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PERTURBATION ANALYSIS OF GLOBAL ERROR BOUNDS FOR PIECEWISE LINEAR CONIC INEQUALITIES

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Abstract. For a piecewise linear system of conic inequalities, we establish the stability of its global error bound when the piecewise linear vector-valued objective function undergoes small linear perturbations. In particular, our results improve the corresponding ones about linear conic inequalities in the literature.

Keywords. Global error bound; Piecewise linearity; Stability.

1. Introduction

Let X,Y be normed spaces and let $\mathscr{L}(X,Y)$ denote the space of all continuous linear operators from X to Y. For $(T,b)\in\mathscr{L}(X,Y)\times Y$ and a closed convex cone $K\subset Y$, consider the following linear conic inequality:

(LCIE) $$Tx\le_K b.$$

Recall that (LCIE) has a global error bound if there exists $\tau\in(0,\,+\infty)$ such that

$$d(x,S(T,b,K))\le\tau d(Tx,b-K)\quad\forall x\in X,$$

where $S(T,b,K)=T^{-1}(b-K)$. In 1994, in the finite-dimensional case, Luo and Tseng [1] first studied stability of the global error bound for (LCIE) and proved the following result.

Theorem 1.1. *Let $T\in\mathscr{L}(\mathbb{R}^m,\mathbb{R}^n)$ and $b\in\mathbb{R}^n$. Then the following statements are equivalent:*
(i) Either $Tx<_{\mathbb{R}^n_+}0$ is solvable or $Tx<_{\mathbb{R}^n_+}b$ is solvable and the solution set $S(T,b,\mathbb{R}^n_+)$ of (LCIE) *is bounded.*
(ii) There exist $\tau,\delta\in(0,\,+\infty)$ such that

$$d(x,S(T',b',\mathbb{R}^n_+))\le\tau d(T'x,b'-\mathbb{R}^n_+)\quad\forall x\in\mathbb{R}^m$$

whenever $(T',b')\in\mathscr{L}(\mathbb{R}^m,\mathbb{R}^n)\times\mathbb{R}^n$ satisfies $\|T'-T\|+\|b-b'\|<\delta$.

In 2005, Zheng and Ng [2] extended the theorem above to the case that X and Y are infinite-dimensional spaces and $K\subset Y$ is a general closed convex cone with a nonempty interior. Undoubtedly, the linearity assumption is restrictive. To overcome the restriction of linearity, one sometimes adopts the piecewise linear functions (cf. [3, 4, 5, 6, 7, 8]). The family of all piecewise linear functions is much larger than that of all linear functions and there exists a very

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wide class of functions that can be approximated by piecewise linear functions. Moreover, in practical applications, one often uses piecewise linear functions for math modeling. Hence it is valuable to study piecewise linearity in both theory and application. This paper is devoted to extend Luo and Tseng's result to the piecewise linear case. Recall that a subset P of X is called a convex polyhedron if there exist $x_i^* \in X^*$ and $r_i \in \mathbb{R}$ ($i \in \overline{1n} := \{1, \cdots, n\}$) such that

$$P = \{x \in X : \langle x_i^*, x\rangle \leq r_i, i \in \overline{1n}\}.$$

Recall that a vector-valued function $f : X \to Y$ is piecewise linear if there exist convex polyhedra P_i in X, continuous linear operators $T_i \in \mathscr{L}(X,Y)$ and points $b_i \in Y$ ($i \in \overline{1m}$) such that

$$X = \bigcup_{i=1}^{m} P_i \text{ and } f(x) = T_i(x) - b_i \ \ \forall x \in P_i \text{ and } i \in \overline{1m}. \tag{1.1}$$

For a piecewise linear function $f : X \to Y$, consider the following piecewise linear conic inequality:

$$\text{(PLCIE)} \qquad f(x) \leq_K 0.$$

We say that piecewise linear conic inequality (PLIE) has a global error bound if there exists $\kappa \in (0, +\infty)$ such that

$$d(x, S(f,K)) \leq \kappa d(f(x), -K) \quad \forall x \in X,$$

where $S(f,K) := f^{-1}(-K)$ is the solution set of (PLIE). In this paper, we consider the stability of global error bounds for (PLIE) in the following sense: there exist $\kappa, \delta \in (0, +\infty)$ such that for any $T \in \mathscr{L}(X,Y)$ with $\|T\| < \delta$,

$$d(x, S(f+T,K)) \leq \kappa d(f(x)+T(x), -K) \quad \forall x \in X.$$

We will provide some sufficient conditions to ensure that piecewise linear conic inequality (PLIE) has a stable global error bound and extend the main results in Luo and Tseng [1] and Zheng and Ng [2] from the linear case to the piecewise linear one.

2. Preliminaries

For normed spaces X and Y, recall that $\mathscr{L}(X,Y)$ is the space of all continuous linear operators from X to Y. In the remainder, let $\mathscr{PL}(X,Y)$ denote the family of all piecewise linear functions from X to Y in the sense of (1.1). For $f \in \mathscr{PL}(X,Y)$, take $T_i \in \mathscr{L}(X,Y)$, $b_i \in Y$ and convex polyhedra P_i in X ($i = 1, \cdots, m$) such that (1.1) holds. If the interior $\text{int}(P_i)$ of P_i is empty, then the first equality of (1.1) implies $X = \text{cl}\left(\bigcup_{j \in \overline{1m}\setminus\{i\}} P_j\right)$, and so $X = \bigcup_{j \in \overline{1m}\setminus\{i\}} P_j$ (because every polyhedron is closed). Hence we can assume that each P_i in (1.1) is of a nonempty interior. In fact, it is known that for each $f \in \mathscr{PL}(X,Y)$ there exist convex polyhedra P_i in X, $T_i \in \mathscr{L}(X,Y)$ and $b_i \in Y$ ($i \in \overline{1m}$) such that

$$X = \bigcup_{i=1}^{m} P_i,\ P_i \cap \text{int}(P_{i'}) = \emptyset \text{ and } f|_{P_i} = T_i|_{P_i} - b_i \quad \forall i, i' \in \overline{1m} \text{ with } i' \neq i \tag{2.1}$$

(see [9, Proposition 3.2]).

