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ON THE WELL-POSEDNESS OF VARIATIONAL-HEMIVARIATIONAL INEQUALITIES AND ASSOCIATED FIXED POINT PROBLEMS

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Abstract. We consider an elliptic variational-hemivariational inequality P in a p-uniformly smooth Banach space. We prove that the inequality is governed by a multivalued maximal monotone operator, and, for each $\lambda > 0$, we use the resolvent of this operator to construct an auxiliary fixed point problem, denoted P_λ. Next, we perform a parallel study of problems P and P_λ based on their intrinsic equivalence. In this way, we prove existence, uniqueness, and well-posedness results with respect to specific Tykhonov triples. The existence of a unique common solution to problems P and P_λ is proved by using the Banach contraction principle in the study of Problem P_λ. In contrast, the well-posedness of the problems is obtained by using a monotonicity argument in the study of Problem P. Finally, the properties of Problem P_λ allow us to deduce a convergence criterion in the study of Problem P.

Keywords. Duality map; Maximal monotone operator; Resolvent operator; Tykhonov well-posedness; Variational-hemivariational inequality.

1. INTRODUCTION

In this paper, we deal with the well-posedness of variational-hemivariational inequalities in p-uniformly smooth Banach spaces. The analysis we perform is carried out by using an approach based on the intrinsic equivalence between the inequality problem and an associated fixed point problem. Variational-hemivariational inequalities were introduced in the pioneering work of Panagiotopoulos [1]. These are inequalities which have both convex and nonconvex structures and represent a useful mathematical tool in the study of boundary value problems which arise in physics, mechanics, and engineering sciences. The theory grew up rapidly and, currently, the literature in the field is extensive. Basic references are [2–4]. Recent existence, uniqueness, and convergence results, obtained by different functional arguments can be found in [5–7]. Results on the numerical analysis of variational-hemivariational inequalities were obtained in [8–13] for instance.

The concept of well-posedness for a minimization problem was introduced in [14] based on two ingredients: the existence of a unique minimizer and the convergence to it of any minimizing sequence. Various extensions were considered in [15,16], [17–19], and [20–22] in the study

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of optimization problems, variational and hemivariational inequalities, and fixed point problems, respectively. A general well-posedness concept for abstract problems in metric spaces was introduced in the recent paper [23]. Based on the notion of Tykhonov triple, this concept was used in [24] in the study of hemivariational inequalities.

Recall also that the nonlinear analysis abound in equivalence results which allow us to study a problem by performing the analysis of an auxiliary problem which, in general, has a different structure. Some simple examples are the equivalence between a Cauchy problem for a differential equation and an integral equation, the equivalence between a variational inequality and a minimization problem, and the equivalence between a nonlinear operator equation and a fixed point problem, for instance.

The inequality problem we consider in this paper is stated as follows.

Problem P. Find u such that

$$u\in K,\quad \langle Au,v-u\rangle+\varphi(v)-\varphi(u)+j^0(u;v-u)\geq\langle f,v-u\rangle,\quad \forall\, v\in K. \tag{1.1}$$

Here and everywhere below, unless stated otherwise, X is a real reflexive Banach space, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and its dual X^*. Moreover, K is a nonempty subset of X, $A:X\to X^*$, $\varphi:X\to\mathbb{R}\cup\{+\infty\}$, $j:X\to\mathbb{R}$ is a locally Lipschitz function, and $f\in X^*$. In addition, $j^0(u;v)$ represents the Clarke directional derivative of j at the point u, in the direction $v\in X$.

Existence and uniqueness results in the study of Problem P have been obtained in many papers under various assumptions on the data. For example, Problem P was considered in [6] under the assumptions that A is a pseudomonotone and strongly monotone operator, φ is a convex lower semicontinuous function, and the Clarke subdifferential of the function j satisfies a growth condition. The unique solvability of Problem P was proved by using a surjectivity result for pseudomonotone multivalued operators. Recently, Problem P was considered in [5, 25] under the assumptions that X is a Hilbert space, A is a strongly monotone Lipschitz continuous operator, and φ is a convex continuous function. The unique solvability of the problem was obtained in [5] by using a minimization principle and in [25] by using a fixed point argument associated to the resolvent of a maximal monotone operator which governs the variational-hemivariational inequality. Finally, recall that well-posedness results in the study of Problem P were obtained in [24].

Our aim in the present paper is three folds. The first aim is to present a new existence and uniqueness result in the study of variational-hemivariational inequality (1.1). The second and the third aims are to prove the well-posedness of the inequality with respect to a given Tykhonov triple and to obtain a convergence criterion in the study of this inequality, respectively. To this end, we use the arguments based on the equivalence between Problem P and a fixed point problem P_λ, which will be described below. The existence and uniqueness result we present here is obtained by using the assumptions on the data, which are different to those used in [5, 6] and, therefore, our work parallels [5, 6]. On the other hand it extends the results in [25] since, for instance, here we work in the framework of p-uniformly smooth Banach spaces. In addition to the unique solvability of Problem P, we provide its well-posedness as well as necessary and sufficient conditions, which guarantee the convergence of a sequence to its solution.

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions and preliminary material, which are be used in the rest of the paper. In Section 3, we introduce a fixed point problem P_λ associated to the resolvent of a maximal monotone operator constructed with the data of Problem P. In Section 4, we state and prove the equivalence of problems P and P_λ, and then we prove the existence of unique common solution for these problems. Next, in Section 5, we study the well-posedness of problems P and P_λ with respect to appropriate Tykhonov triples. We start with an equivalence result, and then we prove the corresponding well-posedness results. In Section 6, we state and prove a convergence criterion to the solution of Problem P. We end this paper with Section 7 in which we present some concluding remarks.

2. Preliminaries

In this section, we introduce some notations and preliminary results, which are used in the rest of the paper. For more details and proofs we refer the reader to [26–28]. Everywhere below, we use $\|\cdot\|_X$ and $\|\cdot\|_{X^*}$ for the norm on the spaces X and X^*, respectively, and the symbols "$\to$", "$\rightharpoonup$" represent the strong and the weak convergence in X or X^*. Moreover, we use notation 0_X and 0_{X^*} for the zero element of X and X^*, respectively, X^*_w for the space X^* equipped with weak topology and *int* B for the interior of the set $B \subset X$, in the strong topology of X. All the limits, lower limits and upper limits below are considered when $n \to \infty$ even if we do not mention it explicitly.

We start with recalling some basic definitions for single-valued operators.

Definition 2.1. The operator $A : X \to X^*$ is said to be:

(1) demicontinuous if $u_n \to u$ in X implies $Au_n \rightharpoonup Au$ in X^*;

(2) p-monotone for some $p > 0$ if there exists constant $m_A > 0$ such that

$$\langle Au - Av, u - v\rangle \geq m_A \|u - v\|^p_X, \quad \forall u, v \in X;$$

(3) a contraction if there exists constant $0 \leq k < 1$ such that

$$\|Au - Av\|_{X^*} \leq k\|u - v\|_X, \quad \forall u, v \in X.$$

Next, we move to some definitions concerning nonsmooth functions defined on the space X.

