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OPTIMAL FEEDBACK CONTROL FOR A CLASS OF SECOND-ORDER EVOLUTION DIFFERENTIAL INCLUSIONS WITH CLARKE'S SUBDIFFERENTIAL

JUN CHEN[1], ZHENHAI LIU[1,2,*], FIODAR E. LOMOVTSEV[3], VALERI OBUKHOVSKII[4]

[1]*Guangxi Colleges and Universities Key Laboratory of Optimization Control and Engineering Calculation, Guangxi Minzu University, Nanning 530006, China*
[2]*Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing, Yulin Normal University, Yulin 537000, China*
[3]*Department of Mathematical Cybernetic, Faculty of Mechanics and mathematics, Belarusian State University, Belarus*
[4]*Faculty of Physics and Mathematics, Voronezh State Pedagogical University, 394043 Voronezh, Russia*

Abstract. The goal of this paper is to study optimal feedback control for a class of non-autonomous second-order evolution inclusions with Clarke's subdifferential in a separable reflexive Banach space. We only assume that the second order evolution operator involved satisfies the strong continuity condition instead of the compactness, which was used in previous literature. By using the properties of multimaps and Clarke's subdifferential, we assume some sufficient conditions to ensure the existence of feasible pairs of the feedback control systems. Furthermore, we also prove the existence of optimal control pairs.
Keywords. Clarke's subdifferential; Feedback control; Feasible pair; Optimal control; Second order evolution inclusion.

1. Introduction

In recent years, feedback control theory has made great progress due to its wide applications in science, engineering, and other real world. Feedback control systems are everywhere around us, including trajectory planning of robotic manipulators, guidance of moving targets by tactical missiles, and control of room temperature and chord vibration; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and the references therein for more details.

The hemi-variational inequality is a popular research field in applied mathematics. At present, more and more authors begin to pay attention to the optimal control problems of hemi-variational inequalities and subdifferential inclusions, one can find [11, 12, 13, 16, 17, 18] for more details.

It is worth mentioning that the applications of linear operator semigroup theory to the optimal control of first-order nonlinear systems have yielded good results. Feedback control of first-order evolution equations was studied by many people; see, e.g., [5, 6, 11, 12, 13] and the references therein. For example, the existence results of pulse feedback control systems were

*Corresponding author.
E-mail addresses: 761832439@qq.com (J.Chen); zhhliu@hotmail.com (Z. Liu); lomovcev@bsu.by (F.E. Lomovtsev); valerio-ob2000@mail.ru (V. Obukhovskii).
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given in [3]. With the deepening of the research, the research of differential inclusion type feedback controls is sought after by people. For example, new results of optimal feedback control and controllability for hyperbolic evolution inclusions of Clarke's subdifferential type were given in [5]. In fact, the Clarke's subdifferential plays an important role in the control theory; see, e.g., [6, 19, 20, 21, 22, 23, 24, 25, 26] for more properties and knowledge about Clarke's subdifferential.

Let X be a separable reflexive Banach space and X^* be the dual space of X. In this paper, we firstly consider the following second order evolution differential inclusion:

$$\begin{cases} x''(t) \in A(t)x(t) + Bu(t) + \partial F(t,x(t)), \quad t \in J = [0,b], \\ x(0) = x_0, x'(0) = y_0, \end{cases} \tag{1.1}$$

where $x_0, y_0 \in X$ and $A(t) : D(A(t)) \subset X \to X$ is a closed densely defined operator. The notation ∂F stands for the Clarke's subdifferential of a locally Lipschitz function $F(t,\cdot) : X \to \mathbb{R}$. Further, $B \in \mathscr{L}(V,X)$, where $\mathscr{L}(V,X)$ is the collection of all bounded linear operators from a separable reflexive Banach space V to X, and $u \in L^2(J,V)$ is a control function.

Furthermore, we deal with the existence of feasible pairs of second-order evolution inclusions with feedback control of the following form:

$$\begin{cases} x''(t) \in A(t)x(t) + Bu(t) + \partial F(t,x(t)), & \text{a.e. } t \in J = [0,b], \\ u(t) \in U(t,x(t)), & \text{a.e. } t \in J = [0,b], \\ x(0) = x_0, x'(0) = y_0, \end{cases} \tag{1.2}$$

where $U : [0,b] \times X \to \mathscr{P}(V)$ is a feedback multifunction.

The novelties of this paper are following: First, for the first time, we study an abstract feedback control system described by a second-order non-autonomous system with Clarke's subdifferential, which is still poorly discussed in the literature.

Second, we consider the optimal feedback control systems of second-order non-autonomous systems with Clarke's subdifferential. As far as we know the optimal control of feedback control system (1.2) have not been studied before in the literature. Here, we fill a gap in the studies and provide an existence result of optimal pairs for general abstract feedback control system (1.2).

Third, we work under general assumptions on the data. The second-order evolution operator only satisfies the strong continuity, the assumption that evolution operators often used in previous literature are compact operators is ignored.

The rest of this paper is organized as follows. In Section 2, we introduce some useful preliminaries for our main results. In Section 3, we introduce some important lemmas and theorems, and present the existence of mild solutions of problem (1.1). In Section 4, we present some sufficient conditions to guarantee the existence of feasible pairs of problem (1.2). Furthermore, we give the existence results of the optimal control pair of problem (1.2). In the last section, Section 5, an example will be shown to illustrate our main results.

