamma u^*(t); \gamma(u(t) - u^*(t))) \tag{3.11}$$

Notice that $\varphi(u,v) + \varphi(v,u) \le 0$ for all $v,u \in K$. One has

$$-\varphi(u^*(t), u(t)) \ge \varphi(u(t), u^*(t)). \tag{3.12}$$

Combining (3.8)-(3.12), we obtain

$$\begin{aligned}
&\Xi_\xi(t,u) \quad (3.13)\\
\geq&\left(m_g - m_J\|\gamma\|^2_{L(V,X)} - \frac{1}{2\xi}\right)\|u(t)-u^*(t)\|_V^2 - L_{\mathscr{R}}\int_0^t \|u(s)-u^*(s)\|_V\,ds\,\|u(t)-u^*(t)\|_V\\
&+\langle g(t,u^*(t)) + (\mathscr{R}u^*)(t), u(t)-u^*(t)\rangle + J^0(\gamma u^*(t);\gamma(u(t)-u^*(t))) + \varphi(u(t),u^*(t))\\
&+\langle f(t), u^*(t)-u(t)\rangle\\
\geq&\left(m_g - m_J\|\gamma\|^2_{L(V,X)} - \frac{1}{2\xi}\right)\|u(t)-u^*(t)\|_V^2 - L_{\mathscr{R}}\int_0^t \|u(s)-u^*(s)\|_V\,ds\,\|u(t)-u^*(t)\|_V.
\end{aligned}$$

Employing Young's inequality with $\varepsilon = \frac{m_g - m_J\|\gamma\|^2_{L(V,X)} - \frac{1}{2\xi}}{2}$, (3.13), and Hölder's inequality gives

$$\begin{aligned}
&L_{\mathscr{R}}\int_0^t \|u(s)-u^*(s)\|_V\,ds\,\|u(t)-u^*(t)\|_V\\
\leq&\varepsilon\|u(t)-u^*(t)\|_V^2 + \frac{L_{\mathscr{R}}^2}{4\varepsilon}\left(\int_0^t \|u(s)-u^*(s)\|_V\,ds\right)^2\\
\leq&\varepsilon\|u(t)-u^*(t)\|_V^2 + \frac{L_{\mathscr{R}}^2 T}{4\varepsilon}\int_0^t \|u(s)-u^*(s)\|_V^2\,ds \qquad (3.14)
\end{aligned}$$

for all $t \in [0,T]$. Inserting (3.14) into (3.13), we derive

$$\|u(t)-u^*(t)\|_V^2 \leq \frac{\Xi_\xi(t,u)}{c_0} + \frac{TL_{\mathscr{R}}^2}{4c_0^2}\int_0^t \|u(s)-u^*(s)\|_V^2\,ds$$

for all $t \in [0,T]$, where $c_0 = \frac{m_g - m_J\|\gamma\|^2_{L(V,X)} - \frac{1}{2\xi}}{2}$. Invoking Gronwall's inequality for the above inequality yields

$$\begin{aligned}
&\|u^*(t)-u(t)\|_V^2\\
\leq&\frac{\Xi_\xi(t,u)}{c_0} + \frac{TL_{\mathscr{R}}^2}{4c_0^2}\int_0^t \frac{\Xi_\xi(s,u)}{c_0}\cdot exp\left\{\frac{TL_{\mathscr{R}}^2}{4c_0^2}(t-s)\right\}ds
\end{aligned}$$

for all $t \in [0,T]$. In addition, Theorem 3.2(ii) indicates that $t \mapsto \Xi_\xi(t,u)$ belongs to $L^\infty_+(0,T)$. Then, from the above inequality, we have

$$\begin{aligned}
&\|u^*(t)-u(t)\|_V\\
\leq&\sqrt{\frac{\Xi_\xi(t,u)}{c_0} + \frac{TL_{\mathscr{R}}^2}{4c_0^2}\int_0^t \frac{\Xi_\xi(s,u)}{c_0}\cdot exp\left\{\frac{TL_{\mathscr{R}}^2}{4c_0^2}(t-s)\right\}ds}
\end{aligned}$$

for all $t \in [0,T]$. Consequently, for each function $u \in C([0,T];V)$, we are able to find function $\Pi_u\colon [0,T] \to R_+$ defined by

