J. Nonlinear Var. Anal. 7 (2023), No. 1, pp. 1-22
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.7.2023.1.01

ACCELERATED CYCLIC ITERATIVE ALGORITHMS FOR THE MULTIPLE-SET SPLIT COMMON FIXED-POINT PROBLEM OF QUASI-NONEXPANSIVE OPERATORS

JING ZHAO[1], HAIXIA WANG[1], NINGNING ZHAO[2,*]

[1]*College of Science, Civil Aviation University of China, Tianjin 300300, China*
[2]*College of Air Traffic Management, Civil Aviation University of China, Tianjin 300300, China*

Abstract. In the paper, we introduce two accelerated cyclic iterative algorithms for solving the multiple-set split common fixed-point problem of quasi-nonexpansive operators in real Hilbert spaces. Inspired by the primal-dual algorithm, our proposed algorithms combine inertial technique with the self-adaptive stepsizes such that the implementation of the algorithms does not need any prior information about bounded linear operator norms. The weak and strong convergence of the proposed algorithms are established under suitable assumptions. As applications, we obtain several iterative algorithms to solve the multiple-set split feasibility problem. Finally, numerical results are included to demonstrate the efficiency of the proposed iterative algorithms.
Keywords. The multiple-set split common fixed-point problem, quasi-nonexpansive operators, inertial technique, the weak and strong convergence.

1. INTRODUCTION

The split feasibility problem (SFP) was first introduced by Censor and Elfving [1] for modelling some inverse problems. Since then, it has played an important role in many real-world problems, such as medical image reconstruction [2] and intensity-modulated radiation therapy [3, 4]. Some generalizations of the SFP have also been studied, such as the multiple-set split feasibility problem (MSFP), the split common fixed point problem (SCFP), the multiple-set split common fixed point problem (MSCFP), the split equality common fixed-point problem (SECFP), and the multiple-set split equality common fixed-point problem (MSECFP). The MSCFP is depicted to find a point in the intersection of a family of fixed point sets such that its image under a bounded linear operator belongs to the intersection of another family of fixed point sets. Let H_1 and H_2 be real Hilbert spaces, and let $A : H_1 \to H_2$ be a bounded linear operator. Given integers $p,\ r \geq 1$, the MSCFP is formulated as finding a point x^* satisfying the

*Corresponding author.
E-mail addresses: zhaojing200103@163.com (J. Zhao), whx1224794892@163.com (H. Wang), xianyuer315@163.com (N. Zhang).
Received April 14, 2022; Accepted September 7, 2022.

property:

$$x^* \in \bigcap_{i=1}^{p} F(U_i) \text{ such that } Ax^* \in \bigcap_{j=1}^{r} F(T_j),$$

where, for $1 \le i \le p$ and $1 \le j \le r$, $F(U_i)$ and $F(T_j)$ are the fixed point sets of $U_i : H_1 \to H_1$ and $T_j : H_2 \to H_2$, respectively. In particular, if $p = r = 1$, then the MSCFP becomes the SCFP which was originally introduced by Censor and Segal [5] in finite-dimensional Hilbert space. The SCFP is to find

$$x^* \in F(U) \text{ such that } Ax^* \in F(T),$$

where $F(U)$ and $F(T)$ are the fixed point sets of $U : H_1 \to H_1$ and $T : H_2 \to H_2$, respectively. For $1 \le i \le p$ and $1 \le j \le r$, when U_i and T_j are the projection operators on the nonempty closed convex subsets C_i and Q_j, respectively, the MSCFP becomes the MSFP which is depicted as finding

$$x^* \in \bigcap_{i=1}^{p} C_i \text{ such that } Ax^* \in \bigcap_{j=1}^{r} Q_j.$$

The above problem was first introduced by Censor et al. [6]. If $p = r = 1$, then the MSFP is reduced to the SFP which is described as finding

$$x^* \in C \text{ such that } Ax^* \in Q,$$

where C and Q are the nonempty closed convex subsets of H_1 and H_2, respectively. Such problem arises in the field of intensity-modulated radiation therapy when one attempts to describe physical dose constraints and equivalent uniform dose constraints within a single model. Note that, if the SFP is consistent, then x^* is a solution to the SFP if and only if it is a solution to the following fixed point equation:

$$x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*,$$

where P_C and P_Q are the projections onto C and Q, respectively, $\gamma > 0$ is any positive constant, and A^* denotes the adjoint of A. For solving the SFP, Byrne [2] proposed the well-known CQ algorithm which generates iterative sequence $\{x_k\}$ by

$$x_{k+1} = P_C(I - \gamma_k A^*(I - P_Q)A)x_k, \tag{1.1}$$

where, for all $k \ge 1$, $\gamma_k \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A. It is observed that, in this algorithm, the stepsize γ_k depends on the bounded linear operator (matrix) norm $\|A\|$ (or the largest eigenvalue of A^*A). It is not always easy in practice to compute the matrix norm of A. To avoid this difficulty, there have been many self-adaptive algorithms that the stepsize does not depend on the norm of the bounded linear operator A. In [7], Lopez et al. improved CQ algorithm (1.1), which selects the stepsize by the following way:

$$\gamma_k = \frac{\rho_k f(x_k)}{\|\nabla f(x_k)\|^2},$$

where $\inf_k \rho_k(4 - \rho_k) > 0$ and $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$.

In 2009, for solving the SCFP, Censor and Segal [5] replaced projection operators P_C and P_Q with directed operators U and T, respectively, and CQ-algorithm (1.1) becomes the following iterative scheme:

$$x_{k+1} = U(x_k - \gamma_k A^*(I - T)Ax_k),$$

where $\gamma_k \in (0, \frac{2}{\|A\|^2})$.

In [8], a self-adaptive priml-dual iterative algorithm was proposed for solving the SCFP of the averaged operators U and T, where U is $\alpha_1 - averaged$ and T is $\alpha_2 - averaged$. The sequence $\{x_k\}$ is generated by the following way:

$$\begin{cases} y_k = x_k - \gamma_k A^*(I-T)Ax_k, \\ \omega_{k+1} = (I-U)(y_k + (1-\lambda)\omega_k), \\ x_{k+1} = y_k - \lambda \omega_{k+1}, \end{cases}$$

where γ_k is chosen by

$$\gamma_k := \begin{cases} \frac{\rho_k \|(I-T)Ax_k\|^2}{\|A^*(I-T)Ax_k\|^2}, & (I-T)Ax_k \neq 0, \\ \gamma, & (I-T)Ax_k = 0 \end{cases}$$

with $\gamma > 0$ and $0 < \rho_k < \frac{1}{\alpha_2}$. Under appropriate conditions, the sequence $\{x_k\}$ converges weakly to a solution of the SCFP. For recent results on the SFP, the MSFP, the SCFP and the MSCFP, one refers to [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] and the references therein.

In [20], Wang and Xu proposed the following cyclic iterative algorithm for solving the MSCFP of directed operators:

$$x_{k+1} = U_{[k]_1}(x_k + \gamma A^*(T_{[k]_2} - I)Ax_k), \tag{1.2}$$

where $0 < \gamma < 2/\rho(A^*A)$, $[k]_1 := k(\mod p)$, and $[k]_2 := k(\mod r)$. They proved the weak convergence of the sequence $\{x_k\}$ generated by (1.2). In [21], Reich, Tuyen and Trang introduced a parallel iterative algorithm. For $1 \leq i \leq p$ and $1 \leq j \leq r$, let U_i and T_j be nonexpansive operators. Two sequences $\{a_{i,k}\}$ and $\{b_{j,k}\}$ are taken in $[a,b] \subset (0,1)$ and $\sum_{i=1}^{p} a_{i,k} = \sum_{j=1}^{r} b_{j,k} = 1$ for all $k \geq 1$. The iterative sequence $\{x_k\}$ is generated by the following way:

$$\begin{cases} y_k = \sum_{i=1}^{p} a_{i,k} \tilde{U}_i x_k, \\ x_{k+1} = \alpha_k u + (1-\alpha_k)(y_k + \sum_{j=1}^{r} b_{j,k} \delta A^*(I-T_j)Ay_k), \end{cases}$$

where $\delta \in (0, \frac{1}{\|A\|^2})$, $u \in H_1$, and $\tilde{U}_i = \beta_{i,k} I + (1-\beta_{i,k})U_i$, $\{\beta_{i,k}\} \subset [c,d] \subset (0,1)$ for $1 \leq i \leq p$. If $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$, then sequence $\{x_k\}$ converges strongly to $x^* = P_\Gamma u$, where Γ denotes the solution set of the MSCFP.