For a polyhedron $P=\{x\in X:\ \langle x_i^*,x\rangle\le r_i,\, i\in\overline{1n}\}$, let P^∞ denote the recession cone of P, that is,

$$P^\infty=\{x\in X:\ \langle x_i^*,x\rangle\le 0,\, i=1\cdots,n\}.$$

Let $X_1:=\{x\in X:\ \langle x_i^*,x\rangle=0,\ i=1\ \cdots,n\}$. Then X_1 is a closed subspace of X with codimension $\operatorname{codim}(X_1)\le n$, that is, there exists a finite subspace X_2 of X with $\dim(X_2)\le n$ such that $X_1\cap X_2=\{0\}$ and $X=X_1+X_2$. Let $\hat{P}=\{x_2\in X_2:\ \langle x_i^*,x_2\rangle\le r_i,\ i=1\cdots,n\}$. Then $\hat{P}$ is a convex polyhedron in the finite dimensional space X_2 and $P=X_1+\hat{P}$. Noting that every convex polyhedron in a finite dimensional space is finitely generated (cf. [10, Theorem 19.1]), we have the following lemma.

Lemma 2.1. *Let P be a convex polyhedron in a normed space X. Then there exist $x_1,\cdots,x_m\in X$ such that $P=\operatorname{co}\{x_1,\cdots,x_m\}+P^\infty$.*

For $(T,b)\in\mathscr{L}(X,Y)\times Y$ and a convex polyhedron P in X, let

$$\gamma(T,P,K):=\sup_{x\in B_X\cap P^\infty}\sup\{r>0:\ B_Y(Tx,r)\subset -K\} \tag{2.2}$$

and

$$\gamma(T,P,b,K):=\sup_{x\in T^{-1}(b-\operatorname{int}(K))\cap P}\sup\{r>0:\ B_Y(Tx-b,r)\subset -K\}. \tag{2.3}$$

The following proposition is immediate from (2.2) and (2.3).

Proposition 2.1. *Let $T:X\to Y$ be a continuous linear operator, P be a convex polyhedron in X and b be a point in Y. The following statements hold:*
(i) The conic strict inequality $Tx<_K 0$ is solvable on P^∞ if and only if $\gamma(T,P,K)>0$.
(ii) The conic strict inequality $Tx<_K b$ is solvable on P if and only if $\gamma(T,P,b,K)>0$.

In the case when $P=X$, $\gamma(T,P,K)>0$ if and only if $Tx<_K 0$ is solvable, while $\gamma(T,P,b,K)>0$ if and only if $Tx<_K b$ is solvable.

We conclude this section with the following lemma, which will play a key in the proof of our main results.

Lemma 2.2. *Let S be a nonempty closed convex set in a Banach space X and let A be a subset of S such that $S=A+S^\infty$, where $S^\infty:=\{h\in X:\ S+\mathbb{R}_+h\subset S\}$ is the recession cone of S. Then, for any $\gamma\in(0,1)$ and $x\in X\setminus S$, there exist $a\in A$ and $h\in S^\infty$ such that*

$$d\left(a+th+\frac{t'(x-a-h)}{\|x-a-h\|},S\right)\ge\gamma t'\quad\forall t,t'\in\mathbb{R}_+. \tag{2.4}$$

Lemma 2.2 is a consequence of [11, Lemma 3.1]. In (2.4), setting $t=1$ and $t'=s\|x-a-h\|$ and noting that $u:=a+h\in S$, we have the following corollary.

Corollary 2.1. *Let S be a nonempty closed convex set in a Banach space X. Then, for any $\gamma\in(0,1)$ and $x\in X\setminus S$, there exists $u\in S$ such that*

$$d(u+s(x-u),S)\ge\gamma s\|x-u\|\quad\forall s\in\mathbb{R}_+.$$

3. Nonconvex Case

Recall that a multifunction F between normed spaces X and Y is convex if its graph $\mathrm{gph}(F) := \{(x,y):\ x \in X \text{ and } y \in F(x)\}$ is a convex set in the product $X \times Y$. It is known and easy to verify that F is convex if and only if

$$tF(x_1) + (1-t)F(x_2) \subset F(tx_1 + (1-t)x_2) \quad \forall x_1, x_2 \in X \text{ and } \forall t \in [0,\ 1].$$

To prove the main results in this paper, we need the following lemma, which is a variant of [12, Theorem 2].

Lemma 3.1. *Let F be a closed convex multifunction between Banach spaces X and Y. Let $(x_0, 0) \in \mathrm{gph}(F)$ and $r \in (0, +\infty)$ be such that*

$$rB_Y \subset F(x_0 + B_X). \tag{3.1}$$

Suppose that A is a convex subset of $F^{-1}(0)$ such that $F^{-1}(0) = A + F^{-1}(0)^\infty$. Then, for any $x \in X \setminus F^{-1}(0)$, there exists $a \in A$ such that

$$d(x, F^{-1}(0)) \le \frac{d(0, F(x))}{r}(1 + \|a - x_0\|).$$

Proof. Since F is a closed convex multifunction, $F^{-1}(0)$ is a closed convex set. Let $\gamma \in (0,\ 1)$ and $x \in X \setminus F^{-1}(0)$. Then, by Lemma 2.2, there exist $a \in A$ and $h \in F^{-1}(0)^\infty$ such that

$$d\left(a + th + \frac{t'(x - a - h)}{\|x - a - h\|}, F^{-1}(0)\right) \ge \gamma t' \quad \forall t, t' \in \mathbb{R}_+.$$

Setting $t' = t\|x - a - h\|$, one has

$$d(a + t(x - a), F^{-1}(0)) \ge \gamma t\|x - a - h\| \quad \forall t \in \mathbb{R}_+.$$

Since $a + h \in A + F^{-1}(0)^\infty = F^{-1}(0)$,

$$d(a + t(x - a), F^{-1}(0)) \ge \gamma t d(x, F^{-1}(0)) \quad \forall t \in \mathbb{R}_+. \tag{3.2}$$

By (3.1) and [12, Theorem 2], one has

$$d(a + t(x - a), F^{-1}(0)) \le \frac{d(0, F(a + t(x - a)))}{r}(1 + \|a + t(x - a) - x_0\|) \quad \forall t \in \mathbb{R}_+. \tag{3.3}$$