$$\Pi_u(t) := \sqrt{\frac{\Xi_\xi(t,u)}{c_0} + \frac{TL_{\mathscr{R}}^2}{4c_0^2}\int_0^t \frac{\Xi_\xi(s,u)}{c_0}\cdot exp\left\{\frac{TL_{\mathscr{R}}^2}{4c_0^2}(t-s)\right\}ds} \text{ for all } t \in [0,T]$$

such that

$$\|u^*(t)-u(t)\|_V \leq \Pi_u(t)$$

for all $t \in [0,T]$. Whereas, from Theorem 3.2(ii), it is not difficult to see that $\Pi_u \in L_+^\infty(0,T)$. Therefore, we conclude that inequality (3.7) is valid. □

4. A Frictional Contact Problem with Long Memory Effect

In order to illustrate the application of our theoretical results established in Section 3, this section is concerned with the study of a quasistatic contact mechanics problem with long memory effect in which the contact problem can be formulated by a variational-hemivariational inequality with history-dependent operator (see Problem 4.2).

The physical setting of the contact problem is described as follows. An elastic body occupies an open bounded connected set $\Omega \subset R^d (d = 2,3)$ with a Lipschitz boundary $\partial\Omega = \Gamma$. The boundary Γ is divided into three disjoint measurable parts Γ_D, Γ_N and Γ_C such that meas $(\Gamma_D) > 0$. Set $\mathscr{Q} = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$, $\Sigma_D = \Gamma_D \times (0,T)$, $\Sigma_N = \Gamma_N \times (0,T)$, and $\Sigma_C = \Gamma_C \times (0,T)$, and denote by $\boldsymbol{x} = (x_i)$ the spatial variable in $\overline{\Omega}$. Let $\boldsymbol{\nu} = (\nu_i)$ be the outward unit normal on Γ. Here and below the indices i and j run between 1 and d. We denote by $\boldsymbol{u} = (u_i)$, $\boldsymbol{\sigma} = (\sigma_{ij})$, and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ the displacement vector, the stress tensor and the linearized strain tensor, respectively, where the component ε_{ij} of the linearized strain tensor $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is given by

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

with $u_{i,j} = \partial u_i / \partial x_j$. For a vector field, the normal and tangential components of $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ on the boundary are given by

$$u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu},$$
$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}.$$

Note that, from time to time, we do not indicate the dependence of various functions on the spatial variable $\boldsymbol{x}$. Furthermore, the symbol $\mathbb{S}^d$ stands for the space of symmetric matrices of order d. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{u}\|_{\mathbb{R}^d} = (\boldsymbol{u} \cdot \boldsymbol{u})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{u} = (u_i), \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau} : \boldsymbol{\tau})^{\frac{1}{2}} \quad \text{for all } \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively. With these preliminaries, the classical formulation of the quasi-static contact problem is stated as follows.

Problem 4.1. Find a displacement field $\boldsymbol{u} \colon \mathscr{Q} \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} \colon \mathscr{Q} \to \mathbb{S}^d$ such that

$$\boldsymbol{\sigma}(t) \in \mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds + \partial_C I_M(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \quad \text{in } \mathscr{Q}, \tag{4.1}$$
$$\mathrm{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = 0 \quad \text{in } \mathscr{Q}, \tag{4.2}$$
$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on } \Sigma_D, \tag{4.3}$$
$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_N(t) \quad \text{on } \Sigma_N, \tag{4.4}$$
$$-\sigma_\nu(t) \in \partial j_\nu(u_\nu(t)) \quad \text{on } \Sigma_C, \tag{4.5}$$
$$-\boldsymbol{\sigma}_\tau(t) \in \partial j_\tau(\boldsymbol{u}_\tau(t)) \quad \text{on } \Sigma_C. \tag{4.6}$$