In optimization theory, the inertial technique is an important method to speed up the convergence rate. In [22], Dang, Sun and Xu proposed the inertial relaxed CQ algorithm for solving the SFP in Hilbert space, which is formulated as

$$\begin{cases} y_k = x_k + \alpha_k(x_k - x_{k-1}), \\ x_{k+1} = P_{C_k}(y_k - \gamma_k A^*(I - P_{Q_k})Ay_k), \end{cases}$$

where $\gamma_k \in (0, \frac{2}{\lambda})$ for all $k \geq 1$ and λ is the spectral radius of the operator A^*A.

For solving the MSCFP, Thong and Hieu [23] combined the Mann iteration with the inertial method and proposed the following iterative algorithm:

$$\begin{cases} y_k = x_k + \alpha_k(x_k - x_{k-1}), \\ x_{k+1} = (1-\beta_k)y_k + \beta_k \sum_{i=1}^{p} \omega_i U_i(I - \sum_{j=1}^{r} \eta_j \gamma A^*(T_j - I)A)y_k, \end{cases} \tag{1.3}$$

where $U_i : H_1 \to H_1$ $(1 \leq i \leq p)$ are quasi-nonexpansive operators and $T_j : H_2 \to H_2$ $(1 \leq j \leq r)$ are demicontractive operators. They proved that the sequence $\{x_k\}$ generated by (1.3) converges weakly to a solution of the MSCFP under approximate conditions. For recent inertial accelerated iterative algorithms for solving the MSFP and the MSCFP, one refers to [24, 25, 26] and the references therein.

Inspired and motivated by the above research works, in the paper, we construct two new self-adaptive cyclic iterative algorithms for solving the MSCFP of quasi-nonexpansive operators. The proposed algorithms combine inertial technique with the primal-dual method. As applications, we obtain several iterative algorithms to solve the MSFP. The contents of this paper are as follows. We give some useful definitions and results for the convergence analysis of the iterative algorithms in Section 2. We prove the weak convergence of the proposed algorithm with the dual variable and inertial technique in Section 3. In Section 4, we modify the proposed algorithm and obtain the strong convergence result. Numerical experiments are provided to illustrate the effectiveness of our proposed algorithms in the last section, Section 5.

2. Preliminaries

Throughout this paper, we denote the inner product by $\langle \cdot, \cdot \rangle$ and the norm by $\|\cdot\|$. Let I denote the identity operator on Hilbert space H. We denote the fixed point set of an operator T by $F(T)$. We use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. And we use $\omega_w(x_k)$ to denote the weak ω-limit set of $\{x_k\}$.

Let $T : H \to H$ be an operator. Recall that T is said to be

(i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x,\ y \in H$;

(ii) *firmly nonexpansive* if $2T - I$ is nonexpansive or, equivalently,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2$$

for all $x,\ y \in H$;

(iii) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - q\| \leq \|x - q\|$ for all $x \in H$ and $q \in F(T)$;

(iv) *firmly quasi-nonexpansive* (also called directed operator) if $F(T) \neq \emptyset$ and

$$\|Tx - q\|^2 \leq \|x - q\|^2 - \|x - Tx\|^2$$

or, equivalently,

$$\langle x - q, Tx - q \rangle \geq \|Tx - q\|^2$$

for all $x \in H$ and $q \in F(T)$;

(v) *demiclosed at the origin* if, for any sequence $\{x_n\}$, which converges weakly to x, $\{Tx_n\}$ converges strongly to 0, then $Tx = 0$.

Let H be a real Hilbert space. For all $x,\ y \in H$, $\alpha \in \mathbb{R}$, one has

$$2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2$$

and

$$\|\alpha x + (1 - \alpha) y\|^2 = \alpha \|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.$$

Lemma 2.1. [27] *Let H be a real Hilbert space, and let $T : H \to H$ be a quasi-nonexpansive operator. Set $T_\alpha = (1 - \alpha)I + \alpha T$ for $\alpha \in (0, 1)$. For all $x \in H$, $q \in F(T)$, then the following results hold:*

(i) $\langle x - Tx, x - q \rangle \geq \frac{1}{2}\|x - Tx\|^2$ and $\langle x - Tx, q - Tx \rangle \leq \frac{1}{2}\|x - Tx\|^2$;

(ii) $\|T_\alpha x - q\|^2 \leq \|x-q\|^2 - \alpha(1-\alpha)\|Tx - x\|^2$;

(iii) $\langle x - T_\alpha x, x - q\rangle \geq \frac{\alpha}{2}\|Tx-x\|^2$.

Remark 2.1. Let $T_\alpha = (1-\alpha)I + \alpha T$, where $T : H \to H$ is a quasi-nonexpansive operator and $\alpha \in (0,1)$. We have $F(T_\alpha) = F(T)$ and $\|T_\alpha x - x\|^2 = \alpha^2\|Tx-x\|^2$. It follows form (ii) of Lemma 2.1 that $\|T_\alpha x - q\|^2 \leq \|x-q\|^2 - \frac{1-\alpha}{\alpha}\|T_\alpha x - x\|^2$, which implies that T_α is firmly quasi-nonexpansive when $\alpha = \frac{1}{2}$. On the other hand, if $\hat{\mathrm{T}}$ is a firmly quasi-nonexpansive operator, we can easily obtain $\hat{\mathrm{T}}= \frac{1}{2}I + \frac{1}{2}T$, where T is quasi-nonexpansive operator.

It follows from (iii) of Lemma 2.1 that the following result is easily obtained.

Proposition 2.1. *Let T be a quasi-nonexpansive operator and $\alpha \in (0,1)$. If $T_\alpha = (1-\alpha)I + \alpha T$, then $\|(I-T_\alpha)x\|^2 \leq 2\alpha\langle x - q, (I-T_\alpha)x\rangle$ for all $x \in H$, $q \in F(T)$.*

Lemma 2.2. [28] *Let the sequences $\{\phi_k\}_{k=1}^\infty \subset [0,\infty)$ and $\{\delta_k\}_{k=1}^\infty \subset [0,\infty)$ which satisfy:*

(i) $\phi_{k+1} - \phi_k \leq \theta_k(\phi_k - \phi_{k-1}) + \delta_k$;

(ii) $\Sigma_{k=1}^\infty \delta_k < \infty$;

(iii) $\{\theta_k\} \subset [0,\theta]$, *where* $\theta \in [0,1)$.

Then $\{\phi_k\}$ is a convergent sequence and $\Sigma_{k=1}^\infty[\phi_{k+1} - \phi_k]_+ < \infty$, where $[t]_+ = max\{t,0\}$ for any $t \in \mathbb{R}$.

Lemma 2.3. [29] *Let K be a nonempty closed convex subset of the real Hilbert space. Let $\{x_k\}$ be a bounded sequence which satisfies the following properties:*

(i) every weak limit point of $\{x_k\}$ lies in K;

(ii) $\lim_{k\to\infty}\|x_k - x\|$ *exists for every* $x \in K$.

Then $\{x_k\}$ converges weakly to a point in K.

Lemma 2.4. [30] *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E and $T : K \to K$ be a nonexpansive operator. Then $I - T$ is demiclosed at the origin.*

Lemma 2.5. [31] *Let K be a nonempty closed convex subset in H, then for any x, $y \in H$ and $z \in K$,*

(i) $\langle P_K x - x, z - P_K x\rangle \geq 0$;

(ii) $\|P_K x - P_K y\|^2 \leq \langle P_K x - P_K y, x - y\rangle$;

(iii) $\|x - P_K x\|^2 \leq \|x - z\|^2 - \|z - P_K x\|^2$.

3. The Weak Convergence

In this section, we introduce an accelerated cyclic iterative algorithm that the stepsize does not depend on the bounded linear operator norm $\|A^*A\|$ for solving the MSCFP of quasi-nonexpansive operators. The proposed algorithm combines the dual variable and inertial technique, and the weak convergence is obtained. In this paper, we make use of the following assumptions:

(A1) H_1, H_2, and H_3 are real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator with $A \neq 0$, $U_i : H_1 \to H_1$ $(1 \leq i \leq p)$, and $T_j : H_2 \to H_2$ $(1 \leq j \leq r)$ are quasi-nonexpansive operators;

(A2) $\forall\, k \geq 1$, $[k]_1 = k(\mod p) + 1$, $[k]_2 = k(\mod r) + 1$. Let $\{\alpha_k\} \subset (0,1)$, $\{\beta_k\} \subset (0,1)$, $U_k = (1-\alpha_k)I + \alpha_k U_{[k]_1}$ and $T_k = (1-\beta_k)I + \beta_k T_{[k]_2}$, $\alpha = \sup_{k\geq 1}\{\alpha_k\}$, and $\beta = \sup_{k\geq 1}\{\beta_k\}$;

(A3) Γ denotes the solution set of the MSCFP of quasi-nonexpansive operators and Γ is nonempty.