Definition 2.2. Let $j : X \to \mathbb{R}$ be a locally Lipschitz function. Then, the Clarke directional derivative of the j at the point $u \in X$ in the direction $v \in X$ is defined by

$$j^0(u; v) = \limsup_{w \to u, \lambda \downarrow 0} \frac{j(w + \lambda v) - j(w)}{\lambda}.$$

Moreover, the Clarke gradient of j is the set-valued operator $\partial j : X \to 2^{X^*}$ defined by

$$\partial j(u) = \Big\{ \xi \in X^* \ : \ j^0(u; v) \geq \langle \xi, v\rangle \quad \forall v \in X \Big\}, \quad \forall u \in X.$$

For the Clarke directional derivative and gradient of a locally Lipschitz function, we have the following properties.

Proposition 2.1. *Let $j : X \to \mathbb{R}$ be a locally Lipschitz function on X. Then*

(1) *for all $u \in X$, the Clarke gradient $\partial j(u)$ is a nonempty convex and weakly compact subset of X^*;*

(2) *the graph of the Clarke gradient ∂j is closed in $X \times X^*_w$ topology;*

(3) *for all $u, v \in X$, $j^0(u; v) = \max\Big\{ \langle \xi, v\rangle \ : \ \xi \in \partial j(u) \Big\}$.*

For a function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, we use the notation $D(\varphi)$ for its effective domain, i.e.,

$$D(\varphi) = \Big\{ u \in X \ : \ \varphi(u) < +\infty \Big\}.$$

In addition, we say that φ is proper if $D(\varphi) \neq \emptyset$.

Definition 2.3. A function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is said to be lower semicontinuous (l.s.c.) if, for any sequence $\{u_n\} \subset X$ such that $u_n \to u$ in X, the following inequality holds:

$$\liminf \varphi(u_n) \geq \varphi(u).$$

Definition 2.4. Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Then the subdifferential of φ is the set-valued operator $\partial^c \varphi : X \to 2^{X^*}$, defined by

$$\partial^c \varphi(u) = \Big\{ \eta \in X^* \ : \ \varphi(v) - \varphi(u) \geq \langle \eta, v - u \rangle \quad \forall\, v \in X \Big\} \quad \forall u \in X.$$

Definition 2.5. Given a nonempty subset K of X, the function $I_K : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K \end{cases}$$

is called the indicator function of set K.

It is well known that if subset K of X is nonempty closed and convex, then the indicator function I_K is proper, convex, and lower semicontinuous. In the rest of this paper, we use the notation $\partial^c(\varphi + I_K)(u)$ for the subdifferential of the convex function $\varphi + I_K$ at point u. We also recall the following two results, proved in [10] and [29], respectively.

Proposition 2.2. *Let X be a Banach space, $\varphi_1 : X \to \mathbb{R} \cup \{+\infty\}$, and $\varphi_2 : X \to \mathbb{R} \cup \{+\infty\}$ be proper convex and lower semicontinuous functions. Assume that there exists an element $u_0 \in D(\varphi_1) \cap D(\varphi_2)$ at which φ_1 or φ_2 is continuous. Then $\partial^c(\varphi_1 + \varphi_2)(u) = \partial^c \varphi_1(u) + \partial^c \varphi_2(u)$, $\forall u \in X$.*

Proposition 2.3. *Let C be a nonempty closed convex subset of X, C^* a nonempty closed convex bounded subset of X^*, $\psi : X \to \mathbb{R} \cup \{+\infty\}$ a proper convex and lower semicontinuous function, and $u \in C$. Assume that, for each $v \in C$, there exists $u^*(v) \in C^*$ such that $\langle u^*(v), v - u \rangle \geq \psi(u) - \psi(v)$. Then, there exists $u^* \in C^*$ such that $\langle u^*, v - u \rangle \geq \psi(u) - \psi(v)$, $\forall v \in C$.*

Consider now a multivalued operator $T : X \to 2^{X^*}$, and recall that its domain $D(T)$, range $R(T)$, and graph $Gr(T)$ are the sets defined by

$$D(T) = \Big\{ u \in X \ : \ Tu \neq \emptyset \Big\},$$

$$R(T) = \Big\{ u^* \in X^* \ : \ \exists u \in D(T) \ s.t.\ u^* \in T(u) \Big\},$$

$$Gr(T) = \Big\{ (u, u^*) \in D(T) \times X^* \ : \ u^* \in Tu \Big\}.$$

Moreover, recall the following definitions.

Definition 2.6. The operator $T : X \to 2^{X^*}$ is said to be

(1) p-relaxed monotone for some $p > 0$ if there exists a constant $\alpha_T > 0$ such that

$$\langle u_1^* - u_2^*, u_1 - u_2 \rangle \geq -\alpha_T \|u_1 - u_2\|_X^p, \quad \forall (u_1, u_1^*),\ (u_2, u_2^*) \in Gr(T);$$

(2) monotone if $\langle u_1^* - u_2^*, u_1 - u_2\rangle \geq 0, \forall (u_1, u_1^*), (u_2, u_2^*) \in Gr(T)$;
(3) maximal monotone if it is monotone and for any $v \in X$ and $v^* \in X^*$

$$\langle u^* - v^*, u - v\rangle \geq 0 \quad \forall\, u \in D(T),\ u^* \in Tu,$$

it follows that $v \in D(T)$ and $v^* \in Tv$.

Below we shall use the following results concerning the maximal monotone operators.

Proposition 2.4. *Assume that $\psi : X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, and lower semicontinuous function. Then the subdifferential operator $\partial^c \psi : X \to 2^{X^*}$ is maximal monotone.*

Proposition 2.5. *Assume that $T : X \to 2^{X^*}$ is a monotone operator such that, for every $u \in X$, Tu is nonempty convex and weakly closed set of X^*. Moreover, assume that ,for all $u, v \in X$, the mapping $\lambda \mapsto T(\lambda u + (1-\lambda)v)$ has a graph which is closed in $[0,1] \times X_w^*$. Then the operator T is maximal monotone.*

Proposition 2.6. *Let $T_1, T_2 : X \to 2^{X^*}$ be two maximal monotone operators such that int $D(T_1) \cap D(T_2) \neq \emptyset$. Then the sum $T_1 + T_2 : X \to 2^{X^*}$ is a maximal monotone operator, too.*

Denote in what follows by $U = \{\, u \in X : \|u\|_X = 1 \,\}$ the unit sphere in X. The properties of the Banach space X, including those of its duality map, play an important role in the rest of the paper. We now proceed with the following additional definitions and results.

Definition 2.7. The Banach space X is said to be strictly convex if

$$\left\| \frac{u+v}{2} \right\|_X < 1$$

whenever $u,\, v \in U$, $u \neq v$.

Definition 2.8. The modulus of smoothness of X is the real-valued function ρ_X defined by

$$\rho_X(\tau) = \sup\left\{ \frac{1}{2}\big(\|u+v\|_X + \|u-v\|_X\big) - 1 : u \in U, \|v\|_X \leq \tau \right\} \quad \forall\, \tau > 0.$$

Definition 2.9. Let $p > 1$ be fixed. A Banach space X is said to be p-uniformly smooth if there exists a constant $k > 0$ such that $\rho_X(\tau) \leq k\tau^p$ for all $\tau > 0$.

