J. Nonlinear Var. Anal. 6 (2022), No. 6, pp. 605-618
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.6.2022.6.02

A STEEPEST DESCENT-LIKE METHOD FOR VECTOR OPTIMIZATION PROBLEMS WITH VARIABLE DOMINATION STRUCTURE

GEMAYQZEL BOUZA[1,*], CHRISTIANE TAMMER[2]

[1] *Universidad de la Habana, Habana, Cuba*
[2] *Institut für Mathematik, Martin-Luther-Universität Halle-Wittenberg, Halle 06099, Germany*

Abstract. In many real world problems, the simultaneous minimization of many certain goals is needed. Such problems lead to vector optimization problems with variable domination structure, where the domination structure is given by a set-valued map. In this framework, one of the most important solution concepts is the concept of nondominated points. In this paper, we propose a steepest descent-like method for computing nondominated solutions of smooth unconstrained vector optimization problems with variable domination structure. We obtain that every accumulation point of the generated sequence satisfies a first order necessary condition. We discuss the consequence of this fact in the convex case.

Keywords. Nondominated solution; Steepest descent-like method; Vector optimization with variable domination structures.

1. INTRODUCTION

Nowadays vector optimization models do not provide good solutions to certain new economical and medical applications. A fixed comparison cone may provide solutions which are mathematically correct but meaningless from the viewpoint of applications. A classical example appears in the process of establishing the diagnosis of the diseases using information from images. The idea is to transform the collected data with the help of a function T and, depending on its values, the diagnosis is done. Given the original data A, the desired pattern B and the set of transformations $\mathscr{T}$, the problem is to find the best transformation. To this aim, m different distance measures are considered and the goal is to find the transformation $T \in \mathscr{T}$ that minimizes them. The model can be described as:

$$\min F(T,A,B), \text{ s.t. } T \in \mathscr{T} \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^m$ describes the measuring criteria.

The solution of Problem (1.1) via classical weighting technique leads to wrong results. Indeed, solutions with large values at some objectives functions are obtained. However, if the set of weights depends on the point T, better results are reported; see [1]. This can be re-interpreted in terms of finding a point T^* such that $F(T^*,A,B) \notin F(T,A,B) - [K(F(T,A,B)) \setminus \{0\}]$. Here

*Corresponding author.
E-mail addresses: gema@matcom.uh.cu (G.B. Allende), christiane.tammer@mathematik.uni-halle.de (C. Tammer)

Received June 30, 2022; Accepted August 10, 2022.

$K(F(T,A,B))$ denotes the dual cone of the feasible weights at point T. So, the ordering in the space $\mathbb{R}^m$ is variable.

In general, the variable domination structure in $\mathbb{R}^m$ is given by the set-valued map $K : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, where $K(y)$ is a pointed, convex, and closed cone, for all $y \in \mathbb{R}^m$. We consider the following vector optimization problem with variable domination structure given by the map K and the objective function $F : \mathbb{R}^n \to \mathbb{R}^m$ in our paper:

$$K - \min F(x). \tag{VVOP}$$

The concept of domination structures was established by Yu in [2, 3], where the sets $K(y)$ are assumed to be cones. Yu considered a domination structure as a family of cones $K(y)$, while Engau [4] defined it as a set-valued map. Furthermore, Bergstresser, Charnes, and Yu [5] implemented domination factors in a finite-dimensional framework with convex domination sets.

We now define two binary relations in $\mathbb{R}^m$ with respect to the choice of domination sets, in contrast to vector optimization with a fixed ordering cone. These binary relations will be employed in the formulation of the solution concepts for (VVOP).

Definition 1.1 (binary relations). Given a domination structure $K : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, and given vectors $y, v \in \mathbb{R}^m$, we introduce the following binary relations:

(i) The **nondomination-like binary relation** denoted by $\leq_{NL,K}$ is defined by

$$v \leq_{NL,K} y :\Longleftrightarrow v \in y - K(y).$$

(ii) The **nondomination binary relation** denoted by $\leq_{N,K}$ is defined by

$$v \leq_{N,K} y :\Longleftrightarrow y \in v + K(v).$$

In the following, we use the notions $\operatorname{dom} F := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$ and $F(M) := \cup_{x \in M} F(x)$, $M \subseteq \mathbb{R}^n$.

Definition 1.2. (nondominated and nondominated-like solutions with respect to domination structures). Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function and let $K : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ be a domination structure in the image space $\mathbb{R}^m$. Given $x^* \in \operatorname{dom} F$, we say that:

(i) x^* is a K**-nondominated-like solution** of F with respect to K if

$$\forall x \in \operatorname{dom} F,\ F(x) \neq F(x^*) \Longrightarrow F(x) \nleq_{NL,K} F(x^*),$$

which is equivalent to the condition

$$F(\mathbb{R}^n) \cap (F(x^*) - K(F(x^*))) = \{F(x^*)\}.$$

(ii) x^* is a K**-nondominated solution** of F with respect to K if

$$\forall x \in \operatorname{dom} F,\ F(x) \neq F(x^*) \Longrightarrow F(x) \nleq_{N,K} F(x^*).$$

The set of all nondominated solutions of Problem (VVOP) is denoted by S^*.

Remark 1.1. Note that for a K-nondominated solution x^* of (VVOP), we have (see Definition 1.2(ii))

$$\text{for all } x \in \mathbb{R}^n:\ F(x^*) - F(x) \notin K(F(x)) \setminus \{0\}.$$

This means that for a K-nondominated solution x^*, we have $F(x^*)$ is not larger than $F(x)$ with respect to the order given by $K(F(x))$.

Although the concept of K-nondominated-like solutions, *i.e.,* $x^* : F(x) - F(x^*) \notin K(F(x^*)) \setminus \{0\}$ (see Definition 1.2(i)), is complex, practical problems where the need of computing nondominated points arise.

We have already presented the practical application in medical diagnosis [1]. Other important applications of optimization problems with variable domination structure appear in economics, portfolio optimization, ecology, radiotherapy treatment in medicine, environmental and behavioral sciences, and location theory; see, e.g., [4, 6].

Iterative methods, such as steepest descent algorithm, proximal points, weighting techniques, Newton-like, and sub-gradient methods for classical vector optimization models were defined and studied in [7, 8, 9, 10, 11, 12, 13, 14]. In [15, 16, 17, 18, 19], these approaches were extended to vector optimization problems with variable domination structures where the solution concept is the concept of K-nondominated-like solutions; see Definition 1.2(i). However, the concept of a nondominated solution x^* (Definition 1.2(ii)) is more complex because the domination set $K(F(x))$ is a set associated with another element x and $F(x^*) - F(x)$ shall not belong to $K(F(x)) \setminus \{0\}$ whereas a K-nondominated-like solution x^* is an element which is not dominated by another element x with respect to the associated set $K(F(x^*))$ at the K-nondominated-like solution x^*.