2. Preliminaries

We first recall some useful notations and well-known results which are used in the article. The norm of a Banach space X is denoted by $\|\cdot\|_X$, and $\langle\cdot,\cdot\rangle_X$ stands for the duality pairing between X and its topological dual X^*. The notation $C(J,X)$ represents the Banach space of all continuous functions from $J = [0,b]$ into X equipped with the norm $\|x\|_C := \|x\|_{C(J,X)} = sup_{t\in J}\|x(t)\|_X$. We denote by " $\to$ " the strong convergence and " $\rightharpoonup$ " the weak convergence in any Banach spaces. Let E_w be the space E endowed with the weak topology. For a set $D \subset E$, the symbol $\overline{D}^w$ denotes the weak closure of D. Denote by $\mathscr{P}(Y)\,[\mathscr{P}_c(Y), \mathscr{P}_{cv}(Y)]$ the collections of all nonempty [respectively, nonempty closed, nonempty closed convex] subsets of any set Y.

Definition 2.1. Let X and Y be two Banach spaces. A multi-valued map $F: X \to \mathscr{P}_c(Y)$ is said to be
(i) sequentially closed if, for any $(x_n, y_n) \in Gr(F) := \{(x,y) \in X \times Y : x \in X, y \in F(x)\}$ with $x_n \to \bar{x}$ in X and $y_n \to \bar{y}$ in Y, $(\bar{x}, \bar{y}) \in Gr(F)$;
(ii) weakly sequentially closed if, for any $(x_n, y_n) \in Gr(F)$ with $x_n \rightharpoonup \bar{x}$ in X and $y_n \rightharpoonup \bar{y}$ in Y, $(\bar{x}, \bar{y}) \in Gr(F)$;
(iii) weakly compact if it maps bounded set in X into relatively compact set in Y_w.

We recall the following well-known result [27, Theorem 4.3.].

Proposition 2.1. *Let X be a reflexive Banach space. A sequence $\{x_n\} \subset C(J,X)$ weakly converges to an element $x \in C(J,X)$ if and only if*

(i) *There exists $N > 0$ such that, for every $n \in \mathbb{N}$ and $t \in J$, $\|x_n(t)\| \le N$;*
(ii) *For every $t \in F, x_n(t) \rightharpoonup x(t)$.*

A function $h: X \to \mathbb{R}$ defined on a Banach space X is called locally Lipschitz if, for all $u \in X$, there exists a neighborhood $\mathscr{N}(u)$ of u such that

$$|h(y) - h(z)| \le K_u \|y - z\|_X \text{ for all } y, z \in \mathscr{N}(u) \text{ with } K_u > 0.$$

Then, we denote by $h^0(u;v)$ the Clarke's generalized directional derivative of h at the point $u \in X$ in the direction $v \in X$ defined by

$$h^0(u;v) = \limsup_{y \to u, \lambda \downarrow 0} \frac{h(y + \lambda v) - h(y)}{\lambda}.$$

Recall also that the Clarke's subdifferential or the generalized gradient of h at the point u, denoted by $\partial h(u)$, is a subset of X^* given by

$$\partial h(u) := \{u^* \in X^* : h^0(u;v) \ge \langle u^*, v \rangle, \quad \forall v \in X\}.$$

Moreover, we collect some significant properties of the generalized gradient and generalized directional derivative of a locally Lipschitz function which will be used in the next sections.

Proposition 2.2. [28] *Let X be a Banach space. If $h\colon X \to \mathbb{R}$ is a locally Lipschitz function on X, then*

(i) *for every $u \in X$, the set $\partial h(u)$ is a nonempty, convex, and weakly* compact subset of X^*. More precisely, $\partial h(u)$ is bounded by the Lipschitz constant $K_u > 0$ of h near u;*
(ii) *the graph of ∂h is closed in $X \times (w^* - X^*)$ topology, namely, if $\{u_k\} \subset X$ and $\{\zeta_k\} \subset X^*$ are sequences such that $\zeta_k \in \partial h(u_k)$ and $u_k \to u$ in X, $\zeta_k \to \zeta$ weakly* in X^*, then $\zeta \in \partial h(u)$, where $w^* - X^*$ denotes the space X^* equipped with weak* topology;*
(iii) *the multivalued map $X \ni u \mapsto \partial h(u) \subseteq X^*$ is upper semicontinuous from X into $w^* - X^*$;*
(iv) *for each $v \in X$, there exists $z_v \in \partial h(u)$ such that*

$$h^0(u;v) = \max\{\langle z, v \rangle \mid z \in \partial h(u)\} = \langle z_v, v \rangle;$$

(v) *the function* $X \ni v \mapsto h^0(u;v) \in \mathbb{R}$ *is finite, positively homogeneous, and subadditive on S, and satisfies* $|h^0(u;v)| \le K_u \|v\|_X$*;*

(vi) $h^0(u;v)$ *as a function of* (u,v) *is upper semicontinuous, and as a function of* v *alone is Lipschitz of rank* K_u *on* X.

(vii) $h^0(u;-v) = (-h)^0(u;v)$ *for all* $u,v \in X$.

At first, we study the following Cauchy problem

$$\begin{cases} x''(t) = A(t)x(t) + \sigma(t), \quad t \in J = [0,b], \\ x(0) = x_0, x'(0) = y_0. \end{cases} \tag{2.1}$$

We assume that the operator $A(t) : D(A(t)) \subset X \to X$ is closed and densely defined, the domain of $A(t)$ does not depend on $t \in J$ and is denoted by $D(A)$, and for each $x \in D(A)$, the function $t \to A(t)x$ is continuous on J. Moreover, there exists an evolution operator $\{S(t,s)|t,s \in J\}$ associated with the family $\{A(t)|t \in J\}$ possessing the following properties.

Definition 2.2. [29] A family $\{S(t,s)|t,s \in J\}$, where $S(t,s) : X \to X$ is a bounded linear operator, is said to be an evolution operator generated by the family $\{A(t)|t \in J\}$ if the following conditions are fulfilled.

(**S1**) For each $x \in X$, the mapping $(t,s) \to S(t,s)x$ is of *class* C^1, and

(a) for each $t \in J, S(t,t) = 0$;

(b) for all $t,s \in J$ and $x \in X$, $\frac{\partial}{\partial t} S(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s} S(t,s)x|_{t=s} = -x$.

