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EXISTENCE OF SOLUTIONS TO A NEW CLASS OF COUPLED VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

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Abstract. The objective of this paper is to introduce and study a complicated nonlinear system, called coupled variational-hemivariational inequalities, which is described by a highly nonlinear coupled system of inequalities on Banach spaces. We establish the nonemptiness and compactness of the solution set to the system. We apply a new proof based on a multivalued version of the Tychonoff fixed point principle in a Banach space combined with the generalized monotonicity arguments, and the elements of the nonsmooth analysis. Our results improve and generalize some earlier theorems obtained for a particular form of the system.

Keywords. Clarke subgradient; Coupled variational-hemivariational inequalities; Nonemptiness and compactness; Tychonoff fixed point theorem.

1. Introduction

In this paper, we study the existence of solutions to a new class of systems of two nonlinear coupled variational-hemivariational inequalities with constraints. Each inequality involves a nonlinear operator, the generalized (Clarke) directional derivative of a locally Lipschitz function, a convex potential, and a constraint set. The main feature of the system is a strong coupling which appears in the nonlinear operators and the generalized directional derivatives. Our results concern the existence and compactness of the solution set to the system, and generalize the results obtained recently in [1] by using a different method.

To introduce the problem we need the following functional framework which is used throughout the paper. Let $(V, \|\cdot\|_V)$ and $(E, \|\cdot\|_E)$ be real reflexive Banach spaces, and $C \subset V$ and $D \subset E$ be nonempty, closed, and convex sets. We are given two nonlinear operators $A\colon E \times V \to V^*$ and $B\colon V \times E \to E^*$, two convex functions $\psi\colon V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ and $\theta\colon E \to \overline{\mathbb{R}}$, two nonlinear functions $J\colon Z_1 \times X \to \mathbb{R}$ and $H\colon Z_2 \times Y \to \mathbb{R}$ (which are locally Lipschitz continuous with

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respect to their second variables), four linear operators $\gamma_1 \colon V \to X$, $\gamma_2 \colon E \to Y$, $\delta_1 \colon E \to Z_1$, and $\delta_2 \colon V \to Z_2$, and two elements $h \in V^*$ and $l \in E^*$. The system of two coupled nonlinear variational-hemivariational inequalities reads as follows.

Problem 1.1. Find $u \in C$ and $w \in D$ satisfying the following inequalities

$$\langle A(w,u), v-u\rangle_V + J^0(\delta_1 w, \gamma_1 u; \gamma_1(v-u)) + \psi(v) - \psi(u) \geq \langle h, v-u\rangle_V \tag{1.1}$$

for all $v \in C$, and

$$\langle B(u,w), z-w\rangle_E + H^0(\delta_2 u, \gamma_2 w; \gamma_2(z-w)) + \theta(z) - \theta(w) \geq \langle l, z-w\rangle_E \tag{1.2}$$

for all $z \in D$.

It should be mentioned that Problem 1.1 is new and contains many challenging and important problems as its special cases. We point out below several interesting particular cases of Problem 1.1.

(i) If J and H are independent of their first variables, respectively, then Problem 1.1 reduces to the following coupled system: find $(u,w) \in C \times D$ such that

$$\langle A(w,u), v-u\rangle_V + J^0(\gamma_1 u; \gamma_1(v-u)) + \psi(v) - \psi(u) \geq \langle h, v-u\rangle_V \tag{1.3}$$

for all $v \in C$, and

$$\langle B(u,w), z-w\rangle_E + H^0(\gamma_2 w; \gamma_2(z-w)) + \theta(z) - \theta(w) \geq \langle l, z-w\rangle_E \tag{1.4}$$

for all $z \in D$. This kind of coupled inequalities (1.3)–(1.4) has not been studied in the literature.

(ii) When $\psi = \theta \equiv 0$, then Problem 1.1 takes the following form of two coupled hemivariational inequalities: find $(u,w) \in C \times D$ satisfying

$$\langle A(w,u), v-u\rangle_V + J^0(\delta_1 w, \gamma_1 u; \gamma_1(v-u)) \geq \langle h, v-u\rangle_V \tag{1.5}$$

for all $v \in C$, and

$$\langle B(u,w), z-w\rangle_E + H^0(\delta_2 u, \gamma_2 w; \gamma_2(z-w)) \geq \langle l, z-w\rangle_E \tag{1.6}$$

for all $z \in D$. To the best of our knowledge, there is no results available in the literature for system (1.5)–(1.6).

(iii) If $J \equiv 0$ and $H \equiv 0$, then Problem 1.1 becomes the following system of coupled variational inequalities: find $(u,w) \in C \times D$ satisfying

$$\langle A(w,u), v-u\rangle_V + \psi(v) - \psi(u) \geq \langle h, v-u\rangle_V \tag{1.7}$$

for all $v \in C$, and

$$\langle B(u,w), z-w\rangle_E + \theta(z) - \theta(w) \geq \langle l, z-w\rangle_E \tag{1.8}$$

for all $z \in D$. This system was considered and investigated in [1] in which the authors applied the Kakutani-Ky Fan fixed point theorem for multivalued operators to prove the existence of the solutions of system (1.7)–(1.8). In this paper, in contrast to [1], we give a new proof which is based on a multivalued version of the Tychonoff fixed point principle in a Banach space combined with the theory of nonsmoth analysis, generalized monotonicity arguments, and the Minty approach.

(iv) When $\theta \equiv 0$, $H \equiv 0$ and $D = E$, then Problem 1.1 can be reformulated as the following variational-hemivariational inequality subjected to a nonlinear equation constraint: find $(u,w) \in C \times D$ such that

$$\begin{cases} \langle A(w,u), v-u\rangle_V + J^0(\delta_1 w, \gamma_1 u; \gamma_1(v-u)) + \psi(v) - \psi(u) \geq \langle h, v-u\rangle_V, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } v \in C, \\ B(u,w) = l. \end{cases}$$

(v) Assume that $\psi \equiv 0$, $\theta \equiv 0$, $J \equiv 0$, $H \equiv 0$, $C = V$, and $D = E$. Then Problem 1.1 is equivalent to the following nonlinear system of two coupled equations: find $(u,w) \in C \times D$ such that

$$\begin{cases} A(w,u) = h, \\ B(u,w) = l. \end{cases}$$

(vi) Suppose that $\theta \equiv 0$, $H \equiv 0$, $D = E$, and B is independent of its first variable. Then Problem 1.1 can be reformulated as the following parameter control system driven by a variational-hemivariational inequality: find $u \in C$ and $w \in W$ such that

$$\langle A(w,u), v-u\rangle_V + J^0(\delta_1 w, \gamma_1 u; \gamma_1(v-u)) + \psi(v) - \psi(u) \geq \langle h, v-u\rangle_V$$

for all $v \in C$, where the admissible set W is defined by $W := \{w \in E \mid B(w) = l\}$.

(vii) If A and J are independent of their first variables, $B \equiv 0$, $\theta \equiv 0$, $H \equiv 0$, and $l = 0$, then Problem 1.1 reduces to the following elliptic variational-hemivariational inequality: find $x \in C$ such that

$$\langle Au, v-u\rangle_V + J^0(\gamma_1(u); \gamma_1(v-u)) + \psi(v) - \psi(u) \geq \langle h, v-u\rangle_V \tag{1.9}$$

for all $v \in C$.

