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APPROXIMATE EFFICIENCY IN SET-VALUED OPTIMIZATION WITH VARIABLE ORDER

MARIUS DUREA[1,2,*], ELENA-ANDREEA FLOREA[3], DIANA-ELENA MAXIM[3], RADU STRUGARIU[4]

[1]*Faculty of Mathematics, "Alexandru Ioan Cuza" University, 700506, Iaşi, Romania*
[2]*"Octav Mayer" Institute of Mathematics of the Romanian Academy, Iaşi, Romania*
[3]*Faculty of Mathematics, "Alexandru Ioan Cuza" University, 700506, Iaşi, Romania*
[4]*Department of Mathematics, "Gheorghe Asachi" Technical University, 700506, Iaşi, Romania*

Abstract. We consider constrained set-valued optimization problems with variable order in view of two main purposes: one is to deduce conditions for stability of minima of such problems at the perturbations of the objective map and the set of constraints, and the second is to study some possibilities of recovering optimization problems with fixed order. We apply our results in order to obtain optimality conditions for such problems. The main tools we use are four types of cone enlargements.

Keywords. Approximate efficiency; Cone enlargements; Cone separation; Fixed order; Variable order.

1. Introduction and Preliminaries

In this paper, we explore several possibilities to define approximate efficiencies for the optimization problems with set-valued maps in the setting of variable ordering structures in the sense defined in [1] and developed in [2]. In this vein, we tackle several issues that are newly considered here. First of all, the problems under investigation are related to the aim of exploring several concepts of approximate efficiency in variable ordering structure (VOS, for short) setting. As it is well known, the approximate efficiency is a highly important topic in all optimization areas, from the scalar to the vectorial case. In order to have a meaningful approach to this subject in the far-reaching generalized case of VOS, we have to study four types of enlargements of a given cone and on this ground we build the basis for approaching the following topics: stability of efficient points, the possibility to recapture the fixed order setting and, finally, optimality conditions.

This paper is structured as follows. After we end the first introductory section by listing the notations we use throughout our work, we discuss, in the second section, about four concepts of cone enlargements, together with their properties and some relations between them. Then, we establish, for infinite dimensional spaces, a cone separation result in the sense given in [3]. More precisely, having two cones C_1 and C_2 whose intersection is equal to $\{0\}$, under some assumptions, we aim to find another cone C_3 which strictly separates them, i.e., the intersection

*Corresponding author.
E-mail addresses: durea@uaic.ro (M. Durea), andreea_acsinte@yahoo.com (E.-A. Florea), plop.diana.elena@gmail.com (D.-E. Maxim), rstrugariu@tuiasi.ro (R. Strugariu).
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of C_3 with C_1 is still $\{0\}$ and the topological interior of C_3 includes the set $C_2 \setminus \{0\}$. In the cone separation result obtained here, the set C_3 is in fact an enlargement of the set C_2. Section 3 deals with the stability of three types of approximate efficiency defined in the context of variable order, for sets and also for set-valued maps. More precisely, we give some results that establish that the limit of a sequence of minimal points for perturbations of a set A is a minimal point of A, and, similarly, the limit of a sequence of minimal points for perturbations of a set-valued map F is a minimal point for F. Although in our paper the main setting of the efficiency concept is with respect to variable orders, in the fourth section we study some possibilities of converting efficiency under variable order into efficiency under fixed order. More precisely, the local robust and nondominated elements of a set-valued map F with respect to an enlargement of an ordering set-valued map K are turned into the proper and approximate minima, respectively, for the same F, with respect to the upper and lower limit of K, respectively. The final section contains an optimality result for problems of optimization with variable order using the conversion to fixed order along with a cone separation result and some properties of the well-known Gerstewitz functional.

The notation we use in this paper is fairly standard. Throughout this paper, X and Y denote normed vector spaces, unless otherwise stated. We denote by $B(x,r)$, $D(x,r)$ and $S(x,r)$ the open ball, the closed ball and the sphere of center $x \in X$ and radius $r > 0$, respectively. For $x = 0$ and $r = 1$, we will write B_X, D_X and S_X, respectively. For $x \in X$, the symbol $\mathscr{V}(x)$ stands for the system of neighborhoods of x with respect to the norm topology, and for a set $A \subset X$, we denote by $\operatorname{int} A$, $\operatorname{cl} A$, and $\operatorname{bd} A$ its topological interior, closure and boundary, respectively. The cone generated by A is designated by $\operatorname{cone} A$, the convex hull of A is $\operatorname{conv} A$ and the distance from a point $x \in X$ to a nonempty set $A \subset X$ is $d(x,A) := \inf\{\|x-a\| \mid a \in A\}$. The topological dual of X is X^*, and we denote by w the weak topology on X and by w^* the weak-star topology on X^*.

For a set-valued map (or multifunction) $F : X \rightrightarrows Y$, we denote the graph of F by $\operatorname{Gr} F = \{(x,y) \in X \times Y \mid y \in F(x)\}$ and the domain of F by $\operatorname{Dom} F = \{x \in X \mid F(x) \neq \emptyset\}$. If A is a nonempty subset of X, then $F(A) = \bigcup_{x \in A} F(x)$.

2. Cone Enlargements and Cone Separation

Let X be a normed vector space, and let $C \subset X$ be a closed convex cone. One says that C is based if there is a convex set B such that $0 \notin \operatorname{cl} B$ and $C = \operatorname{cone} B$. If B is also bounded, one says that C is well-based and in this case, since C is supposed to be closed, one can take B to be closed as well (see [4, Definition 2.2.14] and the subsequent comments). Moreover, it is well known that a cone that has a base is pointed.

We consider and compare four kinds of enlargements for C. Firstly, we note the obvious fact that $C = \operatorname{cone}(C \cap S_X)$, and we denote by S_C the set $C \cap S_X$. The first three enlargements are defined next.

Definition 2.1. Let $\varepsilon > 0$ and $C \subset X$ be a closed convex and pointed cone. Define the following enlargements of the cone C:

(i) the first type enlargement is

$$C^{(1)\varepsilon} = \operatorname{cone}(\{x \in S_X \mid d(x,C) \leq \varepsilon\});$$

(ii) the second type enlargement is

$$C^{(2)\varepsilon} = \operatorname{cone}\left(\{x \in S_X \mid d(x, S_C) \leq \varepsilon\}\right);$$

(iii) if C is based with the base B, the third type enlargement is

$$C^{(3)\varepsilon} = \operatorname{cone}\left(\{x \in X \mid d(x, B) \leq \varepsilon\}\right).$$

Remark 2.1. It is easy to see that all three types of enlargements are cones that contain the cone C. Moreover, these constructions are of real interest for small ε since otherwise they coincide with the whole space. To be more specific, this surely happens as follows:
- for $C^{(1)\varepsilon}$ if $\varepsilon \geq 1$;
- for $C^{(2)\varepsilon}$ if $\varepsilon \geq 2$;
- for $C^{(3)\varepsilon}$ if $\varepsilon > d(0, B)$.

Remark 2.2. These kinds of enlargements are not new and we mention some bibliographic sources for them.

(i) Observe that, in fact, for all $\varepsilon > 0$, $C^{(1)\varepsilon} = \{x \in X \mid d(x, C) \leq \varepsilon \|x\|\}$. It is clear that $C \setminus \{0\} \subset \operatorname{int} C^{(1)\varepsilon}$ since, according to [5, Proposition 4], for every $x \in C$, $D(x, (1+\varepsilon)^{-1}\varepsilon \|x\|) \subset C^{(1)\varepsilon}$.

(ii) The second type enlargement was introduced and studied in [6] in relation with some directional vector optimization problems. It is known ([6, Proposition 2.9]) that $C \setminus \{0\} \subset \operatorname{int} C^{(2)\varepsilon}$, for all $\varepsilon > 0$.

(iii) The third type enlargement is the well-known Henig cone dilating procedure (see [4, Lemma 3.2.51]). For every $\varepsilon \in (0, \delta)$, where $\delta = d(0, B) > 0$, $C^{(3)\varepsilon}$ is a closed convex cone and $C \setminus \{0\} \subset \operatorname{int} C^{(3)\varepsilon}$.

To define the fourth type of enlargement, one needs the following simple lemma. This auxiliary result is not new (see [4, Section 2.2]), but we provide here its proof for completeness and for reasons related to the development of the subsequent results.

Lemma 2.1. *Let $C \subset X$ be a closed and convex cone. Then*

(i) C is based if and only if there is $x^* \in X^*$ such that $x^*(x) > 0$ for all $x \in C \setminus \{0\}$. In this case, $C \cap \{x \in X \mid x^*(x) = 1\}$ is a base for C;

(ii) C is well-based if and only if there are $x^* \in X^*$ and $\alpha > 0$ such that $x^*(x) \geq \alpha \|x\|$ for all $x \in C$. In this case $C \cap \{x \in X \mid x^*(x) = 1\}$ is a bounded base for C.

Proof. (i) If C is based with the base B, a standard separation result proves that there is $x^* \in X^*$ such that $x^*(b) > 0$ for all $b \in B$, whence $x^*(x) > 0$ for all $x \in C \setminus \{0\}$. Conversely, it is easy to see that $B := C \cap \{x \in X \mid x^*(x) = 1\}$ is a base for C.
(ii) If C is well-based with the base B, again there are $x^* \in X^*$ and $\rho > 0$ such that $x^*(b) > \rho$ for all $b \in B$. Take $M > 0$ such that $\|b\| \leq M$ for all $b \in B$. The inequality $x^*(x) \geq \alpha \|x\|$ holds for $x = 0$ and any α. For an arbitrary $x \in C \setminus \{0\}$, there is $t > 0$ and $b \in B$ such that $x = tb$, whence

$$x^*(x) = tx^*(b) \geq t\rho = \|b\|^{-1} \|x\| \rho \geq M^{-1}\rho \|x\|.$$

Taking $\alpha = M^{-1}\rho$, one has the conclusion. Conversely, the set $B := C \cap \{x \in X \mid x^*(x) = 1\}$ is closed, convex, and does not contain zero. Moreover, it is bounded since otherwise there would exist an unbounded sequence (x_n) such that, for all n, $x^*(x_n) = 1$ and $x^*(x_n) \geq \alpha \|x_n\|$, which is clearly impossible. Finally, one observes that $C = \operatorname{cone} B$, whence the conclusion. $\square$

Definition 2.2. Let $C \subset X$ be a closed, convex, well-based cone, and $\varepsilon > 0$. Let $x^* \in X^*$ be the functional from Lemma 2.1 (ii) and $A := \{u \in X \mid x^*(u) = 1\}$. The fourth type enlargement is:

$$C^{(4)\varepsilon} = \operatorname{cone}\left(\{x \in A \mid d(x, C \cap A) \leq \varepsilon\}\right).$$

Remark 2.3. Since A and $C \cap A$ are convex and closed, the set $B_\varepsilon := \{x \in A \mid d(x, C \cap A) \leq \varepsilon\}$ enjoys the same properties. Moreover, clearly $0 \notin B_\varepsilon$ since $0 \notin A$. The boundedness of $C \cap A$ implies the boundedness of B_ε as well. One can prove that all elements $c \in -C \setminus \{0\}$ are not in $C^{(4)\varepsilon}$, therefore $C^{(4)\varepsilon}$ is always a proper, closed, convex, and well-based cone.