(**S2**) For all $t,s \in J$, if $x \in D(A)$, then $S(t,s)x \in D(A)$, the mapping $(t,s) \to S(t,s)x$ is of *class* C^2, and

$(a) \frac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x$;

$(b) \frac{\partial^2}{\partial s^2} S(t,s)x = S(t,s)A(s)x$;

$(c) \frac{\partial^2}{\partial s \partial t} S(t,s)x|_{t=s} = 0$.

(**S3**) For all $t,s \in J$, if $x \in D(A)$, then $\frac{\partial}{\partial t} S(t,s)x \in D(A)$. Furthermore, there exist $\frac{\partial^3}{\partial t^2 \partial s} S(t,s)x$ and $\frac{\partial^3}{\partial s^2 \partial t} S(t,s)x$, and

$(a) \frac{\partial^3}{\partial t^2 \partial s} S(t,s)x = A(t) \frac{\partial}{\partial s} S(t,s)x$;

$(b) \frac{\partial^3}{\partial s^2 \partial t} S(t,s)x = \frac{\partial}{\partial t} S(t,s)A(s)x$,

and the mapping $(t,s) \to A(t) \frac{\partial}{\partial s} S(t,s)x$ is continuous.

For the sake of conveniences, we introduce the operator $C(t,s) = -\frac{\partial}{\partial s}S(t,s)$. In addition, we may find that there exists a positive constant K such that

$$\sup_{0\le t,s\le b} \|C(t,s)\| \le K \quad \textit{and} \sup_{0\le t,s\le b} \|S(t,s)\| \le K. \tag{2.2}$$

And, we denote by K_1 a positive constant such that $\|S(t+\tau,s)-S(t,s)\| \le K_1|\tau|$ for all $s,t,s+\tau \in J$. Following [20, 29], if $\sigma : J \to X$ is integrable, then a mild solution of problem (2.1) is given by $x(t) = C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)\sigma(s)ds$.

3. Main Results

In this section, we provide an existence result of mild solutions for Problem (1.1).

Definition 3.1. Given $u \in L^2(J,V)$ and $B \in \mathscr{L}(V,X)$, a function $x \in C(J,X)$ is called a mild solution to Problem (1.1) if there exists $f \in L^2(J,X)$ with $f(t) \in \partial F(t,x(t))$ for $a.e.t \in J$ such that

$$x(t) = C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)[Bu(s)+f(s)]ds \;\text{ for all } t \in J.$$

To obtain our main results, in what follows, we make the following hypotheses on the data of our problem.

$H(F)$: The function $F : J \times X \to \mathbb{R}$ satisfies the following assumptions:

- (i) for a.e. $t \in J, x \to F(t,x)$ is locally Lipschitz on X;
- (ii) for all $x \in X, t \to F(t,x)$ is measurable on J;
- (iii) there exist a function $\alpha \in L^2(J,\mathbb{R}^+)$ and a constant $\beta \in \mathbb{R}^+$ such that
$$\|\partial F(t,x)\|_{X^*} := \sup\{\|f\|_{X^*} | f \in \partial F(t,x)\} \le \alpha(t) + \beta\|x\|_X,$$
for a.e. $t \in J$ and all $x \in X$;
- (iv) the multimap $\partial F(t,\cdot) : X \to \mathscr{P}(X^*)$ is weakly sequentially closed for a.e.$t \in J$, i.e., it has a weakly sequentially closed graph.

Next, we define a multi-valued map $\mathscr{Q} : C(J,X) \to \mathscr{P}(L^2(J,X^*))$ by

$$\mathscr{Q}(x) = \{w \in L^2(J,X^*) |\, w(t) \in \partial F(t,x(t)) \text{ for a.e. } t \in J\} \text{ for } x \in C(J,X).$$

Also, we know the following properties about the multi-valued map $\mathscr{Q}$.

Lemma 3.1. [28, Lemma 5.3] *If $H(F)$ holds, then, for each function $x \in C(J,X)$, the multifunction $\mathscr{Q}$ has nonempty, convex, and weakly compact values.*

Lemma 3.2. [30, Lemma 11] *If $H(F)$ holds, then the operator $\mathscr{Q}$ satisfies the following closeness property: if $x_n \rightharpoonup x$ in $C(J,X)$, $w_n \rightharpoonup w$ in $L^2(J,X^*)$ and $w_n \in \mathscr{Q}(x_n)$, then $w \in \mathscr{Q}(x)$.*

Let $\{S(t,s)\}_{t,s\in J}$ be an evolution operator generated by the family $\{A(t)\}_{t\in J}$ in a separable reflexive Banach space X. We consider the following fundamental Cauchy operator $G_S : L^1(J,X) \to C(J,X)$ defined by

$$G_S f(t) = \int_0^t S(t,s)f(s)ds, \quad t \in J = [0,b], f \in L^1(J,X).$$

Lemma 3.3. [31] *The fundamental Cauchy operator G_S satisfies the following properties:*
(i) $\|G_S f(t) - G_S g(t)\| \le K \int_0^t \|f(s) - g(s)\| ds$, $t \in J$, $\forall f, g \in L^1(J,X)$;
(ii) The map G_S is weakly continuous from $L^2(J,X)$ to $C(J,X)$.