In order to have a structure of the set-valued domination map K, we assume that, for $y \in \mathbb{R}^m$, $K(y)$ is a **Bishop-Phelps (BP) cone**, that is,

$$K(y) := \{z \in \mathbb{R}^m \ : \ \|z\| \leq l(y)^T z\}, \tag{1.2}$$

where $l : \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function. It is well known that the Bishop-Phelps cone is a pointed, closed, convex cone with nonempty interior if $\|l\|^* > 1$, where $\|\cdot\|^*$ denotes the norm in the space of continuous functions $l : \mathbb{R}^m \to \mathbb{R}^m$.

Due to its simplicity and the adaptability to the structure of the vector optimization problem, in this work, we present a steepest descent-like algorithm for solving vector optimization problems with variable domination structure given by BP-cones. We obtain the properties of the limit points of the sequence generated by the proposed algorithm. Then, under a convex like hypothesis, we guarantee that the sequence is bounded and all its accumulation points are solutions of (VVOP).

The paper is organized as follows: After some preliminary results, we extend the concept of convexity of a function to the variable domination case and obtain some properties of this class of functions. Section 4 is devoted to the presentation of the algorithm and the continuity of the involved operators. Finally, the convergence of the steepest descent method is demonstrated in Section 5.

2. Preliminaries

In this section, we present some previous results and definitions. We begin with some notations.

The inner product is denoted by $\langle \cdot, \cdot \rangle$ and the norm is denoted by $\|\cdot\|$. The ball centered at x with radius r is $B(x,r) = \{y : \|y - x\| \leq r\}$. Given two sets A and B, we consider $\mathrm{dist}(A,B)$ as the Hausdorff distance, *i.e.,*

$$\mathrm{dist}(A,B) = \max\left\{ \sup_{a\in A} \inf_{b\in B} d(a,b), \sup_{b\in B} \inf_{a\in A} d(a,b) \right\}.$$

bd(A) denotes the boundary of the set A and A^c, its complement.

As we have already mentioned, we assume that the domination structure is given by the set-valued map $K:\mathbb{R}^m \rightrightarrows \mathbb{R}^m$, $K(y)=\{z\in\mathbb{R}^m : \|z\|\leq l(y)^T z\}$, where $l:\mathbb{R}^m\to\mathbb{R}^m$ is a continuous function. This means that $K(y)$ is a Bishop-Phelps (BP) cone; see (1.2). So, the domination structure is defined by the functional l. Recall that the norm is the same for all y. As $K(y)$ is a convex, closed, pointed cone such that $\operatorname{int}(K(y))\neq\emptyset$ (see [20]), we will assume $\|l\|^*>1$.

Proposition 2.1. *Consider the BP-cone given by* (1.2) *and suppose that* $\|l\|^*>1$. *Then,* $K(y)$ *is a convex, closed, pointed set, and* $\operatorname{int}(K(y))\neq\varnothing$ *for all* $y\in\mathbb{R}^m$. *Furthermore, if* l *is continuous, then* K *is B-lsc.*

Proof. In [20], it was shown that $K(y)$ is a convex, closed, pointed set, and $\operatorname{int}(K(y))\neq\varnothing$ for all $y\in\mathbb{R}^m$. For the second assertion, take $y^*\in\mathbb{R}^m$, $k\in K(y^*)$, $\varepsilon>0$, and $B(k,\varepsilon)$. As $\operatorname{int}K\neq\varnothing$, we take $k^0\in\operatorname{int}(K(y^*))$. Recalling that $K(y)$ is convex for all $y\in\mathbb{R}^m$ and $\|k\|\leq l(y^*)^T k$, we obtain, for $\alpha<\varepsilon$, $k+\alpha(k^0-k)\in\operatorname{int}(K(y^*)\cap B(k,\varepsilon)$. Because of $\|l\|^*>1$, we conclude $\|k+\alpha(k^0-k)\|<l(y^*)^T[k+\alpha(k^0-k)]$. Since l is continuous, we have $\|k+\alpha(k^0-k)\|<l(y)^T[k+\alpha(k^0-k)]$ for y close enough to y^*. □

For each $y\in\mathbb{R}^m$, the dual cone of $K(y)$ is defined as $K^*(y)=\{y^*\in\mathbb{R}^m : \langle y^*,z\rangle\geq 0$, for all $z\in K(y)\}$. As usual, the graph of a set-valued application K is the set $Gr(K)=\{(y,z)\in\mathbb{R}^m\times\mathbb{R}^m :\ z\in K(y)\}$. We recall that the mapping K is closed if $Gr(K)$ is a closed subset of $\mathbb{R}^m\times\mathbb{R}^m$.

As in the case of classical vector optimization, related solution concepts such as weakly efficient and stationary points can be extended. We introduce the weak solution concept corresponding to nondominated solutions in Definition 1.2(ii).

Definition 2.1. The point x^* is a weakly nondominated solution of Problem (VVOP) if

$$\text{for all }\ x\in\mathbb{R}^n:\ F(x)-F(x^*)\notin -\operatorname{int}(K(F(x))).$$

S^w denotes the set of all weakly nondominated solutions.

We point out that this definition corresponds with the concept of weakly nondominated points given in [21].

On the other hand, if F is a continuously differentiable function, we introduce the following concept of stationary points.

Definition 2.2. Suppose that F is a continuously differentiable function. The point x^* is stationary if

$$\forall y\in\mathbb{R}^m:\ \nabla(F(x^*))^T y\cap -\operatorname{int}(K(F(x^*)))=\emptyset.$$

S^s denotes the set of all stationary points.

A necessary condition for weakly nondominated solutions of (VVOP) in the sense of Definition 2.1 is given as follows.

Proposition 2.2. *Let* x^* *be a weakly nondominated solution of Problem* (VVOP) *in the sense of Definition* 2.1. *If* F *is a continuously differentiable function, and* K *is given by the functional* l *and a norm* $\|\cdot\|$ *(see* (1.2)*), then there exists*

$$y^*\in K^*(F(x^*))=\{y\in\mathbb{R}^m\ :\ \|y-l(F(x^*))\|\leq 1\}$$

such that $\nabla F(x^*)^T y^*=0$. *In particular,* x^* *is a stationary point.*

Proof. Consider the weakly nondominated solution of (VVOP). From [22, Theorem 7.10], we have that there exists $y^* \in K^*(F(x^*))$ with $\nabla F(x^*)^T y^* = 0$. Furthermore, from [22, Lemma 1.16], we obtain

$$K^*(F(x^*)) = \{y \in \mathbb{R}^m \;:\; \|y - l(F(x^*))\| \leq 1\}.$$

In order to show that x^* is a stationary point, we argue by contradiction. We suppose that there is $d \in \mathbb{R}^m$ with $\nabla F(x^*)d \in -\operatorname{int}(K(F(x^*)))$, i.e., x^* is not a stationary point. Then, we have $y^* \nabla F(x)d < 0$ for $y^* \in K^*(F(x^*))$, which a contradiction to $\nabla F(x^*)^T y^* = 0$. □

Next, we deal with the so-called quasi-Fejér convergence and its properties.