Let us analyze some links between these types of enlargements. Notice that $C^{(3)\varepsilon}$ is always convex, but it is defined only if C is based, while $C^{(1)\varepsilon}$ and $C^{(2)\varepsilon}$ can be nonconvex, but they are always well defined.

The utility of the next result is that it shows that, up to a change of constants, several of the subsequent results that are proved for one of the four kinds of enlargements of the cone C can be transferred to the other three as well.

Proposition 2.1. *Let $C \subset X$ be a closed convex cone, and let $\varepsilon > 0$. Then*

(i) $C^{(2)\varepsilon} \subset C^{(1)\varepsilon}$ and there is $\delta > 0$ such that $C^{(1)\delta} \subset C^{(2)\varepsilon}$;

(ii) if C is well-based, then there exists $\delta > 0$ such that $C^{(3)\delta} \subset C^{(1)\varepsilon}$ and there exists $\eta > 0$ such that $C^{(1)\eta} \subset C^{(3)\varepsilon}$;

(iii) if C is well-based, then there exists $\delta > 0$ such that $C^{(4)\delta} \subset C^{(1)\varepsilon}$ and there exists $\eta > 0$ such that $C^{(1)\eta} \subset C^{(4)\varepsilon}$.

Proof. (i) Since $S_C \subset C$, for all $x \in X$, $d(x, C) \leq d(x, S_C)$, whence $d(x, S_C) \leq \varepsilon$ implies $d(x, C) \leq \varepsilon$. The first inclusion follows.

For the second inclusion, it is enough to prove that there is $\delta > 0$ such that

$$\{x \in S_X \mid d(x, C) \leq \delta\} \subset \{x \in S_X \mid d(x, S_C) \leq \varepsilon\}.$$

Suppose, by contradiction, that, for all $n \in \mathbb{N}^*$, there is $x_n \in S_X$ such that $d(x_n, C) \leq n^{-1}$ and $d(x_n, S_C) > \varepsilon$. From the first relation above, for all n, one finds $u_n \in C$ such that $\|x_n - u_n\| < 2n^{-1}$, whence $(u_n - x_n) \to 0$. This fact and the inequality

$$\begin{aligned} 1 + \|u_n - x_n\| &= \|x_n\| + \|u_n - x_n\| \\ &\geq \|u_n\| \geq \|x_n\| - \|u_n - x_n\| \\ &= 1 - \|u_n - x_n\|, \ \forall n, \end{aligned}$$

show that $\|u_n\| \to 1$. So, for all n large enough, $\|u_n\| > 2^{-1}$ and, in particular, u_n is not 0.

On the one hand, from the second relation, for large n, one has

$$\left\| x_n - \frac{u_n}{\|u_n\|} \right\| > \varepsilon,$$

whence

$$\left\| \|u_n\| x_n - u_n \right\| > \varepsilon \|u_n\| > 2^{-1}\varepsilon.$$

On the other hand,

$$\begin{aligned} \left\| \|u_n\| x_n - u_n \right\| &= \left\| \|u_n\| x_n - u_n \|u_n\| + u_n \|u_n\| - u_n \right\| \\ &\leq \left\| \|u_n\| x_n - u_n \|u_n\| \right\| + \left\| u_n \|u_n\| - u_n \right\| \\ &= \|u_n\| \|x_n - u_n\| + \left| \|u_n\| - 1 \right| \|u_n\| \to 0. \end{aligned}$$

These two assertions are in contradiction. Therefore, the conclusion holds.

(ii) Let B be a closed bounded base for C. As seen in Remark 2.2, for all $x \in C$, $D(x,(1+\varepsilon)^{-1}\varepsilon\|x\|) \subset C^{(1)\varepsilon}$. Since $0 \notin B$, there exists $\mu := d(0,B) > 0$ such that $\|b\| \geq \mu$ for every $b \in B$. Then, from the above inclusion, for every $b \in B$, $D(b,(1+\varepsilon)^{-1}\varepsilon\mu) \subset C^{(1)\varepsilon}$, whence

$$\left\{x \in X \mid d(x,B) \leq 2^{-1}(1+\varepsilon)^{-1}\varepsilon\mu\right\} \subset C^{(1)\varepsilon},$$

whence the first inclusion of this point follows for $\delta = 2^{-1}(1+\varepsilon)^{-1}\varepsilon\mu$.

For the second inclusion, we proceed by contradiction. Suppose that for all $n \in \mathbb{N}^*$, there is $x_n \in S_X$ such that $d(x_n,C) \leq n^{-1}$ and $x_n \notin C^{(3)\varepsilon} = \operatorname{cone}(\{x \in X \mid d(x,B) \leq \varepsilon\})$. The first relation gives us the possibility to choose $u_n \in C$ such that $\|x_n - u_n\| < 2n^{-1}$ for all n. For every such element, there are $\alpha_n \geq 0$ and $b_n \in B$ such that $u_n = \alpha_n b_n$. If $\alpha_n \to 0$ on a subsequence then, from the boundedness of B, on that subsequence, $u_n \to 0$ and this is in contradiction with $(x_n - u_n) \to 0$ and $(x_n) \subset S_X$. So, 0 is not a cluster point of (α_n). Moreover, $+\infty$ is not a cluster point of (α_n) because otherwise (u_n) would be unbounded and we would obtain the same contradiction as before. Therefore, on a subsequence, $\alpha_n \to \alpha \in (0,\infty)$. Without loss of generality, we simply put $\alpha_n \to \alpha$. Consequently, for all n,

$$\begin{aligned}\left\|\frac{1}{\alpha}x_n - b_n\right\| &= \left\|\frac{1}{\alpha}x_n - \frac{1}{\alpha_n}x_n + \frac{1}{\alpha_n}x_n - b_n\right\| \\ &\leq \left|\frac{1}{\alpha} - \frac{1}{\alpha_n}\right| \cdot \|x_n\| + \frac{1}{\alpha_n}\|x_n - u_n\|.\end{aligned} \tag{2.1}$$

Now, since the right-hand side of (2.1) converges to 0 for $n \to \infty$, for large n,

$$\frac{1}{\alpha}x_n \in \{x \in X \mid d(x,B) \leq \varepsilon\},$$

that is, $x_n \in C^{(3)\varepsilon}$, and this is a contradiction. The proof of (ii) is complete.

(iii) Let $x^* \in X^*$ from Lemma 2.1 (ii) and $A := \{u \in X \mid x^*(u) = 1\}$. Take $\delta = \|x^*\|^{-1}\varepsilon$ and $u \in C^{(4)\delta}$. If $u = 0$, clearly $u \in C^{(1)\varepsilon}$. Otherwise, there is $t > 0$ and $y \in B_\delta := \{x \in A \mid d(x, C \cap A) \leq \delta\}$ such that $u = ty$. Therefore, $x^*(u) = t$ and then $x^*(u)^{-1}u = y \in B_\delta$. It follows that

$$d\left(x^*(u)^{-1}u, C\right) \leq d\left(x^*(u)^{-1}u, C \cap A\right) \leq \delta,$$

whence $d(u,C) \leq x^*(u)\delta \leq \delta\|x^*\|\|u\| \leq \varepsilon\|u\|$, which means that $u \in C^{(1)\varepsilon}$.

For the second inclusion, consider

$$\rho := \frac{\varepsilon\alpha^2}{\|x^*\|(2+\varepsilon\alpha)},\ \nu := \frac{\rho}{1+\rho},$$

and $\eta \in (0,\nu)$, where α is the constant from Lemma 2.1 (ii). We show that $C^{(1)\eta} \subset C^{(4)\varepsilon}$.

The zero element belongs to both terms, so let us consider $x \in X \setminus \{0\}$ such that $d(x,C) \leq \eta\|x\|$. So there is $c \in C$ with $\|x - c\| < \nu\|x\|$. Notice that $c \neq 0$ since $\nu < 1$, whence $x^*(c) \geq \alpha\|c\| > 0$. In particular, $\|x - c\| < \nu\|x - c\| + \nu\|c\|$, that is

$$\|x - c\| < \frac{\nu}{1-\nu}\|c\| = \rho\|c\|.$$

Observe now that

$$\begin{aligned} x^*(x) &= x^*(c) + x^*(x-c) \geq \alpha \|c\| - \|x^*\| \|x-c\| \\ &\geq \alpha \|c\| - \rho \|c\| \|x^*\| \\ &= \|c\| (\alpha - \rho \|x^*\|) > 0, \end{aligned}$$

where the last inequality uses the obvious fact that

$$\rho = \frac{\varepsilon\alpha^2}{\|x^*\| (2+\varepsilon\alpha)} < \frac{\alpha}{\|x^*\|}.$$

Therefore,

$$\frac{x}{x^*(x)} \in A \text{ and } \frac{c}{x^*(c)} \in C \cap A.$$

We aim to show that

$$\left\| \frac{x}{x^*(x)} - \frac{c}{x^*(c)} \right\| \leq \varepsilon.$$

By denoting $y := x - c$, this becomes

$$\left\| \frac{y+c}{x^*(y) + x^*(c)} - \frac{c}{x^*(c)} \right\| \leq \varepsilon,$$

that is,

$$\|x^*(c) y - x^*(y) c\| \leq \varepsilon (x^*(y) + x^*(c)) x^*(c). \tag{2.2}$$

Now, the left-hand side of (2.2) satisfies

$$\begin{aligned} \|x^*(c) y - x^*(y) c\| &\leq 2 \|x^*\| \|y\| \|c\| \\ &< 2\rho \|x^*\| \|c\|^2 = 2 \frac{\varepsilon\alpha^2}{\|x^*\| (2+\varepsilon\alpha)} \|x^*\| \|c\|^2 \\ &= 2 \frac{\varepsilon\alpha^2}{2+\varepsilon\alpha} \|c\|^2. \end{aligned}$$

The right-hand side of (2.2) satisfies

$$\begin{aligned} \varepsilon (x^*(y) + x^*(c)) x^*(c) &\geq \varepsilon\alpha \|c\| (\alpha \|c\| - \|x^*\| \|y\|) \\ &\geq \varepsilon\alpha \|c\| (\alpha \|c\| - \rho \|x^*\| \|c\|) \\ &= \varepsilon\alpha \|c\|^2 \left(\alpha - \frac{\varepsilon\alpha^2}{\|x^*\| (2+\varepsilon\alpha)} \|x^*\| \right) \\ &= \frac{2\varepsilon\alpha^2}{2+\varepsilon\alpha} \|c\|^2. \end{aligned}$$

These two inequalities prove the conclusion. □

Remark 2.4. (i) The use of the fourth enlargement and the proof of the second part of (iii) from the above result are inspired by [7] (see also [8, Proposition 3.2.1]), where the property of a cone "to allow plastering", which is close to the construction we consider here, is considered.
(ii) Proposition 2.1 (iii) shows as well that $C \setminus \{0\} \subset \operatorname{int} C^{(4)\varepsilon}$ for every $\varepsilon > 0$.