The variational-hemivariational inequalities of form (1.9) were studied from various perspectives. For example, the results on noncoercive hemivariational inequalities can be found in [2] where the equilibrium problems were employed, and in [3] where an application to contact problems in mechanics were treated. Several classes of variational-hemivariational and hemivariational inequalities that model the problems in contact mechanics were also studied in [4, 5]. The nonconvex star-shaped constraints sets in evolution hemivariational inequalities were studied in [6], and singular perturbations of inequality problems were analyzed in [7]. The optimal control problems and inverse problems for the aforementioned inequalities were investigated in [8, 9]. The elliptic variational-hemivariational inequalities were treated in [10, 11], differential hemivariational inequalities in [12], and related double phase obstacle problems were considered in [13, 14]. For other recent results on hemivariational inequalities, we refer, for example, to [15, 16, 17, 18, 19, 20] and the references therein.

The rest of the paper is organized as follows. In Section 2, we recall a preliminary material needed in the sequel. Section 3 is devoted to stating the hypotheses on the data of Problem 1.1, and to delivering the main results of this paper, which contain the nonemptiness and compactness of the solution set to Problem 1.1. In Section 4, the last section, we end this paper with a concluding remark.

2. Mathematical Background

In this section, we recall a necessary preliminary material, which is used throughout the paper. More details can be found in [5, 21, 22, 23, 24].

Let $(E, \|\cdot\|_E)$ be a Banach space, E^* be its dual space, and $\langle\cdot,\cdot\rangle_E$ denote the duality brackets between E^* and E. We adopt the symbols " $\xrightarrow{w}$ " and "$\to$" to denote the weak convergence and the strong convergence in various spaces, respectively.

We recall the definitions and properties of upper semicontinuous multivalued operators.

Definition 2.1. Let Y and Z be topological spaces, $D \subset Y$ be a nonempty set, and $G\colon Y \to 2^Z$ be a multivalued map.

- **(i):** The map G is called upper semicontinuous (u.s.c., for short) at $y \in Y$ if, for each open set $O \subset Z$ such that $G(y) \subset O$, there exists a neighborhood $N(y)$ of y satisfying $G(N(y)) := \cup_{z\in N(y)} G(z) \subset O$. If it holds for each $y \in D$, then G is called to be upper semicontinuous in D.
- **(ii):** The map G is closed at $y \in Y$ if, for every sequence $\{(y_n, z_n)\} \subset \mathrm{Gr}(G)$ satisfying $(y_n, z_n) \to (y, z)$ in $Y \times Z$, it holds $(y, z) \in \mathrm{Gr}(G)$, where $\mathrm{Gr}(G)$ is the graph of the map G defined by $\mathrm{Gr}(G) := \{(y, z) \in Y \times Z \mid z \in G(y)\}$. If it holds for each $y \in Y$, then G is called to be closed (or G has a closed graph).

Let X_1 and X_2 be two Banach spaces. A multivalued map $F\colon X_1 \to 2^{X_2}$ is called sequentially weakly-weakly closed if F is sequentially closed from X_1 endowed with the weak topology into the subsets of X_2 with the weak topology.

The following result provides two useful criteria for the upper semicontinuity of a multivalued map.

Proposition 2.1. *Let $F\colon X \to 2^Y$ with X and Y topological spaces. The following conditions are equivalent:*

- (i): *F is upper semicontinuous;*
- (ii): *for each closed set $C \subset Y$, $F^-(C) := \{x \in X \mid F(x) \cap C \neq \emptyset\}$ is closed in X;*
- (iii): *for each open set $O \subset Y$, $F^+(O) := \{x \in X \mid F(x) \subset O\}$ is open in X.*

The following definitions provide useful notions from the theory of nonsmooth analysis.

Definition 2.2. Let V be a reflexive Banach space, $\psi\colon V \to \overline{\mathbb{R}}$ be a proper, convex and l.s.c. function, and $A\colon V \to 2^{V^*}$ be a multivalued operator. The operator A is said to be

(i) ψ-pseudomonotone if, for any u, $v \in V$ fixed, there exists an element $u^* \in Au$ such that

$$\langle u^*, v-u\rangle_X + \psi(v) - \psi(u) \geq 0,$$

then

$$\langle v^*, v-u\rangle_X + \psi(v) - \psi(u) \geq 0$$

for all $v^* \in A(v)$;

(ii) stable ψ-pseudomonotone with respect to the set $W \subset V^*$ if A and $V \ni u \mapsto Au - w \subset V^*$ are ψ-pseudomonotone for each $w \in W$.

Definition 2.3. Let X be a Banach space with the dual space X^* and norm $\|\cdot\|_X$. A function $J\colon X \to \mathbb{R}$ is called to be a locally Lipschitz continuous at $u \in X$ if there are a neighborhood

$N(u)$ of u and a constant $L_u > 0$ with

$$|J(w) - J(v)| \le L_u \|w - v\|_X \ \text{ for all } \ w, v \in N(u).$$

Given a locally Lipschitz function $J\colon X \to \mathbb{R}$, the generalized (Clarke) directional derivative of J at the point $u \in X$ in the direction $v \in X$, denoted by $J^0(u;v)$, is defined by

$$J^0(u;v) = \limsup_{\lambda \to 0^+,\, w \to u} \frac{J(w + \lambda v) - J(w)}{\lambda}.$$

Further, the generalized subgradient of $J\colon X \to \mathbb{R}$ at $u \in X$ is given by

$$\partial J(u) = \{\, \xi \in X^* \mid J^0(u;v) \ge \langle \xi, v\rangle_{X^* \times X} \ \text{ for all } \ v \in X \,\}.$$

The generalized subgradient and generalized directional derivative of a locally Lipschitz function enjoy nice properties and rich calculus. Here, we summarize some basic results (see, e.g., [5, Proposition 3.23]).

Proposition 2.2. *Let $J\colon X \to \mathbb{R}$ be a locally Lipschitz function. Then*

(i): *for every $x \in X$, the function $X \ni v \mapsto J^0(x;v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $J^0(x;\lambda v) = \lambda J^0(x;v)$ for all $\lambda \ge 0$, $v \in X$, and $J^0(x;v_1+v_2) \le J^0(x;v_1) + J^0(x;v_2)$ for all v_1, $v_2 \in X$, respectively;*

(ii): *for every $x \in X$, the set $\partial J(x)$ is a nonempty, convex, and weakly* compact subset of X^*, which is bounded by the Lipschitz constant $L_x > 0$ of J near x;*

(iii): *the graph of the generalized subgradient operator ∂J of J is closed in $X \times (w^*\text{–}X^*)$ topology, i.e., if $\{x_n\} \subset X$ and $\{\xi_n\} \subset X^*$ are sequences such that $\xi_n \in \partial J(x_n)$ and $x_n \to x$ in X, $\xi_n \xrightarrow{w} \xi$ in X^*, then $\xi \in \partial J(x)$, where, recall, $(w^*\text{–}X^*)$ denotes the space X^* equipped with weak* topology.*

We end this section by recalling a multivalued version of the Tychonoff fixed point principle in a Banach space, its proof can be found in [25, Theorem 8.6].

Theorem 1. *Let C be a bounded, closed, and convex subset of a reflexive Banach space E, and $S\colon C \to 2^C$ be a multivalued map such that*

(i): *S has bounded, closed and convex values;*

(ii): *S is weakly-weakly u.s.c..*

Then S has a fixed point in C.

3. Main Results

This section is devoted to the main results of the paper which include the nonemptiness and compactness of the solution set to Problem 1.1.

Before we state and prove the results, we make the following hypotheses on the data of Problem 1.1.

$\underline{H(0)}$: C and D are nonempty, closed, and convex subsets of V and E, respectively.