Algorithm 3.1. (Self-adaptive inertial cyclic iterative algorithm (I))
Initialization: Choose two sequences $\{\rho_k\}_{k=1}^{\infty} \subset [0,+\infty)$ and $\{\varepsilon_k\}_{k=1}^{\infty} \subset [0,+\infty)$ satisfying

$$\sum_{k=1}^{\infty} \varepsilon_k < \infty.$$

Select arbitrary starting points x_0, x_1, $\omega_0 \in H_1$, $\eta \in [0,1)$, $\lambda \in (0,1]$, $\gamma > 0$, and set $\omega_1 = \omega_0$.

Iterative step: For $k \geq 1$, given the iterates x_{k-1}, x_k, ω_k, choose a_k such that $0 \leq a_k \leq \bar{a}_k$, where

$$\bar{a}_k := \begin{cases} \min\{\eta, \frac{\varepsilon_k}{\|x_k - x_{k-1}\|^2 + \|\omega_{k-1}\|^2}\}, & \text{if } x_k \neq x_{k-1} \text{ or } \omega_{k-1} \neq 0, \\ \eta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Compute

$$\begin{cases} y_k = x_k + a_k(x_k - x_{k-1}), \\ v_k = y_k - \gamma_k A^*(I - T_k)Ay_k, \\ \omega_{k+1} = (I - U_k)(v_k + (1-\lambda)\omega_k), \\ x_{k+1} = v_k - \lambda\omega_{k+1}, \end{cases}$$

where the stepsize γ_k is chosen in such a way that

$$\gamma_k := \begin{cases} \frac{\rho_k\|(I-T_k)Ay_k\|^2}{\|A^*(I-T_k)Ay_k\|^2}, & (I - T_k)Ay_k \neq 0, \\ \gamma, & (I - T_k)Ay_k = 0. \end{cases} \tag{3.2}$$

Remark 3.1. In our proposed Algorithm 3.1, the inertial extrapolation factor a_k and the stepsize γ_k are chosen by a self-adaptive way. We give a way of selecting the stepsize such that the implementation of the algorithm does not need any prior information about the norm of the bounded linear operator.

Remark 3.2. From (3.1), we have that

$$a_k(\|x_k - x_{k-1}\|^2 + \|\omega_{k-1}\|^2) \leq \bar{a}_k(\|x_k - x_{k-1}\|^2 + \|\omega_{k-1}\|^2) \leq \varepsilon_k,$$

and so

$$\Sigma_{k=1}^{\infty} a_k(\|x_k - x_{k-1}\|^2 + \|\omega_{k-1}\|^2) < \infty. \tag{3.3}$$

For example, we take $\varepsilon_k = \frac{1}{k^2}$, i.e.,

$$\bar{a}_k := \begin{cases} \min\{\eta, \frac{1}{k^2(\|x_k - x_{k-1}\|^2 + \|\omega_{k-1}\|^2)}\}, & \text{if } x_k \neq x_{k-1} \text{ or } \omega_{k-1} \neq 0, \\ \eta, & \text{otherwise,} \end{cases}$$

then

$$\Sigma_{k=1}^{\infty} a_k(\|x_k - x_{k-1}\|^2 + \|\omega_{k-1}\|^2) < \infty.$$

From the following lemma, we see that γ_k is well-defined.

Lemma 3.1. *γ_k defined by (3.2) is well-defined.*

Proof. Taking $x \in \Gamma$, one has $x \in \cap_{i=1}^{p} F(U_i)$ and $Ax \in \cap_{j=1}^{r} F(T_j)$. According to Proposition 2.1, we have

$$\begin{aligned}\|A^*(I-T_k)Ay_k\| \cdot \|y_k - x\| \geq & \langle A^*(I-T_k)Ay_k, y_k - x\rangle \\ = & \langle (I-T_k)Ay_k, Ay_k - Ax\rangle \\ \geq & \frac{1}{2\beta_k}\|(I-T_k)Ay_k\|^2 \\ \geq & \frac{1}{2\beta}\|(I-T_k)Ay_k\|^2.\end{aligned}$$

Consequently, when $\|(I-T_k)Ay_k\| \neq 0$, we have $\|A^*(I-T_k)Ay_k\| > 0$. This leads that γ_k is well-defined. □

Theorem 3.1. *Let $\{(x_k, \omega_k)\}$ be the sequence generated by Algorithm 3.1. Assume the following conditions hold:*

(i) $I - U_i$ and $I - T_j$ are demiclosed at origin for $1 \leq i \leq p$ and $1 \leq j \leq r$;

(ii) $0 < \liminf_{k\to\infty} \alpha_k \leq \alpha_k \leq \alpha \leq \frac{1}{2}$ and $0 < \liminf_{k\to\infty} \beta_k \leq \beta_k \leq \beta < 1$;

(iii) $0 < \liminf_{k\to\infty} \rho_k \leq \limsup_{k\to\infty} \rho_k < \frac{1}{\beta}$.

Then the sequence $\{x_k\}$ converges weakly to $x^* \in \Gamma$, and the sequence $\{(x_k, \omega_k)\}$ converges weakly to the point $(x^*, 0)$. Moreover, $\{x_k\}$ and $\{Ax_k\}$ are asymptotically regular.

Proof. **Step 1 .** We prove that $\lim_{k\to\infty} \|x_k - x\|$ exists for any $x \in \Gamma$.

Taking $x \in \Gamma$, we have $x \in \cap_{i=1}^{p} F(U_i)$ and $Ax \in \cap_{j=1}^{r} F(T_j)$. For $1 \leq i \leq p$ and $1 \leq j \leq r$, it follows from the definitions of U_k and T_k, and Remark 2.1 that $x \in \cap_{k=1}^{\infty} F(U_k)$ and $Ax \in \cap_{k=1}^{\infty} F(T_k)$. Thus, from Algorithm 3.1 and Proposition 2.1, we have

$$\begin{aligned}\|\omega_{k+1}\|^2 &= \|(I-U_k)(v_k + (1-\lambda)\omega_k) - (I-U_k)x\|^2 \\ &\leq 2\alpha_k \langle \omega_{k+1}, v_k - x + (1-\lambda)\omega_k\rangle \\ &\leq 2\alpha \langle \omega_{k+1}, v_k - x + (1-\lambda)\omega_k\rangle\end{aligned}$$

and

$$\begin{aligned}\|x_{k+1} - x\|^2 &= \|v_k - \lambda\omega_{k+1} - x\|^2 \\ &= \|v_k - x\|^2 - 2\lambda\langle v_k - x, \omega_{k+1}\rangle + \lambda^2\|\omega_{k+1}\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\|x_{k+1} - x\|^2 + \lambda\|\omega_{k+1}\|^2 \\ =&\|v_k - x\|^2 - 2\lambda\langle v_k - x, \omega_{k+1}\rangle + \lambda^2\|\omega_{k+1}\|^2 + \lambda\|\omega_{k+1}\|^2 \\ =&\|v_k - x\|^2 - 2\lambda\langle v_k - x, \omega_{k+1}\rangle + \frac{\lambda}{\alpha}\|\omega_{k+1}\|^2 - \lambda(\frac{1}{\alpha} - \lambda - 1)\|\omega_{k+1}\|^2 \\ \leq&\|v_k - x\|^2 - 2\lambda\langle v_k - x, \omega_{k+1}\rangle + 2\lambda\langle \omega_{k+1}, v_k - x + (1-\lambda)\omega_k\rangle - \lambda(\frac{1}{\alpha} - \lambda - 1)\|\omega_{k+1}\|^2 \\ =&\|v_k - x\|^2 + 2\lambda(1-\lambda)\langle \omega_{k+1}, \omega_k\rangle - \lambda(\frac{1}{\alpha} - \lambda - 1)\|\omega_{k+1}\|^2.\end{aligned}$$

Since

$$2\lambda(1-\lambda)\langle \omega_{k+1}, \omega_k\rangle = \lambda(1-\lambda)(\|\omega_{k+1}\|^2 + \|\omega_k\|^2 - \|\omega_{k+1} - \omega_k\|^2),$$

we obtain

$$
\begin{aligned}
&\|x_{k+1}-x\|^2+\lambda\|\omega_{k+1}\|^2\\
\leq&\|v_k-x\|^2+\lambda(1-\lambda)\|\omega_{k+1}\|^2+\lambda(1-\lambda)\|\omega_k\|^2\\
&-\lambda(1-\lambda)\|\omega_{k+1}-\omega_k\|^2-\lambda(\frac{1}{\alpha}-\lambda-1)\|\omega_{k+1}\|^2\\
=&\|v_k-x\|^2+\lambda(1-\lambda)\|\omega_k\|^2-\lambda(1-\lambda)\|\omega_{k+1}-\omega_k\|^2-\lambda(\frac{1}{\alpha}-2)\|\omega_{k+1}\|^2.
\end{aligned}
\tag{3.4}
$$