Lemma 3.4. [6, Lemma 2.3] *Let X be a Banach space and let C be a nonempty, weakly compact, and convex subset of X. If $F : C \to \mathscr{P}_v(C)$ is weakly sequentially closed, then F has a fixed point.*

In order to show the existence of mild solutions of system(1.1), for a given $u \in L^2(J,V)$, we may introduce a multi-valued map $\Gamma : C(J,X) \to 2^{C(J,X)}$ as follows:

$$\Gamma(x) = \{\varphi \in C(J,X) | \varphi(t) = C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)[Bu(s) + f(s)]ds, f \in \mathscr{Q}(x)\}$$

and an equivalent norm on the Banach space $C(J,X)$ defined by

$$\|x\|_r = \sup_{t \in J} \|x(t)\| e^{-rt}, \quad \forall x \in C(J,X)$$

where $r > K\beta$ (K and β are defined in (2.2) and H(F)(iii), respectively). We also denote by $B_R := \{x \in C(J,X) | \ \|x\|_r \le R\}$ the closed ball of radius R at the origin in $C(J,X)$, where

$$R \ge \frac{Kr}{r - K\beta}(\|x_0\|_X + \|y_0\|_X + \|\alpha\|_{L^2(J,\mathbb{R}^+)}\sqrt{b} + \|B\|\|u\|_{L^2(J,V)}\sqrt{b}).$$

In what follows, we denote $\Gamma_R = \Gamma|_{B_R} : B_R \to C(J,X)$ the restriction of the multi-valued map Γ on the set B_R.

Lemma 3.5. *If $B \in \mathscr{L}(V,X)$ and $H(F)$ hold, then, for each $u \in L^2(J,V)$, $\Gamma(B_R) \subset B_R$.*

Proof. In fact, for all $x \in B_R, \varphi \in \Gamma_R(x)$, using $H(F)$ and the Hölder's inequality, there exists $f \in \mathscr{Q}(x)$ such that

$$\begin{aligned}
& e^{-rt}\|\varphi(t)\|_X = e^{-rt}\|C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)[f(s) + Bu(s)]ds\|_X \\
\le\ & K\|x_0\|_X + K\|y_0\|_X + e^{-rt}K\int_0^t \|f(s)\|_X ds + e^{-rt}K\|B\|\int_0^t \|u(s)\|_V ds \\
\le\ & K(\|x_0\|_X + \|y_0\|_X) + K\|\alpha\|_{L^2(J,\mathbb{R}^+)}\sqrt{b} + e^{-rt}K\beta\int_0^t \|x(s)\|_X ds \\
& + K\|B\|\|u\|_{L^2(J,V)}\sqrt{b} \\
\le\ & K(\|x_0\|_X + \|y_0\|_X + \|\alpha\|_{L^2(J,\mathbb{R}^+)}\sqrt{b} + \|B\|\|u\|_{L^2(J,V)}\sqrt{b}) \\
& + e^{-rt}K\beta\int_0^t \|x(s)\|_X ds \\
\le\ & K(\|x_0\|_X + \|y_0\|_X + \|\alpha\|_{L^2(J,\mathbb{R}^+)}\sqrt{b} + \|B\|\|u\|_{L^2(J,V)}\sqrt{b}) \\
& + e^{-rt}K\beta\int_0^t e^{rs}e^{-rs}\|x(s)\|_X ds \\
\le\ & K(\|x_0\|_X + \|y_0\|_X + \|\alpha\|_{L^2(J,\mathbb{R}^+)}\sqrt{b} + \|B\|\|u\|_{L^2(J,V)}\sqrt{b}) + \frac{K\beta R}{r} \\
\le\ & R.
\end{aligned}$$

Hence, $\|\varphi\|_r \le R$, which implies $\Gamma(B_R) \subset B_R$. □

Theorem 3.1. *If $B \in \mathscr{L}(V,X)$ and $H(F)$ hold, then, for each $u \in L^2(J,V)$, system (1.1) has at least a mild solution on J.*

Proof. From Lemma 3.5, one has $\Gamma(B_R) \subset B_R$. In virtue of [32, Proposition 4.2. and Proposition 4.4.], the multi-valued map Γ_R has convex, weakly compact values, and a weakly sequentially closed graph. According to the Lemma 3.4, we obtain that the operator Γ has a fixed point, i.e., system (1.1) has at least a mild solution on J. The proof is complete. □

4. Optimal Feedback Control

In this section, we consider the following feedback control problem.

$$\begin{cases} x''(t) \in A(t)x(t) + Bu(t) + \partial F(t,x(t)), & \text{a.e. } t \in J, \\ u(t) \in U(t,x(t)), & \text{a.e. } t \in J, \\ x(0) = x_0, x'(0) = y_0, \end{cases}$$

where the control function u takes values in a separable reflexive Banach space V, and $U : J \times X \to \mathscr{P}(V)$ is a feedback multifunction.

Set

$$V_J = \{u : J \to V | u(\cdot) \text{ is measurable } \},$$
$$X_J = \{(x,u) \in C(J,X) \times V_J | (x,u) \text{ satisfies } (1.2)\}.$$

We call a pair (x,u) is feasible if $(x,u) \in X_J$.

Now we introduce the following concepts of weakly-open neighborhoods of x_0 in X: $\forall f_i \in X^*, i = 1,2,...k$ and $\varepsilon > 0$, we define

$$o_w(f_1, f_2, ..., f_k; \varepsilon) := \{x \in X \mid |\langle f_i, x - x_0\rangle| < \varepsilon, \}$$
$$\mathscr{O}_w(x_0) = \{o_w(f_1, f_2, ..., f_k; \varepsilon) \mid \forall f_i \in X^*, i = 1,2,...k, \forall k \in \mathbb{N}, \forall \varepsilon > 0\}.$$

For more details, we refer to [33, Proposition 3.4].

Similar to [14, 34], we may introduce a weak-Cesari property as follows.

Definition 4.1. Let X be a Banach space and Y be a metric space. Let $F : X \to \mathscr{P}(Y)$ be a multifunction. We say that F possesses the weak-Cesari property at $x_0 \in X$ if

$$\bigcap_{o_w \in \mathscr{O}_w(x_0)} \overline{co} F(o_w) = F(x_0),$$

where $\overline{co} F$ denotes the closed convex hull of F. If F has the weak-Cesari property at every point $x \in Z \subset X$, we simply say that F has the weak-Cesari property on Z.