$\underline{H(1)}$: $h \in V^*$ and $l \in E^*$.

$\underline{H(2)}$: $\gamma_1\colon V \to X$, $\gamma_2\colon E \to Y$, $\delta_1\colon E \to Z_1$, and $\delta_2\colon V \to Z_2$ are bounded, linear, and compact.

$\underline{H(\psi)}$: $\psi\colon V \to \overline{\mathbb{R}}$ is a convex and lower semicontinuous function such that

$\text{dom}(\psi) := \{u \in V \mid \psi(u) < +\infty\} \cap C \neq \emptyset$.

$\underline{H(\theta)}$: $\theta\colon E \to \overline{\mathbb{R}}$ is a convex and lower semicontinuous function such that $\text{dom}(\theta) := \{w \in E \mid \theta(w) < +\infty\} \cap D \neq \emptyset$.

$\underline{H(J)}$: $J\colon Z_1 \times X \to \mathbb{R}$ is such that

(i) for every $w \in Z_1$, $X \ni u \mapsto J(w,u) \in \mathbb{R}$ is locally Lipschitz continuous;

(ii) there exists a constant $c_J \geq 0$ such that

$$\|\xi\|_{X^*} \leq c_J\,(1+\|u\|_X + \|w\|_{Z_1})$$

for all $\xi \in \partial J(w,u)$, $u \in X$ and $w \in Z_1$;

(iii) the following inequality is valid

$$\limsup_{n\to\infty} J^0(w_n,u;v) \leq J^0(w,u;v),$$

whether u, v, and sequence $\{w_n\} \subset Z_1$ are such that $w_n \to w$ in Z_1 as $n \to \infty$ for some $w \in Z_1$.

$\underline{H(H)}$: $H\colon Z_2 \times Y \to \mathbb{R}$ is such that

(i) for every $u \in Z_2$, $Y \ni w \mapsto H(u,w) \in \mathbb{R}$ is locally Lipschitz continuous;

(ii) there exists a constant $c_H \geq 0$ such that

$$\|\eta\|_{Y^*} \leq c_H\,(1+\|u\|_{Z_2} + \|w\|_Y)$$

for all $\eta \in \partial H(u,w)$, $u \in Z_2$, and $w \in Y$;

(iii) the following inequality is valid

$$\limsup_{n\to\infty} H^0(u_n,w;z) \leq H^0(u,w;z),$$

whether w, $z \in Y$ and sequence $\{w_n\} \subset Z_2$ is such that $w_n \to w$ in Z_2 as $n \to \infty$ for some $w \in Z_2$.

$\underline{H(A)}$: $A\colon E \times V \to V^*$ satisfies the following conditions:

(i) for any $w \in E$ and v, $u \in V$, the following inequality holds

$$\limsup_{\lambda\to 0} \langle A(w,tv+(1-t)u), v-u\rangle_V \leq \langle A(w,u), v-u\rangle_V;$$

(ii) for any $w \in E$ fixed, the multivalued mapping $V \ni u \mapsto A(w,u) + \gamma_1^*\partial J(\delta_1 w, \gamma_1 u) \subset V^*$ is stable ψ-pseudomonotone with respect to $\{h\}$;

(iii) if $\{w_n\} \subset E$ and $\{u_n\} \subset V$ are such that

$$w_n \xrightarrow{w} w \text{ in } E \text{ and } u_n \xrightarrow{w} u \text{ in } V \text{ as } n \to \infty$$

for some $(w,u) \in E \times V$, then

$$\limsup_{n\to\infty} \langle A(w_n,v), v-u_n\rangle_V \leq \langle A(w,v), v-u\rangle_V;$$

(iv) the following growth condition is satisfied

$$\|A(w,u)\|_{V^*} \leq b_A(1+\|u\|_V + \|w\|_E)$$

for all $(w,u) \in E \times V$ with some $b_A > 0$;

(v) there exists a function $r_A\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\langle A(w,u),u\rangle_V - J^0(\delta_1 w,\gamma_1 u;-\gamma_1 u) \geq r_A(\|u\|_V,\|w\|_E)\|u\|_V \text{ for all } u\in V \text{ and } w\in E,$$

and

- for every nonempty and bounded set $O\subset\mathbb{R}_+$, we have $r_A(t,s)\to+\infty$ as $t\to+\infty$ for all $s\in O$,
- for any constants c_1, $c_2\geq 0$, it holds $r_A(t,c_1t+c_2)\to+\infty$ as $t\to+\infty$,
- for sequences $\{s_n\}\subset\mathbb{R}_+$ and $\{t_n\}\subset$ such that

$$s_n\to+\infty,\ t_n\to+\infty \text{ and } \tfrac{t_n}{s_n}\to 0 \text{ as } n\to\infty,$$

  we have $r_A(s_n,t_n)\to+\infty$ as $n\to\infty$.

<u>$H(B)$</u>: $B\colon V\times E\to E^*$ satisfies the following conditions:

(i) for any $u\in V$ and z, $w\in E$, it holds

$$\limsup_{\lambda\to 0}\langle B(u,tz+(1-t)w),z-w\rangle_E \leq \langle B(u,w),z-w\rangle_E;$$

(ii) for each $u\in V$ fixed, the multivalued mapping $E\ni w\mapsto B(u,w)+\gamma_2^*\partial H(\delta_2 u,\gamma_2 w)\subset E^*$ is stable θ-pseudomonotone with respect to $\{l\}$;

(iii) if $\{w_n\}\subset E$ and $\{u_n\}\subset V$ are such that

$$w_n \xrightarrow{w} w \text{ in } E \text{ and } u_n \xrightarrow{w} u \text{ in } V \text{ as } n\to\infty$$

for some $(w,u)\in E\times V$, then

$$\limsup_{n\to\infty}\langle B(u_n,z),z-w_n\rangle_E \leq \langle B(u,z),z-w\rangle_E;$$

(iv) the following growth condition is satisfied

$$\|B(u,w)\|_{E^*}\leq b_B(1+\|u\|_V+\|w\|_E)$$

for all $(w,u)\in E\times V$ with some $b_B>0$;

(v) there exists a function $r_B\colon \mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ such that

$$\langle B(u,w),w\rangle_E - H^0(\delta_2 u,\gamma_2 w;-\gamma_2 w)\geq r_B(\|w\|_E,\|u\|_V)\|w\|_E \text{ for all } u\in V \text{ and } w\in E,$$

and

- for every nonempty and bounded set $O\subset\mathbb{R}_+$, we have $r_B(t,s)\to+\infty$ as $t\to+\infty$ for all $s\in O$,
- for any constants c_1, $c_2\geq 0$, it holds $r_B(t,c_1t+c_2)\to+\infty$ as $t\to+\infty$,
- for sequences $\{s_n\}\subset\mathbb{R}_+$ and $\{t_n\}\subset$ such that

$$s_n\to+\infty,\ t_n\to+\infty \text{ and } \tfrac{t_n}{s_n}\to 0 \text{ as } n\to\infty,$$

  we have $r_B(s_n,t_n)\to+\infty$ as $n\to\infty$.

Theorem 2. *Under the assumptions $H(A)$, $H(B)$, $H(0)$, $H(1)$, $H(2)$, $H(J)$, $H(H)$, $H(\psi)$, and $H(\theta)$, the set of solutions to Problem 1.1, denoted by $\mathbb{S}(h,l)$, is nonempty and weakly compact in $V\times E$.*

Proof. The proof of this theorem is divided into five steps.