Remark 2.1. Let Ω be a smooth domain in $\mathbb{R}^d$. Following [30] and using standard notation, we recall that, if $1 < p < 2$, then the spaces $l^p(\Omega)$, $L^p(\Omega)$ and $W^{k,p}(\Omega)$ all are p-uniformly smooth. Moreover, if $p \geq 2$ then the spaces $l^p(\Omega)$, $L^p(\Omega)$, and $W^{k,p}(\Omega)$ are 2-uniformly smooth.

Definition 2.10. The duality map of X is the multivalued operator $J : X \to 2^{X^*}$ defined by

$$J(u) = \left\{ u^* \in X^* : \langle u^*, u\rangle = \|u\|_X^2 = \|u^*\|_{X^*}^2 \right\}.$$

The properties of the space X provide important properties of the duality map as it causes from the following result, proved in [30].

Proposition 2.7. *Let X be a p-uniformly smooth Banach space. Then there exists a positive constant L_J such that $\|J(u) - J(v)\|_{X^*} \leq L_J \|u - v\|_X^{p-1}$, $\forall u, v \in X$.*

Moreover, it is known that every reflexive Banach space X can be endowed with an equivalent norm such that both X and X^* are strictly convex. In addition, the proof of the following results can be found in [27].

Proposition 2.8. *Let the dual space X^* be strictly convex. Then the duality map J is single-valued, surjective, and maximal monotone.*

Proposition 2.9. *Let X be a strictly convex reflexive Banach space with strictly convex dual X^*, and let $T : D(T) \subseteq X \to 2^{X^*}$ be a maximal monotone operator. Then, for each $\lambda > 0$, the inverse operator $(J+\lambda T)^{-1} : X^* \to D(T)$ is well-defined and single-valued.*

3. A Fixed Point Problem

In this section, we provide an equivalence result between Problem P and a family of fixed point problems. To this end, we consider the following assumptions on the data.

$$X \text{ is a strictly convex reflexive Banach space with strictly convex dual } X^*. \tag{3.1}$$

$$X \text{ is a } p\text{-uniformly smooth Banach space with } p > 1. \tag{3.2}$$

$$K \text{ is a closed convex subset of } X \text{ and } int\, K \neq \emptyset. \tag{3.3}$$

$$A\colon X \to X^* \text{ is demicontinuous and } p\text{-monotone with } m_A > 0. \tag{3.4}$$

$$\varphi\colon X \to \mathbb{R}\cup\{+\infty\} \text{ is convex lower semicontinuous such that } K \subset D(\varphi). \tag{3.5}$$

$$\begin{cases} j\colon X \to \mathbb{R} \text{ is such that} \\ \text{(a) } j \text{ is locally Lipschitz continuous;} \\ \text{(b) there exists } \alpha_j > 0 \text{ such that} \\ \qquad j^0(u_1; u_2 - u_1) + j^0(u_2; u_1 - u_2) \leq \alpha_j \|u_1 - u_2\|_X^p \quad \forall\, u_1, u_2 \in X. \end{cases} \tag{3.6}$$

$$m_A > \alpha_j. \tag{3.7}$$

$$f \in X^*. \tag{3.8}$$

We note that (3.6)(b) is equivalent to the p-relaxed monotonicity condition of the multivalued operator $\partial j : X \to 2^{X^*}$ with constant α_j. A proof of the equivalence can be found in [4, p.124].

We now proceed with a result which shows that Problem P is governed by a maximal monotone operator.

Theorem 3.1. *Assume (3.3)–(3.8). Then the operator $S : X \to 2^{X^*}$ defined by*

$$Su = Au + \partial j(u) + \partial^c(\varphi + I_K)(u) - f, \quad \forall\, u \in X \tag{3.9}$$

is a maximal monotone operator.

Proof. We use the arguments similar to those used in the proof of the Lemma 3.1 in [25]. Nevertheless, for the convenience of readers we provide the details, structured in three steps, as follows.

(i) *The operator $A + \partial j : X \to 2^{X^*}$ is maximal monotone*. To prove this statement, we start by using the p-monotonicity of the operator A with constant m_A, the p-relaxed monotonicity of the operator ∂j with constant α_j, and the smallness condition (3.7) to see that

$$\begin{aligned}\langle (Au+u^*)-(Av+v^*),u-v\rangle &= \langle Au-Av,u-v\rangle + \langle u^*-v^*,u-v\rangle \\ &\geq (m_A-\alpha_j)\|u-v\|_X^p \\ &\geq 0,\end{aligned}$$

for all $u,v\in X$, $u^*\in\partial j(u)$ and $v^*\in\partial j(v)$. We conclude from here that operator $A+\partial j: X\to 2^{X^*}$ is monotone.

Moreover, we deduce from Proposition 2.1(1) that $\{Au+\partial j(u)\}$ is a nonempty, convex, and weakly closed subset in X^*, for every $u\in X$.

In addition, let $u,v\in X$ and, for each $n\in\mathbb{N}$, consider an element x_n such that

$$x_n\in (A+\partial j)(\lambda_n u+(1-\lambda_n)v).$$

Suppose that $\lambda_n\to\lambda$ in $[0,1]$, $x_n\rightharpoonup x$ in X^* as $n\to\infty$. Since the operator A is demicontinuous, one obtains

$$x_n-A(\lambda_n u+(1-\lambda_n)v)\rightharpoonup x-A(\lambda u+(1-\lambda)v)\quad \text{in } X^*.$$

Moreover, the closedness of the graph of $\partial j(\cdot)$ in $X\times X_w^*$, guaranteed by Proposition 2.1(2), implies that $x\in(A+\partial j)(\lambda u+(1-\lambda)v)$. We conclude from here that the mapping $\lambda\mapsto(A+\partial j)(\lambda u+(1-\lambda)v)$ has a closed graph in $[0,1]\times X_w^*$.

We are now in a position to use Proposition 2.5 in order to see that the operator $A+\partial j: X\to 2^{X^*}$ is maximal monotone.

(ii) *The operator $A+\partial j+\partial^c(\varphi+I_K): X\to 2^{X^*}$ is maximal monotone*. Indeed, we deduce from (3.3), (3.5), and Proposition 2.4 that operator $\partial^c(\varphi+I_K)$ is maximal monotone. Furthermore, it is easy to see that $int D(A+\partial j)\cap D(\partial^c(\varphi+I_K))=K\neq\emptyset$. Thus, using Proposition 2.6, we deduce that the operator $A+\partial j+\partial^c(\varphi+I_K): X\to 2^{X^*}$ is maximal monotone, as claimed.

(iii) *End of proof.* We now use step (ii) to see that the operator S defined by (3.9) is maximal monotone, which completes the proof. □

Theorem 3.1 and Proposition 2.9 show that, under the assumptions (3.1) and (3.3)–(3.8), for each $\lambda>0$ we are in a position to consider the resolvent operator $S_\lambda: X\to K$ defined by equality

$$S_\lambda u=(J+\lambda S)^{-1}(Ju),\quad \forall\, u\in X. \tag{3.10}$$

We associate to this operator the following fixed point problem.

Problem P_λ. Find an element $u\in K$ such that $u=S_\lambda u$.

In the next section we prove that there exists an intrinsic link between problems P_λ and P. We shall use this link in order to deduce existence, uniqueness and wll-posedness results.

4. Existence and Uniqueness Results

We start with the following equivalence result.