Since F is a convex multifunction,

$$F(a + t(x - a)) \supset tF(x) + (1 - t)F(a) \supset tF(x) \quad \forall t \in [0,\ 1].$$

Hence $d(0, F(a + t(x - a))) \le d(0, tF(x)) = td(0, F(x))$ for all $t \in [0,\ 1]$. It follows from (3.2) and (3.3) that

$$d(x, F^{-1}(0)) \le \frac{d(0, F(x))}{\gamma r}(1 + \|a + t(x - a) - x_0\|) \quad \forall t \in [0,\ 1].$$

Letting $t \to 0^+$, one has $d(x, F^{-1}(0)) \le \frac{d(0,F(x))}{\gamma r}(1 + \|a - x_0\|)$. Since γ is arbitrary in $(0,\ 1)$, this shows $d(x, F^{-1}(0)) \le \frac{d(0,F(x))}{r}(1 + \|a - x_0\|)$. The proof is complete. □

In the remainder, we assume that X and Y are Banach spaces.

Theorem 3.1. *Let $f : X \to Y$ be a piecewise linear function and let P_i, T_i and b_i ($i \in \overline{1m}$) be such that (2.1) holds. Suppose that $\gamma(f,K) := \min_{i\in\overline{1m}} \gamma(T_i,P_i,K) > 0$, where each $\gamma(T_i,P_i,K)$ is defined by (2.2). Then, for any $\eta \in (0,\ 1)$, $T \in \mathscr{L}(X,Y)$ with $\|T\| < (1-\eta)\gamma(f,K)$ and any $b \in Y$,*

$$d(x,S(f_{Tb},K)) \leq \frac{d(f_{Tb}(x),-K)}{\eta\gamma(f,K)} \quad \forall x \in X, \tag{3.4}$$

where

$$f_{Tb}(x) := f(x) + T(x) - b \quad \forall x \in X. \tag{3.5}$$

Proof. Let $(\eta,T,b) \in (0,\ 1) \times \mathscr{L}(X,Y) \times Y$ be such that $\|T\| < (1-\eta)\gamma(f,K)$. Then, there exists $r_0 \in (0,\ \gamma(f,K))$ such that $\|T\| < (1-\eta)r_0$. For any $r \in (r_0,\ \gamma(f,K))$ and $i \in \overline{1m}$, $\gamma(T_i,P_i,K) > r$, and hence there exists $h_i \in B_X \cap P_i^\infty$ such that $T_i(h_i) + rB_Y \subset -K$. Noting that $\|T(h_i)\| \leq \|T\| < (1-\eta)r$, it follows that

$$(T_i+T)(h_i) + \eta r B_Y \subset T_i(h_i) + rB_Y \subset -K \quad \forall i \in \overline{1m},$$

and hence

$$\eta r B_Y \subset T_i(h_i) + T(h_i) + K \quad \forall i \in \overline{1m}. \tag{3.6}$$

Take a point $\bar{x}_i$ in P_i and $t > 0$ such that $\|T_i(\bar{x}_i) + T(\bar{x}_i) - b_i - b\| < t$. Then $\bar{x}_i + \frac{th_i}{\eta r} \in P_i$ and

$$\begin{aligned}(T_i+T)\left(\bar{x}_i + \frac{th_i}{\eta r}\right) - b_i - b &= T_i(\bar{x}_i) + T(\bar{x}_i) - b_i - b + \frac{t}{\eta r}(T_i(h_i) + T(h_i)) \\ &= \frac{t}{\eta r}\left(T_i(h_i) + T(h_i) + \frac{\eta r(T_i(\bar{x}_i) + T(\bar{x}_i) - b_i - b)}{t}\right) \\ &\in -K.\end{aligned}$$

This shows that $\bar{x}_i + \frac{th_i}{\eta r} \in P_i \cap (T_i+T)^{-1}(b_i+b-K)$, and hence

$$P_i \cap (T_i+T)^{-1}(b_i+b-K) \neq \emptyset. \tag{3.7}$$

For each $i \in \overline{1m}$, we claim that

$$d\left(x, P_i \cap (T_i+T)^{-1}(b_i+b-K)\right) \leq \frac{1}{\eta r} d\big(T_i(x) + T(x) - b_i - b, -K\big) \quad \forall x \in P_i. \tag{3.8}$$

To prove this, let $F_i : X \rightrightarrows Y$ be such that

$$F_i(x) = \begin{cases} T_i(x) + T(x) - b_i - b + K, & \text{if } x \in P_i \\ \emptyset, & \text{if } x \in X \setminus P_i. \end{cases} \tag{3.9}$$

Then, since T_i and T are continuous linear operators, it is easy to verify that $\mathrm{gph}(F_i)$ is a closed convex set in $X \times Y$. Let x be an arbitrary point in P_i and γ be an arbitrary number in $(0,\ 1)$. Then, by (3.7) and Corollary 2.1 (applied to $S = P_i \cap (T_i+T)^{-1}(b_i+b-K)$), there exists $a_i \in P_i \cap (T_i+T)^{-1}(b_i+b-K)$ such that

$$\gamma t\|x-a_i\| \leq d(a_i + t(x-a_i), P_i \cap (T_i+T)^{-1}(b_i+b-K)) \quad \forall t \in [0,\ 1]. \tag{3.10}$$

Thus, $T_i(a_i)+T(a_i)\in b_i+b-K$, and so $T_i(a_i)+T(a_i)-b_i-b+K\supset K$. It follows from (3.6) that

$$\begin{aligned} F_i(a_i+B_X) &= (T_i+T)(a_i+B_X)-b_i-b+K \\ &\supset (T_i+T)(a_i+h_i)-b_i-b+K \\ &\supset T_i(h_i)+T(h_i)+K \\ &\supset \eta r B_Y. \end{aligned}$$

Hence, by [12, Theorem 2], one has

$$d(u,F_i^{-1}(0))\leq \frac{d(0,F_i(u))}{\eta r}(1+\|u-a_i\|)\quad \forall u\in X.$$