Step 1. *If the set $\mathbb{S}(h,l)$ of solutions to Problem 1.1 is nonempty, then $\mathbb{S}(h,l)$ is bounded.*

Assume that $\mathbb{S}(h,l)$ is nonempty. Let $(u,w)\in\mathbb{S}(h,l)$, $u_0\in\operatorname{dom}\psi\cap C$, and $w_0\in\operatorname{dom}\theta\cap D$ be arbitrary fixed. Then,

$$\langle A(w,u),u_0-u\rangle_V+J^0(\delta_1 w,\gamma_1 u;\gamma_1(u_0-u))+\psi(u_0)-\psi(u)\geq\langle h,u_0-u\rangle_V,$$

and

$$\langle B(u,w),w_0-w\rangle_E+H^0(\delta_2 u,\gamma_2 w;\gamma_2(w_0-w))+\theta(w_0)-\theta(w)\geq\langle l,w_0-w\rangle_E.$$

From the subadditivity of $x\mapsto J^0(w,u;x)$ and $x\mapsto H^0(u,w;x)$, we have

$$\begin{aligned}&\langle A(w,u),u\rangle_V-J^0(\delta_1 w,\gamma_1 u;-\gamma_1 u)\\ \leq&\langle A(w,u),u_0\rangle_V+J^0(\delta_1 w,\gamma_1 u;\gamma_1 u_0)+\psi(u_0)-\psi(u)-\langle h,u_0-u\rangle_V,\end{aligned}$$

and

$$\begin{aligned}&\langle B(u,w),w\rangle_E-H^0(\delta_2 u,\gamma_2 w;-\gamma_2 w)\\ \leq&\langle B(u,w),w_0\rangle_E+H^0(\delta_2 u,\gamma_2 w;\gamma_2 w_0)+\theta(w_0)-\theta(w)-\langle l,w_0-w\rangle_E.\end{aligned}$$

Let $\xi\in X^*$ and $\eta\in Y^*$ be such that

$$\begin{aligned}&\xi\in\partial J(\delta_1 w,\gamma_1 u)\ \text{ and }\ \langle\xi,\gamma_1 u_0\rangle_X=J^0(\delta_1 w,\gamma_1 u;\gamma_1 u_0),\\ &\eta\in\partial H(\delta_2 u,\gamma_2 w)\ \text{ and }\ \langle\eta,\gamma_2 w_0\rangle_Y=H^0(\delta_2 u,\gamma_2 w;\gamma_1 w_0).\end{aligned}$$

Recall that ψ and θ are convex and l.s.c., so we invoke [22, Proposition 5.2.25] to find constants α_ψ, α_θ, β_ψ, $\beta_\theta\geq 0$ such that

$$\psi(u)\geq-\alpha_\psi\|u\|_V-\beta_\psi\ \text{ and }\ \theta(w)\geq-\alpha_\theta\|w\|_E-\beta_\theta$$

for all $(u,w)\in V\times E$. We apply the inequalities above and hypotheses $H(A)$(iv)–(v) and $H(B)$(iv)–(v) to infer that

$$\begin{aligned}&r_A(\|u\|_V,\|w\|_E)\|u\|_V\\ \leq&\langle A(w,u),u\rangle_V-J^0(\delta_1 w,\gamma_1 u;-\gamma_1 u)\\ \leq&\langle A(w,u),u_0\rangle_V+\langle\xi,\gamma_1 u_0\rangle_X+\psi(u_0)-\psi(u)-\langle h,u_0-u\rangle_V\\ \leq&b_A(1+\|u\|_V+\|w\|_E)\|u_0\|_V+c_J(1+\|\gamma_1 u\|_X+\|\delta_1 w\|_{Z_1})\|\gamma_1 u_0\|_X\\ &+\psi(u_0)-\psi(u)+\|h\|_{V^*}(\|u_0\|_V+\|u\|_V)\\ \leq&b_A(1+\|u\|_V+\|w\|_E)\|u_0\|_V+c_J(1+\|\gamma_1\|\|u\|_V+\|\delta_1\|\|w\|_E)\|\gamma_1\|\|u_0\|_V\\ &+\psi(u_0)+\alpha_\psi\|u\|_V+\beta_\psi+\|h\|_{V^*}(\|u_0\|_V+\|u\|_V),\end{aligned}$$

and

$$\begin{aligned}&r_B(\|w\|_E,\|u\|_V)\|w\|_E\\ \leq&\langle B(u,w),w\rangle_E-H^0(\delta_2 u,\gamma_2 w;-\gamma_2 w)\\ \leq&\langle B(u,w),w_0\rangle_E+H^0(\delta_2 u,\gamma_2 w;\gamma_w w_0)+\theta(w_0)-\theta(w)-\langle l,w_0-w\rangle_E\\ \leq&b_B(1+\|u\|_V+\|w\|_E)\|w_0\|_E+c_H(1+\|\delta_2\|\|u\|_V+\|\gamma_2\|\|w\|_E)\|\gamma_2\|\|w_0\|_E\\ &+\theta(w_0)+\alpha_\theta\|w\|_E+\beta_\theta+\|l\|_{E^*}(\|w_0\|_E+\|w\|_E).\end{aligned}$$

Hence,

$$\begin{aligned} & r_A(\|u\|_V,\|w\|_E) \\ \le & \frac{b_A(1+\|u\|_V+\|w\|_E)\|u_0\|_V + c_J(1+\|\gamma_1\|\|u\|_V+\|\delta_1\|\|w\|_E)\|\gamma_1\|\|u_0\|_V}{\|u\|_V} \\ & + \frac{\psi(u_0)+\alpha_\psi\|u\|_V+\beta_\psi+\|h\|_{V^*}(\|u_0\|_V+\|u\|_V)}{\|u\|_V}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} & r_B(\|w\|_E,\|u\|_V) \\ \le & \frac{b_B(1+\|u\|_V+\|w\|_E)\|w_0\|_E + c_H(1+\|\delta_2\|\|u\|_V+\|\gamma_2\|\|w\|_E)\|\gamma_2\|\|w_0\|_E}{\|w\|_E} \\ & + \frac{\theta(w_0)+\alpha_\theta\|w\|_E+\beta_\theta+\|l\|_{E^*}(\|w_0\|_E+\|w\|_E)}{\|w\|_E}. \end{aligned} \tag{3.2}$$

Assume that $\mathbb{S}(h,l)$ is unbounded. Without any loss of generality, we may suppose that there exists a sequence $\{(u_n,w_n)\}\subset\mathbb{S}(h,l)$ satisfying $\|u_n\|_V+\|w_n\|_E\to\infty$ as $n\to\infty$, namely, one of the following conditions holds:

$$\|u_n\|_V\uparrow+\infty \text{ as } n\to\infty \text{ and } \{w_n\} \text{ is bounded in } E, \tag{3.3}$$

or

$$\|w_n\|_E\uparrow+\infty \text{ as } n\to\infty \text{ and } \{u_n\} \text{ is bounded in } V, \tag{3.4}$$

or

$$\|w_n\|_E\uparrow+\infty \text{ and } \|u_n\|_V\uparrow+\infty \text{ as } n\to\infty. \tag{3.5}$$

If (3.3) is true, then we obtain from (3.1) that

$$\begin{aligned} & r_A(\|u_n\|_V,\|w_n\|_E) \\ \le & \frac{b_A(1+\|u_n\|_V+\|w_n\|_E)\|u_0\|_V + c_J(1+\|\gamma_1\|\|u_n\|_V+\|\delta_1\|\|w_n\|_E)\|\gamma_1\|\|u_0\|_V}{\|u_n\|_V} \\ & + \frac{\psi(u_0)+\alpha_\psi\|u_n\|_V+\beta_\psi+\|h\|_{V^*}(\|u_0\|_V+\|u_n\|_V)}{\|u_n\|_V}. \end{aligned}$$