Theorem 4.1. *Assume (3.1), (3.3)–(3.8), and let $\lambda>0$. Then u is a solution of Problem P if and only if it is a solution of Problem P_λ.*

Proof. It follows from the definition (3.10) that, for any $u \in X$, the following equivalences hold:

$$\begin{aligned} w = S_\lambda u \quad &\Longleftrightarrow \quad w \in K,\ \ Ju \in Jw + \lambda Sw \\ &\Longleftrightarrow \quad w \in K,\ \ \frac{1}{\lambda}(Ju - Jw) \in Sw. \end{aligned}$$

Moreover, using the definition (3.9) of the operator S combined with the definitions of Clarke gradient ∂j and subdifferential $\partial^c \varphi$, we deduce that

$$\begin{aligned} w \in K,\ \ \frac{1}{\lambda}(Ju - Jw) \in Sw \quad &\Longleftrightarrow \quad w \in K, \\ \langle Aw, v - w\rangle + \varphi(v) - \varphi(w) + j^0(w; v - w) &\geq \langle f + \frac{Ju - Jw}{\lambda}, v - w\rangle, \quad \forall\, v \in K. \end{aligned}$$

We deduce from above that

$$\begin{aligned} w = S_\lambda u \quad &\Longleftrightarrow \quad w \in K, \\ \langle Aw, v - w\rangle + \varphi(v) - \varphi(w) + j^0(w; v - w) &\geq \langle f + \frac{Ju - Jw}{\lambda}, v - w\rangle, \quad \forall\, v \in K. \end{aligned} \tag{4.1}$$

Theorem 4.1 is now a direct consequence of equivalence (4.1). □

Note that Theorem 4.1 represents an equivalence result which does not guarantee the unique solvability of problems P and P_λ. For this reason, we continue with the following existence and uniqueness result.

Theorem 4.2. *Assume (3.1)–(3.8). Then Problem P has a unique solution $u \in K$. Moreover, u is the unique solution to Problem P_λ for each $\lambda > 0$.*

The proof of the Theorem is based on the following preliminary result.

Lemma 4.1. *Assume (3.1)–(3.8), and let $\lambda > \frac{L_J}{m_A - \alpha_j}$. Then, Problem P_λ has a unique solution $u \in K$.*

Proof. We shall use the Banach fixed point principle. To this end, we prove that, for each $\lambda > \frac{L_J}{m_A - \alpha_j}$, the operator $S_\lambda : X \to K$ defined by (3.10) is a contraction on X. Let $u_1, u_2 \in X$ be arbitrary, denote $w_1 = S_\lambda u_1$, $w_2 = S_\lambda u_2$, and let $v \in K$. Using (4.1), it follows that $w_1, w_2 \in K$ and, for any $v \in K$, the inequalities below hold:

$$\langle Aw_1, v - w_1\rangle + \varphi(v) - \varphi(w_1) + j^0(w_1; v - w_1) \geq \langle f + \frac{Ju_1 - Jw_1}{\lambda}, v - w_1\rangle, \tag{4.2}$$

$$\langle Aw_2, v - w_2\rangle + \varphi(v) - \varphi(w_2) + j^0(w_2; v - w_2) \geq \langle f + \frac{Ju_2 - Jw_2}{\lambda}, v - w_2\rangle. \tag{4.3}$$

We now take $v = w_2$ in (4.2) and $v = w_1$ in (4.3), and then we add the resulting inequalities to see that

$$\begin{aligned} &\langle Aw_1 - Aw_2, w_2 - w_1\rangle + j^0(w_1; w_2 - w_1) + j^0(w_2; w_1 - w_2) \\ &\geq \frac{1}{\lambda}\langle Ju_1 - Ju_2, w_2 - w_1\rangle + \frac{1}{\lambda}\langle Jw_2 - Jw_1, w_2 - w_1\rangle. \end{aligned} \tag{4.4}$$

Next, we use the p-monotonicity of operator A and assumption (3.6)(b) to deduce that

$$\langle Aw_1 - Aw_2, w_2 - w_1\rangle + j^0(w_1; w_2 - w_1) + j^0(w_2; w_1 - w_2) \leq (\alpha_j - m_A)\|w_1 - w_2\|_X^p. \tag{4.5}$$

In addition, using the Cauchy-Schwarz inequality, Proposition 2.7 and the monotonicity of duality map, we find that

$$\frac{1}{\lambda}\langle Ju_1 - Ju_2, w_2 - w_1\rangle + \frac{1}{\lambda}\langle Jw_2 - Jw_1, w_2 - w_1\rangle \geq -\frac{L_J}{\lambda}\|u_1 - u_2\|_X^{p-1}\|w_1 - w_2\|_X. \quad (4.6)$$

We now combine inequalities (4.4)–(4.6) and use assumption (3.7) to see that

$$\|w_1 - w_2\|_X \leq \left(\frac{L_J}{\lambda(m_A - \alpha_j)}\right)^{\frac{1}{p-1}}\|u_1 - u_2\|_X. \quad (4.7)$$

Recall that $p > 1$ and $\lambda > \frac{L_J}{m_A - \alpha_j} > 0$. Then, it follows that

$$0 < \left(\frac{L_J}{\lambda(m_A - \alpha_j)}\right)^{\frac{1}{p-1}} < 1.$$

Thus inequality (4.7) shows that operator $S_\lambda : X \to K$ is a contraction on X. We now use the Banach fixed point theorem to obtain the unique solvability of Problem P_λ. □

We now proceed with the proof of Theorem 4.2.

Proof. Let $\lambda_0 > \frac{L_J}{m_A - \alpha_j}$. Then we use Lemma 4.1 to see that Problem P_{λ_0} has a unique solution $u \in K$. Therefore, it follows from Theorem 4.1 that u represents the unique solution of Problem P, which concludes the first part of the theorem. The second part follows from the unique solvability of Problem P. Indeed, since Problem P has a unique solution u, Theorem 4.1 guarantees that u is also the unique solution of Problem P_λ for each $\lambda > 0$. □

We end this section with the following remark.

Remark 4.1. A careful analysis of the proof of Lemma 4.1 reveals that, under assumptions (3.1)–(3.8), the operator S_λ is Lipschitz continuous for each $\lambda > 0$. Indeed, (4.7) shows that

$$\|S_\lambda u_1 - S_\lambda u_2\|_X \leq \left(\frac{L_J}{\lambda(m_A - \alpha_j)}\right)^{\frac{1}{p-1}}\|u_1 - u_2\|_X \quad (4.8)$$

for each $u_1, u_2 \in X$. We use inequality (4.8) in Section 6 below.

5. Well-Posedness Results

In this section, we investigate the $\mathscr{T}$-well-posedness of problems P and P_λ. To this end, for a generic Problem $\mathscr{P}$ defined in the Banach space X, we recall the following definitions, introduced in [23].

Definition 5.1. A Tykhonov triple is a mathematical object of the form $\mathscr{T} = (I, \Omega, \mathscr{C})$, where I is a given nonempty set, $\Omega : I \to 2^X$ is a set-valued map such that $\Omega(\varepsilon) \neq \emptyset$ for each $\varepsilon \in I$, and $\mathscr{C}$ is a nonempty subset of sequences with elements in I.