Since

$$
\langle y_k-x,A^*(I-T_k)Ay_k\rangle=\langle Ay_k-Ax,(I-T_k)Ay_k\rangle\geq\frac{1}{2\beta_k}\|(I-T_k)Ay_k\|^2\geq\frac{1}{2\beta}\|(I-T_k)Ay_k\|^2,
$$

we have

$$
\begin{aligned}
\|v_k-x\|^2=&\|y_k-\gamma_kA^*(I-T_k)Ay_k-x\|^2\\
=&\|y_k-x\|^2-2\gamma_k\langle y_k-x,A^*(I-T_k)Ay_k\rangle+\gamma_k^2\|A^*(I-T_k)Ay_k\|^2\\
\leq&\|y_k-x\|^2-\frac{\gamma_k}{\beta}\|(I-T_k)Ay_k\|^2+\gamma_k^2\|A^*(I-T_k)Ay_k\|^2\\
=&\|y_k-x\|^2-\gamma_k(\frac{1}{\beta}\|(I-T_k)Ay_k\|^2-\gamma_k\|A^*(I-T_k)Ay_k\|^2).
\end{aligned}
\tag{3.5}
$$

It follows from Algorithm 3.1 that

$$
\begin{aligned}
\|y_k-x\|^2=&\|x_k+a_k(x_k-x_{k-1})-x\|^2\\
=&\|(1+a_k)(x_k-x)-a_k(x_{k-1}-x)\|^2\\
=&(1+a_k)\|x_k-x\|^2-a_k\|x_{k-1}-x\|^2+a_k(1+a_k)\|x_k-x_{k-1}\|^2.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\|v_k-x\|^2\leq&(1+a_k)\|x_k-x\|^2-a_k\|x_{k-1}-x\|^2+a_k(1+a_k)\|x_k-x_{k-1}\|^2\\
&-\gamma_k(\frac{1}{\beta}\|(I-T_k)Ay_k\|^2-\gamma_k\|A^*(I-T_k)Ay_k\|^2).
\end{aligned}
\tag{3.6}
$$

From (3.4) and (3.6), we obtain

$$
\begin{aligned}
&\|x_{k+1}-x\|^2+\lambda\|\omega_{k+1}\|^2\\
\leq&(1+a_k)\|x_k-x\|^2-a_k\|x_{k-1}-x\|^2+a_k(1+a_k)\|x_k-x_{k-1}\|^2+\lambda(1-\lambda)\|\omega_k\|^2\\
&-\lambda(1-\lambda)\|\omega_{k+1}-\omega_k\|^2-\lambda(\frac{1}{\alpha}-2)\|\omega_{k+1}\|^2-\gamma_k(\frac{1}{\beta}\|(I-T_k)Ay_k\|^2-\gamma_k\|A^*(I-T_k)Ay_k\|^2)\\
\leq&\|x_k-x\|^2+\lambda\|\omega_k\|^2+a_k(\|x_k-x\|^2-\|x_{k-1}-x\|^2)+a_k(1+a_k)\|x_k-x_{k-1}\|^2\\
&-\lambda^2\|\omega_k\|^2-\gamma_k(\frac{1}{\beta}\|(I-T_k)Ay_k\|^2-\gamma_k\|A^*(I-T_k)Ay_k\|^2).
\end{aligned}
$$

Let $c_k=\|x_k-x\|^2+\lambda\|\omega_k\|^2$. It follows that

$$
\begin{aligned}
c_{k+1}\leq&c_k+a_k(\|x_k-x\|^2-\|x_{k-1}-x\|^2)+a_k(1+a_k)\|x_k-x_{k-1}\|^2\\
&-\lambda^2\|\omega_k\|^2-\gamma_k(\frac{1}{\beta}\|(I-T_k)Ay_k\|^2-\gamma_k\|A^*(I-T_k)Ay_k\|^2),
\end{aligned}
$$

which implies that

$$\begin{aligned}
&c_{k+1}-c_k\\
\leq & a_k(\|x_k-x\|^2+\lambda\|\omega_k\|^2-\|x_{k-1}-x\|^2-\lambda\|\omega_{k-1}\|^2)+a_k\lambda\|\omega_{k-1}\|^2+a_k(1+a_k)\|x_k-x_{k-1}\|^2\\
&-\lambda^2\|\omega_k\|^2-\gamma_k(\frac{1}{\beta}\|(I-T_k)Ay_k\|^2-\gamma_k\|A^*(I-T_k)Ay_k\|^2)\\
\leq & a_k(c_k-c_{k-1})+2a_k(\|x_k-x_{k-1}\|^2+\|\omega_{k-1}\|^2)\\
&-\lambda^2\|\omega_k\|^2-\gamma_k(\frac{1}{\beta}\|(I-T_k)Ay_k\|^2-\gamma_k\|A^*(I-T_k)Ay_k\|^2).
\end{aligned}\tag{3.7}$$

For the case $(I-T_k)Ay_k=0$, we have

$$\begin{aligned}
c_{k+1}-c_k \leq & a_k(c_k-c_{k-1})+2a_k(\|x_k-x_{k-1}\|^2+\|\omega_{k-1}\|^2)-\lambda^2\|\omega_k\|^2\\
\leq & a_k(c_k-c_{k-1})+2a_k(\|x_k-x_{k-1}\|^2+\|\omega_{k-1}\|^2).
\end{aligned}\tag{3.8}$$

Otherwise, we deduce from (3.2) and (3.7) that

$$\begin{aligned}
c_{k+1}-c_k \leq & a_k(c_k-c_{k-1})+2a_k(\|x_k-x_{k-1}\|^2+\|\omega_{k-1}\|^2)\\
&-\lambda^2\|\omega_k\|^2-\rho_k(\frac{1}{\beta}-\rho_k)\frac{\|(I-T_k)Ay_k\|^4}{\|A^*(I-T_k)Ay_k\|^2}.
\end{aligned}\tag{3.9}$$

By the assumption conditions on ρ_k and λ, and (3.8)–(3.9), we see that

$$c_{k+1}-c_k\leq a_k(c_k-c_{k-1})+2a_k(\|x_k-x_{k-1}\|^2+\|\omega_{k-1}\|^2).\tag{3.10}$$

Let $\phi_k=c_k$, $\theta_k=a_k$, and $\delta_k=2a_k(\|x_k-x_{k-1}\|^2+\|\omega_{k-1}\|^2)$. It follows from (3.3) that

$$\Sigma_{k=1}^{\infty}2a_k(\|x_k-x_{k-1}\|^2+\|\omega_{k-1}\|^2)<\infty.$$

Applying Lemma 2.2 to (3.10), we obtain that $\lim_{k\to\infty}c_k$ exists. Thus it implies that $\{c_k\}$ is bounded and $\{x_k\}$ is bounded. From (3.8)-(3.9), we also have

$$\lambda^2\|\omega_k\|^2\leq a_k(c_k-c_{k-1})-(c_{k+1}-c_k)+2a_k(\|x_k-x_{k-1}\|^2+\|\omega_{k-1}\|^2),$$

which indicates that

$$\lim_{k\to\infty}\|\omega_k\|=0\tag{3.11}$$

by taking into account that $\lambda>0$, the convergence of $\{c_k\}$, and $\delta_k\to 0$ as $k\to\infty$. So $\lim_{k\to\infty}\|x_k-x\|^2=\lim_{k\to\infty}(c_k-\lambda\|\omega_k\|^2)=\lim_{k\to\infty}c_k$ exists.

Step 2. We prove $\lim_{k\to\infty}\|(I-T_k)Ay_k\|=0$. Moreover, we have that $\{x_k\}$ and $\{Ax_k\}$ are asymptotically regular.

When $(I-T_k)Ay_k=0$, it is clear that $y_k-v_k=\gamma_kA^*(I-T_k)Ay_k=0$. Otherwise, it follows from (3.9) that

$$\begin{aligned}
&\rho_k(\frac{1}{\beta}-\rho_k)\frac{\|(I-T_k)Ay_k\|^4}{\|A^*(I-T_k)Ay_k\|^2}\\
&\leq a_k(c_k-c_{k-1})-(c_{k+1}-c_k)+2a_k(\|x_k-x_{k-1}\|^2+\|\omega_{k-1}\|^2)-\lambda^2\|\omega_k\|^2,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\frac{\|(I-T_k)Ay_k\|^4}{\|A^*(I-T_k)Ay_k\|^2}=0.$$

Further, we obtain

$$\lim_{k\to\infty} \frac{\|(I-T_k)Ay_k\|^2}{\|A^*(I-T_k)Ay_k\|} = 0. \tag{3.12}$$