Taking the limit as $n\to\infty$ in the inequality above and using assumption $H(A)$(v), it yields

$$\begin{aligned} & +\infty = \lim_{n\to\infty} r_A(\|u_n\|_V,\|w_n\|_E) \\ \le & \lim_{n\to\infty}\Bigg[\frac{b_A(1+\|u_n\|_V+\|w_n\|_E)\|u_0\|_V + c_J(1+\|\gamma_1\|\|u_n\|_V+\|\delta_1\|\|w_n\|_E)\|\gamma_1\|\|u_0\|_V}{\|u_n\|_V} \\ & + \frac{\psi(u_0)+\alpha_\psi\|u_n\|_V+\beta_\psi+\|h\|_{V^*}(\|u_0\|_V+\|u_n\|_V)}{\|u_n\|_V}\Bigg] \\ = & b_A\|u_0\|_V + c_J\|\gamma_1\|^2\|u_0\|_V+\alpha_\psi+\|h\|_{V^*}. \end{aligned}$$

This leads to a contradiction. So, we conclude that $\mathbb{S}(h,l)$ is bounded. Likewise, we can employ the same argument to obtain a contradiction when (3.4) occurs. If (3.5) holds, we distinguish further the following cases:

i) $\frac{\|u_n\|_V}{\|w_n\|_E}\to+\infty$ or $\frac{\|w_n\|_E}{\|u_n\|_E}\to+\infty$ as $n\to\infty$.

ii) there exists $n_0 \in \mathbb{N}$ such that $0 < c_0 \le \frac{\|u_n\|_V}{\|w_n\|_E} \le c_1$ for all $n \ge n_0$ for some $c_0, c_1 > 0$.

Concerning the case i), we only examine the situation if $\frac{\|u_n\|_V}{\|w_n\|_E} \to +\infty$ because the same conclusion can be obtained by using a similar proof when $\frac{\|w_n\|_E}{\|u_n\|_E} \to +\infty$ as $n \to \infty$. Keeping in mind (3.1), one has

$$\begin{aligned}
& r_A(\|u_n\|_V, \|w_n\|_E) \\
\le & \frac{b_A(1+\|u_n\|_V+\|w_n\|_E)\|u_0\|_V + c_J(1+\|\gamma_1\|\|u_n\|_V+\|\delta_1\|\|w_n\|_E)\|\gamma_1\|\|u_0\|_V}{\|u_n\|_V} \\
& + \frac{\psi(u_0)+\alpha_\psi\|u_n\|_V+\beta_\psi+\|h\|_{V^*}(\|u_0\|_V+\|u_n\|_V)}{\|u_n\|_V}.
\end{aligned}$$

By virtue of (3.5) and the condition i), we infer

$$\begin{aligned}
& +\infty = \lim_{n\to\infty} r_A(\|u_n\|_V, \|w_n\|_E) \\
\le & \lim_{n\to\infty} \Bigg[\frac{b_A(1+\|u_n\|_V+\|w_n\|_E)\|u_0\|_V + c_J(1+\|\gamma_1\|\|u_n\|_V+\|\delta_1\|\|w_n\|_E)\|\gamma_1\|\|u_0\|_V}{\|u_n\|_V} \\
& + \frac{\psi(u_0)+\alpha_\psi\|u_n\|_V+\beta_\psi+\|h\|_{V^*}(\|u_0\|_V+\|u_n\|_V)}{\|u_n\|_V} \Bigg] \\
= & b_A\|u_0\|_V + c_J\|\gamma_1\|^2\|u_0\|_V + \alpha_\psi + \|h\|_{V^*}.
\end{aligned}$$

This leads to a contradiction too. This implies that $\mathbb{S}(h,l)$ is bounded. Moreover, if condition ii) holds, then we can use hypothesis $H(B)$(v) to see

$$\begin{aligned}
& +\infty = \lim_{n\to\infty} r_B(\|w_n\|_E, \|u_n\|_V) \\
\le & \frac{b_B(1+\|u_n\|_V+\|w_n\|_E)\|w_0\|_E + c_H(1+\|\delta_2\|\|u_n\|_V+\|\gamma_2\|\|w_n\|_E)\|\gamma_2\|\|w_0\|_E}{\|w_n\|_E} \\
& + \frac{\theta(w_0)+\alpha_\theta\|w_n\|_E+\beta_\theta+\|l\|_{E^*}(\|w_0\|_E+\|w_n\|_E)}{\|w_n\|_E} \\
\le & b_B(c_1+1)\|w_0\|_E + c_H(\|\delta_2\|c_1+\|\gamma_2\|)\|\gamma_2\|\|w_0\|_E + \theta(w_0) + \alpha_\theta + \|l\|_{E^*}.
\end{aligned}$$

This implies that the set $\mathbb{S}(h,l)$ is bounded.

Step 2. *For each $w \in E$ (resp. $u \in V$) fixed, the solution set of inequality problem* (1.1) *(resp.* (1.2)*) is nonempty, bounded, and closed.*

Let $w \in E$ be arbitrary fixed. By the definition of generalized subgradient in the sense of Clarke, we have

$$\begin{aligned}
\langle A(w,u)+\gamma_1^*\xi, u\rangle_V = & \langle A(w,u),u\rangle_V - \langle \xi, -\gamma_1 u\rangle_X \\
\ge & \langle A(w,u),u\rangle_V - J^0(\delta_1 y, \gamma_1 u; -\gamma_1 u)
\end{aligned}$$

for all $\xi \in \partial J(\delta_1 w, \gamma_1 u)$. Taking into account hypothesis $H(A)$(v) and the inequality above, it gives

$$\begin{aligned}
& \frac{\inf_{\xi\in\partial J(\delta_1 w,\gamma_1 u)}\langle A(w,u)+\gamma^*\xi, u\rangle_V}{\|u\|_V} \ge \frac{\langle A(w,u),u\rangle_V - J^0(\delta_1 y,\gamma_1 u;-\gamma_1 u)}{\|u\|_V} \\
\ge & r_A(\|u\|_V, \|w\|_E)
\end{aligned}$$

for all $u \in V$. This means that multivalued operator

$$V \ni u \mapsto A(w,u) + \gamma_1^* \partial J(\delta_1 w, \gamma_1 u) \subset V^*$$

is coercive. In an analogous way, we can verify that, for every $u \in V$ fixed, multivalued operator

$$E \ni w \mapsto B(u,w) + \gamma_2^* \partial H(\delta_2 u, \gamma_2 w) \subset E^*$$

is coercive as well. Therefore, we can invoke the same arguments as in the proof of [3, Theorem 3] to conclude that the solution set of inequality problem (1.1) (resp. (1.2)) is nonempty, bounded, and closed.

Next, we introduce the multivalued map $\Gamma\colon C \times D \to 2^{C \times D}$ defined by

$$\Gamma(u,w) := (\mathscr{P}(w), \mathscr{Q}(u)) \text{ for all } (u,w) \in C \times D,$$

where $\mathscr{P}\colon E \to 2^C$ and $\mathscr{Q}\colon V \to 2^D$ stand for the solution mappings problems (1.1) and (1.2), respectively, namely, $\mathscr{P}(w)$ and $\mathscr{Q}(u)$ are the solution sets of problems (1.1) and (1.2) corresponding to $w \in E$ and $u \in V$, respectively. From the definition of Γ, it is not difficult to show that $(u,w) \in C \times D$ is a fixed point of Γ if and only if it is a solution to Problem 1.1. Based on this property, we are going to verify that Γ has at least one fixed point in $C \times D$.