Noting that $F_i^{-1}(0)=P_i\cap(T_i+T)^{-1}(b_i+b-K)$ and

$$F_i(a_i+t(x-a_i))\supset tF_i(x)+(1-t)F_i(a_i)\supset tF_i(x)=t(T_i(x)+T(x)-b_i-b+K)$$

for all $t\in[0,\ 1]$, this and (3.10) imply that

$$\begin{aligned} \gamma t\|x-a_i\| &\leq \frac{d(0,t(T_i(x)+T(x)-b_i-b+K)}{\eta r}(1+\|a_i+t(x-a_i)-a_i\|) \\ &= \frac{td(0,T_i(x)+T(x)-b_i-b+K)}{\eta r}(1+t\|x-a_i\|) \end{aligned}$$

for all $t\in[0,\ 1]$. Hence

$$\begin{aligned} d(x,P_i\cap(T_i+T)^{-1}(b_i+b-K)) &\leq \|x-a_i\| \\ &\leq \frac{d(0,T_i(x)+T(x)-b_i-b+K)}{\gamma\eta r}(1+t\|x-a_i\|) \end{aligned}$$

for all $t\in[0,\ 1]$. Letting $t\to 0^+$, one has

$$\begin{aligned} d(x,P_i\cap(T_i+T)^{-1}(b_i+b-K)) &\leq \frac{d(0,T_i(x)+T(x)-b_i-b+K)}{\gamma\eta r} \\ &= \frac{d(T_i(x)+T(x)-b_i-b,-K)}{\gamma\eta r}. \end{aligned}$$

Since γ is arbitrary in $(0,\ 1)$, this shows that (3.8) holds. For any $x\in X$, there exists $i_0\in\overline{1m}$ such that $x\in P_{i_0}$. Hence, by (2.1) and (3.5),

$$\frac{d(f_{Tb}(x),-K)}{\eta r}=\frac{d(T_{i_0}(x)+T(x)-b_{i_0}-b,-K)}{\eta r}.$$

Noting that $S(f_{Tb},K)=\bigcup_{i=1}^{m}P_i\cap(T_i+T)^{-1}(b_i+b-K)$, it follows from (3.8) that

$$d(x,S(f_{Tb},K))\leq d(x,P_{i_0}\cap(T_{i_0}+T)^{-1}(b_{i_0}+b-K))\leq\frac{d(f_{Tb}(x),-K)}{\eta r}.$$

Since r is arbitrary in $(r_0,\ \gamma(f,K))$,

$$d(x,S(f_{Tb},K))\leq d(x,P_{i_0}\cap(T_{i_0}+T)^{-1}(b_{i_0}+b-K))\leq\frac{d(f_{Tb}(x),-K)}{\eta\gamma(f,K)}.$$

The proof is complete. $\square$

Under the boundedness assumption on the solution set, we have the following sufficient condition for the stability of global error bounds for piecewise linear conic inequalities.

Theorem 3.2. *Let f, P_i and (T_i,b_i) $(i\in\overline{1m})$ be such that (2.1) holds. Suppose that*

$$M:=\sup\{\|x\|:\ x\in S(f,K)\}<+\infty \ \text{ and } \ \widetilde{\gamma}(f,K):=\min_{i\in\overline{1m}}\gamma(T_i,P_i,b_i,K)>0,$$

where each $\gamma(T_i,P_i,b_i,K)$ is defined by (2.3). Then, for any $\eta\in(0,\ 1)$,

$$d(x,S(f_{Tb},K))\leq\frac{(6M+2)}{\eta\widetilde{\gamma}(f,K)}d(f_{Tb}(x),-K)\quad\forall x\in X \tag{3.11}$$

whenever $(T,b)\in\mathscr{L}(X,Y)\times Y$ satisfies $\|T\|+\|b\|<\frac{(1-\eta)\widetilde{\gamma}(f,K)}{6M+1}$, where f_{Tb} is as (3.5).

Proof. Let $(\eta,T,b)\in(0,\ 1)\times\mathscr{L}(X,Y)\times Y$ be such that $\|T\|+\|b\|<\frac{(1-\eta)\widetilde{\gamma}(f,K)}{6M+1}$. Take $r\in(0,\ \widetilde{\gamma}(f,K))$ sufficiently close to $\widetilde{\gamma}(f,K)$ such that

$$\|T\|+\|b\|<\frac{(1-\eta)r}{6M+1}. \tag{3.12}$$

Then, by the definition of $\widetilde{\gamma}(f,K)$, for each $i\in\overline{1m}$ there exists $\bar{u}_i\in S(f,K)\cap P_i$ such that $B_Y(T_i(\bar{u}_i)-b_i,r)\subset -K$, and so

$$rB_Y\subset T_i(\bar{u}_i)-b_i+K. \tag{3.13}$$

We claim that

$$\sup\{\|u\|:\ u\in S(f_{Tb},K)\}\leq\max\{5M,1\}. \tag{3.14}$$

Indeed, if this is not the case, there exists $u\in S(f_{Tb},K)$ such that

$$\|u\|>\max\{5M,1\}. \tag{3.15}$$

Then $\frac{\|u\|}{6M}\leq\frac{\|u\|-2M}{3M}$. Since $X=\bigcup\limits_{i=1}^{m}P_i$, there exists $i\in\overline{1m}$ such that $u\in P_i$. Hence $f_{Tb}(u)=(T_i+T)(u)-(b_i+b)\in -K$ and

$$T_i\left(\bar{u}_i+\frac{3M(u-\bar{u}_i)}{\|u\|+M}\right)-b_i=\frac{\|u\|-2M}{\|u\|+M}(T_i(\bar{u}_i)-b_i)+\frac{3M}{\|u\|+M}(T_i(u)-b_i).$$

By (3.12), one has

$$\begin{aligned}T_i(u)-b_i &\in (T_i+T)(u)-(b_i+b)+(\|T\|\|u\|+\|b\|)B_Y\\ &\subset -K+\frac{\|u\|(1-\eta)rB_Y(0,1)}{6M}\\ &\subset -K+\frac{(\|u\|-2M)(1-\eta)rB_Y(0,1)}{3M},\end{aligned}$$

and so

$$T_i\left(\bar{u}_i+\frac{3M(u-\bar{u}_i)}{\|u\|+M}\right)-b_i\in\frac{\|u\|-2M}{\|u\|+M}(T_i(\bar{u}_i)-b_i+B_Y(0,(1-\eta)r))-K.$$