We present a short description of the equations and relations in Problem 4.1. Inclusion (4.1) represents the constitutive law of the locking material with long memory effect in which $\mathscr{A}$ stands for the elasticity operator, $\mathscr{B}$ is the relaxation tensor, and I_M is the indicator function of the set $M \subset \mathbb{S}^d$, and $\partial_C I_M$ is the convex subdifferential of I_M, where M is defined by

$$M := \{\boldsymbol{xi} \in \mathbb{S}^d \mid \|\boldsymbol{xi}\|_{\mathbb{S}^d} \leq k_0\} \tag{4.7}$$

for some $k_0 > 0$. Particularly, if $\mathscr{B} = 0$, then inclusion (4.1) becomes the constitutive law of the locking material $\boldsymbol{\sigma}(t) \in \mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \partial_C I_M(\varepsilon(\boldsymbol{u}(t)))$, which was studied by Sofonea [42]. Equation (4.2) is the equilibrium equation where Div represents the divergence operator, i.e., $\mathrm{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$, and $\boldsymbol{f}_0$ denotes the density of the body forces. Conditions (4.3) and (4.4) are the classical displacement and traction boundary conditions, respectively.

The contact condition (4.5) is called a multivalued normal contact boundary condition, which is described by the subgradient of a nonconvex functional j_ν, where j_ν is assumed to be locally Lipschitz in its last variable. On the other hand, the general tangential contact condition (4.6) is governed by the subgradient of a nonconvex functional j_τ . In fact, such kind of contact conditions have been treated in many papers; see e.g., [43, 44].

In order to establish the variational formulation of contact problem (4.1)–(4.6), we introduce the following function spaces

$$V = \{\boldsymbol{v} \in H^1(\Omega;\mathbb{R}^d) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}, \;\; H = L^2(\Omega;\mathbb{R}^d), \;\; \mathscr{H} = L^2(\Omega;\mathbb{S}^d).$$

It is known that $\mathscr{H}$ is a Hilbert space equipped with inner products

$$\langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_{\mathscr{H}} = \int_\Omega \sigma_{ij}(\boldsymbol{x})\tau_{ij}(\boldsymbol{x})\, d\boldsymbol{x} \quad \text{for all } \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathscr{H}$$

and associated norm $\|\cdot\|_{\mathscr{H}}$. Besides, since meas $(\Gamma_D) > 0$, from Korn's inequality, we can see that the space V is also a Hilbert space with the inner product given by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_V = \langle \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_{\mathscr{H}} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V$$

and the associated norm $\|\cdot\|_V$. Also, we denote by $\gamma\colon V \to L^2(\Gamma_C;\mathbb{R}^d)$ the trace operator. Subsequently, the trace of an element $\boldsymbol{v} \in V$ is denoted by $\boldsymbol{v}$. In addition, we consider the space of the forth-order tensor fields defined by

$$\mathscr{Q}_\infty := \{\mathscr{E} = (\mathscr{E}_{ijkl}) \mid \mathscr{E}_{ijkl} = \mathscr{E}_{jikl} = \mathscr{E}_{klij}, 1 \leq i,j,k,l \leq d\}.$$

It is obvious that $\mathscr{Q}$ is a real Banach space with the usual norm

$$\|\mathscr{E}\|_{\mathscr{Q}_\infty} = \max_{1 \leq i,j,k,l \leq d} \|\mathscr{E}_{ijkl}\|_{L^\infty(\Omega)} \quad \text{for all } \mathscr{E} \in \mathscr{Q}_\infty.$$

We list now the assumption on the data to Problem 4.1. Assume that the elasticity operator $\mathscr{A}$ and relaxation tensor $\mathscr{B}$ satisfy the following conditions:

$\underline{H(\mathscr{A})}$: $\mathscr{A}\colon \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ is such that

(i) for each $\boldsymbol{\varepsilon} \in \mathbb{S}^d$, function $\boldsymbol{x} \mapsto \mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})$ is measurable on Ω;
(ii) for a.e. $\boldsymbol{x} \in \Omega$, function $\boldsymbol{\varepsilon} \mapsto \mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})$ is continuous in $\mathbb{S}^d$;
(iii) there exist a function $a_0 \in L^2(\Omega)$ with $a_0(\boldsymbol{x}) \geq 0$ for a.e. $\boldsymbol{x} \in \Omega$ and a constant $a_1 > 0$ such that

$$\|\mathscr{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})\|_{\mathbb{S}^d} \leq a_0(\boldsymbol{x}) + a_1 \|\boldsymbol{\varepsilon}\|_{\mathbb{S}^d}$$

for all $\boldsymbol{\varepsilon} \in \mathbb{S}^d$ and a.e. $\boldsymbol{x} \in \Omega$;

(iv) there exists a constant $m_{\mathscr{A}} > 0$ such that

$$(\mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathscr{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathscr{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2_{\mathbb{S}^d}$$

for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$ and a.e. $\boldsymbol{x} \in \Omega$.