Step 3. *There exists a bounded, closed, and convex subset $\mathscr{X}$ of $C \times D$ such that Γ maps $\mathscr{X}$ into itself.*

Indeed, it is sufficient to demonstrate that there exists a constant $m_0 > 0$ such that

$$\Gamma(\mathscr{O}(m_0)) \subset \mathscr{O}(m_0), \tag{3.6}$$

where $\mathscr{O}(m_0)$ is defined by

$$\mathscr{O}(m_0) := \{(u,w) \in C \times D \mid \|u\|_V \le m_0 \text{ and } \|w\|_E \le m_0\}.$$

Arguing by contradiction, there is no $m_0 > 0$ such that (3.6) holds. So, for each $n \in \mathbb{N}$, there exist sequences (u_n, w_n), $(v_n, z_n) \in C \times D$ satisfying

$$(u_n, w_n) \in \mathscr{O}(n),\ (v_n, z_n) \in \Gamma(u_n, w_n) \text{ and } \|v_n\|_V > n \text{ or } \|z_n\|_E > n.$$

Now, we suppose that $\|v_n\|_V > n$. It follows from (3.1) that

$$\begin{aligned} &r_A(\|v_n\|_V, \|w_n\|_E) \\ &\le \frac{b_A(1 + \|v_n\|_V + \|w_n\|_E)\|u_0\|_V + c_J(1 + \|\gamma_1\|\|v_n\|_V + \|\delta_1\|\|w_n\|_E)\|\gamma_1\|\|u_0\|_V}{\|v_n\|_V} \\ &\quad + \frac{\psi(u_0) + \alpha_\psi\|v_n\|_V + \beta_\psi + \|h\|_{V^*}(\|u_0\|_V + \|v_n\|_V)}{\|v_n\|_V}. \end{aligned}$$

Recalling that $\|w_n\|_E \le n < \|v_n\|_V$, we apply hypothesis $H(A)$(v) to find that

$$\begin{aligned} &+\infty = \lim_{n\to\infty} r_A(\|v_n\|_V, \|w_n\|_E) \\ &\le \lim_{n\to\infty} \Bigg[\frac{b_A(1 + \|v_n\|_V + \|w_n\|_E)\|u_0\|_V + c_J(1 + \|\gamma_1\|\|v_n\|_V + \|\delta_1\|\|w_n\|_E)\|\gamma_1\|\|u_0\|_V}{\|v_n\|_V} \\ &\quad + \frac{\psi(u_0) + \alpha_\psi\|v_n\|_V + \beta_\psi + \|h\|_{V^*}(\|u_0\|_V + \|v_n\|_V)}{\|v_n\|_V} \Bigg] \\ &\le 2b_A\|u_0\|_V + c_J(\|\gamma_1\| + \|\delta_1\|)\|\gamma_1\|\|u_0\|_V + \alpha_\psi + \|h\|_{V^*}. \end{aligned}$$

Hence we reach a contradiction. As we did before, it can also lead to a contraction when the case $\|z_n\|_E > n$ occurs. This means that there exists a constant $m_0 > 0$ such that (3.6) is valid. Thus we conclude that there exists a bounded, closed, and convex subset $\mathscr{X}$ of $C \times D$ such that Γ maps $\mathscr{X}$ into itself.

Step 4. Γ *is weakly-weakly upper semicontinuous.*

Let $\mathscr{M} \subset C \times D$ be an arbitrary weakly closed set such that $\Gamma^-(\mathscr{M}) \neq \emptyset$. From Proposition 2.1, it is sufficient to verify that $\Gamma^-(\mathscr{M})$ is weakly closed in $V \times E$. Let $\{(v_n, z_n)\} \subset \Gamma^-(\mathscr{M})$ be such that

$$(u_n, w_n) \xrightarrow{w} (u, w) \text{ in } V \times E \text{ as } n \to \infty$$

for some $(u, w) \in V \times E$. Hence, for every $n \in \mathbb{N}$, we are able to find $(u_n, w_n) \in V \times E$ satisfying

$$(v_n, z_n) \in \Gamma(u_n, w_n) \cap \mathscr{M}.$$

From (3.1) and (3.2), we have

$$\begin{aligned}
&r_A(\|v_n\|_V, \|w_n\|_E) \\
\leq &\frac{b_A(1 + \|v_n\|_V + \|w_n\|_E)\|u_0\|_V + c_J(1 + \|\gamma_1\|\|v_n\|_V + \|\delta_1\|\|w_n\|_E)\|\gamma_1\|\|u_0\|_V}{\|v_n\|_V} \\
&+ \frac{\psi(u_0) + \alpha_\psi\|v_n\|_V + \beta_\psi + \|h\|_{V^*}(\|u_0\|_V + \|v_n\|_V)}{\|v_n\|_V},
\end{aligned}$$

and

$$\begin{aligned}
&r_B(\|z_n\|_E, \|u_n\|_V) \\
\leq &\frac{b_B(1 + \|u_n\|_V + \|z_n\|_E)\|w_0\|_E + c_H(1 + \|\delta_2\|\|u_n\|_V + \|\gamma_2\|\|z_n\|_E)\|\gamma_2\|\|w_0\|_E}{\|z_n\|_E} \\
&+ \frac{\theta(w_0) + \alpha_\theta\|z_n\|_E + \beta_\theta + \|l\|_{E^*}(\|w_0\|_E + \|z_n\|_E)}{\|z_n\|_E}.
\end{aligned}$$

Combining the latter with $H(A)$(v), $H(B)$(v), we infer that sequence $\{(v_n,\, z_n)\}$ is bounded in $V \times E$. Passing to a relabeled subsequence if necessary, we may suppose that

$$(v_n, z_n) \xrightarrow{w} (v, z) \text{ in } V \times E \text{ as } n \to \infty$$

for some $(v, z) \in V \times E$.

Next, for each $n \in \mathbb{N}$, let $\xi_n \in X^*$ and $\eta_n \in Y^*$ be such that

$$\begin{aligned}
\langle \xi_n, \gamma_1(x - v_n)\rangle_X &= J^0(\delta_1 w_n, \gamma_1 v_n; \gamma_1(x - v_n)), \\
\langle \eta_n, \gamma_2(y - z_n)\rangle_Y &= H^0(\delta_2 u_n, \gamma_2 z_n; \gamma_2(y - z_n)).
\end{aligned}$$