This and (3.13) imply that $T_i\left(\bar{u}_i+\frac{3M(u-\bar{u}_i)}{\|u\|+M}\right)-b_i\in -K$. Noting that $u,\bar{u}_i\in P_i$ and $\frac{3M}{\|u\|+M}\in (0,\ 1)$ (thanks to (3.15)), one has $\bar{u}_i+\frac{3M(u-\bar{u}_i)}{\|u\|+M}\in P_i$. Hence

$$f\left(\bar{u}_i+\frac{3M(u-\bar{u}_i)}{\|u\|+M}\right)\in -K \text{ and } \bar{u}_i+\frac{3M(u-\bar{u}_i)}{\|u\|+M}\in S(f,K).$$

It follows from the definition of M that

$$M\geq\left\|\bar{u}_i+\frac{3M(u-\bar{u}_i)}{\|u\|+M}\right\|\geq\frac{3M\|u-\bar{u}_i\|}{\|u\|+M}-\|\bar{u}_i\|\geq\frac{3M(\|u\|-M)}{\|u\|+M}-M,$$

contradicting (3.15). This shows that (3.14) holds. For each $i\in\overline{1m}$, let F_i be as in (3.9). Then, by (3.13),

$$F_i(\bar{u}_i)=T_i(\bar{u}_i)+T(\bar{u}_i)-b_i-b+K\supset B_Y(0,r)+T(\bar{u}_i)-b. \tag{3.16}$$

By the definition of M and $\bar{u}_i\in S(f,K)$, one has

$$\|T(\bar{u}_i)-b\|\leq\|T\|\|\bar{u}_i\|+\|b\|\leq(\|T\|+\|b\|)(\|\bar{u}_i\|+1)\leq(1+M)(\|T\|+\|b\|).$$

This and (3.12) imply that $\|T(\bar{u}_i)-b\|\leq\frac{(1+M)(1-\eta)r}{6M+1}<(1-\eta)r$. Thus, by (3.16), one has $F_i(\bar{u}_i)\supset B_Y(0,\eta r)$. This and Lemma 3.1 imply that for any $x\in X$ there exists $u_i\in F_i^{-1}(0)$ such that

$$d(x,F_i^{-1}(0))\leq\frac{d(0,F_i(x)))}{\eta r}(1+\|u_i-\bar{u}_i\|).$$

Since $F_i^{-1}(0)=P_i\cap S(f_{Tb},K)$, we have $\|u_i\|\leq 5M+1$. Noting that $\|\bar{u}_i\|\leq M$, it follows that

$$d(x,P_i\cap S(f_{Tb},K))\leq\frac{(6M+2)d(0,F_i(x)))}{\eta r}\quad\forall x\in X.$$

By the definition of f_{Tb} and F_i, one has

$$F_i(x)=T_i(x)+T(x)-b_i-b+K=f_{Tb}(x)+K\quad\forall x\in P_i.$$

It follows that $d(x,S(f_{Tb},K))\leq\frac{(6M+2)d(f_{Tb}(x),-K)}{\eta r}$ for all $x\in P_i$. Thus, since $X=\bigcup_{i=1}^{m}P_i$,

$$d(x,S(f_{Tb},K))\leq\frac{(6M+2)d(f_{Tb}(x),-K)}{\eta r}\quad\forall x\in X.$$

Noting that r is arbitrary in $(0,\ \widetilde{\gamma}(f,K))$, this shows that (3.11) holds. The proof is complete. □

4. Convex Case

In Theorem 3.1, the assumption

$$\gamma(f,K):=\min_{i\in\overline{1m}}\gamma(T_i,P_i,K)>0$$

means that there exist $r>0$ and $h_i\in P_i^\infty$ ($i\in\overline{1m}$) such that

$$B_Y(T_i(h_i),r)\subset -K\quad\forall i\in\overline{1m}.$$

For each $i\in\overline{1m}$, take a point $\bar{x}_i$ in P_i and $t>0$ such that $\|T_i(\bar{x}_i)-b_i\|<t$. Then $\bar{x}_i+\frac{th_i}{r}\in P_i$ and

$$f\left(\bar{x}_i+\frac{th_i}{r}\right)=T_i(\bar{x}_i)-b_i+\frac{t}{r}T_i(h_i)=\frac{t}{r}\left(T_i(h_i)+\frac{r(T_i(\bar{x}_i)-b_i)}{t}\right)\in -K.$$

It follows that $\bar{x}_i + \frac{th_i}{r} \in P_i \cap S(f,K)$ for all $i \in \overline{1m}$. Hence the assumption $\gamma(f,K) > 0$ implies

$$P_i \cap S(f,K) \neq \emptyset \quad \forall i \in \overline{1m}. \tag{4.1}$$

Similar to the above proof, it is easy to verify that the assumption

$$\widetilde{\gamma}(f,K) := \min_{i \in \overline{1m}} \gamma(T_i, P_i, b_i, K) > 0$$

in Theorem 3.2 also implies (4.1), which is restrictive. Let

$$I(f,K) := \{i \in \overline{1m} : \; P_i \cap S(f,K) \neq \emptyset\},$$

where f and P_i is as in (2.1). It is natural to ask the following question: *Are Theorems 3.1 and 3.2 still true if* $\gamma(f,K) > 0$ *and* $\widetilde{\gamma}(f,K) > 0$ *are weakened respectively to* $\min\limits_{i \in I(f,K)} \gamma(T_i, P_i, K) > 0$ *and* $\min\limits_{i \in I(f,K)} \gamma(T_i, P_i, b_i, K) > 0$?

The following examples show that the answer to this question is in general negative.

Example 4.1. Let $X = Y = \mathbb{R}$, $K = \mathbb{R}_+$ and define $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} x & x \in P_1 := (-\infty, \, 1], \\ 1 & x \in P_2 := [1, \, +\infty). \end{cases}$$

Then f is piecewise linear, $I(f,K) = \{1\}$, $-1 \in P_1^\infty$ and $\min\limits_{i \in I(f,K)} \gamma(T_i, P_i, K) = 1 > 0$. On the other hand, since $S(f,K) = (-\infty, \, 0]$, $d(k, S(f,K)) = k$ and $d(f(k), -K) = 1$ for all $k \in \mathbb{N}$. This shows that (3.4) does not hold with $T = 0$ and $b = 0$. Hence Theorem 3.1 does not necessarily hold if $\gamma(f,K) > 0$ is weakened to $\min\limits_{i \in I(f,K)} \gamma(T_i, P_i, K) > 0$.