Definition 2.3. Let S be a nonempty subset of $\mathbb{R}^n$. A sequence $\{z^k\}$ is said to be quasi-Fejér convergent to S if and only if, for all $x \in S$, there exists $\bar{k}$ and a summable sequence $\{\delta_k\} \subset \mathbb{R}_+$ such that $\|z^{k+1} - x\|^2 \leq \|z^k - x\|^2 + \delta_k$ for all $k \geq \bar{k}$.

This definition originates in [23] and was further elaborated in [24]. A useful result on the quasi-Fejér sequences is the following.

Theorem 2.1. *If $\{z^k\}$ is quasi-Fejér convergent to S, then*

i) *the sequence $\{z^k\}$ is bounded;*
ii) *if a cluster point of the sequence $\{z^k\}$ belongs to S, then the whole sequence $\{z^k\}$ converges.*

Proof. See [23, Lemma6] and [25, Theorem 1]. □

3. A Steepest Descent-like Method

This section is devoted to a steepest descent-like algorithm for solving unconstrained smooth vector optimization problems with variable domination structure. Some definitions, the algorithm, and some basic properties of the involved functions are given in this section.

Our algorithm makes use of the set-valued mapping $G : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, which defines, for each $y \in \mathbb{R}^m$, the set of the normalized generators of $K^*(y)$, *i.e.,* $G(y) \subseteq K^*(y) \cap \operatorname{bd}(B(0,1))$ is a compact set such that the cone generated by its convex hull is $K^*(y)$.

Although $K^*(y) \cap \operatorname{bd}(B(0,1))$ fulfills those properties, it is possible to take smaller sets in general; see [26, 27, 28].

On the other hand, we consider the function $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\phi(x,y) := \|y^T \nabla F(x)\|^2,$$

and for each $x \in \mathbb{R}^n$, the auxiliary problem

$$\min_{y \in G(F(x^k))} \{\phi(x,y)\}. \tag{P^k}$$

We denote the set of integers by $\mathbb{Z}$ and the set of nonnegative integers by $\mathbb{Z}_+$. For fixed constants $\sigma \in (0,1)$, $\delta > 0$, and $\beta_k \geq \delta > 0$ for all k, the algorithm is defined as follows.

Algorithm A for generating stationary points of problem (VVOP):

Initialization: Take $x^0 \in \mathbb{R}^n$ and β_0.

Iterative step: Given x^k and β_k, compute a solution y^k of (P^k).
If $\nabla F(x^k)^T y^k = 0$, then stop. Otherwise compute a solution v^k of the problem

$$\inf_{v\in\mathbb{R}^n} l[F(x^k)]\nabla F(x^k)^T v + \|\nabla F(x^k)^T v\| + \|v\|_2^2/2 \tag{3.1}$$

and

$$j(k) := \min\left\{ j\in\mathbb{Z}_+ : F(x^k) + \sigma 2^{-j}\nabla F(x^k)^T v^k - F(x^k + 2^{-j}v^k) \in K(F(x^k+2^{-j}v^k))\right\}. \tag{3.2}$$

Set

$$x^{k+1} := x^k + \gamma_k v^k,$$

with $\gamma_k = 2^{-j(k)}$.

Note that the stopping criteria is given by the necessary condition involved in Proposition 2.2. Furthermore, the iterative step is well defined. First, note that problem (3.1) is convex, so there are algorithms to solve it. The next proposition shows important properties of generated solutions v^k in (3.1).

Proposition 3.1. *If $\nabla F(x^k)y^k \neq 0$, then there exists v_0^k such that*

$$l(F(x^k))\nabla F(x^k)^T v_0^k + \|\nabla F(x^k)^T v_0^k\| < 0.$$

Moreover, $v^k \neq 0$ is unique and

$$\inf_{v\in\mathbb{R}^n} l(F(x^k))\nabla F(x^k)^T v + \|\nabla F(x^k)^T v\| + \|v\|^2/2 < 0.$$

Proof. Take the set

$$C := \{u\in\mathbb{R}^n \ : \ u = \nabla F(x^k)^T l(F(x^k)) + \nabla F(x^k)^T e : e\in B(0,1)\}.$$

C is a convex set and by the hypothesis, it does not contain 0. So, we can strongly separate C from 0. This means that there exists $v : v^T y < \varepsilon < 0$ for all $y \in C$. W.l.o.g., we take $\|v\| = 1$. Then, for $e = 0$, $v^T\nabla F(x^k)^T l(F(x^k)) < 0$. Thus $\nabla F(x^k)^T v \neq 0$ and in particular $v \neq 0$. Since the objective function of (3.1) is a strong convex function and its limit is $+\infty$ when $\|v\| \to +\infty$, the existence and the uniqueness of the solution is clear.

Now, we take $e := \nabla F(x^k)v/\|\nabla F(x^k)v\|$. Observe that

$$\begin{aligned} &v^T\nabla F(x^k)^T l(F(x^k)) + v^T\nabla F(x^k)^T\nabla F(x^k)v/\|\nabla F(x^k)v\| \\ &= v^T\nabla F(x^k)^T l(F(x^k)) + \|\nabla F(x^k)^T v\| < 0 \end{aligned}$$

Thus $v \in \operatorname{int}(-K(F(x^k)))$. Taking $\alpha > 0$ small enough, we have

$$\alpha(v^T\nabla F(x^k)^T l(F(x^k)) + \|\nabla F(x^k)v\|) + \alpha^2\|v\|^2/2 < 0.$$

But v^k solves (P^k) and αv^k is a feasible point. So,

$$\begin{aligned} &(v^k)^T\nabla F(x^k)^T l(F(x^k)) + \|\nabla F(x^k)v^k\| + \|v^k\|^2/2 \\ &\leq \alpha(v^T\nabla F(x^k)^T l(F(x^k)) + \|\nabla F(x^k)v\|) + \alpha^2\|v\|^2/2 < 0. \end{aligned}$$

□

As a consequence, we obtain $\nabla F(x^k)v^k \in \operatorname{int}(-K(F(x^k)))$. With respect to the existence of $j(k)$, the next proposition guarantees it.