Separation results are useful tools in establishing optimality conditions. It is known that it is easier to work with weak Pareto minima instead of Pareto minima, but in several situations the ordering cones may have empty interior.

We record here such a result which follows from [5, Theorem 3] (see the comments concerning the assumptions of the quoted theorem). The importance of this result comes from the fact that it allows the transition from a cone with possible empty interior to a solid one and thus, the possibility to work with weak efficiency.

Theorem 2.1. *Let X be a reflexive Banach space. Let $P,Q \subset X$ be cones such that P,Q are weakly closed and P is well-based. If $P \cap Q = \{0\}$, then there exists $\varepsilon > 0$ such that $P^{(1)\varepsilon} \cap Q = \{0\}$.*

Remark 2.5. In view of Proposition 2.1 (note that every based cone is convex), the conclusion reads as follows: there are some constants α, β, γ such that $P^{(2)\alpha} \cap Q = \{0\}$, $P^{(3)\beta} \cap Q = \{0\}$, $P^{(4)\gamma} \cap Q = \{0\}$.

A comparable result holds under some additional convexity assumptions.

Proposition 2.2. *Let X be a reflexive Banach space and P,Q be closed convex cones such that $P \cap Q = \{0\}$. If P is well-based with the base B, then there is U, a weak neighborhood of the origin, such that*

$$\operatorname{cone}(B+U) \cap Q = \{0\}.$$

In particular, there are $\varepsilon_i > 0$ such that $P^{(i)\varepsilon_i} \cap Q = \{0\}$ for all $i \in \overline{1,4}$.

Proof. Since P is convex, closed, and well-based, the base B is bounded, convex, closed, and satisfies that $0 \notin B$ and $P = \operatorname{cone} B$. The properties of B and the reflexivity of X ensure that B is $w-$compact (Kakutani's theorem). On the other hand, the relation $0 \notin B$ and $P \cap Q = \{0\}$ imply that $B \cap Q = \emptyset$. In particular, for all $b \in B$, $b \notin Q$. Now we can use a standard separation theorem to deduce the existence of some $x_b^* \in X^* \setminus \{0\}$, $\alpha_b \in \mathbb{R}$, and $\varepsilon_b > 0$ such that

$$x_b^*(b) < \alpha_b < \alpha_b + \varepsilon_b < \inf_{u \in Q} x_b^*(u).$$

Consider $U_{x_b^*,\varepsilon_b} := \{u \in X \mid |x_b^*(u)| < \varepsilon_b\}$ which is a $w-$neighborhood of the origin. From the above relation, one has $\left(b + U_{x_b^*,\varepsilon_b}\right) \cap Q = \emptyset$. Taking into account that $B \subset \bigcup_{b \in B} \left(b + U_{x_b^*,2^{-1}\varepsilon_b}\right)$ and B is $w-$compact, there exist $n \in \mathbb{N}^*$ and $b_i \in B$ for $i \in \overline{1,n}$ such that

$$B \subset \bigcup_{i=1}^{n} \left(b_i + U_{x_{b_i}^*,2^{-1}\varepsilon_{b_i}}\right).$$

Choose $U = \bigcap_{i=1}^{n} U_{x_{b_i}^*,2^{-1}\varepsilon_{b_i}}$, which is a $w-$neighborhood of the origin and this yields

$$B \subset B + U \subset \bigcup_{i=1}^{n} \left(b_i + U_{x_{b_i}^*,\varepsilon_{b_i}}\right).$$

Moreover, $(B+U) \cap Q = \emptyset$, which yields the conclusion $\operatorname{cone}(B+U) \cap Q = \{0\}$.

Finally, since U is a $w-$neighborhood of the origin, it is as well a neighborhood of 0 in the strong topology, whence there is $\eta > 0$ such that $D(0,\eta) \subset U$. Then, for all $\delta \in (0,\eta)$,

$$\{x \in X \mid d(x,B) \le \delta\} \subset B + D(0,\eta),$$

so

$$\begin{aligned}\{0\} &\subset P^{(3)\delta} \cap Q = \operatorname{cone}\{x \in X \mid d(x,B) \leq \delta\} \cap Q \\ &\subset \operatorname{cone}(B + D(0,\eta)) \cap Q = \{0\}.\end{aligned}$$

Using Proposition 2.1 we obtain all the conclusions. □

Remark 2.6. We mention that the use of the third type enlargement cone in this separation result has two great advantages that could prove to be useful when applying the result, namely that it is always convex and, for $\varepsilon < d(0,B)$, it is also pointed since $\{x \in X \mid d(x,B) \leq \varepsilon\}$ is a base for it. The same remark is in order for the fourth type enlargement.

3. Stability of Approximate Efficiency

The setting for this section is the following. Let Y be a normed vector space, $A \subset Y$ a nonempty set, and $K : Y \rightrightarrows Y$ a set-valued map such that $K(y)$ is a closed, convex, proper, pointed cone in Y for every $y \in Y$. This leads to the following three different (pseudo-)orders on Y :

$$\begin{aligned} y_1 \leq_1 y_2 &\iff y_2 - y_1 \in K(y_2), \\ y_1 \leq_2 y_2 &\iff y_2 - y_1 \in K(y_1), \\ y_1 \leq_3 y_2 &\iff y_2 - y_1 \in K(y),\ \forall y \in A. \end{aligned}$$

If the set-valued map K has solid values, then one can consider the strict corresponding (pseudo-)order relations on Y, namely $<_1, <_2$ and $<_3$. These strict pseudo-orders lead to the introduction of three different concepts of weak minimality (see, [9, 10]), which in fact represent extensions of the well-known proper efficiency from fixed order setting to variable order setting.

Definition 3.1. Let $\overline{x} \in A$.

(i) Suppose that $\operatorname{int} K(\overline{x}) \neq \emptyset$. The point $\overline{x}$ is a weak minimal element of A with respect to K if $x \not<_1 \overline{x}$ for every $x \in A$, i.e.,

$$\{x - \overline{x}\} \cap (-\operatorname{int} K(\overline{x})) = \emptyset,\ \forall x \in A.$$

(ii) Suppose that $\operatorname{int} K(x) \neq \emptyset$ for every $x \in A$. The point $\overline{x}$ is a weak nondominated element of A with respect to K if $x \not<_2 \overline{x}$ for every $x \in A$, i.e.,

$$\{x - \overline{x}\} \cap (-\operatorname{int} K(x)) = \emptyset,\ \forall x \in A.$$

(iii) Suppose that $\operatorname{int} K(x) \neq \emptyset$ for every $x \in A$. The point $\overline{x}$ is a weak robust element of A with respect to K if $x \not<_3 \overline{x}$ for every $x \in A$, i.e.,

$$\{x - \overline{x}\} \cap (-\operatorname{int} K(z)) = \emptyset,\ \forall x, z \in A.$$

The corresponding concepts of the approximate minima are introduced as follows.

Definition 3.2. Let $\overline{x} \in A$, $c \in Y \setminus \{0\}$ and $\varepsilon \geq 0$.

(i) Suppose that $\operatorname{int} K(\overline{x}) \neq \emptyset$. The point $\overline{x}$ is a weak $\varepsilon c-$minimal element of A with respect to K if $x + \varepsilon c \not<_1 \overline{x}$ for every $x \in A$, i.e.,

$$\{x - \overline{x} + \varepsilon c\} \cap (-\operatorname{int} K(\overline{x})) = \emptyset,\ \forall x \in A.$$

(ii) Suppose that $\operatorname{int} K(x) \neq \emptyset$ for every $x \in A$. The point $\bar{x}$ is a weak $\varepsilon c-$nondominated element of A with respect to K if $x \not<_2 \bar{x} - \varepsilon c$ for every $x \in A$, i.e.,

$$\{x - \bar{x} + \varepsilon c\} \cap (-\operatorname{int} K(x)) = \emptyset, \ \forall x \in A.$$

(iii) Suppose that $\operatorname{int} K(x) \neq \emptyset$ for every $x \in A$. The point $\bar{x}$ is a weak $\varepsilon c-$robust element of A with respect to K if $x \not<_3 \bar{x} - \varepsilon c$ for every $x \in A$, i.e.,

$$\{x - \bar{x} + \varepsilon c\} \cap (-\operatorname{int} K(z)) = \emptyset, \ \forall x, z \in A.$$

The aim of this section is to prove that, under some assumptions, a sequence of approximate minima of a sequence of sets converges, in a certain sense, to an approximate minimum for the limit of the sequence of sets. That is, the concept of approximate efficiency for sets is stable at the perturbation of the underlying set. We also prove that such a stability holds in the case of approximate efficiency for set-valued maps.

Remark 3.1. (i) Observe that the non-approximate minimality notions from Definition 3.1 could be obtained by taking $\varepsilon = 0$ in the corresponding concepts from Definition 3.2. Moreover, if $K(y) := Q$ for every $y \in A$, where by Q we denote a closed, convex, proper, and pointed cone in Y, then the notions given in the previous definitions reduce to the classical ones, namely the weak Pareto minimality of the set A and the weak approximate Pareto minimality of the set A.
(ii) All the efficiency concepts given above are global. For the local versions one has to ask that the strict pseudo-order relations hold for all $x \in A \cap U$ for some U a neighborhood of $\bar{x}$.