$\underline{H(\mathscr{B})}$: $\mathscr{B} \in C([0,T];\mathscr{Q}_\infty)$.

The potential functions j_ν and j_τ enjoy the following properties:

$\underline{H(j_\nu)}$: $j_\nu\colon \Gamma_C \times R \to R$ is such that

(i) for each $s \in R$, $\boldsymbol{x} \mapsto j_\nu(\boldsymbol{x},s)$ is measurable on Γ_C, and $\boldsymbol{x} \mapsto j_\nu(\boldsymbol{x},0)$ belongs to $L^1(\Gamma_C)$;
(ii) for a.e. $\boldsymbol{x} \in \Gamma_C$, $s \mapsto j_\nu(\boldsymbol{x},s)$ is locally Lipschitz continuous;
(iii) there exist a function $c_{0\nu} \in L^2(0,T)$ with $c_{0\nu}(\boldsymbol{x}) \geq 0$ for a.e. $\boldsymbol{x} \in \Gamma_C$ and a constant $c_{1\nu} \geq 0$ satisfying

$$|\partial j_\nu(\boldsymbol{x},s)| \leq c_{0\nu}(\boldsymbol{x}) + c_{1\nu}|s|$$

for all $s \in R$ and a.e. $\boldsymbol{x} \in \Gamma_C$;
(iv) there exists a constant $m_{j_\nu} \geq 0$ such that

$$j^0_\nu(\boldsymbol{x},s_1,s_2 - s_1) + j^0_\nu(\boldsymbol{x},s_2,s_1 - s_2) \leq m_{j_\nu}|s_1 - s_2|^2$$

for all $s_1, s_2 \in R$ and a.e. $\boldsymbol{x} \in \Gamma_C$.

$\underline{H(j_\tau)}$: $j_\tau\colon \Gamma_C \times R^d \to R$ is such that

(i) for each $\boldsymbol{\xi} \in R^d$, $\boldsymbol{x} \mapsto j_\tau(\boldsymbol{x},\boldsymbol{\xi})$ is measurable on Γ_C, and $\boldsymbol{x} \mapsto j_\tau(\boldsymbol{x},0)$ belongs to $L^1(\Gamma_C)$;
(ii) for a.e. $\boldsymbol{x} \in \Gamma_C$, $\boldsymbol{\xi} \mapsto j_\tau(\boldsymbol{x},\boldsymbol{\xi})$ is locally Lipschitz continuous;
(iii) there exist a function $c_{0\tau} \in L^2(\Gamma_C)$ with $c_{0\tau}(\boldsymbol{x}) \geq 0$ for a.e. $\boldsymbol{x} \in \Gamma_C$ and a constant $c_{1\tau} \geq 0$ satisfying

$$\|\partial j_\tau(\boldsymbol{x},\boldsymbol{\xi})\|_{R^d} \leq c_{0\tau}(\boldsymbol{x}) + c_{1\tau}\|\boldsymbol{\xi}\|_{R^d}$$

for all $\boldsymbol{\xi} \in R^d$ and a.e. $\boldsymbol{x} \in \Gamma_C$;
(iv) there exists a constant $m_{j_\tau} \geq 0$ such that

$$j^0_\tau(\boldsymbol{x},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + j^0_\tau(\boldsymbol{x},\boldsymbol{\xi}_2,\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \leq m_{j_\tau}\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2_{R^d}$$

for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in R^d$ and a.e. $\boldsymbol{x} \in \Gamma_C$.