Proposition 3.2. *$j(k)$ is finite.*

Proof. As $\nabla F(x^k)v^k \in \mathrm{int}(-K(F(x^k)))$, that is, $l(F(x^k))\nabla F(x^k)v^k + \|\nabla F(x^k)v^k\| < 0$. By continuity arguments, we obtain, for x close to x^k, $l(F(x))\nabla F(x^k)v^k + \|\nabla F(x^k)v^k\| < 0$. Therefore,

$$\nabla F(x^k)v^k \in \mathrm{int}(-K(F(x))). \tag{3.3}$$

On the other hand, $F(x^k+\alpha v^k) - F(x^k) - \sigma\alpha\nabla F(x^k)v^k = (1-\sigma)\alpha\nabla F(x^k)v^k + o(\alpha)$. As $(1-\sigma)\nabla F(x^k)v^k \in \mathrm{int}(-K(F(x^k)))$, by Proposition 2.1 and [19, Lemma 3.8], for α small enough,

$$(1-\sigma)\nabla F(x^k)v^k \in \mathrm{int}(-K(F(x^k+\alpha v^k))).$$

As a consequence, $(1-\sigma)\alpha\nabla F(x^k)v^k + o(\alpha) \in \mathrm{int}(K(F(x^k+\alpha v^k)))$. The desired result follows. □

Proposition 3.3. *If $x \in \mathbb{R}^n$ and fix $\beta > 0$, then $\|v(x^k)\| \leq 2\|l(F(x^k))\nabla F(x^k)\| + \|\nabla F(x^k)\|$.*

Proof. Taking into account Proposition 3.1, it holds that $l(F(x^k))\nabla F(x^k)v^k + \|\nabla F(x^k)v^k\| + \|v^k\|^2/2 \leq 0$. Thus

$$\begin{aligned}\|v^k\|^2/2 &\leq -l(F(x^k))\nabla F(x^k)v^k - \|\nabla F(x^k)v^k\| \\ &\leq \|l(F(x^k))\nabla F(x^k)\|\|v^k\| + \|\nabla F(x^k)\|\|v^k\|\end{aligned}$$

and $\|v^k\| \leq 2\|l(F(x^k))\nabla F(x^k)\| + \|\nabla F(x^k)\|$. □

After discussing the continuity of the involved functions, we now present the convergence results.

4. Convergence of the Method: General Case

In this section, we investigate the convergence of Algorithm A, presented in the previous section. First, we consider the general case and then the result is refined for K-convex functions. From now on, $\{x^k\}$ will denote the sequence generated by Algorithm A. We begin with the following lemma, where we are dealing with a Daniell cone $\mathscr{K}$.

Definition 4.1. Let $\mathscr{K} \subseteq \mathbb{R}^m$ be a proper convex cone. $\mathscr{K}$ is called Daniell if every $\mathscr{K}$-increasing and $\mathscr{K}$-upper bounded sequence from $\mathbb{R}^m$ has a supremum to which it converges.

Furthermore, in the following lemma, we are dealing with the set-valued map $G : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ (the set of the normalized generators of $K^*(F(x))$), where $G(F(x)) \subseteq K^*(F(x)) \cap \mathrm{bd}\,(B(0,1))$ is a compact set such that the cone generated by its convex hull is $K^*(F(x))$ (see Section 3).

Lemma 4.1. *Assume that*

(i) $\cup_{x\in\mathbb{R}^n} K(F(x)) \subset \mathscr{K}$, *where $\mathscr{K} \subseteq \mathbb{R}^m$ is a Daniell cone.*
(ii) *The set-valued map $G : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is closed.*
(iii) *There exists $L > 0$ such that $dist(G(F(x)), G(F(\bar{x}))) \leq L\|x - \bar{x}\|$ for all $x, \bar{x} \in \mathbb{R}^n$.*

If x^* is an accumulation point of $\{x^k\}$, then $\lim_{k\to\infty} F(x^k) = F(x^*)$.

Proof. Take $\lim_{k\to\infty} x^{i_k} = x^*$, where $\{x^{i_k}\}$ is a subsequence of $\{x^k\}$. By its definition, it holds that $F(x^{k+1}) - F(x^k) - \sigma\gamma_k\nabla F(x^k)v^k \in -K(F(x^{k+1}))$. By (3.3), this means that

$$F(x^{k+1}) - F(x^k) \in -\mathrm{int}(K(F(x^{k+1}))). \tag{4.1}$$

As $\cup_{x\in\mathbb{R}^n} K(F(x)) \subset \mathscr{K}$, it holds that, for all x, $\mathrm{int}(K(F(x))) \subset \mathrm{int}(\mathscr{K})$. Thus $F(x^k) - F(x^{k+1}) \in \mathrm{int}(\mathscr{K})$. Equivalently, $F(x^k)$ is a decreasing sequence with respect to the cone $\mathscr{K}$. As F is a continuous function, $\lim_{k\to\infty} F(x^{i_k}) = F(x^*)$. In particular, $F(x^*)$ is an accumulation point, so $\mathscr{K}$ is a Daniell cone; see [29, 30]. It follows that $\lim_{k\to\infty} F(x^k) = F(x^*)$. □

Theorem 4.1. *Suppose that*

(i) $\cup_{x\in\mathbb{R}^n} K(F(x)) \subset \mathscr{K}$, *where* $\mathscr{K}$ *is a Daniell cone,*
(ii) $l(y)$ *is continuous,*
(iii) $dist(G(F(x)), G(F(\hat{x}))) \leq L\|x-\hat{x}\|$, *for all* $x, \hat{x} \in \mathbb{R}^n$,
(iv) $\nabla F(x)$ *is a locally Lipschitz function.*

If $\|l(F(x^k))\nabla F(x^k)\| + \|\nabla F(x^k)\|$ *is bounded, then each accumulation point* x^* *of* $\{x^k\}$ *is a stationary point of Problem* (VVOP).