We denote by $(\varepsilon, c) - \operatorname{WMin}(A, K)$ the set of all weak $\varepsilon c-$minimal elements of A with respect to K, by $(\varepsilon, c) - \operatorname{WNond}(A, K)$ the set of all weak $\varepsilon c-$nondominated elements of A with respect to K and by $(\varepsilon, c) - \operatorname{WRob}(A, K)$ the set of all weak $\varepsilon c-$robust elements of A with respect to K. For the concepts in Definition 3.1 we use the same corresponding notations without mentioning (ε, c). For more details on properties of the minimality concepts above and differences between them, see, for example, [9, 11, 12].

In the following, we will formulate some stability results for the approximate efficiency notions introduced in Definition 3.2 under perturbations of the underlying set A with sequences of sets converging in the sense of the Painlevé-Kuratowski limit. We recall the following definition.

Definition 3.3. Let A and $(A_n)_n$ be nonempty subsets of Y. One says that A is the Painlevé-Kuratowski limit of (A_n) if $A \subset \operatorname*{Liminf}_{n \to \infty} A_n$ and $\operatorname*{Limsup}_{n \to \infty} A_n \subset A$, where

$$\operatorname*{Liminf}_{n \to \infty} A_n = \{x \in Y \mid \exists (x_n), x_n \in A_n, \forall n \in \mathbb{N} : x_n \to x\}$$

and

$$\operatorname*{Limsup}_{n \to \infty} A_n = \{x \in Y \mid \exists (n_k), \exists (x_{n_k}), x_{n_k} \in A_{n_k}, \forall k \in \mathbb{N} : x_{n_k} \to x\}.$$

The notation we use is $A_n \xrightarrow{P-K} A$. One writes $A_n \xrightarrow{P-K_-} A$ if only the inclusion on the left-hand side holds and $A_n \xrightarrow{P-K_+} A$ if only the inclusion on the right-hand side holds.

We give now our result concerning the stability mentioned above for the weak robust minimality concept.

Proposition 3.1. *Let $\emptyset \neq A \subset Y$ be a closed set, $(A_n) \subset Y$ be a sequence of nonempty closed sets, $\varepsilon > 0$ and consider the set $C := \operatorname{int}\left(\bigcap\{K(y_n) \mid y_n \in A_n,\ \text{with } n \in \mathbb{N}^*\}\right) \neq \emptyset$. Take $c \in C$ and suppose that $\operatorname{int} K(x) \neq \emptyset$ for every $x \in A$, $A_n \xrightarrow{P-K_-} A$ and $K(A_n) \xrightarrow{P-K_-} K(A)$. Then, for all $\delta \in [0, \varepsilon)$,*

$$\operatorname*{Limsup}_{n\to\infty}(\delta, c) - \mathrm{WRob}(A_n, K) \subset (\varepsilon, c) - \mathrm{WRob}(A, K).$$

Proof. Take $\delta \in [0, \varepsilon)$ and $x \in \operatorname*{Limsup}_{n\to\infty}(\delta, c) - \mathrm{WRob}(A_n, K)$. There exists $(n_p) \subset \mathbb{N}^*$ a strictly increasing sequence and $x_{n_p} \in (\delta, c) - \mathrm{WRob}\left(A_{n_p}, K\right)$ such that $x_{n_p} \to x$. Hence $x \in \operatorname*{Limsup}_{n\to\infty} A_n$ and since $A_n \xrightarrow{P-K_+} A$ this gives that $x \in A$. Now, suppose by way of contradiction that $x \notin (\varepsilon, c) - \mathrm{WRob}(A, K)$, whence there exist $a, z \in A$ such that $a - x + \varepsilon c \in -\operatorname{int} K(z)$. As $A_n \xrightarrow{P-K_-} A$ and $a \in A$, there exists $(a_n) \subset A_n$ such that $a_n \to a$. Since $x - a - \varepsilon c \in \operatorname{int} K(z) \subset K(z)$ and $K(z) \subset K(A) \subset \operatorname*{Liminf}_{n\to\infty} K(A_n)$ by the assumption $K(A_n) \xrightarrow{P-K_-} K(A)$, it follows that there are sequences (z_n) and (u_n) such that $u_n \to x - a - \varepsilon c$, $z_n \in A_n$ and $u_n \in K(z_n)$ for every n. Therefore, since $\delta - \varepsilon < 0$ and $(a_n) \subset A_n$, we get that

$$a_{n_p} - x_{n_p} + \delta c + u_{n_p} \to (\delta - \varepsilon)c \in -C \subset -\operatorname{int} K(z_{n_p}),$$

which yields that for every p large enough, $a_{n_p} - x_{n_p} + \delta c \in -\operatorname{int} K(z_{n_p}) - u_{n_p} \subset -\operatorname{int} K(z_{n_p}) - K(z_{n_p}) \subset -\operatorname{int} K(z_{n_p})$, and this is in contradiction with the minimality assumption since $a_{n_p} - x_{n_p} + \delta c \in A_{n_p} - x_{n_p} + \delta c$ and $z_{n_p} \in A_{n_p}$. Then the conclusion follows. $\square$

Remark 3.2. Obviously, $(\varepsilon, c) - \mathrm{WRob}(A, K) \subset (\varepsilon, c) - \mathrm{WNond}(A, K)$ for every $\varepsilon > 0$ and $c \in Y \setminus \{0\}$. Thus, in the assumptions of the previous proposition, we get that

$$\operatorname*{Limsup}_{n\to\infty}(\varepsilon_1, c) - \mathrm{WRob}(A_n, K) \subset (\varepsilon_2, c) - \mathrm{WNond}(A, K),$$

for every $0 \leq \varepsilon_1 < \varepsilon_2$ and $c \in C$.

Next, we aim to obtain a similar stability result for the weak nondominated efficiency. In order to do that, we will use the usual lower semicontinuity assumption for the ordering set-valued map K, which we recall next.

Definition 3.4. A set-valued map $G : X \rightrightarrows Y$ is lower semicontinuous at $(x, y) \in \operatorname{Gr} G$ if for every sequence $x_n \to x$, there exists a sequence $y_n \to y$ with $(x_n, y_n) \in \operatorname{Gr} G$ for every n (for more details on this kind of semicontinuity of a multifunction, see for instance, [15]).

Proposition 3.2. *Let $\emptyset \neq A \subset Y$ be a closed set, $(A_n) \subset Y$ be a sequence of nonempty closed sets, $\varepsilon > 0$. Suppose that $C := \operatorname{int}\left(\bigcap\{K(y_n) \mid y_n \in A_n,\ \text{with } n \in \mathbb{N}^*\}\right) \neq \emptyset$, $A_n \xrightarrow{P-K} A$, K is lower semicontinuous at $(x, y) \in \operatorname{Gr} K$ for every $x \in A \cap \operatorname{Dom} K$ and take $c \in C$. Then, for every $\delta \in [0, \varepsilon)$,*

$$\operatorname*{Limsup}_{n\to\infty}(\delta, c) - \mathrm{WNond}(A_n, K) \subset (\varepsilon, c) - \mathrm{WNond}(A, K).$$

Proof. Take $\delta \in [0, \varepsilon)$, and let x be arbitrarily in $\operatorname*{Limsup}_{n\to\infty}(\delta, c) - \mathrm{WNond}(A_n, K)$. Therefore, there exist $(n_p) \subset \mathbb{N}^*$ a strictly increasing sequence and $x_{n_p} \in (\delta, c) - \mathrm{WNond}\left(A_{n_p}, K\right)$ such that $x_{n_p} \to x$. Hence $x \in \operatorname*{Limsup}_{n\to\infty} A_n$. Since $A_n \xrightarrow{P-K_+} A$, one has $x \in A$. Now, suppose by contradiction that $x \notin (\varepsilon, c) - \mathrm{WNond}\,(A, K)$, whence there exists $a \in A$ such that $a - x + \varepsilon c \in -\operatorname{int} K\,(a)$. As $A_n \xrightarrow{P-K_-} A$ and $a \in A$, there exists $(a_n) \subset A_n$ such that $a_n \to a$. Now, using the lower semicontinuity assumption, since $a_n \to a$ and $x - a - \varepsilon c \in \operatorname{int} K\,(a) \subset K\,(a)$, we have that there exists $k_n \to x - a - \varepsilon c$ such that $k_n \in K\,(a_n)$ for every n. Thus

$$a_{n_p} - x_{n_p} + \delta c + k_{n_p} \to (\delta - \varepsilon)\, c \in -C \subset -\operatorname{int} K\left(a_{n_p}\right).$$

It follows that, for every p large enough, $a_{n_p} - x_{n_p} + \delta c + k_{n_p} \in -\operatorname{int} K\left(a_{n_p}\right)$, which implies that

$$a_{n_p} - x_{n_p} + \delta c \in -k_{n_p} - \operatorname{int} K\left(a_{n_p}\right) \subset -K\left(a_{n_p}\right) - \operatorname{int} K\left(a_{n_p}\right) \subset -\operatorname{int} K\left(a_{n_p}\right).$$

The last relation is in contradiction with the minimality assumption due to $a_{n_p} \in A_{n_p}$. Then the conclusion follows. □

As expected, we need stronger hypothesis for the stability of the weak nondominated efficiency rather than for the weak robust one. Indeed, K to be lower semicontinuous at $(x, y) \in \operatorname{Gr} K$ for all $x \in A \cap \operatorname{Dom} K$ implies that $K(A_n) \xrightarrow{P-K_-} K(A)$ provided that $A_n \xrightarrow{P-K} A$, since for every $k \in K(a)$ with $a \in A \cap \operatorname{Dom} K$ the lower semicontinuity assumption gives a sequence (k_n) converging to k, with $k_n \in K(a_n) \subset K(A_n)$, corresponding to a sequence $a_n \to a$, with $a_n \in A_n$ for every n, given by the convergence assumption $A_n \xrightarrow{P-K} A$. The reverse implication does not hold, as can be seen by the next simple example.

Example 3.1. Let $Y = \mathbb{R}^2$, $K_1 = \operatorname{cone}\operatorname{conv}\{(0,2);(1,2)\}$, $K_2 = \operatorname{cone}\operatorname{conv}\{(2,1);(2,0)\}$, and let $K : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be given by

$$K(x) = \begin{cases} K_1, \text{ if } x \notin \mathbb{Q}^2, \\ K_2, \text{ if } x \in \mathbb{Q}^2. \end{cases}$$

K is lower semicontinuous only at points $(x, 0)$. Yet, by choosing $A = [0,1] \times [0,1]$ and $A_n = [-n^{-1}, 1 + n^{-1}] \times [-n^{-1}, 1 + n^{-1}]$, it is easy to see that $A_n \xrightarrow{P-K} A$ and $K(A_n) \xrightarrow{P-K_-} K(A)$ since $K(A_n) = \cup_{a_n \in A_n} K(a_n) = K_1 \cup K_2 = \cup_{a \in A} K(a) = K(A)$.