Since A is a bounded linear operator, we obtain

$$\|A^*(I-T_k)Ay_k\| \le \|A\|\|(I-T_k)Ay_k\|.$$

Hence, we have

$$\frac{1}{\|A\|}\|(I-T_k)Ay_k\| = \frac{\|(I-T_k)Ay_k\|^2}{\|A\|\|(I-T_k)Ay_k\|} \le \frac{\|(I-T_k)Ay_k\|^2}{\|A^*(I-T_k)Ay_k\|} \tag{3.13}$$

by taking into account that $A \neq 0$. From (3.12) and (3.13), we obtain $\lim_{k\to\infty}\|(I-T_k)Ay_k\| = 0$. It follows from the definitions of $y_k,\ v_k,\ \gamma_k$, and (3.12) that

$$\lim_{k\to\infty}\|y_k - v_k\| = \lim_{k\to\infty}\|\gamma_k A^*(I-T_k)Ay_k\| = \lim_{k\to\infty}\rho_k\frac{\|(I-T_k)Ay_k\|^2}{\|A^*(I-T_k)Ay_k\|} = 0.$$

By the above two cases, we obtain

$$\lim_{k\to\infty}\|(I-T_k)Ay_k\| = \lim_{k\to\infty}\|y_k - v_k\| = 0. \tag{3.14}$$

In view of $x_{k+1} = v_k - \lambda\omega_{k+1}$, we have

$$\lim_{k\to\infty}\|x_{k+1} - v_k\| = \lim_{k\to\infty}\lambda\|\omega_{k+1}\| = 0. \tag{3.15}$$

It follows from (3.3) that $\lim_{k\to\infty} a_k(\|x_k - x_{k-1}\|^2 + \|\omega_{k-1}\|^2) = 0$, which implies $\lim_{k\to\infty} a_k\|x_k - x_{k-1}\|^2 = 0$. Since $a_k^2\|x_k - x_{k-1}\|^2 \le a_k\|x_k - x_{k-1}\|^2$, it turns out that $\lim_{k\to\infty} a_k\|x_k - x_{k-1}\| = 0$, and hence

$$\lim_{k\to\infty}\|y_k - x_k\| = \lim_{k\to\infty} a_k\|x_k - x_{k-1}\| = 0. \tag{3.16}$$

From (3.14) and (3.16), we have

$$\lim_{k\to\infty}\|x_k - v_k\| = 0. \tag{3.17}$$

By (3.15) and (3.17), we have $\lim_{k\to\infty}\|x_{k+1} - x_k\| = 0$, which deduces that $\{x_k\}$ is asymptotically regular. So, $\{Ax_k\}$ is asymptotically regular, i.e., $\lim_{k\to\infty}\|Ax_{k+1} - Ax_k\| = 0$.

Step 3. We prove that $\omega_w(x_k) \subset \Gamma$.

Assume that $\hat{x} \in \omega_w(x_k)$, i.e., there exists a subsequence $\{x_{k_l}\}$ of $\{x_k\}$ such that $x_{k_l} \rightharpoonup \hat{x}$ as $l \to \infty$. At the same time, it follows from (3.16) that $y_{k_l} \rightharpoonup \hat{x}$ and $Ay_{k_l} \rightharpoonup A\hat{x}$ as $l \to \infty$. By (3.11) and (3.17), we have $v_{k_l} + (1-\lambda)\omega_{k_l} \rightharpoonup \hat{x}$ as $l \to \infty$. Noting that the pool of indexes is finite and $\{x_k\}$ is asymptotically regular, for any $1 \le i \le p$, we can choose a subsequence $\{k_{i_m}\} \subset \{k\}$ such that $x_{k_{i_m}} \rightharpoonup \hat{x}$, $v_{k_{i_m}} + (1-\lambda)\omega_{k_{i_m}} \rightharpoonup \hat{x}$ as $m \to \infty$, and $[k_{i_m}]_1 = i$ for all m. It turns out that

$$\begin{aligned}
\lim_{m\to\infty}\|(I-U_i)(v_{k_{i_m}} + (1-\lambda)\omega_{k_{i_m}})\| &= \lim_{m\to\infty}\|(I-U_{[k_{i_m}]_1})(v_{k_{i_m}} + (1-\lambda)\omega_{k_{i_m}})\| \\
&= \lim_{m\to\infty}\frac{1}{\alpha_{k_{i_m}}}\|(I-U_{k_{i_m}})(v_{k_{i_m}} + (1-\lambda)\omega_{k_{i_m}})\| \\
&= \lim_{m\to\infty}\frac{1}{\alpha_{k_{i_m}}}\|\omega_{k_{i_m}+1}\| \\
&=0.
\end{aligned} \tag{3.18}$$

Likewise, for any $1 \leq j \leq r$, we can choose a subsequence $\{k_{j_n}\} \subset \{k\}$ such that $Ay_{k_{j_n}} \rightharpoonup A\hat{x}$ as $n \to \infty$ and $[k_{j_n}]_2 = j$ for all n. It turns out that

$$\begin{aligned}\lim_{n\to\infty} \|(I-T_j)Ay_{k_{j_n}}\| &= \lim_{n\to\infty} \|(I-T_{[k_{j_n}]_2})Ay_{k_{j_n}}\| \\ &= \lim_{n\to\infty} \frac{1}{\beta_{k_{j_n}}} \|(I-T_{k_{j_n}})Ay_{k_{j_n}}\| \\ &= 0.\end{aligned} \tag{3.19}$$

Since $I-U_i$ $(1 \leq i \leq p)$ and $I-T_j$ $(1 \leq j \leq r)$ are demiclosed at origin, it follows from (3.18) and (3.19) that $\hat{x} \in \cap_{i=1}^{p} F(U_i), A\hat{x} \in \cap_{j=1}^{r} F(T_j)$. Hence $\hat{x} \in \Gamma$. This proves $\omega_w(x_k) \subset \Gamma$. Using Lemma 2.3, we have $x_k \rightharpoonup x^*$ as $k \to \infty$, where x^* is a solution to the MSCFP. Thus it follows from $\omega_k \to 0$ that $(x_k, \omega_k) \rightharpoonup (x^*, 0)$ as $k \to \infty$. $\square$

Remark 3.3. (i) When $\lambda = 1$, Algorithm 3.1 becomes the following self-adaptive inertial cyclic iterative algorithm for solving the MSCFP of quasi-nonexpansive operators:

$$\begin{cases} y_k = x_k + a_k(x_k - x_{k-1}), \\ x_{k+1} = U_k(y_k - \gamma_k A^*(I-T_k)Ay_k), \end{cases} \tag{3.20}$$

where $0 \leq a_k \leq \bar{a}_k$, $\bar{a}_k$ is chosen in the following way:

$$\bar{a}_k := \begin{cases} \min\{\eta, \frac{\varepsilon_k}{\|x_k - x_{k-1}\|^2}\}, & \text{if } x_k \neq x_{k-1}, \\ \eta, & \text{otherwise,} \end{cases} \tag{3.21}$$

and γ_k is chosen by (3.2).

(ii) When $a_k \equiv 0$, Algorithm 3.1 becomes the following self-adaptive primal-dual cyclic iterative algorithm for solving the MSCFP of quasi-nonexpansive operators:

$$\begin{cases} \omega_k = (I-U_k)(x_k - \gamma_k A^*(I-T_k)Ax_k + (1-\lambda)\omega_k), \\ x_{k+1} = x_k - \gamma_k A^*(I-T_k)Ax_k - \lambda \omega_{k+1}, \end{cases} \tag{3.22}$$

where γ_k is chosen by

$$\gamma_k := \begin{cases} \frac{\rho_k \|(I-T_k)Ax_k\|^2}{\|A^*(I-T_k)Ax_k\|^2}, & (I-T_k)Ax_k \neq 0, \\ \gamma, & (I-T_k)Ax_k = 0. \end{cases} \tag{3.23}$$

(iii) When $\lambda = 1$ and $a_k \equiv 0$, Algorithm 3.1 becomes the following self-adaptive cyclic iterative algorithm for solving the MSCFP of quasi-nonexpansive operators:

$$x_{k+1} = U_k(x_k - \gamma_k A^*(I-T_k)Ax_k), \tag{3.24}$$

where γ_k is chosen by (3.23).

It is well known that the projection operator P_C on the nonempty closed convex subset C is firmly nonexpansive. Thus the projection operator is demisclosed at the origin. Suppose that the solution set of the MSFP is nonempty. Based on Remark 2.1, we take $\alpha_k \equiv \frac{1}{2}$ and $\beta_k \equiv \frac{1}{2}$ in Theorem 3.1, the following results can be obtained easily.