The densities of body forces $\boldsymbol{f}_0$ and surface traction $\boldsymbol{f}_N$ fulfill the following regularity

$$\boldsymbol{f}_0 \in C([0,T];L^2(\Omega;R^d)) \quad \boldsymbol{f}_N \in C([0,T];L^2(\Gamma_N;R^d)). \tag{4.8}$$

We are now in a position to derive the variational formulation of Problem 4.1. Let us consider the admissible set K of displacement fields defined by

$$K = \{\boldsymbol{v} \in V \mid \boldsymbol{\varepsilon}(\boldsymbol{v}) \in M \text{ in } \Omega\}. \tag{4.9}$$

By the definition of M, it is not difficult to see that the admissible set K is nonempty, bounded, closed, and convex in V. Suppose that $\boldsymbol{u}\colon \mathscr{Q} \to R^d$ and $\boldsymbol{\sigma}\colon \mathscr{Q} \to \mathbb{S}^d$ are two smooth functions such that (4.1)–(4.6) hold. Because of $\boldsymbol{u}(t) \in K$, we have $\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \in M$ for a.e. $(\boldsymbol{x},t) \in \mathscr{Q}$, i.e., $\partial_C I_M(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) = \{\boldsymbol{0}\}$ Hence,

$$\boldsymbol{\sigma}(t) = \mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))) + \int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds \quad \text{in } \mathscr{Q}. \tag{4.10}$$

For any $\boldsymbol{v} \in K$ fixed, we multiply the equilibrium equation (4.2) by $\boldsymbol{v} - \boldsymbol{u}(t)$ and then apply the Green formula (see e.g. [11, Theorem 2.25]), to obtain

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\rangle_{\mathscr{H}} = \langle \boldsymbol{f}_0(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_H + \int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, d\Gamma. \tag{4.11}$$

We take into account the boundary conditions (4.3) and (4.4) to see that

$$\begin{aligned}
&\int_\Gamma \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, d\Gamma \\
=& \int_{\Gamma_C} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, d\Gamma + \int_{\Gamma_D} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, d\Gamma + \int_{\Gamma_N} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, d\Gamma \\
=& \int_{\Gamma_C} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, d\Gamma + \int_{\Gamma_N} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, d\Gamma \\
=& \langle \boldsymbol{f}_N(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{L^2(\Gamma_N;R^d)} + \int_{\Gamma_C} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t))\, d\Gamma.
\end{aligned}$$

The latter combined with the equation (4.11) implies

$$\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\rangle_{\mathscr{H}} = \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^* \times V} + \int_{\Gamma_C} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) d\Gamma, \tag{4.12}$$

where the element $\boldsymbol{f} \in C([0,T];V^*)$ is defined by

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^* \times V} = \langle \boldsymbol{f}_0(t), \boldsymbol{v}\rangle_H + \langle \boldsymbol{f}_N(t), \boldsymbol{v}\rangle_{L^2(\Gamma_N;R^d)} \quad \text{for all } \boldsymbol{v} \in V \text{ and } t \in [0,T]. \tag{4.13}$$

Note that

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}(t)) = \sigma_\nu(t)(v_\nu - u_\nu(t)) + \boldsymbol{\sigma}_\tau(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t)). \tag{4.14}$$

Taking into account (4.12) and (4.14), we have

$$\begin{aligned}
\langle \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\rangle_{\mathscr{H}} =& \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^* \times V} + \int_{\Gamma_C} \sigma_\nu(t)(v_\nu - u_\nu(t))\, d\Gamma \\
&+ \int_{\Gamma_C} \boldsymbol{\sigma}_\tau(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))\, d\Gamma.
\end{aligned} \tag{4.15}$$

By the definition of generalized Clarke subgradient and boundary conditions (4.5) and (4.6), we have

$$\begin{aligned}
-\int_{\Gamma_C} \sigma_\nu(t)(v_\nu - u_\nu(t))\, d\Gamma &\le \int_{\Gamma_C} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t))\, d\Gamma \\
-\int_{\Gamma_C} \boldsymbol{\sigma}_\tau(t) \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))\, d\Gamma &\le \int_{\Gamma_C} j_\tau^0(\boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))\, d\Gamma.
\end{aligned} \tag{4.16}$$

Inserting (4.16) and (4.10) into (4.15), we have

$$\begin{aligned}
&\langle \mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\rangle_{\mathscr{H}} + \left\langle \int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})(t) \right\rangle_{\mathscr{H}} \\
&+ \int_{\Gamma_C} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t)) + j_\tau^0(\boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))\, d\Gamma \\
\ge& \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t)\rangle_{V^* \times V}.
\end{aligned}$$

Therefore, we obtain the variational formulation of Problem 4.1 as follows

Problem 4.2. Find a displacement field $\boldsymbol{u} \in C([0,T];K)$ such that

$$\langle \mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \rangle_{\mathscr{H}} + \left\langle \int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})(t) \right\rangle_{\mathscr{H}} \tag{4.17}$$
$$+ \int_{\Gamma_C} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t)) + j_\tau^0(\boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))\,d\Gamma$$
$$\geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t) \rangle_{V^* \times V}.$$

for all $\boldsymbol{v} \in K$ and $t \in [0,T]$.