Proof. It is clear that, for all $a, b \in \mathbb{R}^m$,

$$l(F(x^{k+1}))(a+b) + \|a+b\| \leq l(F(x^{k+1}))a + \|a\| + l(F(x^{k+1}))b + \|b\|.$$

By (3.2), one has

$$l(F(x^{k+1}))[F(x^k) - F(x^{k+1}) + \sigma\gamma_k \nabla F(x^k)v(x^k)] + \|F(x^k) - F(x^{k+1}) + \sigma\gamma_k \nabla F(x^k)v(x^k)\| \geq 0.$$

In particular,

$$\begin{aligned} &l(F(x^{k+1}))[F(x^k) - F(x^{k+1})] + \|F(x^k) - F(x^{k+1})\| + l(F(x^{k+1}))[\sigma\gamma_k \nabla F(x^k)v(x^k)] \\ &\quad + \|\sigma\gamma_k \nabla F(x^k)v(x^k)\| \\ &\geq l(F(x^{k+1}))[F(x^k) - F(x^{k+1}) + \sigma\gamma_k \nabla F(x^k)v(x^k)] + \|F(x^k) - F(x^{k+1}) + \sigma\gamma_k \nabla F(x^k)v(x^k)\| \\ &\geq 0. \end{aligned}$$

Thus

$$\begin{aligned} &l(F(x^{k+1}))[F(x^k) - F(x^{k+1})] + \|F(x^k) - F(x^{k+1})\| \\ &\geq -\sigma\gamma_k[l(F(x^{k+1}))\nabla F(x^k)v(x^k) + \|\nabla F(x^k)v(x^k)\|]. \end{aligned}$$

Consider the subsequence $\{x^{i_k}\} \to x^*$, where x^* is an accumulation point, the corresponding direction $\{v^{i_k}\}$, $v^{i_k} := v(x^{i_k})$ and step size $\gamma^{i^k} := 2^{-j(k)}$. Then,

$$\lim_{k\to\infty} l(F(x^{i_k+1}))\left[F(x^{i_k}) - F(x^{i_k+1}) + \|F(x^{i_k}) - F(x^{i_k+1})\|\right] = 0,$$

due to the facts that l is a continuous function and $F(x^k)$ is a convergent sequence by Lemma 4.1. Therefore, combining the two previous conditions, we have

$$0 \geq \lim -\sigma\gamma_k\left[l(F(x^{i_k+1}))\nabla F(x^{i_k})v(x^{i_k}) + \|\nabla F(x^{i_k})v(x^{i_k})\|\right].$$

It follows from (4.1) that

$$\nabla F(x^{i_k})v(x^{i_k}) \in \mathrm{int}(-K(F(x^{i_k+1}))).$$

So,

$$l(F(x^{i_k+1}))\nabla F(x^{i_k})v(x^{i_k}) + \|\nabla F(x^{i_k})v(x^{i_k})\| < 0.$$

These facts imply that

$$0 \geq -\sigma \lim_{k\to\infty} \gamma_{i_k}\left[l(F(x^{i_k+1}))\nabla F(x^{i_k})v(x^{i_k}) + \|\nabla F(x^{i_k})v(x^{i_k})\|\right] \geq 0.$$

Hence,

$$\lim_{k\to\infty} \gamma_{i_k}[l(F(x^{i_k+1}))\nabla F(x^{i_k})v(x^{i_k}) + \|\nabla F(x^{i_k})v(x^{i_k})\|] = 0.$$

As $\gamma_k \in (0,1)$ for all k, $\limsup_{k\to\infty}\gamma_{i_k} \geq 0$. Since x^{i_k} is bounded, $l \in C^0$, and $F \in C^1$,

$$2\|l(F(x^{i_k-1}))\nabla F(x^{i_k})\| + \|\nabla F(x^{i_k})\| \text{ is bounded.}$$

By Proposition 3.3, $\|v^{i_k}\|$ is also bounded. As x^{i_k} converges and

$$\|x^{i_k} - x^{i_k+1}\| = \|\gamma_{i_k} v^{i_k}\| \leq \gamma_{i_k}\|v^{i_k}\| \leq \|v^{i_k}\|,$$

we can assume that x^{i_k+1} is bounded. Without loss of generality, suppose that, for the subsequence $\{i_k\}$, $\{x^{i_k}\}$, $\{x^{i_k+1}\}$, $\{v^{i_k}\}$, and $\{\gamma_{i_k}\}$ converge to x^*, x^{*+1}, v^*, and to $\gamma^* = \limsup_{k\to\infty}\gamma_{i_k}$, respectively. We will consider two cases $\gamma^* > 0$ and $\gamma^* = 0$.

Case 1: $\gamma^* > 0$. Hence, $\lim_{k\to\infty} l(F(x^{i_k+1}))\nabla F(x^{i_k})v(x^{i_k}) + \|\nabla F(x^{i_k})v(x^{i_k})\| = 0$. By continuity arguments and the facts that l,F are continuous and $x^{i_k}, v^{i_k}, x^{i_k+1}$ converge, it holds that:

$$l(F(x^{*+1}))\nabla F(x^*)v^* + \|\nabla F(x^*)v^*\| = 0. \tag{4.2}$$

But, as $F(x^*)$ and $F(x^{*+1})$ are accumulations points of $F(x^k)$ and it is a convergent sequence, $F(x^*) = F(x^{*+1})$. So,

$$l(F(x^*)) = l(F(x^{*+1})), \tag{4.3}$$

and we can rewrite (4.2) as $l(F(x^*))\nabla F(x^*)v^* + \|\nabla F(x^*)v^*\| = 0$. But, $v = 0$ is also a feasible solution of problem (3.1) for x^*. As $v(x^*)$ solves it,

$$0 \geq l(F(x^*))\nabla F(x^*)v(x^*) + \|\nabla F(x^*)v(x^*)\| + \|v(x^*)\|^2/2 = \|v(x^*)\|^2/2 \geq 0.$$

Therefore,

$$l(F(x^*))\nabla F(x^*)v(x^*) + \|\nabla F(x^*)v(x^*)\| + \|v(x^*)\|^2/2 = \|v(x^*)\|^2/2 = 0.$$

So, $v^* = 0$. Moreover, by Proposition 3.1, x^* is a stationary point.

Case 2: $\gamma^* = 0$.

Using the previously defined convergent sub-sequences $\{x^{i_k}\}$, $\{x^{i_k+1}\}$, $\{\beta_{i_k}\}$, $\{v^{i_k}\}$, and $\{\gamma_{i_k}\}$, we have that

$$\begin{aligned} l(F(x^{i_k+1}))\nabla F(x^{i_k})v^{i_k} + \|\nabla F(x^{i_k})v^{i_k}\| &\leq l(F(x^{i_k+1}))\nabla F(x^{i_k})v^{i_k} + \|\nabla F(x^{i_k})v^{i_k}\| + \|v^{i_k}\|^2/2 \\ &< 0. \end{aligned}$$

Taking limits, it follows

$$\begin{aligned} l(F(x^{*+1}))\nabla F(x^*)v^* + \|\nabla F(x^*)v^*\| &\leq l(F(x^{*+1}))\nabla F(x^*)v^* + \|\nabla F(x^*)v^*\| + \|v^*\|^2/2 \\ &\leq 0. \end{aligned}$$