Corollary 3.1. *Assume that* $0 < \lambda \leq 1$ *and* $0 < \liminf_{k\to\infty} \rho_k \leq \limsup_{k\to\infty} \rho_k < 2$. *Let the sequence* $\{(x_k, \omega_k)\}$ *be arised by*

$$\begin{cases} \text{arbitrarily chosing } x_0,\ x_1,\ \omega_0 \in H_1, \\ \text{set } \omega_1 = \omega_0, \\ y_k = x_k + a_k(x_k - x_{k-1}), \\ v_k = y_k - \gamma_k A^*(I - P_{Q_{[k]_2}})Ay_k), \\ \omega_{k+1} = (I - P_{C_{[k]_1}})(v_k + (1-\lambda)\omega_k), \\ x_{k+1} = v_k - \lambda\omega_{k+1}, \end{cases} \tag{3.25}$$

where $0 \leq a_k \leq \bar{a}_k$, $\bar{a}_k$ *is chosen by* (3.1) *with* $\eta \in [0,1)$ *and* $\sum_{k=1}^{\infty} \varepsilon_k < \infty$, *and* γ_k *is chosen in the following way that*

$$\gamma_k := \begin{cases} \frac{\rho_k \|(I-P_{Q_{[k]_2}})Ay_k\|^2}{\|A^*(I-P_{Q_{[k]_2}})Ay_k\|^2}, & (I - P_{Q_{[k]_2}})Ay_k \neq 0, \\ \gamma, & (I - P_{Q_{[k]_2}})Ay_k = 0 \end{cases} \tag{3.26}$$

with $0 < \gamma < 1$. *Then the sequence* $\{x_k\}$ *converges weakly to a point* x^*, *where* x^* *is a solution to the MSFP, and the sequence* $\{(x_k, \omega_k)\}$ *converges weakly to* $(x^*, 0)$.

Remark 3.4. (i) When $\lambda = 1$, algorithm (3.25) becomes the following self-adaptive inertial cyclic iterative algorithm for solving the MSFP:

$$\begin{cases} y_k = x_k + a_k(x_k - x_{k-1}), \\ x_{k+1} = P_{C_{[k]_1}}(y_k - \gamma_k A^*(I - P_{Q_{[k]_2}})Ay_k), \end{cases} \tag{3.27}$$

where $0 \leq a_k \leq \bar{a}_k$, $\bar{a}_k$ is chosen by (3.21) and γ_k is chosen by (3.26).

(ii) When $a_k \equiv 0$, algorithm (3.25) becomes the following self-adaptive primal-dual cyclic iterative algorithm for solving the MSFP:

$$\begin{cases} \omega_k = (I - P_{C_{[k]_1}})(x_k - \gamma_k A^*(I - P_{Q_{[k]_2}})Ax_k + (1-\lambda)\omega_k), \\ x_{k+1} = x_k - \gamma_k A^*(I - P_{Q_{[k]_2}})Ax_k - \lambda\omega_{k+1}, \end{cases} \tag{3.28}$$

where γ_k is chosen by the following way that

$$\gamma_k := \begin{cases} \frac{\rho_k \|(I-P_{Q_{[k]_2}})Ax_k\|^2}{\|A^*(I-P_{Q_{[k]_2}})Ax_k\|^2}, & (I - P_{Q_{[k]_2}})Ax_k \neq 0, \\ \gamma, & (I - P_{Q_{[k]_2}})Ax_k = 0. \end{cases} \tag{3.29}$$

(iii) When $\lambda = 1$ and $a_k \equiv 0$, algorithm (3.25) becomes the following self-adaptive cyclic iterative algorithm for solving the MSFP:

$$x_{k+1} = P_{C_{[k]_1}}(x_k - \gamma_k A^*(I - P_{Q_{[k]_2}})Ax_k), \tag{3.30}$$

where γ_k is chosen by (3.29).

4. The Strong Convergence

In this section, we modify Algorithm 3.1 so that it has strong convergence. Let us denote the product space $G := H_1 \times H_1$ with the inner product

$$\langle (u_1,v_1),(u_2,v_2)\rangle = \langle u_1,u_2\rangle + \langle v_1,v_2\rangle$$

for $(u_1,v_1),\ (u_2,v_2) \in G$. Let $F = \{(x,0) : x \in \Gamma\} \subset G$, where Γ is the nonempty closed convex solution set of the MSCFP. Then F is a nonempty closed convex subset in G.

Algorithm 4.1. (Self-adaptive inertial cyclic iterative algorithm (II))
Initialization: Choose two sequences $\{\rho_k\}_{k=1}^{\infty} \subset [0,+\infty)$ and $\{a_k\}_{k=1}^{\infty} \subset [0,+\infty)$ satisfying

$$\limsup_{k\to\infty} a_k \le a < +\infty$$

where a is positive constant. Select arbitrary starting points x_0, x_1, $\omega_0 \in H_1$, $\lambda \in (0,1]$, $\gamma > 0$, and set $\omega_1 = \omega_0$
Iterative step: For $k \ge 1$, given the iterates x_{k-1}, x_k, ω_k. Compute

$$\begin{cases} y_k = x_k + a_k(x_k - x_{k-1}), \\ \bar{v}_k = y_k - \gamma_k A^*(I - T_k)Ay_k, \\ \bar{\omega}_k = (I - U_k)(\bar{v}_k + (1-\lambda)\omega_k), \\ \bar{x}_k = \bar{v}_k - \lambda\bar{\omega}_k, \\ C_k = \{(u,v) \in G : \|\bar{x}_k - u\|^2 + \lambda\|\bar{\omega}_k - v\|^2 \le \|y_k - u\|^2 + \lambda\|\omega_k - v\|^2\}, \\ Q_k = \{(u,v) \in G : \langle (x_k,\omega_k) - (u,v), (x_1,\omega_1) - (x_k,\omega_k)\rangle \ge 0\}, \\ (x_{k+1},\omega_{k+1}) = P_{C_k\cap Q_k}(x_1,\omega_1), \end{cases}$$

where the stepsize γ_k is chosen by (3.2).

Theorem 4.1. *Let $\{(x_k,\omega_k)\}$ be the sequence generated by Algorithm 4.1. Assume the following conditions hold:*

(i) $I - U_i$ and $I - T_j$ are demiclosed at origin for $1 \le i \le p$ and $1 \le j \le r$;

(ii) $0 < \liminf_{k\to\infty} \alpha_k \le \alpha_k \le \alpha \le \frac{1}{2}$ and $0 < \liminf_{k\to\infty} \beta_k \le \beta_k \le \beta < 1$;

(iii) $0 < \liminf_{k\to\infty} \rho_k \le \limsup_{k\to\infty} \rho_k < \frac{1}{\beta}$.

Then the sequence $\{(x_k,\omega_k)\}$ converges strongly to $(x^*,0)$, where $(x^*,0) = P_F(x_1,\omega_1)$.

Proof. **Step 1.** $C_k \cap Q_k$ is closed and convex for all $k \ge 1$.

According to the definitions of C_k and Q_k, we have that C_k is closed and Q_k is closed and convex for all $k \ge 1$. Note that the inequality in C_k is equivalent to the inequality

$$2\langle y_k - \bar{x}_k, u\rangle + 2\lambda\langle \omega_k - \bar{\omega}_k, v\rangle \le \|y_k\|^2 - \|\bar{x}_k\|^2 + \lambda\|\omega_k\|^2 - \lambda\|\bar{\omega}_k\|^2,$$

that is,

$$2\langle (y_k - \bar{x}_k, \lambda(\omega_k - \bar{\omega}_k)), (u,v)\rangle \le \|y_k\|^2 - \|\bar{x}_k\|^2 + \lambda\|\omega_k\|^2 - \lambda\|\bar{\omega}_k\|^2.$$

It is seen easily that C_k is convex for all $k \ge 1$. Hence $C_k \cap Q_k$ is closed and convex for all $k \ge 1$.

Step 2. $F \subset C_k \cap Q_k$ for all $k \ge 1$.

Taking $(x,0)\in F$, we have $x\in\Gamma$. It follows from (3.4)-(3.5), Algorithm 4.1 and the conditions on $\{\alpha_k\}$ that

$$
\begin{aligned}
&\|\bar{x}_k - x\|^2 + \lambda\|\bar{\omega}_k\|^2 \\
\leq&\|y_k - x\|^2 + \lambda(1-\lambda)\|\omega_k\|^2 - \lambda(1-\lambda)\|\bar{\omega}_k - \omega_k\|^2 \\
&-\lambda(\frac{1}{\alpha}-2)\|\bar{\omega}_k\|^2 - \gamma_k(\frac{1}{\beta}\|(I-T_k)Ay_k\|^2 - \gamma_k\|A^*(I-T_k)Ay_k\|^2) \\
\leq&\|y_k - x\|^2 + \lambda\|\omega_k\|^2 - \lambda^2\|\omega_k\|^2 - \gamma_k(\frac{1}{\beta}\|(I-T_k)Ay_k\|^2 - \gamma_k\|A^*(I-T_k)Ay_k\|^2),
\end{aligned}
\tag{4.1}
$$

which implies that $\|\bar{x}_k - x\|^2 + \lambda\|\bar{\omega}_k - 0\|^2 \leq \|y_k - x\|^2 + \lambda\|\omega_k - 0\|^2$. Hence $(x,0)\in C_k$ for all $k\geq 1$.