Also, we assume that the feedback multi-valued map $U : J \times X \to \mathscr{P}(V)$ satisfies the following conditions H(U):

(i) there exist a function $\alpha_1 \in L^2(J, \mathbb{R}^+)$ and a constant $\beta_1 > 0$ such that

$$\|U(t,x)\| = \sup_{v \in U(t,x)} \|v\|_V \le \alpha_1(t) + \beta_1 \|x\|_X \text{ for a.e. } t \in J, \ \forall x \in X.$$

(ii) for a.e. $t \in J$, $U(t,\cdot) : X \to 2^V$ has the weak-Cesari property, i.e.,

$$\bigcap_{o_w \in \mathscr{O}_w(t,x)} \overline{co} U(o_w) = U(t,x).$$

Theorem 4.1. *If $H(F)$ and $H(U)$ hold, then the solution set of Problem 4 is nonempty, i.e., $X_J \neq \emptyset$.*

Proof. For any $n \in \mathbb{N}$, let $\tau_j = \frac{j}{n}b$, $0 \leq j \leq n-1$. Set

$$u_n(t) = \sum_{j=0}^{n-1} u^j \chi_{[\tau_j,\tau_{j+1})}(t) \text{ a.e. } t \in J,$$

where $\chi_{[\tau_j,\tau_{j+1})}$ is the characteristic function of interval $[\tau_j,\tau_{j+1})$. The sequence $\{u^j\}$ is constructed as follows.

Firstly, we take $u^0 \in U(0,x_0)$. By Theorem 3.1, there exists $x_n(\cdot)$, which is given by

$$x_n(t) = C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)[f_n(s) + Bu^0\chi_{[\tau_0,\tau_1)}(s)]ds, \quad t \in [\tau_0,\tau_1].$$

where $f_n(t) \in \partial F(t,x_n(t))$ a.e. $t \in [\tau_0,\tau_1]$. Then take $u^1 \in U(\tau_1,x_n(\tau_1))$ such that

$$x_n(t) = C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)[f_n(s) + Bu^0\chi_{[\tau_0,\tau_1)}(s) + Bu^1\chi_{[\tau_1,\tau_2)}(s)]ds, \quad t \in [\tau_0,\tau_2].$$

We can repeat this procedure to obtain x_n on $[\tau_0,\tau_{j+1}]$ for $j = 0,1,\cdots,n-1$. By induction, we end up with the following:

$$\begin{cases} x_n(t) = C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)[f_n(s) + Bu_n(s)]ds, \quad t \in J. \\ u^j \in U(\tau_j,x_n(\tau_j)), \quad 0 \leq j \leq n-1. \end{cases}$$

where $f_n(t) \in \partial F(t,x_n(t))$ a.e. $t \in J$.

Now we prove that there exists a constant $r_0 > 0$ such that $\|x_n\|_C \leq r_0$. Using H(F)(iii), H(U)(i), and the Hölder's inequality, we have

$$\begin{aligned}
&\|x_n(t)\|_X = \|C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)[f_n(s) + Bu_n(s)]ds\|_X \\
\leq\ & K\|x_0\|_X + K\|y_0\|_X + K\int_0^t \|f_n(s)\|_X ds + K\|B\|\int_0^t \|u_n(s)\|_V ds \\
\leq\ & K\|x_0\|_X + K\|y_0\|_X + K\int_0^t (\alpha(t) + \beta\|x_n(s)\|_X)ds + K\|B\|\int_0^t (\alpha_1(t) + \beta_1\|x_n(s)\|_X)ds \\
\leq\ & K\|x_0\|_X + K\|y_0\|_X + K\|\alpha\|_{L^2(J,\mathbb{R}^+)}\sqrt{b} + K\beta\int_0^t \|x_n(s)\|_X ds \\
& +K\|B\|\|\alpha_1\|_{L^2(J,\mathbb{R}^+)}\sqrt{b} + K\|B\|\beta_1\int_0^t \|x_n(s)\|_X ds \\
\leq\ & \zeta + \theta\int_0^t \|x_n\|_X ds,
\end{aligned}$$

where

$$\zeta = K(\|x_0\| + \|y_0\| + \|\alpha\|_{L^2(J,\mathbb{R}^+)}\sqrt{b} + \|B\|\|\alpha_1\|_{L^2(J,\mathbb{R}^+)}\sqrt{b}),$$
$$\theta = K(\beta + \|B\|\beta_1).$$

Then, by the Gronwall inequality, we can easily obtain that $\|x_n(t)\|_X \leq \zeta e^{\theta t}$, Therefore,

$$\|x_n\|_C \leq r_0 := \zeta e^{\theta b}. \tag{4.1}$$

Again by H(F)(iii), H(U)(i), and (4.1) there exist $r_1, r_2 > 0$ such that

$$\|f_n\|_{L^2(J,X^*)} \leq r_1, \qquad \|u_n\|_{L^2(J,V)} \leq r_2.$$

Thus there are two subsequences of $\{f_n\}$ and $\{u_n\}$, denoted still by $\{f_n\}$ and $\{u_n\}$, respectively, such that

$$f_n \rightharpoonup \overline{f} \quad \text{in} \quad L^2(J,X^*), \quad u_n \rightharpoonup \overline{u} \quad \text{in} \quad L^2(J,V). \tag{4.2}$$

Let

$$\overline{x}(t) = C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)[\overline{f}(s) + B\overline{u}(s)]ds.$$

Due to Lemma 3.3(ii), we have from Proposition 2.1 that, for any $t \in J$,

$$\int_0^t S(t,s)[f_n(s) + Bu_n(s)]ds \rightharpoonup \int_0^t S(t,s)[\overline{f}(s) + B\overline{u}(s)]ds \quad \text{in } X.$$