Remark 3.3. Following the same arguments as above, we have that, for all $\delta \in [0, \varepsilon)$,

$$\operatorname*{Limsup}_{n\to\infty}\,(\delta, c) - \mathrm{WMin}\,(A_n, K) \subset (\varepsilon, c) - \mathrm{WMin}\,(A, K).$$

Remark 3.4. (i) Often, in the setting of VOS, when dealing with strong minimality, the hypothesis of nontriviality of the intersection of $K(x)$ over a set, such as a neighborhood in the local case, a closed set in the case of geometric restrictions, or even over Y in the global situation, is widely used (see, e.g., [1, 2, 13]). The assumption $\bigcap \{\operatorname{int} K\,(y_n) \mid y_n \in A_n, \text{ with } n \in \mathbb{N}^*\} \neq \emptyset$ used in the previous propositions appears due to the work with weak minimality concepts, being similar to that used in the strong case, this being nothing but the nontriviality of the intersection of $\operatorname{int} K(x)$ over the perturbation of A (see [11]).

Also, the lower semicontinuity assumption seems to be a natural one in the setting of variable ordering structures since it was used in more papers on this subject, such as [1, 2]. Moreover, in [14], it was mentioned that the upper semicontinuity property is in general not an appropriate notion for cone-valued maps and an example was given in this sense, but in order to underline the fact that this could not be said about the lower semicontinuity, also, an example was given (see, [14, Example 3.5]). Furthermore, it is easy to see that if a set-valued map K has the Aubin property around $(x,y) \in \operatorname{Gr} K$, which is another assumption often used in vector optimization (see [15] for more details), then it is lower semicontinuous at (x,y).

(ii) Remark that in the assumptions of the previous proposition, if we take $\delta = 0$ and $\varepsilon > 0$, we obtain the following inclusion:

$$\operatorname*{Limsup}_{n\to\infty} \operatorname{WNond}(A_n, K) \subset (\varepsilon, c) - \operatorname{WNond}(A, K).$$

(iii) Due to the simple inclusion $K(y) \subset K^{(i)\mu}(y)$ for every $i = 1,2,3,4$, $\mu > 0$ and $y \in Y$, all the stability statements above also hold for any perturbation with an enlargement of the ordering set-valued map K.

We consider next the case of efficiency for set-valued maps. If until now in this section we worked naturally only on the space Y since we dealt with minima for sets, now we will consider $F : X \rightrightarrows Y$ to be a set-valued map between the real normed spaces X and Y, the ordering set-valued map $K : X \rightrightarrows Y$ to act between the same spaces as the objective F, and A a subset of X to be the constraints set. That is, we consider the following set-valued optimization problem

$$\text{minimize } F(x) \text{ s.t. } x \in A. \tag{P_A}$$

The following concept of approximate solution of the problem (P_A) was defined in [16] and represents an extension of the approximate proper efficiency from fixed order setting to variable order setting.

Definition 3.5. Let $\overline{x} \in A$, $c \in Y \setminus \{0\}$, and $\varepsilon \geq 0$. Suppose that $\operatorname{int} K(x) \neq \emptyset$ for every $x \in A$. The point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a weak $\varepsilon c-$nondominated point for F on A with respect to K if

$$(F(x) - \overline{y} + \varepsilon c) \cap (-\operatorname{int} K(x)) = \emptyset, \ \forall x \in A.$$

We denote by $(\varepsilon, c) - \operatorname{WNond}(A, K, F)$ the set of all weak $\varepsilon c-$nondominated points for F on A with respect to K. Notice that if $\varepsilon = 0$, we obtain the classical approximate minimality concept, which was defined and studied in [1]. Of course, the weak $\varepsilon c-$minimal point for F on A with respect to K and the weak $\varepsilon c-$robust point for F on A with respect to K could be defined in a similar manner.

Further, we give a result concerning the stability of a nondominated solution with respect to the perturbations of the set-valued map F and the perturbations of the set A. We mention that the demonstration of it follows the line of the proof of Proposition 3.2, but for the sake of completeness, we include here all the details.

Proposition 3.3. *Let $F : X \rightrightarrows Y$ be a set-valued map with closed graph, (F_n) a sequence of set-valued maps from X to Y with closed graphs, $\emptyset \neq A \subset Y$ a closed set, and $(A_n) \subset Y$ a sequence of nonempty closed sets. Assume that $C := \operatorname{int}\left(\bigcap \{K(y_n) \mid y_n \in A_n, \text{ with } n \in \mathbb{N}^*\}\right) \neq \emptyset$ and take $c \in C$. Suppose that:*

(i) there exist $\varepsilon > 0$ and $(x_n, y_n) \to (\overline{x}, \overline{y})$ such that $(x_n, y_n) \in (\varepsilon, c) - \operatorname{WNond}(A_n, K, F_n)$ for every n large enough;

(ii) $A_n \xrightarrow{P-K_-} A$ and $\operatorname{Gr} F_n \xrightarrow{P-K_-} \operatorname{Gr} F$;

(iii) for every $(x,y) \in \operatorname{Gr} F$, there exists V a neighborhood of y such that for every $x_n^1 \to x$ and $x_n^2 \to x$, there exists $(\lambda_n) \subset (0,\infty)$ with $\lambda_n \left\|x_n^1 - x_n^2\right\| \to 0$ and

$$F_n\left(x_n^1\right) \cap V \subset F_n\left(x_n^2\right) + \lambda_n \left\|x_n^1 - x_n^2\right\| D_Y,$$

for every n large enough;

(iv) K is lower semicontinuous at $(x,y) \in \operatorname{Gr} K$ for every $x \in A \cap \operatorname{Dom} K$.

Then, for every $\delta > \varepsilon$, $(\bar{x},\bar{y}) \in (\delta,c) - \operatorname{WNond}(A,K,F)$.

Proof. Suppose, by way of contradiction, that the conclusion does not hold, i.e., there exist $\delta > \varepsilon$ such that $(\bar{x},\bar{y}) \notin (\delta,c) - \operatorname{WNond}(A,K,F)$. Therefore, there exists $x \in A$, $y \in F(x)$ such that $y - \bar{y} + \delta c \in -\operatorname{int} K(x)$. Now, using the assumption (ii), we obtain that there exist $\left(x_n^1, y_n^1\right) \to (x,y)$ and $x_n^2 \to x$ such that $\left(x_n^1, y_n^1\right) \in \operatorname{Gr} F_n$ and $x_n^2 \in A_n$, for every $n \in \mathbb{N}$.

Further, applying hypothesis (iii) for (x,y) from above, we see that there exist $\lambda_n > 0$ and $y_n^2 \in F_n\left(x_n^2\right)$ such that $\lambda_n \left\|x_n^1 - x_n^2\right\| \to 0$ and $\left\|y_n^1 - y_n^2\right\| \le \lambda_n \left\|x_n^1 - x_n^2\right\|$ for every n large enough. Taking into account that $\lambda_n \left\|x_n^1 - x_n^2\right\| \to 0$ and $y_n^1 \to y$, it follows from the last inequality that $y_n^2 \to y$. But, using the lower semicontinuity assumption, since $x_n^2 \to x$ and $\bar{y} - y - \delta c \in \operatorname{int} K(x) \subset K(x)$, we have that there exists $k_n \to \bar{y} - y - \delta c$ such that $k_n \in K\left(x_n^2\right)$ for every n. Therefore,

$$y_n^2 - y_n + k_n + \varepsilon c \to (\varepsilon - \delta) c \in -C \subset -\operatorname{int} K\left(x_n^2\right).$$

Hence, for every n large enough, $y_n^2 - y_n + k_n + \varepsilon c \in -\operatorname{int} K\left(x_n^2\right)$, so

$$y_n^2 - y_n + \varepsilon c \in -k_n - \operatorname{int} K\left(x_n^2\right) \subset -K\left(x_n^2\right) - \operatorname{int} K\left(x_n^2\right) \subset -\operatorname{int} K\left(x_n^2\right).$$

The last relation is in contradiction with the minimality assumption since $x_n^2 \in A_n$ and $y_n^2 - y_n + \varepsilon c \in F_n\left(x_n^2\right) - y_n + \varepsilon c$. Then the conclusion follows. □

Remark 3.5. Obviously, similar results could be given also for the minimal and the robust efficiency concepts for problem (P_A).

Remark 3.6. The results in this section generalize several results from [17, 18] that we adapted from the fixed case to the variable ordering structures case.

4. Efficiency Under Variable Order Seen as Efficiency under Fixed Order

Let X,Y be normed vector spaces, and let $F,K : X \rightrightarrows Y$ be set-valued maps with K having as values proper closed pointed convex cones. The aim of this section is to give some results that convert robust and nondominated points of F with respect to an enlargement of K into Pareto and approximate Pareto minimal points respectively for F with respect to a limit of the set-valued map K (that is, to convert efficiency under variable order into efficiency under fixed order).

We use the Painlevé-Kuratowski lower and upper limits for the set-valued map F, that are defined as follows: for $\bar{x} \in X$,

$$\begin{aligned}\operatorname*{Liminf}_{x \to \bar{x}} F(x) &= \{y \in Y \mid \forall V \in \mathscr{V}(y), \exists U \in \mathscr{V}(\bar{x}), \forall u \in U, F(u) \cap V \neq \emptyset\} \\ &= \{y \in Y \mid \forall x_n \to \bar{x}, \exists y_n \to y, y_n \in F(x_n), \forall n \in \mathbb{N}\}\end{aligned}$$

and

$$\begin{aligned}\operatorname*{Limsup}_{x \to \bar{x}} F(x) &= \{y \in Y \mid \forall V \in \mathscr{V}(y), \forall U \in \mathscr{V}(\bar{x}), \exists u \in U, F(u) \cap V \neq \emptyset\} \\ &= \{y \in Y \mid \exists x_n \to \bar{x}, \exists y_n \to y, y_n \in F(x_n), \forall n \in \mathbb{N}\}.\end{aligned}$$

These sets are always closed and $\operatorname*{Liminf}_{x \to \bar{x}} F(x) \subset \operatorname{cl} F(\bar{x}) \subset \operatorname*{Limsup}_{x \to \bar{x}} F(x)$. If $\operatorname*{Liminf}_{x \to \bar{x}} F(x)$ is nonempty, then $\bar{x} \in \operatorname{int} \operatorname{Dom} F$.

Remark 4.1. Notice that these notions hold as well in metric spaces, and in this sense they cover those in Definition 3.3 if one considers $F : \overline{\mathbb{N}} \rightrightarrows Y$, $F(n) := A_n$.