Arguing as in the proof of [43, 44], we can apply Theorem 2.1 with $\varphi = 0$ to obtain the unique solvability of Problem 4.2 by the following theorem.

Theorem 4.1. *Assume that $H(\mathscr{A})$, $H(\mathscr{B})$, $H(j_\nu)$, $H(j_\tau)$, and (4.8) are satisfied. If, in addition, the inequality $m_{\mathscr{A}} > \|\gamma\|^2 \max\,\{m_{j_\nu}, m_{j_\tau}\}$ holds, then Problem 4.2 has a unique solution $\boldsymbol{u} \in C([0,T];K)$.*

Moreover, we are going to establish a global error bound to Problem 4.1. Let $\xi > 0$ be a given parameter. Let us consider the regularized gap function $\Xi_\xi \colon [0,T] \times C([0,T];K) \to R$ defined by

$$\Xi_\xi(t,\boldsymbol{u}) = \sup_{\boldsymbol{v} \in K} \Bigg(\langle \mathscr{A}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_{\mathscr{H}} + \left\langle \int_0^t \mathscr{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{v}) \right\rangle_{\mathscr{H}}$$
$$- \int_{\Gamma_C} j_\nu^0(u_\nu(t); v_\nu - u_\nu(t))\,d\Gamma - \int_{\Gamma_C} j_\tau^0(\boldsymbol{u}_\tau(t); \boldsymbol{v}_\tau - \boldsymbol{u}_\tau(t))\,d\Gamma$$
$$+ \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t) \rangle_{V^* \times V} - \frac{1}{2\xi}\|\boldsymbol{u}(t) - \boldsymbol{v}\|_V^2 \Bigg) \tag{4.18}$$

for all $\boldsymbol{u} \in C([0,T];K)$ and $t \in [0,T]$. Employing Theorems 3.1–3.3 and 4.1, we have the following results.

Theorem 4.2. *Let $\boldsymbol{u}^* \in C([0,T];K)$ be the unique solution to Problem 4.1. Assume that $H(\mathscr{A})$, $H(\mathscr{B})$, $H(j_\nu)$, $H(j_\tau)$, and (4.8) are satisfied. If, in addition, $m_{\mathscr{A}} > \|\gamma\|^2 \max\{m_{j_\nu}, m_{j_\tau}\}$ is satisfied, then the following statements hold:*

- (i) *For any parameter $\xi > 0$ fixed, the function $\Xi_\xi \colon [0,T] \times C([0,T];K) \to R$ defined in (4.18) is a regularized gap function for inequality (4.17).*
- (ii) *Let parameter $\xi > 0$ be such that $m_{\mathscr{A}} - \|\gamma\|^2 m_{j_\nu} > \frac{1}{2\xi}$. Then, for each function $\boldsymbol{u} \in C([0,T];K)$ the estimate holds*

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^*(t)\|_V \leq \Pi_{\boldsymbol{u}}(t) \quad \text{for all } t \in [0,T],$$

where the function $\Pi_{\boldsymbol{u}} \colon [0,T] \to R_+$ is defined by

$$\Pi_{\boldsymbol{u}}(t) := \sqrt{\frac{\Xi_\xi(t,\boldsymbol{u})}{c_0} + \frac{TL_{\mathscr{R}}^2}{4c_0^2}\int_0^t \frac{\Xi_\xi(s,\boldsymbol{u})}{c_0} \cdot exp\left\{\frac{TL_{\mathscr{R}}^2}{4c_0^2}(t-s)\right\} ds} \text{ for all } t \in [0,T]$$

with $L_{\mathscr{R}} := \sup_{t \in [0,T]} \|\mathscr{B}(t)\|_{\mathscr{Q}_\infty}$ and $c_0 = \frac{m_{\mathscr{A}} - \|\gamma\|^2 \max\{m_{j_\nu}, m_{j_\tau}\} - \frac{1}{2\xi}}{2}$.

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