Definition 5.2. Given a Tykhonov triple $\mathscr{T} = (I, \Omega, \mathscr{C})$, a sequence $\{u_n\} \subset X$ is called a $\mathscr{T}$-approximating sequence if there exists a sequence $\{\varepsilon_n\} \in \mathscr{C}$, such that $u_n \in \Omega(\varepsilon_n)$ for each $n \in \mathbb{N}$.

Definition 5.3. Given a Tykhonov triple $\mathscr{T} = (I, \Omega, \mathscr{C})$, Problem $\mathscr{P}$ is said to be $\mathscr{T}$-well-posed (or, equivalently, well-posed with $\mathscr{T}$) if it has a unique solution $u \in X$ and every $\mathscr{T}$-approximating sequence converges strongly to u in X.

Given a Tykhonov triple $\mathscr{T} = (I, \Omega, \mathscr{C})$, we refer to I as the set of parameters. A typical element in it is denoted by ε. Moreover, we refer to the family of sets $\{\Omega(\varepsilon)\}_{\varepsilon \in I}$ as the family of approximating sets. It is worth mentioning that $\mathscr{T}$-approximating sequences always exist since, by assumption, $\mathscr{C} \neq \emptyset$. In addition, for any sequence $\{\varepsilon_n\} \in \mathscr{C}$ and any $n \in \mathbb{N}$, the set $\Omega(\varepsilon_n)$ is nonempty. We also remark that the concept of approximating sequence above depends on the Tykhonov triple $\mathscr{T}$ and, for this reason, we use the terminology "$\mathscr{T}$-approximating sequence". As a consequence, the concept of well-posedness for Problem $\mathscr{P}$ depends on the Tykhonov triple $\mathscr{T}$ and, therefore, we refer to it as "well-posedness with $\mathscr{T}$" or "$\mathscr{T}$-well-posedness", as mentioned in Definition 5.3. We shall use this concept for both problems P and P_λ.

In what follows, we assume (3.1)–(3.8) and fix $\lambda > 0$. We introduce two specific Tykhonov triples, in the study of problems P and P_λ, denoted by $\mathscr{T} = (I, \Omega, \mathscr{C})$ and $\mathscr{T}_\lambda = (I, \Omega_\lambda, \mathscr{C})$, respectively. The set of parameters I and the set $\mathscr{C}$ are defined as follows:

$$I = \mathbb{R}_+,$$
$$\mathscr{C} = \Big\{ \{\varepsilon_n\}_n : \varepsilon_n \in I \quad \forall n \in \mathbb{N}, \quad \varepsilon_n \to 0 \quad as \quad n \to \infty \Big\}.$$

Next, for any $\varepsilon \in I$, we define the approximating sets $\Omega(\varepsilon)$ and $\Omega_\lambda(\varepsilon)$ by equalities:

$$u \in \Omega(\varepsilon) \iff u \in K \quad \text{and}$$
$$\langle Au, v-u \rangle + \varphi(v) - \varphi(u) + j^0(u; v-u) + \varepsilon \|v-u\|_X \geq \langle f, v-u \rangle, \quad \forall v \in K,$$

$$u \in \Omega_\lambda(\varepsilon) \iff u \in X \quad \text{and} \quad \|u - S_\lambda u\|_X \leq \varepsilon.$$

Recall that, here and below, S_λ denotes the resolvent operator (3.10). Also, note that Theorem 4.2 guarantees that both problems P_λ and P have a unique common solution $u \in K$. Then, it is easy to see that $u \in \Omega(\varepsilon)$ and $u \in \Omega_\lambda(\varepsilon)$ for any $\varepsilon > 0$, which implies that $\Omega(\varepsilon) \neq \emptyset$ and $\Omega_\lambda(\varepsilon) \neq \emptyset$ for any $\varepsilon > 0$. Therefore, $\mathscr{T} = (I, \Omega, \mathscr{C})$ and $\mathscr{T}_\lambda = (I, \Omega_\lambda, \mathscr{C})$ are Tykhonov triples in the sense of Definition 5.1.

The following result shows the equivalence between the $\mathscr{T}$-well-posedness of Problem P and the $\mathscr{T}_\lambda$-well-posedness of Problem P_λ.

Theorem 5.1. *Assume (3.1)–(3.8) and $\lambda > 0$. Then Problem P is $\mathscr{T}$-well-posed if and only if Problem P_λ is $\mathscr{T}_\lambda$-well-posed.*

The proof of Theorem 5.1 is based on two preliminary results that we state and prove in what follows.

Lemma 5.1. *Assume (3.1)–(3.8), let $\lambda > 0$, and consider the following statements.*

$$\left\{ \begin{array}{l} u \in K \textit{ and there exists } \varepsilon \geq 0 \textit{ such that} \\ \langle Au, v-u \rangle + \varphi(v) - \varphi(u) + j^0(u; v-u) + \varepsilon \|v-u\|_X \geq \langle f, v-u \rangle, \quad \forall v \in K. \end{array} \right. \tag{5.1}$$

$$\left\{ \begin{array}{l} u \in K \textit{ and} \\ \|u - S_\lambda u\|_X \leq \left(\frac{\varepsilon}{m_A - \alpha_j} \right)^{\frac{1}{p-1}}. \end{array} \right. \tag{5.2}$$

Then (5.1) implies (5.2).

Proof. Assume (5.1). Then, we deduce from Proposition 2.1(3) that, for each $v \in K$, there exists $\xi_v \in \partial j(u)$ such that

$$\langle Au + \xi_v, v - u\rangle + \varphi(v) - \varphi(u) + \varepsilon \|v - u\|_X \geq \langle f, v - u\rangle.$$

In addition, from Proposition 2.1(1), we deduce that the set $\{Au + \xi - f : \xi \in \partial j(u)\}$ is a nonempty, closed, convex, and bounded subset in X^*. Since the function

$$\psi(v) = \varphi(v) + \varepsilon \|v - u\|_X$$

is proper, convex, and lower semicontinuous on X, it follows from Proposition 2.3 that there exists $\xi \in \partial j(u)$ such that

$$\langle Au + \xi, v - u\rangle + \varphi(v) - \varphi(u) + \varepsilon \|v - u\|_X \geq \langle f, v - u\rangle, \quad \forall\, v \in K. \tag{5.3}$$

Note that the element ξ (as well as the function ψ above and the functions M, P, Q below) depend on u. Nevertheless, for simplicity, since u is fixed, we do not mention here this dependence. We now use (5.3) to find that

$$\varphi(u) \leq \varphi(v) + \langle Au + \xi - f, v - u\rangle + \varepsilon \|v - u\|_X, \quad \forall v \in K.$$

We also consider the function $M : X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$M(v) = \varphi(v) + P(v) + \varepsilon Q(v), \quad \forall\, v \in X,$$

where $P(v)$ and $Q(v)$ are the functions on X defined by

$$P(v) = \langle Au + \xi - f, v - u\rangle, \qquad Q(v) = \|v - u\|_X, \quad \forall v \in X.$$

It is clear that the function M is proper convex lower semicontinuous, and u is a minimizer of M on K, which implies that $0_{X^*} \in \partial^c(M + I_K)(u)$. Hence, using the definition of M and Proposition 2.2, we deduce that $u \in K$ and

$$0_{X^*} \in \partial^c(\varphi + I_K)(u) + Au + \xi - f + \varepsilon \partial^c Q(u).$$