Combining this with (4.3), we have

$$l(F(x^*))\nabla F(x^*)v^* + \|\nabla F(x^*)v^*\| \leq 0. \tag{4.4}$$

Fix $q \in \mathbb{N}$. Then, for k large enough,

$$F(x^{i_k} + 2^{-q}v^{i_k}) \notin F(x^{i_k}) + \frac{\sigma\nabla F(x^{i_k})v^{i_k}}{2^q} - K(F(x^{i_k} + 2^{-q}v^{i_k})).$$

It follows that

$$\begin{aligned} &l(F(x^{i_k} + 2^{-q}v^{i_k}))\left[F(x^{i_k} + 2^{-q}v^{i_k}) - F(x^{i_k}) - \frac{\sigma\nabla F(x^{i_k})v^{i_k}}{2^q}\right] \\ &\quad + \|F(x^{i_k} + 2^{-q}v^{i_k}) - F(x^{i_k}) - \frac{\sigma\nabla F(x^{i_k})v^{i_k}}{2^q}\| > 0. \end{aligned}$$

Taking limits as k tends to $+\infty$ and recalling that l, F and ∇F are continuous functions, we obtain

$$l(F(x^*+2^{-q}v^*))\left[F(x^*+2^{-q}v^*)-F(x^*)-\frac{\sigma\nabla F(x^*)v^*}{2^q}\right]$$
$$+\left\|F(x^*+2^{-q}v^*)-F(x^*)-\frac{\sigma\nabla F(x^*)v^*}{2^q}\right\|\geq 0.$$

Multiplying by 2^q, we have

$$l(F(x^*+2^{-q}v^*))\left[\frac{F(x^*+2^{-q}v^*)-F(x^*)}{2^{-q}}-\sigma\nabla F(x^*)v^*\right]$$
$$+\left\|\frac{F(x^*+2^{-q}v^*)-F(x^*)}{2^{-q}}-\sigma\nabla F(x^*)v^*\right\|\geq 0.$$

Taking limits as q tends to $+\infty$, we obtain

$$l(F(x^*))\left[(1-\sigma)\nabla F(x^*)v^*\right]+\|(1-\sigma)\nabla F(x^*)v^*\|\geq 0.$$

Dividing by $(1-\sigma)>0$ and using (4.4), we get

$$0\geq l(F(x^*))\left[\nabla F(x^*)v^*\right]+\|\nabla F(x^*)v^*\|\geq 0.$$

Again, as $v=0$ is also a feasible solution of problem (3.1) for x^* and v^* solves it:

$$l(F(x^{i_*}))\nabla F(x^{i_*})v^*+\|\nabla F(x^{i_*})v^*\|+\|v^*\|^2/2=\|v^*\|^2/2\leq 0.$$

So, $v^*=0$ $x^*\in S^s$. □

This result needs the existence of an accumulation point. The convergence of the sequence generated by the algorithm is only possible under stronger assumptions. Now, based on quasi-Féjer theory, we prove the convergence of the sequence in the convex case.

For guaranteeing the convergence of Algorithm A, we need to consider

$$\beta_k=\frac{\alpha_k}{2\|l(F(x^k))\nabla F(x^k)\|+\|\nabla F(x^k)\|},$$

where $\alpha_k\geq 0$, $\sum_{k=1}^{\infty}\alpha_k^2<\infty$ and redefine problem (3.1) as the problem to find a solution v^k of

$$\inf_v \beta_k\left[l(F(x^k))\nabla F(x^k)v+\|\nabla F(x^k)v\|\right]+\|v\|^2/2. \tag{4.5}$$

This is only a technicality. So, we only included here to avoid complicated notations in the proofs of the results presented above. We want to point out that the convergence results are also valid if this parameter is added. We begin with a result that provides an upper bound of $\|x^{k+1}-x^k\|^2-\|x^k-x\|^2$. Here, we suppose that F is a K-convex function in the following sense:

Definition 4.2. Let $F:\mathbb{R}^n\to\mathbb{R}^m$ be a vector-valued function, and let $K:\mathbb{R}^m\rightrightarrows\mathbb{R}^m$. The function F is K-convex if, for all $\lambda\in[0,1]$ and $x^1,x^2\in\mathbb{R}^n$, it holds that

$$F(\lambda x^1+(1-\lambda)x^2)\in\lambda F(x^1)+(1-\lambda)F(x^2)-K(F(\lambda x^1+(1-\lambda)x^2)).$$

Lemma 4.2. *Take $x\in\mathbb{R}^n$. If F is a K-convex function and K is a closed map, then*

$$\|x^{k+1}-x^k\|^2-\|x^k-x\|^2\leq 2\gamma_k[l(F(x^k))+z^T][F(x)-F(x^k)]+\gamma_k^2\alpha_k^2,$$

for some z with $\|z\|^*\leq 1$ and $z^T\nabla F(x^k)v=\|\nabla F(x^k)v\|$.

Proof. First note that

$$\|x^{k+1}-x\|^2-\|x^k-x\|^2=2\gamma_k\langle v^k,x^k-x\rangle+\gamma_k^2\|v^k\|^2, \tag{4.6}$$

but v^k is a solution of (4.5). As already mentioned in Section 3, the problem in (4.5) is convex. So, $0=v^T+l(F(x^k))\nabla F(x^k)+z^T\nabla F(x^k)$ for some z with $\|z\|^*\le 1$ and $z^T\nabla F(x^k)v=\|\nabla F(x^k)v\|$. This together with (4.6) yields

$$\|x^{k+1}-x\|^2-\|x^k-x\|^2=2\gamma_k[l(F(x^k))+z^T]\nabla F(x^k)(x^k-x)+\gamma_k^2\|v^k\|^2. \tag{4.7}$$

From Proposition 2.2, $[l(F(x^k))+z^T]\in K^*(F(x^k))$. As F is a K-convex function, it follows from [15, Proposition 3.4] that $F(x)-F(x^k)-\nabla F(x^k)(x-x^k)\in K(F(x^k))$. Moreover,

$$[l(F(x^k))+z^T]\left(F(x)-F(x^k)\right)\ge 2\gamma_k[l(F(x^k))+z^T]\nabla F(x^k)(x-x^k).$$

Combining the previous inequality with (4.7) and taking into account that $0<\gamma_k<1$, we have

$$\|x^{k+1}-x\|^2-\|x^k-x\|^2\le 2\gamma_k[l(F(x^k))+z^T][F(x)-F(x^k)]+\gamma_k^2\|v(x^k)\|^2. \tag{4.8}$$

On the other hand, by the same arguments as in Proposition 3.3,

$$\|v^k\|\le 2\beta_k\|l(F(x^k))\nabla F(x^k)\|+\|\nabla F(x^k)\|\alpha_k.$$

Combining (4.8) and the previous inequality, it follows that

$$\|x^{k+1}-x^k\|^2-\|x^k-x\|^2\le 2\gamma_k[l(F(x^k))+z^T][F(x)-F(x^k)]+\gamma_k^2\alpha_k^2.$$

□

Now, we prove the convergence of Algorithm A for K-convex functions. First, we define the set

$$T:=\{x\in\mathbb{R}^n:\ \forall k:\quad F(x^k)-F(x)\in K(F(x^k))\}.$$

We assume that $T\neq\emptyset$. This hypothesis is closely related with the completeness of $F(\mathbb{R}^n)$, and as reported in [28], the completeness of $F(\mathbb{R}^n)$ ensures the existence of efficient points. That is why $T\neq\emptyset$ is assumed for proving the convergence of several methods for solving classical vector optimization problems; see, e.g., [10, 12, 31, 32, 33, 34].