Next, we prove that $F\subset Q_k$ for all $k\geq 1$. Here, we use the mathematical induction. For $k=1$, we have $F\subset G = Q_1$. Assume that $F\subset Q_k$ for some $k>1$. Then $F\subset C_k\cap Q_k$. Thus $C_k\cap Q_k$ is nonempty, closed, and convex subset in G. Hence, there exists unique element (x_{k+1},ω_{k+1}) such that $(x_{k+1},\omega_{k+1}) = P_{C_k\cap Q_k}(x_1,\omega_1)$. We have

$$\langle (x_{k+1},\omega_{k+1}) - (u,v), (x_1,\omega_1) - (x_{k+1},\omega_{k+1})\rangle \geq 0,\ \forall (u,v)\in C_k\cap Q_k.$$

By the induction assumption, for any $(x,0)\in F$, we have

$$\langle (x_{k+1},\omega_{k+1}) - (x,0), (x_1,\omega_1) - (x_{k+1},\omega_{k+1})\rangle \geq 0,$$

which implies that $(x,0)\in Q_{k+1}$. Therefore, $F\subset Q_{k+1}$. By the principle of mathematical induction, we obtain $F\subset Q_k$ and $F\subset C_k\cap Q_k$ for all $k\geq 1$. This illustrates that Algorithm 4.1 is well-defined.

Step 3. $\lim_{k\to\infty}\|(x_k,\omega_k)-(x_1,\omega_1)\|^2$ exists and $\{(x_k,\omega_k)\}$ is bounded. Moreover, $\{(x_{k+1},\omega_{k+1})\}$ is asymptotically regular.

From Algorithm 4.1, we can obtain $(x_k,\omega_k) = P_{Q_k}(x_1,\omega_1)$ and $(x_{k+1},\omega_{k+1}) = P_{C_k\cap Q_k}(x_1,\omega_1)\in Q_k$. It follows from Lemma 2.5 (iii) that, for all $k\geq 1$,

$$\|(x_1,\omega_1)-(x_k,\omega_k)\|^2 \leq \|(x_1,\omega_1)-(x_{k+1},\omega_{k+1})\|^2 - \|(x_{k+1},\omega_{k+1})-(x_k,\omega_k)\|^2, \tag{4.2}$$

which implies that, for all ≥ 1,

$$\|(x_1,\omega_1)-(x_k,\omega_k)\|^2 \leq \|(x_1,\omega_1)-(x_{k+1},\omega_{k+1})\|^2, \forall k\geq 1.$$

Hence, we obtain that $\{\|(x_k,\omega_k)-(x_1,\omega_1)\|^2\}$ is nondecreasing. Since $(x^*,0) = P_F(x_1,\omega_1)\in F\subset Q_k$ and $(x_k,\omega_k) = P_{Q_k}(x_1,\omega_1)$, we have

$$\|(x_1,\omega_1)-(x_k,\omega_k)\|^2 \leq \|(x_1,\omega_1)-(x,0)\|^2,$$

which implies that $\{\|(x_k,\omega_k)-(x_1,\omega_1)\|^2\}$ is bounded. Therefore, $\lim_{k\to\infty}\|(x_k,\omega_k)-(x_1,\omega_1)\|^2$ exists and $\{(x_k,\omega_k)\}$ is bounded. From (4.2), we obtain

$$\|(x_{k+1},\omega_{k+1})-(x_k,\omega_k)\|^2 \leq \|(x_1,\omega_1)-(x_{k+1},\omega_{k+1})\|^2 - \|(x_1,\omega_1)-(x_k,\omega_k)\|^2$$

and $\lim_{k\to\infty}\|(x_{k+1},\omega_{k+1})-(x_k,\omega_k)\|^2 = 0$, which implies that

$$\lim_{k\to\infty}\|x_{k+1}-x_k\|^2 = \lim_{k\to\infty}\|\omega_{k+1}-\omega_k\|^2 = 0. \tag{4.3}$$

Step 4. $\omega_k\to 0$, $(I-T_k)Ay_k\to 0$ as $k\to\infty$ and $\omega_w(x_k,\omega_k)\subset F$.

Since $\{a_k\}$ is bounded and $y_k - x_k = a_k(x_k - x_{k-1})$, it follows from (4.3) that

$$\lim_{k\to\infty} \|y_k - x_k\| = 0 \tag{4.4}$$

and

$$\lim_{k\to\infty} \|x_{k+1} - y_k\| = 0. \tag{4.5}$$

Due to $(x_{k+1}, \omega_{k+1}) \in C_k$, we have

$$\|\bar{x}_k - x_{k+1}\|^2 + \lambda\|\bar{\omega}_k - \omega_{k+1}\|^2 \leq \|y_k - x_{k+1}\|^2 + \lambda\|\omega_k - \omega_{k+1}\|^2. \tag{4.6}$$

From $\lambda > 0$, (4.3), (4.5), and (4.6), we obtain

$$\lim_{k\to\infty} \|\bar{x}_k - x_{k+1}\| = \lim_{k\to\infty} \|\bar{\omega}_k - \omega_{k+1}\| = 0, \tag{4.7}$$

which implies that

$$\lim_{k\to\infty} \|\bar{x}_k - y_k\| = \lim_{k\to\infty} \|\bar{\omega}_k - \omega_k\| = 0. \tag{4.8}$$

According to (4.1) and

$$\begin{aligned}
&\|\bar{x}_k - x\|^2 + \lambda\|\bar{\omega}_k\|^2 \\
=&\|\bar{x}_k - y_k + y_k - x\|^2 + \lambda\|\bar{\omega}_k - \omega_k + \omega_k\|^2 \\
=&\|\bar{x}_k - y_k\|^2 + 2\langle \bar{x}_k - y_k, y_k - x\rangle + \|y_k - x\|^2 + \lambda\|\bar{\omega}_k - \omega_k\|^2 + 2\lambda\langle \bar{\omega}_k - \omega_k, \omega_k\rangle + \lambda\|\omega_k\|^2,
\end{aligned} \tag{4.9}$$

we have

$$\begin{aligned}
\lambda^2\|\omega_k\| &\leq \|y_k - x\|^2 + \lambda\|\omega_k\|^2 - \|\bar{x}_k - x\|^2 - \lambda\|\bar{\omega}_k\|^2 \\
&\leq 2\langle y_k - \bar{x}_k, y_k - x\rangle - \|\bar{x}_k - y_k\|^2 + 2\lambda\langle \omega_k - \bar{\omega}_k, \omega_k\rangle - \lambda\|\bar{\omega}_k - \omega_k\|^2 \\
&\leq 2\|y_k - \bar{x}_k\| \cdot \|y_k - x\| - \|\bar{x}_k - y_k\|^2 + 2\lambda\|\omega_k - \bar{\omega}_k\| \cdot \|\omega_k\| - \lambda\|\bar{\omega}_k - \omega_k\|^2.
\end{aligned}$$

Because of the boundedness of $\{(x_k, \omega_k)\}$, we see that $\{x_k\}$, $\{\omega_k\}$, and $\{y_k\}$ are bounded. Moreover, we have

$$\lim_{k\to\infty} \|\omega_k\| = 0 \tag{4.10}$$

by taking into account (4.8). Similarly, it follows from (4.1), (4.8), and (4.9) that

$$\lim_{k\to\infty} \gamma_k\left(\frac{1}{\beta}\|(I - T_k)Ay_k\|^2 - \gamma_k\|A^*(I - T_k)Ay_k\|^2\right) = 0. \tag{4.11}$$

When $(I - T_k)Ay_k = 0$, it is obvious that $y_k - \bar{v}_k = \gamma A^*(I - T_k)Ay_k = 0$. Otherwise, it follows from (3.2) and (4.11) that

$$\lim_{k\to\infty} \rho_k\left(\frac{1}{\beta} - \rho_k\right)\frac{\|(I - T_k)Ay_k\|^4}{\|A^*(I - T_k)Ay_k\|^2} = 0. \tag{4.12}$$

From the condition on ρ_k and (4.12), we have

$$\lim_{k\to\infty} \frac{\|(I - T_k)Ay_k\|^2}{\|A^*(I - T_k)Ay_k\|} = 0.$$

Similar to the proof of the Theorem 3.1, we can obtain

$$\lim_{k\to\infty} \|y_k - \bar{v}_k\| = \lim_{k\to\infty} \|(I - T_k)Ay_k\| = 0. \tag{4.13}$$

From (4.4) and (4.13), we have

$$\lim_{k\to\infty} \|x_k - \bar{v}_k\| = 0. \tag{4.14}$$

Next, we show $\omega_w(x_k,\omega_k)\subset F$. Indeed, we need to show $\omega_w(x_k)\subset\Gamma$ from $\omega_k\to 0$ as $k\to\infty$. According to (4.7) and (4.10), we see that

$$\lim_{k\to\infty}\|\bar{\omega}_k\|=\lim_{k\to\infty}\|(I-U_k)(\bar{v}_k+(1-\lambda)\omega_k)\|=0. \tag{4.15}$$

Similar to the proof of Theorem 3.1, it follows from (4.4), (4.10), (4.13), (4.14), and (4.15) that $\omega_w(x_k)\subset\Gamma$. Therefore, we have $\omega_w(x_k,\omega_k)\subset F$.