Hence, one has

$$x_n(t) \rightharpoonup \overline{x}(t) \ \text{ in } X, \quad \forall t \in J. \tag{4.3}$$

Combining (4.1) and (4.3) and Proposition 2.1, we obtain $x_n \rightharpoonup \overline{x}$ in $C(J,X)$. Therefore, for any weakly open neighborhood $o_w(\overline{x})$ of $\overline{x}$ in $C(J,X)$, there exists a $n_0 > 0$ such that

$$x_n \in o_w(\overline{x}), n \geq n_0. \tag{4.4}$$

Next, we need to prove that $\overline{f}(t) \in \partial F(t,\overline{x}(t))$ for a.e. $t \in J$. By Mazur Theorem (cf. [14, Corollary 2.8]), there exist $\lambda_{in} \geq 0$ and $\sum_{i=1}^{m_n} \lambda_{in} = 1$ such that

$$\widetilde{f}_n(\cdot) = \sum_{i=1}^{m_n} \lambda_{in} f_{i+n}(\cdot) \to \overline{f}(\cdot) \text{ in } L^2(J,X^*),$$

which implies, up to a subsequence,

$$\widetilde{f}_n(t) \to \overline{f}(t) \quad \text{in } X^* \text{ for all } t \in J \setminus J_0, \tag{4.5}$$

where $J_0 \subset J$ with Lebesgue measure zero. Denote by $\overline{B}(r_0)$ the closed ball in X with zero as the center and r_0 as the radius. Now we claim that $\overline{f}(t) \in \partial F(t,\overline{x}(t))$, $\forall t \in J \setminus J_0$. For $\forall \bar{t} \in J \setminus J_0$, we assume, by contradiction, that $\overline{f}(\bar{t}) \notin \partial F(\bar{t},\overline{x}(\bar{t}))$. In virtue of H(F)(iii),(iv), and the reflexivity of X, we easily obtain that $\partial F(\bar{t},\cdot) : B(r_0) \to \mathscr{P}(X^*)$ is weakly compact and a weakly closed multimap. By [35, Theorem1.1.5], it is weakly u.s.c. Due to $\partial F(\bar{t},\overline{x}(\bar{t}))$ being closed and convex and Hahn-Banach Theorem, there exists a weakly open convex set $o_w \supset \partial F(\bar{t},\overline{x}(\bar{t}))$ such that $\overline{f}(\bar{t}) \notin \overline{o}_w$. In virtue of the weak upper semi-continuity of $\partial F(\bar{t},\cdot)$, there is a weak neighborhood $\mathscr{N}$ of $\overline{x}(\bar{t})$ such that $\partial F(\bar{t},x) \subset O_w$ for all $x \in \mathscr{N} \cap \overline{B}(r_0)$. Combining (4.4) and (4.5), there exists $N > 0$ such that $x_n \in \mathscr{N} \cap \overline{B}(r_0)$ for $n \geq N$. And so, $f_n(\bar{t}) \in \partial F(\bar{t},x_n(\bar{t})) \subset o_w$ for $n \geq N$. Therefore, $\widetilde{f}_n(\bar{t}) \in o_w$ for all $n \geq N$ by use of the convexity of o_w, which implies that $\overline{f}(\bar{t}) \in \overline{o}_w$. This is a contradiction. Hence, $\overline{f}(t) \in \partial F(t,\overline{x}(t))$ for a.e. $t \in J$. On the other hand, for any $o_w \in \mathscr{O}_w(t,\overline{x}(t))$, by the definition of $u_n(\cdot)$ for n large enough, we have

$$u_n(t) \in U(\tau_j, x_n(\tau_j)) \subset U(o_w) \text{ for all } t \in [\tau_j, \tau_{j+1}), 0 \leq j \leq n-1. \tag{4.6}$$

Secondly, by (4.2) and Mazur Theorem, there exist $a_{il} \geq 0$ and $\sum_{i=1}^{l_n} a_{il} = 1$ such that

$$\phi_l(\cdot) = \sum_{i\geq 1} a_{il} u_{i+l}(\cdot) \to \overline{u}(\cdot) \text{ in } L^2(J,V).$$

Then, there is a subsequence of $\{\phi_l\}$, denoted by $\{\phi_l\}$ again, such that $\phi_l(t) \to \overline{u}(t)$ in V, a.e. $t \in J$. Hence, from (4.4) and (4.6), for l large enough, $\phi_l(t) \in coU(o_w)$, a.e. $t \in J$. Thus, for any $o_w \in \mathscr{O}_w(t,\overline{x}(t))$, $\overline{u}(t) \in \overline{co}U(o_w)$, a.e. $t \in J$. By $H(U)(ii)$, we have $\overline{u}(t) \in U(t,\overline{x}(t))$, a.e. $t \in J$. Therefore, $(\overline{x},\overline{u})$ is a feasible pair in J. The proof is complete. □

Next, we consider an optimal control problem stated as follows.

Problem (Ψ): find a pair $(x^0,u^0) \in X_J$ such that

$$\Psi(x^0,u^0) \leq \Psi(x,u), \quad \forall (x,u) \in X_J,$$

$$\text{where } \Psi(x,u) = \int_0^b \varphi(t,x(t),u(t))dt.$$

We make the following assumptions on φ :

(φ1) the functional $\varphi : J \times X \times V \to \mathbb{R} \bigcup \{\pm\infty\}$ is Borel measurable in (t,x,u);

(φ2) $\varphi(t,\cdot,\cdot)$ is weakly lower semicontinuous on $X \times V$ for $a.e.t \in J$ and there exists a constant $M_1 > 0$ such that

$$\varphi(t,x,u) \geq -M_1, \qquad (t,x,u) \in J \times X \times V.$$

For any $(t,x) \in J \times X$, we set

$$\varepsilon(t,x) = \{(z^0,z^1) \in \mathbb{R} \times V | z^0 \geq \varphi(t,x,z^1),\ z^1 \in U(t,x)\}.$$