Recall that if $F(\overline{x}) \subset \operatorname*{Liminf}_{x\to\overline{x}} F(x)$, then one says that F is lower semicontinuous at $\overline{x}$. If one compares with Definition 3.4, then F is lower semicontinuous at $\overline{x}$ if and only if it is lower semicontinuous at $(\overline{x}, y)$ for all $y \in F(\overline{x})$.

Proposition 4.1. *Let $(\overline{x},\overline{y}) \in \operatorname{Gr} F$, and suppose there are $U \in \mathscr{V}(\overline{x})$ and $\varepsilon > 0$ such that, for all $x,u \in U$, $(F(x)-\overline{y}) \cap \left(-K^{(1)\varepsilon}(u)\right) \subset \{0\}$. Then there is $U' \in \mathscr{V}(\overline{x})$ such that*

$$\left(F\left(U'\right)-\overline{y}\right) \cap \left(-\operatorname*{Limsup}_{x\to\overline{x}} K(x)\right) \subset \{0\}.$$

Proof. Suppose, by contradiction, that the conclusion does not hold. Then there is a sequence $x_n \to \overline{x}$ such that, for all n, there exits

$$z_n \in \left[(F(x_n)-\overline{y}) \cap \left(-\operatorname*{Limsup}_{x\to\overline{x}} K(x)\right)\right] \setminus \{0\}.$$

Of course, for n large enough, $x_n \in U$. By the definition of Limsup, for all n, there is $u_n \in U$ such that $(-z_n + \varepsilon \|z_n\| D_Y) \cap K(u_n) \neq \emptyset$. Thus one finds $d_n \in D_Y$ such that $-z_n + \varepsilon \|z_n\| d_n \in K(u_n)$, whence $d(-z_n, K(u_n)) \leq \varepsilon \|z_n\| \|d_n\| \leq \varepsilon \|z_n\|$. This shows that $-z_n \in K^{(1)\varepsilon}(u_n)$ for all n, and this is a contradiction. □

Remark 4.2. The (pseudo-)orders discussed in Section 3 determine the following notions of minimality for set-valued maps:

(i) $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ is a local robust element of F with respect to K if there is $U \in \mathscr{V}(\overline{x})$ such that for all $x,z \in U$, $(F(x)-\overline{y}) \cap (-K(z)) \subset \{0\}$ and similarly,

(ii) $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ is a local nondominated element of F with respect to K if there is $U \in \mathscr{V}(\overline{x})$ such that, for all $x \in U$, $(F(x)-\overline{y}) \cap (-K(x)) \subset \{0\}$.

Hence, the relation in the hypothesis of the result above is actually the fact that $(\overline{x},\overline{y})$ is a local robust element of F with respect to $K^{(1)\varepsilon}$, while the conclusion states that the same point $(\overline{x},\overline{y})$ is a local Pareto minimum for F with respect to the cone $\operatorname*{Limsup}_{x\to\overline{x}} K(x)$.

As announced, the next result is about converting a nondominated element for F in variable order to an approximate Pareto minimum for F in fixed order. But the conclusion is in fact stronger than that and, as we will see in the next section, its special form makes it suitable for using cone separation results and by that, useful in obtaining optimality conditions.

Proposition 4.2. *Let $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ such that there are $U \in \mathscr{V}(\overline{x})$ and $\varepsilon > 0$ such that, for all $x \in U$,*

(i) $K^{(1)\varepsilon}(x)$ is convex and $K^{(1)\varepsilon}(x) \cap (-K(x)) = \{0\}$;

(ii) $(F(x)-\overline{y}) \cap \left(-K^{(1)\varepsilon}(x)\right) \subset \{0\}$.

Take $C = \left(\bigcap_{x\in U} K(x)\right)$ and suppose that there exists $e \in C \setminus \{0\}$. Suppose that there exists $L > 0$ such that for all $x,z \in U$, $F(x) \subset F(z) - L\|x-z\| e + K(z)$. Then, for all $\delta > 0$, there is $U_\delta \in \mathscr{V}(\overline{x})$ such that $(F(U_\delta) - \overline{y} + \delta e) \cap \left(-\operatorname*{Liminf}_{x\to\overline{x}} K(x)\right) = \emptyset$.

Proof. Firstly, observe that the minimality property of $(\overline{x},\overline{y})$ written at (ii) is equivalent, in view of (i), to, for all $x \in U, (F(x) - \overline{y} + K(x)) \cap \left(-K^{(1)\varepsilon}(x)\right) \subset \{0\}$. Suppose by contradiction that there exist $\delta > 0$ and a sequence $x_n \to \overline{x}$ such that, for all n,

$$(F(x_n) - \overline{y} + \delta e) \cap \left(-\operatorname*{Liminf}_{x \to \overline{x}} K(x)\right) \neq \emptyset.$$

Clearly, for n large enough, $x_n \in U$. Consider, for each such n, $y_n \in F(x_n)$ such that $y_n - \overline{y} + \delta e \in -\operatorname*{Liminf}_{x \to \overline{x}} K(x)$. We observe that $z_n := y_n - \overline{y} + \delta e$ cannot be 0 since otherwise,

$$\begin{aligned} 0 \neq -\delta e = y_n - \overline{y} &\in (F(x_n) - \overline{y}) \cap (-C) \\ &\subset (F(x_n) - \overline{y}) \cap \left(-K^{(1)\varepsilon}(x_n)\right), \end{aligned}$$

and this contradicts the hypothesis. Therefore, $\|z_n\| > 0$ for all n and by the definition of Liminf, there is a neighborhood W_n of $\overline{x}$ such that, for all $u \in W_n$, $(-z_n + \varepsilon D_Y) \cap K(u) \neq \emptyset$. Since $x_n \to \overline{x}$, for all n there exists $m_n \in \mathbb{N}$ such that, for all $m \geq m_n$, $x_m \in W_n$. The relation above means that $-z_n \in K^{(1)\varepsilon}(x_m), \forall m \geq m_n$. Now, using the Lipschitz property, for n large and $m \geq m_n$, one has

$$\begin{aligned} z_n = y_n - \overline{y} + \delta e \in F(x_n) - \overline{y} + \delta e &\subset F(x_m) - L\|x_n - x_m\| e + K(x_m) - \overline{y} + \delta e \\ &= F(x_m) + (\delta - L\|x_n - x_m\|) e + K(x_m) - \overline{y} \\ &\subset F(x_m) + K(x_m) - \overline{y}, \end{aligned}$$

where for the last inclusion we used the Cauchy property of the sequence (x_n). Therefore,

$$z_n \in (F(x_m) + K(x_m) - \overline{y}) \cap \left(-K^{(1)\varepsilon}(x_m)\right),$$

and this is a contradiction. The conclusion holds. $\square$

Remark 4.3. In the notation of Proposition 4.2, suppose that, for all $x \in U$, the cone $K(x)$ is well-based with the base $B(x)$, $\bigcup_{x \in U} B(x)$ is bounded, and $0 \notin \operatorname{cl} \bigcup_{x \in U} B(x)$. Then, as in Lemma 2.1 (ii), there are $y^* \in Y^*$ and $\alpha > 0$ such that $y^*(y) \geq \alpha \|y\|$ for all $y \in \bigcup_{x \in U} K(x)$ and in this case $E(x) = K(x) \cap \{y \in Y \mid y^*(y) = 1\}$ is a bounded base for $K(x)$ for all $x \in U$. We construct the fourth type enlargement for all cones $K(x)$ using these bounded bases. Now suppose that there is $\varepsilon > 0$ such that $(F(x) - \overline{y}) \cap \left(-K^{(4)\varepsilon}(x)\right) \subset \{0\}$. Then one obtains the conclusion of Proposition 4.2. Indeed, since $K^{(4)\varepsilon}(x)$ is convex and pointed for all $x \in U$, we have

$$(F(x) - \overline{y} + K(x)) \cap \left(-K^{(4)\varepsilon}(x)\right) \subset \{0\}.$$

Now if we look at the proof of Proposition 2.1 (iii), the second part, we obtain a $\eta > 0$, the same for all $x \in U$, such that

$$(F(x) - \overline{y} + K(x)) \cap \left(-K^{(1)\eta}(x)\right) \subset \{0\},$$

and the rest of the argument is the same.

Minimality in the context of variable order structures was studied in [19] with respect to the third type of cone enlargement. The Lipschitz property from above was used to penalize the Henig-type nondominated efficiency and some necessary optimality conditions were given for it by using the incompatibility between minimality and openness. This will be for us also one of the purposes in the next section, that is obtaining necessary optimality conditions for the Henig-type minimality, but we will not follow the line from [19], but instead, using the conclusion above, we will use a cone separation result and the properties of $\operatorname*{Liminf}_{x\to\bar{x}} K(x)$ obtained in the proposition below.

Remark 4.4. Notice that, in general, one can prove that $\operatorname*{Limsup}_{x\to\bar{x}} K(x)$ is a closed cone. In contrast, for $\operatorname*{Liminf}_{x\to\bar{x}} K(x)$, we have that it is a closed cone which is also convex if K has convex cone values. Moreover, since for every neighborhood U of $\bar{x}$

$$\bigcap_{x\in U} K(x) \subset \operatorname*{Liminf}_{x\to\bar{x}} K(x) \subset \operatorname{cl} K(\bar{x}),$$

$\operatorname*{Liminf}_{x\to\bar{x}} K(x)$ is pointed as well provided that $K(\bar{x})$ is pointed.