Theorem 4.2. *Let F be a K-convex function and $T\neq\emptyset$. Assume that*

(i) $\cup_{x\in\mathbb{R}^n}K(F(x))\subset\mathscr{K}$, *where $\mathscr{K}$ is a Daniell cone,*
(ii) *The mapping $G(F(x))$ is closed,*
(iii) $dist(G(F(x)),G(F(\hat{x})))\le L\|x-\hat{x}\|$, *for all $x,\hat{x}\in\mathbb{R}^n$,*
(iv) $\nabla F(x)$ *is a locally Lipschitz function.*

Then, there exists x^* such that $\lim_{k\to\infty}x^{i_k}=x^*$. Moreover, all accumulation points of $\{x^k\}$ are stationary points of Problem (VVOP).

Proof. By Lemma 4.2, we obtain

$$\|x^{k+1}-x^k\|^2-\|x^k-x\|^2\le 2\gamma_k[l(F(x^k))+z^T][F(x)-F(x^k)]+\gamma_k^2\alpha_k^2,$$

where $z\in\mathbb{R}^m$ with $\|z\|\le 1$ and $z^T\nabla F(x^k)v=\|\nabla F(x^k)v\|$. Since $x\in T$, it follows that $[F(x^k)-F(x)]\in K(F(x^k))$ and $2\gamma_k[l(F(x^k))+z^T][F(x)-F(x^k)]\le 0$. Thus $\|x^{k+1}-x^k\|^2-\|x^k-x\|^2\le\gamma_k^2\alpha_k^2$. As $\gamma_k\le 1$ and $\sum_{k=1}^{\infty}\alpha_k^2<\infty$, $\{x^k\}$ is a quasi- Féjer sequence with respect to T, and, by Theorem 2.1(i), x^k is bounded. So, it has an accumulation point, which is denoted as x^*. Finally, x^* is a stationary point by Theorem 4.1. □

Remark 4.1. In order to prove the convergence of the whole sequence using Theorem 2.1 (ii), we need to show that there exists an accumulation point x^* of $\{x^k\}$ such that $x^* \in T$. The main difficulty is that, due to the variability of the cones, we can only guarantee $F(x^*) - F(x) \in \mathrm{bd}(K(F(x^*)))$ for all $x \in T$. However, in vector optimization, the main goal is to reconstruct the set of solutions. So, in the case that the sequence $\{x^k\}$ has different accumulation points, a clustering technique such as K-means (see [35]) can be used to identify the different accumulation points.

5. Illustrative Examples

At the end of this article, we illustrate our results by employing Algorithm A (see Section 3) in two examples. In all cases, we considered 10 randomly generated, feasible points as initial solutions. The algorithm was implemented in MatLab R2012 and ran at a Intel(R) Atom(TM) CPU N270 at 1.6GHz.

Example 5.1. Compute K-nondominated solutions of $F(x) = (x+1, x^2+1)^T$ with respect to K s.t. $x \in [0,1]$, where $K(y) = \{z \in \mathbb{R}^2 : \|z\|_2 \leq y_1 z_1\}$, $y \in \mathbb{R}^2$.

Although the problem has a constraint, the algorithm can be easily adapted. We only need to take the set of feasible directions in point x^k as the set of feasible solutions of problem (3.1). The results were

Instances	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Initial Point	0.2581	0.4087	0.5949	0.2622	0.6028	0.7112	0.2217	0.1174	0.2967	0.3188
Solution	0.8143	0.8065	0.8120	0.8144	0.8121	0.8145	0.8140	0.8131	0.8145	0.8147
CPU Time	1.7188	0.7500	0.7813	0.6875	0.3594	0.3750	0.7500	0.7344	0.6250	0.5781
Nº. Iterations	3	2	2	3	2	2	3	3	3	3

We point out here that all the computed solutions are nondominated points. Although the number of iterations is small, the CPU times are larger due to the optimization auxiliary problems that need to be solved.

Example 5.2. Compute K-nondominated solutions of $F(x) = (x^1, x^2)^T$ with respect to K s.t. $\pi \leq x_1^2 + x_2^2 \leq 2\pi$, where

$$K(y) = \left\{ z \in \mathbb{R}^2 : \|z\|_2 \leq y \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} z_1 \right\}, \quad y \in \mathbb{R}^2.$$

First note that the model has non-linear constraints. So, problem (3.1) has to be adapted. Now, the set of feasible solutions is the set of feasible directions at x^k. The results were

Instances	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Initial Point	1.3392 1.3911	1.6822 1.5305	1.3713 1.4589	0.5814 −1.8980	2.3587 −0.7054	1.7285 −0.9709	−1.6341 1.2554	−1.8039 1.2707	−2.1224 −0.1666	−0.1958 −1.7657
Solution	1.2044 1.3000	1.2416 1.2646	1.1766 1.3257	0.5814 −1.8980	1.5672 −0.8274	1.3952 −1.0932	−1.6341 1.2554	−1.8039 1.2707	−2.1224 −0.1666	−0.1958 −1.7657
CPU Time	3.0000	0.9219	1.7500	0.0781	781.7031	0.5781	0.1719	0.2813	0.1406	0.4375
Nº. Iterations	7	6	7	1	5468	7	1	1	1	1

Note that in 6 cases, the solution is close to the nondominated set, $x_1^2 + x_2^2 = \pi$. Indeed, the norm of the computed points is smaller than 0.015. We realize that if $x < 0$ the algorithm is not working properly. The steps are too small and the initial point is taken as solution. This fact illustrates the need of a more accurate algorithm for the restricted case.

The CPU-times are large. The solution of the auxiliary problems (non linear in general) consumes a lot of computational resources. An inexact variant should be studied.