In order to obtain the existence result of optimal state-control pairs for Problem (Ψ), we also need the assumption:

(H_ε) : for a.e. $t \in J$, the map $\varepsilon(t,\cdot) : X \to \mathscr{P}(\mathbb{R} \times X \times V)$ is such that

$$\bigcap_{o_w \in \mathscr{O}_w(t,x)} \overline{co}\,\varepsilon(t,o_w) = \varepsilon(t,x), \qquad \forall x \in X.$$

Theorem 4.2. *Assume that all the assumptions of $H(F)$, $H(U)$, (φ), and (H_ε) are satisfied. Then Problem (Ψ) admits at least one optimal state-control pair.*

Proof. We only need to assume that $\inf\{\Psi(x,u) | (x,u) \in X_J\} = m < \infty$. By ($\varphi$2), we have $\Psi(x,u) \geq m \geq -M_1 b > -\infty$. Then there exists a subsequence $\{(x_n,u_n)\}_{n\geq 1} \subset X_J$ such that $\Psi(x_n,u_n) \to m$. From the proof of Theorem 4.1, we obtain that $x_n \rightharpoonup \overline{x}$ in $C(J,X)$, and

$$u_n \rightharpoonup \overline{u} \text{ in } L^2(J,V), \quad f_n \rightharpoonup \overline{f} \text{ in } L^2(J,X^*) \text{ with } f_n(t) \in \partial F(t,x_n(t)) \text{ a.e. } t \in J.$$

where

$$\overline{x}(t) = C(t,0)x_0 + S(t,0)y_0 + \int_0^t S(t,s)[\overline{f}(s) + B\overline{u}(s)]ds,\ \overline{f} \in \partial F(t,\overline{x}),\ t \in J.$$

Then, by Mazur Theorem again, there exist $b_{il} \geq 0$ and $\sum_{i=1}^{m_l} b_{il} = 1$ such that

$$\phi_l(\cdot) = \sum_{i=1}^{m_l} b_{il} u_{i+l}(\cdot) \to \overline{u}(\cdot) \text{ in } L^2(J,V).$$

Furthermore, there exist $q_{kl} \geq 0$ with $\sum_{k=1}^{n_l} q_{kl} = 1$ such that

$$\begin{aligned} \psi_l(\cdot) &= \sum_{k=1}^{n_l} q_{kl} \varphi(\cdot, x_{k+l}(\cdot), u_{k+l}(\cdot)), \\ \overline{\varphi}(t) &= \underline{\lim}_{l\to+\infty} \psi_l(t) \geq -M_1, \qquad a.e. t \in J. \end{aligned}$$

For any $o_w \in \mathscr{O}_w(t,\overline{x})$ and l large enough, from $(\varphi 2)$ we have $(\psi_l(t), \phi_l(t)) \in co\varepsilon(t, o_w)$, a.e. $t \in J$. Then $(\overline{\varphi}(t), \overline{u}(t)) \in \overline{co}\varepsilon(t, o_w)$, a.e. $t \in J$. From (H_ε), we have $(\overline{\varphi}(t), \overline{u}(t)) \in \varepsilon(t, \overline{x}(t))$, $a.e. t \in J$. That means,

$$\begin{cases} \overline{\varphi}(t) \geq \varphi(t, \overline{x}(t), \overline{u}(t)), & t \in J, \\ \overline{u}(t) \in U(t, \overline{x}(t)), & t \in J. \end{cases}$$

Since $\overline{f} \in \partial F(t,\overline{x})$, then $(\overline{x},\overline{u}) \in X_J$. By Fatou's Lemma, we obtain

$$\begin{aligned} \int_0^b \overline{\varphi}(t)dt = \int_0^b \underline{\lim}_{l\to+\infty} \psi_l(t)dt &\leq \underline{\lim}_{l\to+\infty} \int_0^b \psi_l(t)dt \\ &= \underline{\lim}_{l\to+\infty} \int_0^b \sum_{k\geq 1}^{n_l} q_{kl} \varphi(t, x_{k+l}(t), u_{k+l}(t))dt \\ &= \underline{\lim}_{l\to+\infty} \sum_{k\geq 1}^{n_l} q_{kl} \int_0^b \varphi(t, x_{k+l}(t), u_{k+l}(t))dt \\ &= \underline{\lim}_{l\to+\infty} \sum_{k\geq 1}^{n_l} q_{kl} \Psi(x_{k+l}, u_{k+l}) \\ &= m \end{aligned}$$

Therefore, $m \leq \varphi(\overline{x},\overline{u}) = \int_0^b \varphi(t, \overline{x}(t), \overline{u}(t))dt \leq m$, *i.e.*,

$$\int_0^b \varphi(t, \overline{x}(t), \overline{u}(t))dt = m = \inf_{(x,u)\in X_J} \Psi(x,u).$$

Thus, $(\overline{x},\overline{u})$ is an optimal state-control pair. The proof is complete. □

5. An Application

In this section we provide an example. We consider

$$A(t) = A + \widetilde{A}(t),$$

where A is the infinitesimal generator of a cosine function $C(t)$ with associated sine function $S(t)$, and $\widetilde{A}(t) : D(\widetilde{A}(t)) \to X$ is a closed linear operator with $D \subseteq D(\widetilde{A}(t)), \forall t \in J$. We take the space $X = L^2(\mathbb{T}, \mathbb{C})$, where the group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$, and we denote by $L^2(\mathbb{T}, \mathbb{C})$ the space of 2π periodic 2-integrable functions from $\mathbb{R}$ to $\mathbb{C}$. Also, we use the identification between functions on $\mathbb{T}$ and 2π periodic functions on $\mathbb{R}$. Furthermore, $H^2(\mathbb{T},\mathbb{C})$ denotes the Sobolev space of 2π periodic from $\mathbb{R}$ to $\mathbb{C}$ such that $x'' \in L^2(\mathbb{T},\mathbb{C})$.