Thus, using the inclusion $\xi \in \partial j(u)$ we conclude that there exists an element $\eta \in \partial^c Q(u)$ (which depends on u) such that

$$0_{X^*} \in \partial^c(\varphi + I_K)(u) + Au - f + \partial j(u) + \varepsilon \eta.$$

Next, from the definition of operator S, one has $-\varepsilon\eta \in Su$, which implies that

$$u = (J + \lambda S)^{-1}(Ju - \lambda \varepsilon \eta). \tag{5.4}$$

We now claim that operator $(J + \lambda S)^{-1} : X^* \to K$ is Lipschitz continuous. Indeed, let $u_1, u_2 \in X^*$ be arbitrary and denote $w_1 = (J + \lambda S)^{-1}(u_1)$ and $w_2 = (J + \lambda S)^{-1}(u_2)$. Using arguments similar to those used in the proof of (4.1), it follows that $w_1, w_2 \in K$. For any $v \in K$, the inequalities below hold:

$$\langle Aw_1, v - w_1\rangle + \varphi(v) - \varphi(w_1) + j^0(w_1; v - w_1) \geq \langle f + \frac{u_1 - Jw_1}{\lambda}, v - w_1\rangle, \tag{5.5}$$

$$\langle Aw_2, v - w_2\rangle + \varphi(v) - \varphi(w_2) + j^0(w_2; v - w_2) \geq \langle f + \frac{u_2 - Jw_2}{\lambda}, v - w_2\rangle. \tag{5.6}$$

We now take $v = w_2$ in (5.5) and $v = w_1$ in (5.6), and then we add the resulting inequalities to see that

$$\begin{aligned}&\langle Aw_1 - Aw_2, w_2 - w_1\rangle + j^0(w_1; w_2 - w_1) + j^0(w_2; w_1 - w_2)\\&\geq \frac{1}{\lambda}\langle u_1 - u_2, w_2 - w_1\rangle + \frac{1}{\lambda}\langle Jw_2 - Jw_1, w_2 - w_1\rangle.\end{aligned} \tag{5.7}$$

Next, we use the p-monotonicity of operator A and assumption (3.6)(b) to deduce that

$$\langle Aw_1 - Aw_2, w_2 - w_1\rangle + j^0(w_1; w_2 - w_1) + j^0(w_2; w_1 - w_2) \leq (\alpha_j - m_A)\|w_1 - w_2\|_X^p. \tag{5.8}$$

Moreover, using the Cauchy-Schwarz inequality and the monotonicity of duality map, one has

$$\frac{1}{\lambda}\langle u_1 - u_2, w_2 - w_1\rangle + \frac{1}{\lambda}\langle Jw_2 - Jw_1, w_2 - w_1\rangle \geq -\frac{1}{\lambda}\|u_1 - u_2\|_{X^*}\|w_1 - w_2\|_X. \tag{5.9}$$

We now combine inequalities (5.7)–(5.9) to find that

$$\|(J+\lambda S)^{-1}(u_1) - (J+\lambda S)^{-1}(u_2)\|_X \leq \Big(\frac{1}{\lambda(m_A - \alpha_j)}\Big)^{\frac{1}{p-1}}\|u_1 - u_2\|_{X^*}^{\frac{1}{p-1}}. \tag{5.10}$$

Finally, we use (5.4) and (3.10) to see that

$$\|u - S_\lambda u\|_X = \|(J+\lambda S)^{-1}(Ju - \lambda\varepsilon\eta) - (J+\lambda S)^{-1}(Ju)\|_X.$$

Using inequality (5.10) with $u_1 = Ju - \lambda\varepsilon\eta$ and $u_2 = Ju$ we deduce that

$$\|u - S_\lambda u\|_X \leq \Big(\frac{\varepsilon}{m_A - \alpha_j}\Big)^{\frac{1}{p-1}}\|\eta\|_{X^*}^{\frac{1}{p-1}}. \tag{5.11}$$

Recall now that $\eta \in \partial^c Q(u)$, which implies that $\langle \eta, v - u\rangle \leq \|v - u\|_X$ for all $v \in X$ or, equivalently, $\langle \eta, w\rangle \leq \|w\|_X$ for all $w \in X$. Then, $\|\eta\|_{X^*} = \sup_{\|w\|_X \leq 1}|\langle \eta, w\rangle| \leq 1$. Thus inequality (5.11) shows that the statement (5.2) holds. □

Lemma 5.2. *Assume (3.1)–(3.8), let $\lambda > 0$, and consider the following statements.*

$$\left\{\begin{array}{l} u \in X \text{ and there exists } \varepsilon > 0 \text{ such that}\\ \|u - S_\lambda u\|_X \leq \varepsilon.\end{array}\right. \tag{5.12}$$

$$\left\{\begin{array}{l} w \in K \text{ and } \langle Aw, v - w\rangle + \varphi(v) - \varphi(w) + j^0(w; v - w)\\ \qquad + \frac{\varepsilon^{p-1}L_J}{\lambda}\|v - w\|_X \geq \langle f, v - w\rangle \quad \forall\, v \in K,\\ \text{where } w = S_\lambda u.\end{array}\right. \tag{5.13}$$

Then (5.12) implies (5.13).

Proof. Assume (5.12), let $w = S_\lambda u$, and let v be an arbitrary element such that $v \in K$. Then (4.1) implies that $u \in K$ and

$$\langle Aw, v - w\rangle + \varphi(v) - \varphi(w) + j^0(w; v - w) \geq \langle \frac{Ju - Jw}{\lambda} + f, v - w\rangle. \tag{5.14}$$

Moreover, the Cauchy-Schwarz inequality, Proposition 2.7 and (5.12) yield

$$\begin{aligned}\langle \frac{Ju - Jw}{\lambda}, v - w\rangle &\geq -\frac{1}{\lambda}\|Ju - Jw\|_{X^*}\|v - w\|_X\\ &\geq -\frac{L_J}{\lambda}\|u - w\|_X^{p-1}\|v - w\|_X \geq -\frac{\varepsilon^{p-1}L_J}{\lambda}\|v - w\|_X.\end{aligned} \tag{5.15}$$

We now combine inequalities (5.14) and (5.15) to deduce that

$$\langle Aw, v-w\rangle + \varphi(v) - \varphi(w) + j^0(w; v-w) + \frac{\varepsilon^{p-1}L_J}{\lambda}\|v-w\|_X \geq \langle f, v-w\rangle,$$

which concludes the proof. □

We are now in a position to provide the proof of Theorem 5.1 .