Example 4.2. Let $X = Y = \mathbb{R}$, $K = \mathbb{R}_+$, $P_1 = (-\infty, -1]$, $P_2 = [-1, 1]$, and $P_3 = [1, +\infty)$. Define $f : X \to Y$ by

$$f(x) = \begin{cases} -x-2 & x \in P_1, \\ x & x \in P_2, \\ 1 & x \in P_3. \end{cases}$$

Then f is piecewise linear, $S(f,K) = [-2, 0]$ and $I(f,K) = \{1, 2\}$. It follows that

$$\begin{aligned} \gamma(T_1, P_1, b_1, K) &= \sup_{x \in T_1^{-1}(b_1 - \mathrm{int}(K)) \cap P_1} \sup\{r > 0 : \; B_Y(T_1 x - b_1, r) \subset -K\} \\ &= \sup_{x \in (-2, -1]} \sup\{r > 0 : \; (-x-2-r, \; -x-2+r) \subset (-\infty, 0)\} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \gamma(T_2, P_2, b_2, K) &= \sup_{x \in T_2^{-1}(b_2 - \mathrm{int}(K)) \cap P_2} \sup\{r > 0 : \; B_Y(T_2 x - b_2, r) \subset -K\} \\ &= \sup_{x \in [-1, 0)} \sup\{r > 0 : \; (x - r, \; x + r) \subset (-\infty, 0)\} \\ &= 1, \end{aligned}$$

where $T_1(x) = -T_2(x) = -x$ for all $x \in \mathbb{R}$, $b_1 = 2$ and $b_2 = 0$. Hence

$$\min_{i \in I(f,K)} \gamma(T_i, P_i, b_i, K) = 1 > 0.$$

On the other hand, $d(k, S(f,K)) = k$ and $d(f(k), -K) = 1$ for all $k \in \mathbb{N}$. This shows that (3.11) does not hold with $T = 0$ and $b = 0$. Hence Theorem 3.2 does not necessarily hold if $\tilde{\gamma}(f,K) > 0$ is weakened to $\min\limits_{i \in I(f,K)} \gamma(T_i, P_i, b_i, K) > 0$.

However, under the K-convexity assumption on f, we will show that Theorem 3.2 is still true even if $\tilde{\gamma}(f,K) > 0$ is weakened to $\max\limits_{i \in I(f,K)} \gamma(T_i, P_i, b_i, K) > 0$ (i.e. $0 \in \mathrm{int}(f(X) + K)$).

Theorem 4.1. *Let X and Y be Banach spaces, K be a closed convex cone, and let $f: X \to Y$ be a closed K-convex function (i.e. $\mathrm{epi}_K(f) := \{(x,y) : f(x) \le_K y\}$ is a closed convex set in $X \times Y$). Suppose that $S(f,K)$ is bounded. Then $0 \in \mathrm{int}(f(X)+K)$ if and only if there exist $\kappa, \delta \in (0, +\infty)$ such that, for any $(T,b) \in \mathscr{L}(X,Y) \times Y$ with $\|T\| + \|b\| < \delta$,*

$$d(x, S(f_{Tb}, K)) \le \kappa d(0, f_{Tb}(x) + K) \quad \forall x \in X. \tag{4.2}$$

Proof. First suppose that there exist $\kappa, \delta \in (0, +\infty)$ such that (4.2) holds for any $(T,b) \in \mathscr{L}(X,Y) \times Y$ with $\|T\| + \|b\| < \delta$. Then, setting $T = 0$, $S(f_{0b}, K) \neq \emptyset$ for all $b \in B_Y(0,\delta)$, that is, $B_Y(0,\delta) \subset f(X) + K$. Hence $0 \in \mathrm{int}(f(X)+K)$.

Now suppose that $0 \in \mathrm{int}(f(X)+K)$. Then, by the Robinson-Ursescu theorem, there exist $\bar{u} \in X$ and $r > 0$ such that

$$0 \in f(\bar{u}) + K \text{ and } rB_Y \subset f(\bar{u} + B_X) + K. \tag{4.3}$$

Let $(\eta, T, b) \in (0, \frac{1}{2}) \times \mathscr{L}(X,Y) \times Y$ be such that

$$\|T\| + \|b\| < \frac{\eta r}{2M+1}, \tag{4.4}$$

where $M := \sup\{\|u\| : u \in S(f,K)\}$. To prove the necessity part, it suffices to show that

$$d(x, S(f_{Tb}, K)) \le \frac{(2M+1) d(0, f_{Tb}(x) + K)}{(1-2\eta)^2 r} \quad \forall x \in X. \tag{4.5}$$

Noting that $\bar{u} \in S(f,K)$, one has

$$\|T(\bar{u}) - b\| \le \|T\| \|\bar{u}\| + \|b\| \le \|T\| M + \|b\| \le \frac{\eta r(1+M)}{2M+1} < \eta r.$$

Thus, by (4.3),

$$(1-\eta) r B_Y \subset f(\bar{u} + B_X) + T(\bar{u}) - b + K \subset G_{Tb}(\bar{u} + B_X), \tag{4.6}$$

where

$$G_{Tb}(x) := f_{Tb}(x) + K = f(x) + T(x) - b + K \quad \forall x \in X. \tag{4.7}$$

Since $\mathrm{gph}_K(f)$ is a closed convex set, it is easy to verify that G_{Tb} is a closed convex multifunction. Let x be an arbitrary element in X. Then, by (4.6) and Lemma 3.1, there exists $a \in G_{Tb}^{-1}(0) = S(f_{Tb}, K)$ such that

$$d(x, G_{Tb}^{-1}(0)) \le \frac{d(0, G_{Tb}(x))}{(1-\eta) r} (1 + \|a - \bar{u}\|).$$