We consider the operator: $A : D(A) \subset X \to X$ defined by $Ax(t) = x''(t)$ with domain $D(A) = H^2(\mathbb{T},\mathbb{C})$, Then, A can be written as

$$Ax = -\sum_{n=1}^{\infty} n^2 \langle x, w_n\rangle w_n, \quad x \in D(A),$$

where $w_n(t) = \frac{1}{\sqrt{2\pi}} e^{int} (n \in \mathbb{Z})$ is an orthonormal basis of X. It is well known that A is the infinitesimal generator of a strongly continuous cosine function $C(t)$ on X. The cosine function $C(t)$ is given by

$$C(t)x = -\sum_{n=1}^{\infty} cos(nt)\langle x, w_n\rangle w_n, \quad t \in \mathbb{R},$$

with associated sine function

$$S(t)x = -\sum_{n=1}^{\infty} \frac{sin(nt)}{n}\langle x, w_n\rangle w_n, \quad t \in \mathbb{R},$$

It is clear that $\|C(t)\| \leq 1$ for all $t \in \mathbb{R}$, so it is uniformly bounded on $\mathbb{R}$.

Consider the following second-order Cauchy problem:

$$\begin{cases} \frac{\partial^2}{\partial t^2}x(t,y) \in \frac{\partial^2}{\partial y^2}x(t,y) + b(t)\frac{\partial}{\partial t}x(t,y) + \mu(t,y) + Q(t,x(t,y)), t \in J = [0,b], y \in [0,\pi], \\ u(t) \in U(t,x(t,y)), \qquad a.e. t \in J = [0,b], y \in [0,\pi], \\ x(t,0) = x(t,\pi) = 0, \qquad t \in J = [0,b], \\ x(0,y) = x_0(y), \frac{\partial}{\partial t}x(0,y) = x_1(y), \qquad y \in [0,\pi], \end{cases} \tag{5.1}$$

where $b : \mathbb{R} \to \mathbb{R}$ is continuous, and $U : [0,b] \times X \to \mathscr{P}(V)$. Now let $t > 0$ and fix $\lambda = sup_{0 \leq t \leq b}|b(t)|$. Here $x(t,y)$ represents the temperature at the point $y \in [0,\pi]$ and $t \in J = [0,b]$. It is supposed that $Q = \mathscr{F}_1 + \mathscr{F}_2$, where $\mathscr{F}_2$ is given, and $\mathscr{F}_1$ is a known function of the temperature of the form

$$-\mathscr{F}_1 \in \partial F(t,x(t,y)), \qquad (t,y) \in J \times [0,\pi],$$

where $F = F(t,x(t,y))$ is a locally Lipschitz energy function that is generally nonsmooth and nonconvex, and ∂F denotes the generalized Clarke's gradient in the third variable ψ ([21]). The simple example of a function F, which satisfies hypothesis $H(F)$ is $F(\psi) = \min\{\mathscr{H}_1(\psi), \mathscr{H}_2(\psi)\}$, where $\mathscr{H}_i : \mathbb{R} \to \mathbb{R} (i = 1,2)$ are convex quadratic functions ([36]).

Now we take $\widetilde{A}(t)x(y) = b(t)x'(y)$ defined on $\mathbb{H}^1(\mathbb{T},\mathbb{C})$. It is easy to see that $A(t) = A + \widetilde{A}(t)$ is a closed linear operator. Initially, we will show that $A + \widetilde{A}(t)$ generates an evolution operator. It is well known that the solution of the scalar initial value problem

$$\begin{aligned} p''(t) &= -n^2 p(t) + q(t), \\ p(s) &= 0, \ p'(s) = p_1, \end{aligned}$$

is given by

$$p(t) = \frac{p_1}{n} sin\, n(t-s) + \frac{1}{n}\int_s^t sin\, n(t-\sigma)q(\sigma)d\sigma.$$

Therefore, the solution of the scalar initial value problem

$$p''(t) = -n^2 p(t) + inb(t)p(t), \tag{5.2}$$

$$p(s) = 0,\ p'(s) = p_1, \tag{5.3}$$

satisfies the following equation

$$p(t) = \frac{p_1}{n} sin\, n(t-s) + i\int_s^t sin\, n(t-\sigma)b(\sigma)p(\sigma)d\sigma.$$

Applying the Gronwall-Bellman lemma, we can see that

$$|p(t)| \le \frac{|p_1|}{n} e^{c(t-s)} \tag{5.4}$$

for $s \le t$ and c is a constant. We denote by $p_n(t,s)$ the solution of (5.2)-(5.3). We define

$$S(t,s)y = \sum_{n=1}^{\infty} p_n(t,s)\langle y, w_n\rangle w_n.$$

It follows from the estimate (5.4) that $S(t,s) : X \to X$ is well defined and satisfies the condition of Definition 2.2. We set $x(t) = x(t,\cdot)$, that is, $x(t)(y) = x(t,y), t \in J, y \in [0,\pi]$, and $u(t) = \mu(t,\cdot)$, where $\mu : J \times [0,\pi] \to [0,\pi]$ is continuous. Then let us define the bounded linear operator $B : V \to X$ by

$$Bu(t)(y) = \mu(t,y), t \in J, y \in [0,\pi].$$

Assume these functions satisfy the requirement of the hypotheses. From the above choices of the functions and evolution operator $A(t)$ with B, system (5.1) can be formulated as system (1.2) in X. Also, all the hypotheses of Theorem 3.1 are satisfied, we know that system (1.1) has a mild solution on J. So, combining the necessary conditions, Theorem 4.1 and Theorem 4.2 can be applied to problem (5.1).

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