Proof. Assume that Problem P is $\mathscr{T}$-well-posed. Then Problem P has a unique solution $u \in K$, and Theorem 4.1 guarantees that u is the unique solution to Problem P_λ, too. Let $\{u_n\} \subset X$ be a $\mathscr{T}_\lambda$-approximating sequence. Then there exists a sequence $0 \leq \varepsilon_n \to 0$ such that $\|u_n - S_\lambda u_n\|_X \leq \varepsilon_n$, $\forall n \in \mathbb{N}$, which implies that

$$\|u_n - S_\lambda u_n\|_X \to 0. \tag{5.16}$$

Moreover, Lemma 5.2 implies that, for any $v \in K$ and $n \in \mathbb{N}$,

$$\langle Aw_n, v-w_n\rangle + \varphi(v) - \varphi(w_n) + j^0(w_n; v-w_n) + \varepsilon_n'\|v-w_n\|_X \geq \langle f, v-w_n\rangle$$

with $w_n = S_\lambda u_n$ and $\varepsilon_n' = \frac{\varepsilon_n^{p-1}L_J}{\lambda}$. It follows from here that $\{w_n\} \subset K$ is a $\mathscr{T}$-approximating sequence for Problem P. Since Problem P is $\mathscr{T}$-well-posed, we deduce that

$$\|w_n - u\|_X \to 0. \tag{5.17}$$

Therefore, writing $\|u_n - u\|_X \leq \|u_n - S_\lambda u_n\|_X + \|S_\lambda u_n - u\|_X$, and, using equality $w_n = S_\lambda u_n$ together with convergences (5.16) and (5.17), we deduce that $u_n \to u$ in X. This shows that Problem P_λ is $\mathscr{T}_\lambda$-well-posed.

Conversely, assume that Problem P_λ is $\mathscr{T}_\lambda$-well-posed. Then Problem P_λ has a unique solution $u \in K$, which is also the unique solution to Problem P. Let $\{u_n\} \subset K$ be a $\mathscr{T}$-approximating sequence. Then, Lemma 5.1 shows that $\{u_n\}$ is a $\mathscr{T}_\lambda$-approximating sequence. Using the $\mathscr{T}_\lambda$-well-posedness of Problem P_λ, one sees that $\{u_n\}$ converges to u in X. one concludes from here that Problem P is $\mathscr{T}$-well-posed, which concludes the proof. □

Note that Theorem 5.1 represents an equivalence result which does guarantee neither the $\mathscr{T}$-well-posedness of Problem P, nor the $\mathscr{T}_\lambda$-well-posedness of Problem P_λ. These well-posedness results are provided below.

Theorem 5.2. *Assume* (3.1)–(3.8) *and* $\lambda > 0$. *Then Problem* P *is* $\mathscr{T}$*-well-posed and Problem* P_λ *is* $\mathscr{T}_\lambda$*-well-posed.*

Proof. The existence of a unique solution $u \in K$ is guaranteed by Theorem 4.2. Let $\{u_n\} \subset X$ be a $\mathscr{T}$-approximating sequence. Then, there exists a sequence $0 \leq \varepsilon_n \to 0$ such that $u_n \in \Omega(\varepsilon_n)$ for each $n \in \mathbb{N}$, i.e., $u_n \in K$ and

$$\langle Au_n, v-u_n\rangle + \varphi(v) - \varphi(u_n) + j^0(u_n; v-u_n) + \varepsilon_n\|v-u_n\|_X \geq \langle f, v-u_n\rangle, \tag{5.18}$$

for all $v \in K$ and $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed. We take $v = u$ in (5.18) and $v = u_n$ in (1.1), and then add the resulting inequalities to find that

$$\langle Au_n - Au, u-u_n\rangle + j^0(u_n; u-u_n) + j^0(u; u_n-u) + \varepsilon_n\|u_n-u\|_X \geq 0. \tag{5.19}$$

We now use the p-monotonicity of operator A and assumption (3.6)(b) to deduce that

$$\langle Au_n - Au, u-u_n\rangle + j^0(u_n; u-u_n) + j^0(u; u_n-u) \leq (\alpha_j - m_A)\|u_n-u\|_X^p. \tag{5.20}$$

Next, we combine inequalities (5.19) and (5.20) and use assumption (3.7) to see that

$$\|u_n - u\|_X \le \Big(\frac{\varepsilon_n}{m_A - \alpha_j}\Big)^{\frac{1}{p-1}}.$$

Therefore, since $\varepsilon_n \to 0$, we deduce that $\|u_n - u\|_X \to 0$. Using Definition 5.3, we deduce that Problem P is $\mathscr{T}$-well-posed. Finally, we use Theorem 5.1 to find that Problem P_λ is $\mathscr{T}_\lambda$-well-posed. □

6. A CONVERGENCE CRITERION

We assume in what follows that (3.1)–(3.8) hold, and we denote by u the solution of Problem P obtained in Theorem 4.2. It follows from Theorem 5.2 and Definition 5.3 that any $\mathscr{T}$-approximating sequence converges in X to u. This property given rise to the following question.

(Q_1) *Are there sequences $\{u_n\} \subset X$ which converge in X to u and which are not $\mathscr{T}$-approximating sequences?*

To provide an answer to this question, we start by considering an elementary example.

Example 6.1. Let X be a Hilbert space endowed with inner product $(\cdot,\cdot)_X$ and associated norm $\|\cdot\|_X$. In addition, assume that K satisfies condition (3.3). It is known that the duality map $J : X \to X^*$ is the isometry given by the Riesz representation theorem and, therefore,

$$\langle Ju, v\rangle = (u,v)_X, \quad \forall\, u,\, v \in X. \tag{6.1}$$

Let a and b be positive constants, $f \in X^*$, and let $A : X \to X^*$, $\varphi : X \to \mathbb{R}$, $j : X \to \mathbb{R}$, $g \in X$ be defined by

$$Au = aJu, \quad \varphi(u) = 0, \quad j(u) = \frac{b}{2}\|u\|_X^2, \quad \forall\, u \in X, \quad f = Jg. \tag{6.2}$$

Then, it is easy to see that

$$j^0(u;v) = b(u,v)_X, \quad \forall\, u,\, v \in X. \tag{6.3}$$

Moreover, conditions (3.1)–(3.8) are satisfied with $p = 2$, $m_A = a$, and $\alpha_j = 0$. We now use (6.1)–(6.3) to deduce that, with the choice (6.2), inequality (1.1) can be written, equivalently,

$$u \in K, \quad \big((a+b)u, v-u\big)_X \ge (g, v-u)_X, \quad \forall\, v \in K.$$

The solution of this inequality is given by

$$u = P_K\Big(\frac{g}{a+b}\Big), \tag{6.4}$$

where, here and below, $P_K : X \to K$ represents the projection operator on K.

Next, let $\lambda > 0$, $u \in X$, and let $w = S_\lambda u$. Then, using similar arguments as above, we deduce from (4.1) that w is the unique element of X which satisfies the inequality

$$w \in K, \quad ((a+b)w, v-w)_X \ge (g + \frac{u-w}{\lambda}, v-w)_X, \quad \forall\, v \in K.$$

Therefore,

$$w = S_\lambda u = P_K\Big(\frac{\lambda g + u}{\lambda(a+b)+1}\Big), \quad \forall\, u \in X. \tag{6.5}$$

The interest in formulas (6.4) and (6.5) arises in the fact that they provide an explicit formula for the solution of inequality (1.1) and the corresponding resolvent operator S_λ, as well.

We now use Example 6.1 to provide a positive answer to question (Q_1) above.