Hence,

$$\langle A(w_n, v_n) + \gamma_1^* \xi_n, x - v_n\rangle_V + \psi(x) - \psi(v_n) \geq \langle h, x - v_n\rangle_V$$

for all $x \in C$, and

$$\langle B(u_n, z_n) + \gamma_2^* \eta_n, y - z_n\rangle_E + \theta(y) - \theta(z_n) \geq \langle l, y - z_n\rangle_E$$

for all $y \in D$. Then, we use the hypotheses $H(A)$(ii) and $H(B)$(ii) to find that

$$\begin{aligned}&\langle h, x-v_n\rangle_V\\ \leq&\langle A(w_n,x)+\gamma_1^*\alpha_n, x-v_n\rangle_V+\psi(x)-\psi(v_n)\\ \leq&\langle A(w_n,x),x-v_n\rangle_V+J^0(\delta_1 w_n,\gamma_1 x;\gamma_1(x-v_n))+\psi(x)-\psi(v_n)\end{aligned}$$

for all $\alpha_n \in \partial J(\delta_1 w_n, \gamma_1 x)$ and $x \in C$, and

$$\begin{aligned}&\langle l, y-z\rangle_E\\ \leq&\langle B(u_n,y)+\gamma_2^*\beta_n, y-z_n\rangle_E+\theta(y)-\theta(z_n)\\ \leq&\langle B(u_n,y),y-z_n\rangle_E+H^0(\delta_2 u_n,\gamma_2 y;\gamma_2(y-z_n))+\theta(y)-\theta(z_n)\end{aligned}$$

for all $\beta_n \in \partial H(\delta_2 u_n, \gamma_2 y)$ and $y \in D$. Passing to the upper limit as $n \to \infty$ in the inequalities above and using hypotheses $H(J)$(iii), $H(H)$(iii), $H(A)$(iii), and $H(B)$(iii), we obtain

$$\begin{aligned}&\langle h, x-v\rangle_V\\ =&\limsup_{n\to\infty}\langle h, x-v_n\rangle_V\\ \leq&\limsup_{n\to\infty}\left[\langle A(w_n,x),x-v_n\rangle_V+J^0(\delta_1 w_n,\gamma_1 x;\gamma_1(x-v_n))+\psi(x)-\psi(v_n)\right]\\ \leq&\limsup_{n\to\infty}\langle A(w_n,x),x-v_n\rangle_V+\limsup_{n\to\infty}J^0(\delta_1 w_n,\gamma_1 x;\gamma_1(x-v_n))\\ &+\psi(x)-\liminf_{n\to\infty}\psi(v_n)\\ \leq&\langle A(w,x),x-v\rangle_V+J^0(\delta_1 w,\gamma_1 x;\gamma_1(x-v))+\psi(x)-\psi(v),\end{aligned}$$

and

$$\begin{aligned}&\langle l, y-z_n\rangle_E\\ =&\limsup_{n\to\infty}\langle l, y-z_n\rangle_E\\ \leq&\limsup_{n\to\infty}\left[\langle B(u_n,y),y-z_n\rangle_E+H^0(\delta_2 u_n,\gamma_2 y;\gamma_2(y-z_n))+\theta(y)-\theta(z_n)\right]\\ \leq&\limsup_{n\to\infty}\langle B(u_n,y),y-z_n\rangle_E+\limsup_{n\to\infty}H^0(\delta_2 u_n,\gamma_2 y;\gamma_2(y-z_n))\\ &+\theta(y)-\liminf_{n\to\infty}\theta(z_n)\\ \leq&\langle B(u,y),y-z\rangle_E+H^0(\delta_2 u,\gamma_2 y;\gamma_2(y-z))+\theta(y)-\theta(z).\end{aligned}$$

Hence

$$\langle A(w,x),x-v\rangle_V+J^0(\delta_1 w,\gamma_1 x;\gamma_1(x-v))+\psi(x)-\psi(v)\geq\langle h,x-v\rangle_V \tag{3.7}$$

for all $x \in C$, and

$$\langle B(u,y),y-z\rangle_E+H^0(\delta_2 u,\gamma_2 y;\gamma_2(y-z))+\theta(y)-\theta(z)\geq\langle l,y-z_n\rangle_E$$

for all $y \in D$. Let $t \in (0,1)$ and $r \in C$ be arbitrary. We insert $x = x_t := tr + (1-t)v$ into (3.7) and apply the positive homogeneity of $v \mapsto J^0(w,u;v)$ and convexity of ψ to see that

$$\begin{aligned} & t\left[\langle A(w,x_t), r-v\rangle_V + J^0(\delta_1 w, \gamma_1 x_t; \gamma_1(r-v)) + \psi(r) - \psi(v)\right] \\ \geq & t\langle A(w,x_t), r-v\rangle_V + J^0(\delta_1 w, \gamma_1 x_t; t\gamma_1(r-v)) + \psi(x_t) - \psi(v) \\ \geq & t\langle h, r-v\rangle_V. \end{aligned}$$

So, we obtain

$$\langle A(w,x_t), r-v\rangle_V + J^0(\delta_1 w, \gamma_1 x_t; \gamma_1(r-v)) + \psi(r) - \psi(v) \geq \langle h, r-v\rangle_V.$$

Passing to the upper limit as $t \downarrow 0$ in the inequality above and using hypothesis $H(A)$(i) and upper semicontinuity of $(u,v) \mapsto J^0(w,u;v)$, we have

$$\begin{aligned} & \langle A(w,v), r-v\rangle_V + J^0(\delta_1 w, \gamma_1 v; \gamma_1(r-v)) + \psi(r) - \psi(v) \\ \geq & \limsup_{t\downarrow 0}\left[\langle A(w,x_t), r-v\rangle_V + J^0(\delta_1 w, \gamma_1 x_t; \gamma_1(r-v)) + \psi(r) - \psi(v)\right] \\ \geq & \langle h, r-v\rangle_V. \end{aligned}$$

Since $r \in C$ is arbitrary, so we have that $v \in C$ is a solution to problem (1.1) corresponding to $w \in D$. Similarly, we also can obtain that $z \in D$ is a solution to problem (1.2) corresponding to $u \in C$. This implies that $(v,z) \in \Gamma(u,w)$. Due to the weak closedness of $\mathscr{M}$, we obtain that $(v,z) \in \mathscr{M}$, that is, $(u,w) \in \Gamma^-(\mathscr{M})$. Therefore, we use Proposition 2.1 to conclude that Γ is weakly-weakly u.s.c..

We conclude that all the conditions of the Tychonoff theorem, Theorem 1, are verified. Using this theorem, we deduce that Γ has at least one fixed point $(u^*,w^*) \in C \times D$ in $\mathscr{X}$. This implies that $(u^*,w^*) \in C \times D$ is also a solution to Problem 1.1.

Step 5. *The set $\mathbb{S}(h,l)$ is weakly compact.*

From Step 1, we can see that the set $\mathbb{S}(h,l)$ is bounded. Because of the reflexivity of $V \times E$, it is sufficient to demonstrate that $\mathbb{S}(h,l)$ is weakly closed. Let $\{(u_n,w_n)\} \subset \mathbb{S}(h,l)$ be a sequence such that

$$(u_n,w_n) \xrightarrow{w} (u,w) \text{ in } V \times E \text{ as } n \to \infty \tag{3.8}$$

for some $(u,w) \in V \times E$. In virtue of the definition of Γ, it yields $(u_n,w_n) \in \Gamma(u_n,w_n)$. Keeping in mind that Γ is weakly-weakly u.s.c. and has nonempty, bounded, closed, and convex values, it follows from [26, Theorem 1.1.4] that Γ is weakly-weakly closed. The latter together with the convergence (3.8) implies that $(u,v) \in \Gamma(u,v)$. From the definition of Γ, we infer that $(u,v) \in \mathbb{S}(h,l)$, i.e., $\mathbb{S}(h,l)$ is weakly closed. Consequently, we conclude that $\mathbb{S}(h,l)$ is weakly compact in $V \times E$. This completes the proof. □

We formulate several corollaries of Theorem 2. To this end, we need the following hypotheses.

<u>$H(J')$</u>: $J\colon X \to \mathbb{R}$ is locally Lipschitz continuous such that there exists a constant $c_J \geq 0$ such that

$$\|\xi\|_{X^*} \leq c_J(1 + \|u\|_X)$$

for all $\xi \in \partial J(u)$ and $u \in X$.