Proposition 4.3. *Suppose that K has well-based closed and pointed cone values in a neighborhood U of $\bar{x}$ and denote by B the set-valued map that associates to every $x \in U$ the corresponding closed bounded base of $K(x)$. If $0 \notin \operatorname{cl} \bigcup_{x\in U} B(x)$ and $\bigcup_{x\in U} B(x)$ is bounded, then $D = \operatorname*{Liminf}_{x\to\bar{x}} B(x)$ is a bounded base for $\operatorname*{Liminf}_{x\to\bar{x}} K(x)$.*

Proof. First of all, we deduce that $0 \notin D$ since otherwise, following Remark 4.4, $0 \in B(\bar{x})$ which cannot happen since $B(\bar{x})$ is a closed base. Then, as $D \subset B(\bar{x})$, it follows that D is bounded. It is an easy matter to observe that the convexity of the values of B ensures the convexity of D. Let $u \in \operatorname*{Liminf}_{x\to\bar{x}} K(x) \setminus \{0\}$. Then for all $(x_n) \to \bar{x}$ there is $(u_n) \to u$ such that $u_n \in K(x_n)$ for all n. Accordingly, there exist $(\alpha_n) \subset [0,\infty)$ and $b_n \in B(x_n)$ such that $u_n = \alpha_n b_n$ for all n. Since $\bigcup_{x\in U} B(x)$ is bounded and $0 \notin \operatorname{cl} \bigcup_{x\in U} B(x)$, the sequence (b_n) is bounded and 0 is not a cluster point of it. Then neither 0, nor ∞ is a cluster point of (α_n). So, at least on a subsequence, (α_n) is converging towards a value $\alpha \in (0,\infty)$. Then

$$b_n = \frac{u_n}{\alpha_n} \to \frac{u}{\alpha} \in \operatorname*{Liminf}_{x\to\bar{x}} B(x).$$

Therefore, $u \in \operatorname{cone} \operatorname*{Liminf}_{x\to\bar{x}} B(x)$, so $\operatorname*{Liminf}_{x\to\bar{x}} K(x) \subset \operatorname{cone} \operatorname*{Liminf}_{x\to\bar{x}} B(x)$. As $B(x) \subset K(x)$ and $\operatorname*{Liminf}_{x\to\bar{x}} K(x)$ is a cone, the other inclusion follows easily and hence $\operatorname*{Liminf}_{x\to\bar{x}} B(x)$ is a bounded base for $\operatorname*{Liminf}_{x\to\bar{x}} K(x)$. □

It is interesting to record the following result that gives the lower semicontinuity of the lower limit set-valued map.

Proposition 4.4. *Let $K : X \rightrightarrows Y$ be a set-valued map with cone values, $\bar{x} \in X$, and U an open neighborhood of $\bar{x}$. Suppose that there is a cone P with nonempty interior such that $P \subset \bigcap_{x\in U} K(x)$. If $P + K(x) \subset K(x)$ for all $x \in U$ (in particular, if K has convex values), then the set-valued map $G : X \rightrightarrows Y$, $G(x) = \operatorname*{Liminf}_{u\to x} K(u)$ is lower semicontinuous at every point $x \in U$.*

Proof. Let $x \in U$ and $y \in G(x)$. We have to show that $y \in \operatorname*{Liminf}_{u\to x} G(u)$. Actually, we will show the stronger fact that $y \in G(u)$ for all u close enough to x. Take $k \in P$ such that $\|k\| = 1$ and $D(k,\theta) \subset P$ for some $\theta > 0$. Of course, such k exists since $\operatorname{int} P \neq \emptyset$.

Firstly, we claim that

$$\forall\varepsilon>0,\ \exists\delta>0 \text{ such that } \forall u\in B(x,\delta), y+\varepsilon k\in K(u). \tag{4.1}$$

Indeed, take $\varepsilon>0$ and $\mu\in(0,\varepsilon\theta)$. Since $y\in\operatorname*{Liminf}_{u\to x}K(u)$, there is $\delta>0$ such that, for all $u\in D(x,\delta)\cap U$, $(y+\mu D_Y)\cap K(u)\neq\emptyset$. Clearly, without loss of generality, one can suppose that $D(x,\delta)\cap U=D(x,\delta)$. Accordingly, for all $u\in D(x,\delta)$, there are $z\in D_Y$ and $v\in K(u)$ such that $y+\mu z=v$. Then $v-(y+\varepsilon k)=\mu z-\varepsilon k=-\varepsilon\left(k-\varepsilon^{-1}\mu z\right)\in -P$, whence $y+\varepsilon k\in K(u)+P\subset K(u)$. Therefore, the claim (4.1) is proved.

Let $u\in B(x,\delta)$. Choose $\rho>0$ such that $B(u,\rho)\subset B(x,\delta)\subset U$. Then, from (4.1), for all $z\in B(u,\rho)$, $y+\varepsilon k\in K(z)$, that is, $D(y,\varepsilon)\cap K(z)\neq\emptyset$. This implies that $y\in\operatorname*{Liminf}_{z\to u}K(z)=G(u)$ and, consequently, $y\in\operatorname*{Liminf}_{u\to x}G(u)$. □

5. Optimality Conditions

The final section of our paper is devoted to some optimality conditions for efficiency with respect to a variable order structure. The optimality conditions we obtain are in terms of generalized differentiation objects lying in dual spaces developed by Mordukhovich and his collaborators. Thus, following the book [20], we give some of the constructions that we use in this section.

Consider S a nonempty subset of a real Banach space X and $\overline{x}\in S$. The Fréchet normal cone to S at $\overline{x}$ is the set

$$\widehat{N}(S,\overline{x}):=\left\{x^*\in X^*\mid \limsup_{x\xrightarrow{S}\overline{x}}\frac{\langle x^*,x-\overline{x}\rangle}{\|x-\overline{x}\|}\leq 0\right\}.$$

The basic (or limiting, or Mordukhovich) normal cone to S at $\overline{x}$ is

$$N(S,\overline{x}):=\{x^*\in X^*\mid \exists\varepsilon_n\xrightarrow{(0,\infty)}0, x_n\xrightarrow{S}\overline{x}, x_n^*\xrightarrow{w^*}x^*, x_n^*\in\widehat{N}_{\varepsilon_n}(S,x_n),\forall n\in\mathbb{N}\},$$

where the notation $\varepsilon_n\xrightarrow{(0,\infty)}0$ means that $\varepsilon_n\to 0$ and $(\varepsilon_n)\subset(0,\infty)$; similarly for $x_n\xrightarrow{S}\overline{x}$. If X is an Asplund space and S is closed around $\overline{x}$ (i.e., there is a neighborhood V of $\overline{x}$ such that $S\cap\operatorname{cl}V$ is closed), the formula for the basic normal cone looks as follows:

$$N(S,\overline{x})=\{x^*\in X^*\mid \exists x_n\xrightarrow{S}\overline{x}, x_n^*\xrightarrow{w^*}x^*, x_n^*\in\widehat{N}(S,x_n),\forall n\in\mathbb{N}\}.$$

For a function $f:X\to\mathbb{R}\cup\{\pm\infty\}$, finite at $\overline{x}\in X$, the basic (or limiting, or Mordukhovich) subdifferential of f at $\overline{x}$ is the set

$$\partial f(\overline{x}):=\{x^*\in X^*\mid (x^*,-1)\in N(\operatorname{epi}f,(\overline{x},f(\overline{x})))\}.$$

If f is convex, then this coincides with the classical Fenchel subdifferential of f at $\overline{x}$. A well known result is the next generalized Fermat rule: if $\overline{x}$ is a local minimum point for f then $0\in\partial f(\overline{x})$.

For a set-valued map $F:X\rightrightarrows Y$, where Y is a real Banach space, the normal coderivative of F at $(\overline{x},\overline{y})\in\operatorname{Gr}F$ is the set-valued map $D^*F(\overline{x},\overline{y}):Y^*\rightrightarrows X^*$ given by

$$D^*F(\overline{x},\overline{y})(y^*):=\{x^*\in X^*\mid (x^*,-y^*)\in N(\operatorname{Gr}F,(\overline{x},\overline{y}))\}.$$

The positive dual cone of a cone $Q\subset Y$ is defined by

$$Q^+:=\{y^*\in Y^*\mid y^*(y)\geq 0,\ \forall y\in Q\}.$$

One of the main tools used here in obtaining the optimality conditions is the well-known Gerstewitz's (Tammer's) scalarizing functional, that we present in the next lemma together with some of the properties of it needed in the following (see, [4, Theorem 2.3.1] and [21, Theorem 3.1, Propositions 3.7 and 4.1]).

Lemma 5.1. *Let $\emptyset \neq A \subset Y$ be a closed proper set and $e \in Y \setminus \{0\}$ be such that $A + [0,\infty)e \subset A$. Then, the functional $s_{e,A} : Y \to \overline{\mathbb{R}}$ given by $s_{e,A}(y) := \inf\{\lambda \in \mathbb{R} \mid \lambda e \in y + A\}$ is lower semicontinuous and for every $\lambda \in \mathbb{R}$, $\{y \in Y \mid s_{e,A}(y) \leq \lambda\} = \lambda e - A$. Moreover, $s_{e,A}$ is convex if and only if A is convex. Furthermore, if A has nonempty interior and $A + (0,\infty)e \subset \operatorname{int} A$, then $s_{e,A}$ is continuous and for every $\lambda \in \mathbb{R}$, $\{y \in Y \mid s_{e,A}(y) < \lambda\} = \lambda e - \operatorname{int} A$. In addition, if there is $K \subset Y$ a closed convex cone with nonempty interior such that the free-disposal property holds, i.e., $A + K = A$, then, for every $e \in \operatorname{int} K$, $s_{e,A}$ is finite and Lipschitz on Y. In the assumptions from above and supposing also that A is convex and $-e \notin A_\infty$, where A_∞ is the recession cone defined by $A_\infty := \{y \in Y \mid v + ty \in A,\ \forall v \in A,\ \forall t \in \mathbb{R}\}$, we have that $\partial s_{e,A}(y) \subset \{y^* \in K^+ \mid y^*(e) = 1\}$ for every $y \in Y$.*

We give now our result concerning optimality conditions.

Theorem 5.1. *Let X be an Asplund space and Y a reflexive Banach space. Let $F, K : X \rightrightarrows Y$ be set-valued maps with $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Suppose that there is a neighborhood U of $\overline{x}$ such that*

(i) K has well-based closed convex and pointed cone values in the neighborhood U;

(ii) $0 \notin \operatorname{cl} \bigcup_{x \in U} B(x)$ and $\bigcup_{x \in U} B(x)$ is bounded, where we denote by B the set-valued map that associates to every $x \in U$ the corresponding closed bounded base of $K(x)$;

(iii) there exist $e \in C = \left(\bigcap_{x \in U} K(x) \right) \setminus \{0\}$ and $L > 0$ such that for all $x, z \in U$,

$$F(x) \subset F(z) - L\|x - z\| e + K(z);$$

(iv) $\operatorname{Gr} F$ is locally closed and $\operatorname{cone}(F(V) - \overline{y} + \rho e)$ is weakly closed for all small $\rho > 0$ and all small closed neighborhood V of $\overline{x}$.

If there exists $\overline{\varepsilon} > 0$ such that $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is a local nondominated element of F with respect to the $K^{(4)\overline{\varepsilon}}$ enlargement obtained as described in Remark 4.3, then, for all $\delta > 0$, there exist $\varepsilon > 0$, $(x, y) \in \operatorname{Gr} F \cap D\left((\overline{x}, \overline{y}), \dfrac{\delta}{1+\delta} \right)$ and $y^* \in \left(Q^{(4)\varepsilon} \right)^+$ with $y^*(e) = \frac{1}{1+\delta}$, such that

$$0 \in D^* F(x, y) \left(y^* + \sqrt{\frac{\delta}{1+\delta}} D_{Y^*} \right) + \sqrt{\frac{\delta}{1+\delta}} D_{X^*},$$

where $Q = \operatorname*{Liminf}_{x \to \overline{x}} K(x)$.