6. Conclusions

In our paper, we derived a steepest descent method for generating approximately nondominated solutions of the vector optimization problems with variable domination structures. For further research, it would be interesting to combine our method with an adaptive solution procedure to obtain an overview on the set of approximately nondominated solutions of the problem. We recall that the steepest descent method is one of the oldest, classical, and more basic schemes for solving optimization problems. Despite its computational shortcomings, it sets the foundations of future more efficient algorithms, like projected gradient, Newton-like method, and inexact variants.

Acknowledgments
This research was developed as part of the project PN223LH010-005 Desarrollo de nuevos modelos y métodos matemáticos para la toma de decisiones, supported by Oficina de Gestión de Fondos y Proyectos Internacionales del Ministerio de Ciencia, Tecnología y Medio Ambiente de la República de Cuba. The authors would like to extend their gratitude to the anonymous referee whose suggestions helped us to improve the presentation of this paper.

References

[1] M.M. Wiecek, Advances in cone-based preference modeling for decision making with multiple criteria, Decis. Mak. Manuf. Serv. 1 (2007), 153-173.
[2] P.L. Yu, Introduction to domination structures in multicriteria decision problems, In: Multicriteria Decision Making, J.L. Cochrane and M. Zeleny (eds.), pp. 249-261, University of South Carolina Press, Columbia, SC, 1973.
[3] P.L. Yu, Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives, J. Optim. Theory Appl. 14 (1974), 319-377.
[4] A. Engau, Variable preference modeling with ideal-symmetric convex cones, J. Global Optim. 42 (2008), 295-311.
[5] K. Bergstresser, A. Charnes, P.L. Yu, Generalization of domination structures and nondominated solutions in multicriteria decision making, J. Optim. Theory Appl. 18 (1976), 3-13.
[6] D. Baatar, Wiecek, M.M.: Advancing equitability in multiobjective programming, Comput. Math. Appl. 2 (2007), 225-234.
[7] J.Y. Bello-Cruz, L.R. Lucambio Pérez, Convergence of a projected gradient method variant for quasiconvex objectives, Nonlinear Anal. 9 (2010), 2917-2922.
[8] H. Bonnel, A.N. Iusem, B.F. Svaiter, Proximal methods in vector optimization, SIAM J. Optim. 15 (2005), 953-970.
[9] J. Fliege, L.M. Graña Drummond, B.F. Svaiter, Newton's method for multiobjective optimization, SIAM J. Optim. 20 (2009), 602-626.
[10] J. Fliege, B.F. Svaiter, Steepest descent methods for multicriteria optimization, Math. Methods Oper. Res. 51 (2000), 479-494.

[11] L.M. Graña Drummond, N. Maculan, B.F. Svaiter, On the choice of parameters for the weighting method in vector optimization, Math. Program. 111 (2008), 201-216.

[12] L.M. Graña Drummond, B.F. Svaiter, A steepest descent method for vector optimization, J. Comput. Appl. Math. 175 (2005), 395-414.

[13] J. Jahn, Scalarization in vector optimization, Math. Program. 29 (1984), 203-218.

[14] D.T. Luc, Scalarization of vector optimization problems, J. Optim. Theory Appl. 55 (1987), 85-102.

[15] J.Y. Bello-Cruz, G. Bouza Allende, A steepest descent-like method for variable order vector optimization problems, J. Optim. Theory Appl. 162 (2014), 371-391.

[16] J.Y. Bello-Cruz, G. Bouza Allende, On inexact projected gradient methods for solving variable vector optimization problems, Optim. Eng. 23 (2020), 1-32.

[17] J.Y. Bello-Cruz, G. Bouza Allende, L.R. Lucambio Pérez, Subgradient algorithms for solving variable inequalities, Appl. Math. Comput. 247 (2014), 1052-1063.

[18] G.C. Bento, G. Bouza Allende, Y.R. Pereira, A Newton-like method for variable order vector optimization problems, J. Optim. Theory Appl. 177 (2018), 201-221.

[19] G. Bouza Allende, C. Tammer, Nonlinear scalarizing functionals for computing minimal points under variable ordering structures, Appl. Anal. Optim. 1 (2017), 67-97.

[20] G. Eichfelder, T.X. Duc Ha, Optimality conditions for vector optimization problems with variable ordering structures, Optimization 62 (2011), 597-627.

[21] G. Eichfelder, Optimal elements in vector optimization with variable ordering structure, J. Optim. Theory Appl. 151 (2011), 217-240.

[22] G. Eichfelder, Variable Ordering Structures in Vector Optimization, Springer, Berlin, 2014.

[23] F.E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, Math. Z. 100 (1967), 201-225.

[24] A.N. Iusem, B.F. Svaiter, M. Teboulle, Entropy-like proximal methods in convex programming, Math. Oper. Res. 19 (1994), 790-814.

[25] R. Burachik, L.M. Graña Drummond, A.N. Iusem, B.F. Svaiter, Full convergence of the steepest descent method with inexact line searches, Optimization 32 (1995), 137-146.

[26] J. Jahn, Mathematical vector optimization in Partially Ordered Linear Spaces, Verlag Peter D. Lang, Frankfurt, 1986.

[27] J. Jahn, Vector optimization - Theory, applications and extensions, Springer, Berlin, 2004.

[28] D.T. Luc, Theory of vector optimization. Lecture Notes in Economics and Mathematical Systems , 319, Springer, Berlin, 1989.

[29] A.L. Peressini, Ordered Topological Vector Space, Harper and Row, 1967.

[30] G. Isac, C. Tammer, Application of a Vector-Valued Ekeland-Type Variational Principle for Deriving Optimality Conditions. In: P. Pardalos, T. Rassias, A. Khan, (eds) Nonlinear Analysis and Variational Problems., Springer Optimization and Its Applications, vol 35. Springer, New York, 2010,

[31] J.Y. Bello Cruz, L.R. Lucambio Pérez, J.G., Melo, Convergence of the projected gradient method for quasiconvex multiobjective optimization, Nonlinear Anal. 74 (2011), 5268-5273.

[32] E.H. Fukuda, L.M. Graña Drummond, On the convergence of the projected gradient method for vector optimization, Optimization, 60 (2011), 1009-1021.

[33] L.M. Graña Drummond, A.N. Iusem, A projected gradient method for vector optimization problems, Comput. Optim. Appl. 28 (2004), 5-30.

[34] K.C. Kiwiel, The efficiency of subgradient projection methods for convex optimization I. General level methods, SIAM J. Control Optim. 34 (1996), 660-676.

[35] J. Kogan, Introduction to Clustering Large and High-dimensional Data, Cambridge University Press, Cambridge, 2007.