Since $d(x, S(f_{Tb}, K)) = d(x, G_{Tb}^{-1}(0))$ and $d(0, G_{Tb}(x)) = d(0, f_{Tb}(x) + K)$, one has

$$d(x, S(f_{Tb}, K)) \leq \frac{d(0, f_{Tb}(x) + K)}{(1-\eta)r}(1 + \|a - \bar{u}\|). \tag{4.8}$$

We claim that

$$\sup\{\|u\| : u \in S(f_{Tb}, K)\} \leq \frac{\eta + (1-\eta)M}{1-2\eta}. \tag{4.9}$$

Granting this and noting that $a \in S(f_{Tb}, K)$ and $\bar{u} \in S(f, K)$, (4.8) implies that

$$\begin{aligned} d(x, S(f_{Tb}, K)) &\leq \frac{d(0, f_{Tb}(x) + K)}{(1-\eta)r}(1 + \|a\| + \|\bar{u}\|) \\ &\leq \frac{d(0, f_{Tb}(x) + K)}{(1-\eta)r}\left(1 + \frac{\eta + (1-\eta)M}{1-2\eta} + M\right) \\ &\leq \frac{(2M+1)d(0, f_{Tb}(x) + K)}{(1-2\eta)^2 r}. \end{aligned}$$

This shows that (4.5) holds. It remains to prove that (4.9) holds. To prove this, set $T = 0$ and $b = 0$ in (4.8). Then, $a \in S(f, K)$ and hence

$$d(x, S(f, K)) \leq \frac{d(0, f(x) + K)}{(1-\eta)r}(1 + \|a\| + \|\bar{u}\|) \leq \frac{(2M+1)d(0, f(x) + K)}{(1-\eta)r} \quad \forall x \in X.$$

Let u be an arbitrary point in $S(f_{Tb}, K)$. Then $b - T(u) \in f(u) + K$ and so

$$d(u, S(f, K)) \leq \frac{(2M+1)\|b - T(u)\|}{(1-\eta)r} \leq \frac{(2M+1)(\|b\| + \|T\|)(1 + \|u\|)}{(1-\eta)r}.$$

This and (4.4) imply that $d(u, S(f, K)) \leq \frac{\eta(1+\|u\|)}{1-\eta}$. By the definition of M, one has $\|u\| - M \leq d(u, S(f, K))$. Hence $\|u\| - M \leq \frac{\eta(1+\|u\|)}{1-\eta}$, that is, $\|u\| \leq \frac{\eta + (1-\eta)M}{1-2\eta}$. Since u is arbitrary in $S(f_{Tb}, K)$, this shows that (4.9) holds. The proof is complete. $\square$

For two sets A_1 and A_2 in a Banach space X, we use $\mathscr{H}(A_1, A_2)$ to denote the Hausdorff distance between A_1 and A_2, that is,

$$\mathscr{H}(A_1, A_2) := \max\left\{\sup\{d(a_1, A_2) : a_1 \in A_1\}, \quad \sup\{d(a_2, A_1) : a_2 \in A_2\}\right\}.$$

The following theorem establishes the continuity of solution sets.

Theorem 4.2. *Let X and Y be Banach spaces, K be a closed convex cone, and let $f : X \to Y$ be a closed K-convex function. Suppose that $S(f, K)$ is bounded and $0 \in \operatorname{int}(f(X) + K)$. Then there exist $L, \delta \in (0, +\infty)$ such that*

$$\mathscr{H}(S(f_{T'b'}, K), S(f_{Tb}, K)) \leq L(\|T - T'\| + \|b - b'\|) \tag{4.10}$$

for all $T, T' \in B_{\mathscr{L}(X,Y)}(0, \delta)$ and $b, b' \in B_Y(0, \delta)$.

Proof. By Theorem 4.1, there exist $\kappa, \delta \in (0, +\infty)$ such that for all $T \in B_{\mathscr{L}(X,Y)}(0, \delta)$ and $b \in B_Y(0, \delta)$

$$d(x, S(f_{Tb}, K) \leq \kappa d(0, f_{Tb}(x) + K) \quad \forall x \in X.$$

By (4.9), without loss of generality, we can assume that

$$\beta := \sup\{M_{Tb} : \; T \in B_{\mathscr{L}(X,Y)}(0, \delta) \text{ and } b \in B_Y(0, \delta)\} < +\infty,$$

where $M_{Tb} := \sup\{\|x\| : x \in S(f_{Tb}, K)\}$. Let $T, T' \in B_{\mathscr{L}(X,Y)}(0,\delta)$ and $b, b' \in B_Y(0,\delta)$. Noting that $T(u') - b - (T'(u') - b') \in f_{Tb}(u') + K$ for any $u' \in S(f_{T'b'}, K)$, one has

$$\begin{aligned} d(u', S(f_{Tb}, K)) &\leq \kappa\|T(u') - b - (T'(u') - b')\| \\ &\leq \kappa(1 + \|u'\|)(\|T - T'\| + \|b - b'\|) \\ &\leq \kappa(1 + \beta)(\|T - T'\| + \|b - b'\|) \end{aligned}$$

for any $u' \in S(f_{T'b'}, K)$. Similarly, one also has

$$d(u, S(f_{T'b'}, K)) \leq \kappa(1 + \beta)(\|T - T'\| + \|b - b'\|) \quad \forall u \in S(f_{Tb}, K).$$

It follows that (4.10) holds with $L = \kappa(1 + \beta)$. □

Corollary 4.1. *Let f, P_i and (T_i, b_i) $(i \in \overline{1m})$ be such that (2.1) holds. Suppose that f is K-convex and that $\widetilde{\gamma}(f, K) := \max\limits_{i \in I(f,K)} \gamma(T_i, P_i, b_i, K) > 0$, where each $\gamma(T_i, P_i, b_i, K)$ is defined by (2.3). Further suppose that X is finite dimensional and $S(f, K)$ is bounded. Then exists $\delta \in (0, +\infty)$ such that*

$$S(f_{Tb}, K) \subset \bigcup_{i \in I(f,K)} P_i \quad \forall (T, b) \in \mathscr{L}(X, Y) \times Y \text{ with } \|T\| + \|b\| < \delta, \tag{4.11}$$

where $I(f, K) := \{i \in \overline{1m} : S(f, K) \cap P_i \neq \emptyset\}$.