Example 6.2. Keep the assumptions and notation in Example 6.1 and, in addition, assume that K is the closed ball of radius 1 centered on 0_X, i.e.,

$$K = \Big\{ v \in X \,:\, \|v\|_X \le 1 \Big\}. \tag{6.6}$$

It is known that the projection operator P_K on the set (6.6) is given by

$$P_K z = \begin{cases} \frac{z}{\|z\|_X} & \text{if } \ \|z\|_X > 1, \\ z & \text{if } \ \|z\|_X \le 1. \end{cases} \tag{6.7}$$

Take $g \in X$ such that $\|g\|_X = a + b$, and consider the sequence $\{u_n\} \subset X$ defined by

$$u_n = \Big(1 + \frac{1}{n}\Big) \frac{g}{a+b}, \qquad \forall\, n \in \mathbb{N}. \tag{6.8}$$

Then, it is easy to see that $\|u_n\|_X = 1 + \frac{1}{n} > 1$ and, therefore $u_n \notin K$ for any $n \in \mathbb{N}$. This implies that $\{u_n\} \subset X$ is not a $\mathscr{T}$-approximating sequence. Nevertheless (6.4) and (6.7) imply that $u = \frac{g}{a+b}$. Using (6.8), we deduce that the convergence $u_n \to u$ in X holds.

Since the answer to question (Q_1) is positive, we are in a position to formulate in the second question, as follows.

(Q_2) *Which is the necessary and sufficient condition for a sequence $\{u_n\} \subset X$ to converge in X to u ?*

An answer to this question is given by the result below.

Theorem 6.1. *Assume (3.1)–(3.8) and let $\{u_n\} \subset X$. Then $u_n \to u$ in X if and only if, for each $\lambda > 0$, the sequence $\{u_n\}$ is a $\mathscr{T}_\lambda$-approximating sequence, i.e., $S_\lambda u_n - u_n \to 0_X$ in X as $n \to \infty$.*

Proof. Assume that $u_n \to u$ in X, and let $\lambda > 0$. Moreover, for each $n \in \mathbb{N}$, denote $w_n = S_\lambda u_n$. We use Theorem 4.1 to see that $u = S_\lambda u$. Thus

$$\|u_n - S_\lambda u_n\|_X \le \|u_n - u\|_X + \|S_\lambda u - S_\lambda u_n\|_X, \quad \forall\, n \in \mathbb{N}. \tag{6.9}$$

Now, it follows from (4.8) that there exists $k_\lambda > 0$ such that

$$\|S_\lambda u - S_\lambda u_n\|_X \le k_\lambda \|u_n - u\|_X, \quad \forall\, n \in \mathbb{N}. \tag{6.10}$$

We now combine inequalities (6.9) and (6.10) to deduce that

$$\|u_n - S_\lambda u_n\|_X \le (1 + k_\lambda)\|u_n - u\|_X, \quad \forall\, n \in \mathbb{N}.$$

Using notation $\varepsilon_n = (1 + k_\lambda)\|u_n - u\|_X$, we obtain that

$$\|u_n - S_\lambda u_n\|_X \le \varepsilon_n, \quad \forall\, n \in \mathbb{N}. \tag{6.11}$$

Now, since $u_n \to u$ in X, we find that $\varepsilon_n \to 0$. Therefore, (6.11) shows that the sequence $\{u_n\}$ is a $\mathscr{T}_\lambda$-approximating sequence.

Conversely, assume now that $\lambda > 0$, and $\{u_n\}$ is a $\mathscr{T}_\lambda$-approximating sequence. Then,

$$S_\lambda u_n - u_n \to 0_X \quad \text{in } X. \tag{6.12}$$

Next, the $\mathscr{T}_\lambda$-well-posedness of Problem P_λ, guaranteed by Theorem 5.2, implies that

$$S_\lambda u_n \to u \quad \text{in } X. \tag{6.13}$$

We now write $\|u_n - u\|_X \le \|u_n - S_\lambda u_n\|_X + \|S_\lambda u_n - u\|_X$ and use the convergences (6.12), (6.13) to see that $u_n \to u$ in X, which concludes the proof. □

We underline that Theorem 6.1 represents a criterion of convergence since it indicates a necessary and sufficient condition, which guarantees the convergence of a sequence $\{u_n\} \subset X$ to the solution of variational-hemivariational inequality (1.1). Below we provide an explicit form of this convergence criterion within the framework of Example 6.1.

Example 6.3. Keep the assumptions and notation in Example 6.1 and let $\{u_n\} \subset X$. We now use (6.5) to see that

$$S_\lambda u_n = P_K\Big(\frac{\lambda g + u_n}{\lambda(a+b)+1}\Big), \quad \forall n \in \mathbb{N}. \tag{6.14}$$

It follows now from Theorem 6.1 that the sequence $\{u_n\}$ converges to the solution (6.4) if and only if

$$u_n - P_K\Big(\frac{\lambda g + u_n}{\lambda(a+b)+1}\Big) \to 0_X \quad \text{in } X. \tag{6.15}$$

We now move to the framework of Example 6.2 and check the validity of the criterion (6.15) for the sequence (6.8).

Example 6.4. Keep the assumptions and notation in Example 6.2, and let $\{u_n\} \subset X$ be given by (6.8). We use (6.14) to see that in this case

$$S_\lambda u_n = P_K\Big(\frac{\lambda(a+b)+1+\frac{1}{n}}{\lambda(a+b)+1} \cdot \frac{g}{a+b}\Big), \quad \forall n \in \mathbb{N}. \tag{6.16}$$

Therefore, since $\|g\|_X = a+b$, equalities (6.16) and (6.7) imply that

$$S_\lambda u_n = \frac{g}{a+b}. \tag{6.17}$$

We now use (6.8) and (6.17) to see that $u_n - S_\lambda u_n \to 0_X$ which represents a validation of the convergence results $u_n \to u$ in X, obtained in Example 6.2.

7. Conclusion

In this paper, we considered an elliptic variational-hemivariational inequality (Problem P) in a p-uniformly smooth Banach space X. We proved that this problem is governed by a maximal monotone operator S (Theorem 3.1). This allowed us to define the resolvent operator S_λ for any $\lambda > 0$ and to introduce the problem of finding a fixed point of the operator S_λ (Problem P_λ), as well. We then performed an analysis of problems P and P_λ, carried out in several steps. First, we proved that the unique solvability of Problem P is equivalent with the unique solvability of Problem P_λ (Theorem 4.1), then we proved the unique solvability of Problem P_λ and deduced the existence of a unique solution u, common for problems P and P_λ (Theorem 4.2). Based on this result, we introduced the Tykhonov triples $\mathscr{T}$ and $\mathscr{T}_\lambda$ and we proved that well-posedness of Problem P with $\mathscr{T}$ is equivalent with the well-posedness of Problem P_λ with $\mathscr{T}_\lambda$ (Theorem 5.1). Moreover, we proved the $\mathscr{T}$-well-posedness of Problem P and deduced from here the $\mathscr{T}_\lambda$-well-posedness of Problem P_λ (Theorem 5.2). Finally, we used Problem P_λ, again, in order to deduce a convergence criterion to the solution of Problem P.

A careful analysis based on the description above indicates that the results on Problem P have been obtained by proving the corresponding results for Problem P_λ and vice versa. In other words, the analysis of problems P and P_λ was carried out in parallel, based on the intrinsic relationship which exists between these problems. Besides the existence, uniqueness and well-posedness results presented, the method we used, based on the parallel study of problems P and P_λ, represents the main trait of novelty of the current paper.

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