$\underline{H(H')}$: $H\colon Y\to\mathbb{R}$ is locally Lipschitz continuous such that there exists a constant $c_H\geq 0$ such that

$$\|\eta\|_{Y^*}\leq c_H\,(1+\|w\|_Y)$$

for all $\eta\in\partial H(w)$ and $w\in Y$.

$\underline{H(2')}$: $\gamma_1\colon V\to X$ and $\gamma_2\colon E\to Y$ are bounded, linear, and compact.

$\underline{H(A)}$(ii)': for each $w\in E$ the multivalued mapping $V\ni u\mapsto A(w,u)+\gamma_1^*\partial J(\gamma_1 u)\subset V^*$ is stable ψ-pseudomonotone with respect to $\{h\}$.

$\underline{H(A)}$(v)': there exists a function $r_A\colon\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ such that

$$\langle A(w,u),u\rangle_V - J^0(\gamma_1 u;-\gamma_1 u)\geq r_A(\|u\|_V,\|w\|_E)\|u\|_V \text{ for all } u\in V \text{ and } w\in E,$$

and

- for every nonempty and bounded set $O\subset\mathbb{R}_+$, we have $r_A(t,s)\to+\infty$ as $t\to+\infty$ for all $s\in O$,
- for any constants $c_1,c_2\geq 0$, it holds $r_A(t,c_1t+c_2)\to+\infty$ as $t\to+\infty$,
- for sequences $\{s_n\}\subset\mathbb{R}_+$ and $\{t_n\}\subset$ such that

  $$s_n\to+\infty,\ t_n\to+\infty \text{ and } \tfrac{t_n}{s_n}\to 0 \text{ as } n\to\infty,$$

  we have

  $$r_A(s_n,t_n)\to+\infty \text{ as } n\to\infty.$$

$\underline{H(B)}$(ii)': for each $u\in V$ the multivalued mapping $E\ni w\mapsto B(u,w)+\gamma_2^*\partial H(\gamma_2 w)\subset E^*$ is stable θ-pseudomonotone with respect to $\{l\}$.

$\underline{H(B)}$(v)': there exists a function $r_B\colon\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ such that

$$\langle B(u,w),w\rangle_E - H^0(\gamma_2 w;-\gamma_2 w)\geq r_B(\|w\|_E,\|u\|_V)\|w\|_E \text{ for all } u\in V \text{ and } w\in E,$$

and

- for every nonempty and bounded set $O\subset\mathbb{R}_+$, we have $r_B(t,s)\to+\infty$ as $t\to+\infty$ for all $s\in O$,
- for any constants $c_1,c_2\geq 0$, it holds $r_B(t,c_1t+c_2)\to+\infty$ as $t\to+\infty$,
- for sequences $\{s_n\}\subset\mathbb{R}_+$ and $\{t_n\}\subset$ such that

  $$s_n\to+\infty,\ t_n\to+\infty \text{ and } \tfrac{t_n}{s_n}\to 0 \text{ as } n\to\infty,$$

  we have

  $$r_B(s_n,t_n)\to+\infty \text{ as } n\to\infty.$$

Corollary 3.1. *Suppose that $H(A)$(i), (ii)', (iii), (iv)', $H(B)$(i), (ii)', (iii), (iv)', $H(0)$, $H(1)$, $H(2')$, $H(J')$, $H(H')$, $H(\psi)$, and $H(\theta)$ hold. Then, the set of solutions to problem (1.3)–(1.4) is nonempty and weakly compact in $V\times E$.*

Corollary 3.2. *Suppose that $H(A)$, $H(B)$, $H(0)$, $H(1)$, $H(2)$, $H(J)$, and $H(H)$ are fulfilled. Then, the set of solutions to problem (1.5)–(1.6) is nonempty and weakly compact in $V\times E$.*

$\underline{H(A)}$(ii)”: for each $w \in E$, $V \ni u \mapsto A(w,u) \in V^*$ is stable ψ-pseudomonotone with respect to $\{h\}$.

$\underline{H(A)}$(v)”: there exists a function $r_A \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\langle A(w,u), u\rangle_V \geq r_A(\|u\|_V, \|w\|_E)\|u\|_V \text{ for all } u \in V \text{ and } w \in E,$$

and

- for every nonempty and bounded set $O \subset \mathbb{R}_+$, we have $r_A(t,s) \to +\infty$ as $t \to +\infty$ for all $s \in O$,
- for any constants $c_1, c_2 \geq 0$, it holds $r_A(t, c_1 t + c_2) \to +\infty$ as $t \to +\infty$,
- for sequences $\{s_n\} \subset \mathbb{R}_+$ and $\{t_n\} \subset$ such that

  $$s_n \to +\infty,\ t_n \to +\infty \text{ and } \tfrac{t_n}{s_n} \to 0 \text{ as } n \to \infty,$$

  we have

  $$r_A(s_n, t_n) \to +\infty \text{ as } n \to \infty.$$

$\underline{H(B)}$(ii)”: for each $u \in V$, $E \ni w \mapsto B(u,w) \in E^*$ is stable θ-pseudomonotone with respect to $\{l\}$.

$\underline{H(B)}$(v)”: there exists a function $r_B \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\langle B(u,w), w\rangle_E \geq r_B(\|w\|_E, \|u\|_V)\|w\|_E \text{ for all } u \in V \text{ and } w \in E,$$

and

- for every nonempty and bounded set $O \subset \mathbb{R}_+$, we have $r_B(t,s) \to +\infty$ as $t \to +\infty$ for all $s \in O$,
- for any constants $c_1, c_2 \geq 0$, it holds $r_B(t, c_1 t + c_2) \to +\infty$ as $t \to +\infty$,
- for sequences $\{s_n\} \subset \mathbb{R}_+$ and $\{t_n\} \subset$ such that

  $$s_n \to +\infty,\ t_n \to +\infty \text{ and } \tfrac{t_n}{s_n} \to 0 \text{ as } n \to \infty,$$

  we have

  $$r_B(s_n, t_n) \to +\infty \text{ as } n \to \infty.$$

Corollary 3.3. *Suppose that $H(A)$(i), (ii)”, (iii), (iv)”, $H(B)$(i), (ii)”, (iii), (iv)”, $H(0)$, $H(1)$, $H(\psi)$, and $H(\theta)$ are satisfied. Then, the set of solutions to problem (1.7)–(1.8) is nonempty and weakly compact in $V \times E$.*

Remark 3.1. Corollary 3.3 coincides with a result of Liu-Yang-Zeng-Zhao [1, Theorem 7]. In comparision with that result, in the present paper, we give a new proof, which is based on a multivalued version of the Tychonoff fixed point principle in a Banach space along with the theory of nonsmoth analysis, the generalized monotonicity arguments, and the Minty approach. We conclude that our results are much more general, improve the former one in several directions, and are proved by using a new approach.

4. Conclusions

In the present paper, a coupled system which consists of two nonlinear variational-hemivariational inequalities with constraints in Banach spaces was investigated. A general existence result to the system was established by using a multivalued version of the Tychonoff fixed point principle in a Banach space together with the theory of nonsmoth analysis, generalized monotonicity arguments, and the Minty approach. Our result extends the recent ones obtained in [1, Theorem 7]. There are plenty of problems arising in engineering applications, which can be formulated as a system of coupled variational-hemivariational inequalities. With this motivation, in the future, we plan to utilize the theoretical results established in this paper to study various real engineering problems. Also, we will further develop the mathematical theory for the systems of the variational-hemivariational inequalities, to cover, for instance, stability analysis, optimal control, sensitivity, and homogenization.

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