Proof. By the definition of $I(f, K)$, one has $S(f, K) \cap \bigcup\limits_{i \in \overline{1m} \setminus I(f,K)} P_i = \emptyset$. Noting that $S(f, K)$ is a bounded closed set and $\bigcup\limits_{i \in \overline{1m} \setminus I(f,K)} P_i$ is a closed set in the finite dimensional space X, it follows that $r := d\left(S(f, K), \bigcup\limits_{i \in \overline{1m} \setminus I(f,K)} P_i\right) > 0$. Hence, by the first equality in (2.1),

$$S(f, K) + B_X(0, r) \subset \bigcup_{i \in I(f,K)} P_i. \tag{4.12}$$

By Theorem 4.2, there exist $L, \delta \in (0, +\infty)$ such that $L\delta < r$ and (4.10) holds for all $T, T' \in B_{\mathscr{L}(X,Y)}(0, \delta)$ and $b, b' \in B_Y(0, \delta)$. Hence, setting $(T', b') = (0, 0)$ in (4.10), one has

$$\mathscr{H}(S(f, K), S(f_{Tb}, K)) \leq L(\|T\| + \|b\|) < r \quad \forall T \in B_{\mathscr{L}(X,Y)}(0, \delta) \text{ and } b \in B_Y(0, \delta).$$

This and (4.12) imply that (4.11) holds. □

Relaxing the boundedness assumption on the solution set $S(f, K)$, we have the following result.

Theorem 4.3. *Let f, P_i and (T_i, b_i) $(i \in \overline{1m})$ be such that (2.1) holds. Suppose that f is K-convex and that there exists $i_0 \in \overline{1m}$ such that $\gamma(T_{i_0}, P_{i_0}, K) > 0$, where $\gamma(T_{i_0}, P_{i_0}, K)$ is defined by (2.2). Further suppose that K be a convex polyhedral cone. Then, for any $\eta \in (0, 1)$, $b \in Y$ and $T \in \mathscr{L}(X, Y)$ with $\|T\| < (1 - \eta)\gamma(T_{i_0}, P_{i_0}, K)$, there exists $\kappa_{Tb} \in (0, +\infty)$ such that*

$$d(x, S(f_{Tb}, K)) \leq \kappa_{Tb} d(f_{Tb}(x), -K) \quad \forall x \in X. \tag{4.13}$$

Proof. Let $(\eta,T,b)\in(0,1)\times\mathscr{L}(X,Y)\times Y$ be such that $\|T\|<(1-\eta)\gamma(T_{i_0},P_{i_0},K)$. Then, there exist $r\in(0,\ \gamma(T_{i_0},P_{i_0},K))$ and $h_{i_0}\in B_X\cap P_{i_0}^{\infty}$ such that $\|T\|<(1-\eta)r$ and $B_Y(0,r)\subset T_{i_0}(h_{i_0})+K$. Hence $\|T(h_{i_0})\|<(1-\eta)r$ and

$$B_Y(0,\eta r)\subset T_{i_0}(h_{i_0})+T(h_{i_0})+K. \tag{4.14}$$

Thus, similar to (3.7) in the proof of Theorem 3.1, one has

$$P_{i_0}\cap S(f_{Tb},K)=P_{i_0}\cap(T_{i_0}+T)^{-1}(b_{i_0}+b-K)\neq\emptyset.$$

Since K is a convex polyhedral cone in Y, b_i+b-K is a convex polyhedron in Y and hence there exist $(y_1^*,t_1),\cdots,(y_k^*,t_k)\in Y^*\times\mathbb{R}$ such that

$$b_i+b-K=\{y\in Y:\ \langle y_j^*,y\rangle\le t_j,\ j=1,\cdots,k\}.$$

Hence

$$(T_i+T)^{-1}(b_i+b-K)=\{x\in X:\ \langle (T_i+T)^*(y_j^*),x\rangle\le t_j,\ j=1,\cdots,k\}.$$

Since P_i is a convex polyhedron in X, $P_i\cap(T_i+T)^{-1}(b_i+b-K)$ is a convex polyhedron in X. By the K-convexity of f, $S(f_{Tb},K)$ is closed convex set in X. Noting that

$$S(f_{Tb},K)=\bigcup_{i=1}^{m}P_i\cap(T_i+T)^{-1}(b_i+b-K),$$

it follows from [10, Theorem 19.6] that $S(f_{Tb},K)$ is a convex polyhedron in X. Thus, by Lemma 2.1, there exist $u_1,\cdots,u_k\in S(f_{Tb},K)$ such that

$$S(f_{Tb},K)=\mathrm{co}\{u_1,\cdots,u_k\}+S(f_{Tb},K)^{\infty}. \tag{4.15}$$

Take a $\bar u_{i_0}\in P_{i_0}\cap S(f_{Tb},K)$. Then,

$$f_{Tb}(\bar u_{i_0})=T_{i_0}(\bar u_{i_0})+T(\bar u_{i_0})-b_{i_0}-b\in -K,$$

and so $K\subset T_{i_0}(\bar u_{i_0})+T(\bar u_{i_0})-b_{i_0}-b+K$. This and (4.14) imply that

$$\begin{aligned}G_{Tb}(\bar u_{i_0}+h_{i_0})&=(T_{i_0}+T)(\bar u_{i_0}+h_{i_0})-b_{i_0}-b+K\\&\supset T_{i_0}(h_{i_0})+T(h_{i_0})+K\\&\supset B_Y(0,\eta r),\end{aligned}$$

where G_{Tb} is as in (4.7). Thus, by (4.15) and Lemma 3.1, for any $x\in X$, there exists $a\in\mathrm{co}\{u_1,\cdots,u_k\}$ such that

$$d(x,G_{Tb}^{-1}(0))\le\frac{d(0,G_{Tb}(x))}{\eta r}(1+\|a-\bar u_{i_0}-h_{i_0}\|)\le\tau_{Tb}\frac{d(0,G_{Tb}(x))}{\eta r},$$

where $\tau_{Tb}:=\sup\{\|u\|:\ u\in\mathrm{co}\{u_1,\cdots,u_k\}\}+\|\bar u_{i_0}\|+2$. Hence (4.13) holds with $\kappa_{Tb}=\frac{\tau_{Tb}}{\eta r}$. □

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