Proof. The minimality assumption means that there is $U' \in \mathscr{V}(\overline{x})$ such that, for all $x \in U'$,

$$(F(x) - \overline{y}) \cap \left(-K^{(4)\overline{\varepsilon}}(x) \right) \subset \{0\}.$$

According to Proposition 4.2 and Remark 4.3, for all $\delta > 0$, there exits $U_\delta \in \mathscr{V}(\overline{x})$ such that

$$(F(U_\delta) - \overline{y} + \delta e) \cap \left(-\operatorname*{Liminf}_{x \to \overline{x}} K(x) \right) = \emptyset,$$

whence

$$\operatorname{cone}\left(F\left(U_{\delta}\right)-\bar{y}+\delta e\right)\cap\left(-\operatorname*{Liminf}_{x\to\bar{x}}K\left(x\right)\right)=\{0\}.$$

Applying Proposition 4.3, we have that $D:=\operatorname*{Liminf}_{x\to\bar{x}}B\left(x\right)$ is a bounded base for $Q:=\operatorname*{Liminf}_{x\to\bar{x}}K\left(x\right)$. Thus Q is well-based. Using Remark 4.4, we obtain that Q is a closed, convex, and well-based cone.

Therefore, using Theorem 2.1 and Remark 2.5, there exists $\varepsilon>0$ such that

$$\operatorname{cone}\left(F\left(U_{\delta}\right)-\bar{y}+\delta e\right)\cap\left(-Q^{(4)\varepsilon}\right)=\{0\},$$

so $\left(F\left(U_{\delta}\right)-\bar{y}+\delta e\right)\cap\left(-Q^{(4)\varepsilon}\right)\subset\{0\}$. One obtains

$$\left(F\left(U_{\delta}\right)-\bar{y}+\delta e\right)\cap\left(-\operatorname{int}Q^{(4)\varepsilon}\right)=\emptyset,$$

whence, for every $x\in U_{\delta}$, $F\left(x\right)-\bar{y}\notin-\operatorname{int}Q^{(4)\varepsilon}-\delta e=-\operatorname{int}\left(Q^{(4)\varepsilon}+\delta e\right)$. Further, applying Lemma 5.1 for $Q^{(4)\varepsilon}+\delta e$ and the element $e+\delta e\in Q\setminus\{0\}\subset\operatorname{int}Q^{(4)\varepsilon}$, we obtain

$$s_{e+\delta e,Q^{(4)\varepsilon}+\delta e}\left(y-\bar{y}\right)\geq 0,\ \forall\left(x,y\right)\in\left(U_{\delta}\times Y\right)\cap\operatorname{Gr}F.$$

Now, observe that

$$s_{e+\delta e,Q^{(4)\varepsilon}+\delta e}\left(0\right)=\frac{\delta}{1+\delta},$$

so one can write

$$s_{e+\delta e,Q^{(4)\varepsilon}+\delta e}\left(y-\bar{y}\right)+\frac{\delta}{1+\delta}\geq s_{e+\delta e,Q^{(4)\varepsilon}+\delta e}\left(\bar{y}-\bar{y}\right),\ \forall\left(x,y\right)\in\left(U_{\delta}\times Y\right)\cap\operatorname{Gr}F.$$

i.e., $\left(\bar{x},\bar{y}\right)$ is a $\frac{\delta}{1+\delta}-$local minimum for the function $f:X\times Y\to\mathbb{R}$, $f\left(x,y\right)=s_{e+\delta e,Q^{(4)\varepsilon}+\delta e}\left(y-\bar{y}\right)$, on $\operatorname{Gr}F$. By the Ekeland Variational Principle, for $\mu:=\sqrt{\frac{\delta}{1+\delta}}$ there exists $\left(x_{\mu},y_{\mu}\right)\in\operatorname{Gr}F\cap D\left(\left(\bar{x},\bar{y}\right),\sqrt{\frac{\delta}{1+\delta}}\right)$ which is a minimum on $\operatorname{Gr}F$ for

$$\left(x,y\right)\longmapsto s_{e+\delta e,Q^{(4)\varepsilon}+\delta e}\left(y-\bar{y}\right)+\sqrt{\frac{\delta}{1+\delta}}\left\|\left(x,y\right)-\left(x_{\mu},y_{\mu}\right)\right\|.$$

Therefore, $\left(x_{\mu},y_{\mu}\right)$ is a local minimum without constraints for

$$\left(x,y\right)\longmapsto s_{e+\delta e,Q^{(4)\varepsilon}+\delta e}\left(y-\bar{y}\right)+\sqrt{\frac{\delta}{1+\delta}}\left\|\left(x,y\right)-\left(x_{\mu},y_{\mu}\right)\right\|+\delta_{\operatorname{Gr}F}\left(x,y\right),$$

where $\delta_{\operatorname{Gr}F}$ denotes the indicator function associated with the set $\operatorname{Gr}F$, i.e.,

$$\delta_{\operatorname{Gr}F}\left(x,y\right)=\begin{cases}0, & \text{if }\left(x,y\right)\in\operatorname{Gr}F,\\ \infty, & \text{otherwise.}\end{cases}$$

Using now the generalized Fermat rule and the sum rule from [20, Theorem 3.36], we can write

$$
\begin{aligned}
(0,0) &\in \partial\left(\sqrt{\frac{\delta}{1+\delta}}\|(\cdot,\cdot)-(x_\mu,y_\mu)\|+s_{e+\delta e,Q^{(4)\varepsilon}+\delta e}(\cdot-\bar{y})+\delta_{\operatorname{Gr}F}\right)(x_\mu,y_\mu)\\
&\subset \sqrt{\frac{\delta}{1+\delta}}(D_{X^*}\times D_{Y^*})+\partial(s_{e+\delta e,Q^{(4)\varepsilon}+\delta e}(\cdot-\bar{y})+\delta_{\operatorname{Gr}F})(x_\mu,y_\mu)\\
&\subset \sqrt{\frac{\delta}{1+\delta}}(D_{X^*}\times D_{Y^*})+\partial(s_{e+\delta e,Q^{(4)\varepsilon}+\delta e}(\cdot-\bar{y}))(x_\mu,y_\mu)+\partial\delta_{\operatorname{Gr}F}(x_\mu,y_\mu).
\end{aligned}
$$

Notice that f is Lipschitz on $X\times Y$ according to Lemma 5.1, by taking $K=Q^{(4)\varepsilon}$, and $\delta_{\operatorname{Gr}F}$ is lower semicontinuous around (x_μ,y_μ). Again by Lemma 5.1, we have that

$$
\partial f(x_\mu,y_\mu)=\{0\}\times\partial s_{e+\delta e,Q^{(4)\varepsilon}+\delta e}(y_\mu-\bar{y})\subset\{0\}\times\left\{y^*\in\left(Q^{(4)\varepsilon}\right)^+\mid y^*(e+\delta e)=1\right\}.
$$

Also, since $\partial\delta_A(\bar{x})=N(A,\bar{x})$ for every set A and $\bar{x}\in A$, we have that

$$
\partial\delta_{\operatorname{Gr}F}(x_\mu,y_\mu)=N(\operatorname{Gr}F,(x_\mu,y_\mu)).
$$

Therefore,

$$
(0,0)\in\sqrt{\frac{\delta}{1+\delta}}D_{X^*}\times D_{Y^*}+\{0\}\times\left\{y^*\in\left(Q^{(4)\varepsilon}\right)^+|y^*(e)=\frac{1}{1+\delta}\right\}+N(\operatorname{Gr}F,(x_\mu,y_\mu)).
$$

Hence, there are $x^*\in D_{X^*}$, $z^*\in D_{Y^*}$ and $y^*\in\left(Q^{(4)\varepsilon}\right)^+$ with $y^*(e)=\dfrac{1}{1+\delta}$, such that

$$
\left(-\sqrt{\frac{\delta}{1+\delta}}x^*,-\sqrt{\frac{\delta}{1+\delta}}z^*-y^*\right)\in N(\operatorname{Gr}F,(x_\mu,y_\mu)),
$$

or, equivalently,

$$
\begin{aligned}
0 &\in D^*F(x_\mu,y_\mu)\left(y^*+\sqrt{\frac{\delta}{1+\delta}}z^*\right)+\sqrt{\frac{\delta}{1+\delta}}x^*\\
&\subset D^*F(x_\mu,y_\mu)\left(y^*+\sqrt{\frac{\delta}{1+\delta}}D_{Y^*}\right)+\sqrt{\frac{\delta}{1+\delta}}D_{X^*}.
\end{aligned}
$$

The proof is complete. □

Remark 5.1. (i) Notice that the set for which we applied Lemma 5.1 in the proof above, namely $Q^{(4)\varepsilon}+\delta e$, is not a cone, so the scalarization functional does not satisfy the sublinearity property (see [4, Theorem 2.3.1] for more details). Therefore we can not use the continuity together with the sublinearity to obtain the Lipschitz property of the scalarization functional on the whole space. However, due to the nonemptiness of the interior of $Q^{(4)\varepsilon}$ and since the set $Q^{(4)\varepsilon}+\delta e$ satisfies the free-disposal property, the scalarization functional is Lipschitz on the whole space and thus we are able to apply a sum rule for the Mordukhovich subdifferential. In addition, observe that inasmuch as $Q^{(4)\varepsilon}+\delta e$ is not a cone, it means that $Q^{(4)\varepsilon}+\delta e\neq Y$ and also that $Q^{(4)\varepsilon}+\delta e\neq\{0\}$, so it results that the set $Q^{(4)\varepsilon}+\delta e$ is proper.

(ii) In many papers in the literature (see, e.g., [1, 2, 13, 22]), in order to obtain a nontrivial Lagrange multiplier for the objective, some generalized compactness hypothesis were used, while in Theorem 5.1 we obtain a multiplier different from 0 without using any kind of generalized compactness assumptions. Of course, the obtaining of a nontrivial Lagrange multiplier here is due essentially to the use of the scalarizing functional. Observe as well that the result above is proved for a set-valued map K whose values are possibly non-solid cones; thus, the fact that we were able to use the Gerstewitz scalarizing functional emphasizes one of the advantages of using enlargements of a cone, as they have nonempty interior.

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