Step 5. $(x_k,\omega_k)\to(x^*,0)\in F$ as $k\to\infty$, where $(x^*,0)=P_F(x_1,\omega_1)$.

Since $\{(x_k,\omega_k)\}$ is bounded, there exists a subsequence $\{(x_{k_j},\omega_{k_j})\}\subset\{(x_k,\omega_k)\}$ such that $(x_{k_j},\omega_{k_j})\rightharpoonup(z,v)$ as $j\to\infty$. Therefore, we have $(z,v)\in F$ by Step 4. Moreover, $z\in\Gamma$ and $v=0$. According to $(x_{k+1},\omega_{k+1})\in Q_k$ and $(x_k,\omega_k)=P_{Q_k}(x_1,\omega_1)$, we can obtain

$$\|(x_1,\omega_1)-(x_k,\omega_k)\|^2\le\|(x_1,\omega_1)-(x_{k+1},\omega_{k+1})\|^2. \tag{4.16}$$

By $(x^*,0)=P_F(x_1,\omega_1)\in F\subset C_k\cap Q_k$ and $(x_{k+1},\omega_{k+1})=P_{C_k\cap Q_k}(x_1,\omega_1)$, we have

$$\|(x_1,\omega_1)-(x_{k+1},\omega_{k+1)}\|^2\le\|(x_1,\omega_1)-(x^*,0)\|^2. \tag{4.17}$$

It follows from (4.16) and (4.17) that

$$\begin{aligned}
&\|(x^*,0)-(x_{k_j},\omega_{k_j})\|^2\\
&=\|(x^*,0)-(x_1,\omega_1)+(x_1,\omega_1)-(x_{k_j},\omega_{k_j})\|^2\\
&=\|(x^*,0)-(x_1,\omega_1)\|^2+2\langle(x^*,0)-(x_1,\omega_1),(x_1,\omega_1)-(x_{k_j},\omega_{k_j})\rangle+\|(x_1,\omega_1)-(x_{k_j},\omega_{k_j})\|^2\\
&\le\|(x^*,0)-(x_1,\omega_1)\|^2+\|(x_1,\omega_1)-(x^*,0)\|^2+2\langle(x^*,0)-(x_1,\omega_1),(x_1,\omega_1)\\
&-(x^*,0)+(x^*,0)-(x_{k_j},\omega_{k_j})\rangle\\
&=2\langle(x^*,0)-(x_1,\omega_1),(x^*,0)-(x_{k_j},\omega_{k_j})\rangle,
\end{aligned} \tag{4.18}$$

which indicates that

$$\begin{aligned}
\limsup_{k\to\infty}\|(x^*,0)-(x_{k_j},\omega_{k_j})\|^2\le&\limsup_{k\to\infty}2\langle(x^*,0)-(x_1,\omega_1),(x^*,0)-(x_{k_j},\omega_{k_j})\rangle\\
=&2\langle(x^*,0)-(x_1,\omega_1),(x^*,0)-(z,0)\rangle\le 0
\end{aligned}$$

by taking into account that $(x^*,0)=P_F(x_1,\omega_1)\in F$ and $(z,0)\in F$. Hence, $(x_{k_j},\omega_{k_j})\to(x^*,0)=P_F(x_1,\omega_1)$ as $j\to\infty$. Moreover, we have that the weak limit point of $\{(x_k,\omega_k)\}$ is unique. So, $(x_k,\omega_k)\rightharpoonup(x^*,0)=P_F(x_1,\omega_1)$ as $k\to\infty$. From (4.18), we have

$$\|(x^*,0)-(x_k,\omega_k)\|^2\le 2\langle(x^*,0)-(x_1,\omega_1),(x^*,0)-(x_k,\omega_k)\rangle\to 0$$

as $k\to\infty$. We obtain $(x_k,\omega_k)\to(x^*,0)=P_F(x_1,\omega_1)\in F$ as $k\to\infty$. So, $x_k\to x^*\in\Gamma$ as $k\to\infty$. □

Remark 4.1. (i) When $\lambda=1$, Algorithm 4.1 becomes the following self-adaptive inertial cyclic iterative algorithm for solving the MSCFP of quasi-nonexpansive operators:

$$\begin{cases}
y_k=x_k+a_k(x_k-x_{k-1}),\\
\bar{x}_k=U_k(y_k-\gamma_k A^*(I-T_k)Ay_k),\\
C_k=\{u\in G:\|\bar{x}_k-u\|^2\le\|y_k-u\|^2\},\\
Q_k=\{u\in G:\langle x_k-u,x_1-x_k\rangle\ge 0\},\\
x_{k+1}=P_{C_k\cap Q_k}x_1,
\end{cases}$$

where $a_k \subset [0,+\infty)$ with $\limsup_{k\to\infty} a_k \le a < +\infty$ and γ_k is chosen by (3.2).

(ii) When $a_k \equiv 0$, Algorithm 4.1 becomes the following self-adaptive primal-dual cyclic iterative algorithm for solving the MSCFP of quasi-nonexpansive operators:

$$\begin{cases} \bar{\omega}_k = (I-U_k)(x_k - \gamma_k A^*(I-T_k)Ax_k + (1-\lambda)\omega_k), \\ \bar{x}_k = x_k - \gamma_k A^*(I-T_k)Ax_k - \lambda\bar{\omega}_k, \\ C_k = \{(u,v) \in G : \|\bar{x}_k - u\|^2 + \lambda\|\bar{\omega}_k - v\|^2 \le \|x_k - u\|^2 + \lambda\|\omega_k - v\|^2\}, \\ Q_k = \{(u,v) \in G : \langle (x_k,\omega_k) - (u,v), (x_1,\omega_1) - (x_k,\omega_k)\rangle \ge 0\}, \\ (x_{k+1},\omega_{k+1}) = P_{C_k\cap Q_k}(x_1,\omega_1), \end{cases}$$

where γ_k is chosen by (3.23).

(iii) When $\lambda = 1$ and $a_k \equiv 0$, Algorithm 4.1 becomes the following self-adaptive cyclic iterative algorithm for solving the MSCFP of quasi-nonexpansive operators:

$$\begin{cases} \bar{x}_k = U_k(x_k - \gamma_k A^*(I-T_k)Ax_k), \\ C_k = \{u \in G : \|\bar{x}_k - u\|^2 \le \|x_k - u\|^2\}, \\ Q_k = \{u \in G : \langle x_k - u, x_1 - x_k\rangle \ge 0\}, \\ x_{k+1} = P_{C_k\cap Q_k}x_1, \end{cases}$$

where γ_k is chosen by (3.23).

Corollary 4.1. *Assume that $0 < \lambda \le 1$, $a_k \subset [0,+\infty)$, $\limsup_{k\to\infty} a_k \le a < +\infty$, and $0 < \liminf_{k\to\infty}\rho_k \le \limsup_{k\to\infty}\rho_k < 2$. Let the sequence $\{(x_k,\omega_k)\}$ be defined by*

$$\begin{cases} arbitrarily\ chosing\ x_0,\ x_1,\ \omega_0 \in H_1\ and\ set\ \omega_1 = \omega_0, \\ y_k = x_k + a_k(x_k - x_{k-1}), \\ \bar{v}_k = y_k - \gamma_k A^*(I - P_{Q_{[k]_2}})Ay_k), \\ \bar{\omega}_k = (I - P_{C_{[k]_1}})(v_k + (1-\lambda)\omega_k), \\ \bar{x}_k = v_k - \lambda\bar{\omega}_k, \\ C_k = \{(u,v) \in G : \|\bar{x}_k - u\|^2 + \lambda\|\bar{\omega}_k - v\|^2 \le \|y_k - u\|^2 + \lambda\|\omega_k - v\|^2\}, \\ Q_k = \{(u,v) \in G : \langle (x_k,\omega_k) - (u,v), (x_1,\omega_1) - (x_k,\omega_k)\rangle \ge 0\}, \\ (x_{k+1},\omega_{k+1}) = P_{C_k\cap Q_k}(x_1,\omega_1), \end{cases} \tag{4.19}$$

where the stepsize γ_k is chosen by (3.26). Then $\{(x_k,\omega_k)\}$ converges strongly to $(x^*,0)$, where x^* is a solution of the MSFP.

Remark 4.2. (i) When $\lambda = 1$, algorithm (4.19) becomes the following self-adaptive inertial cyclic iterative algorithm for solving the MSFP:

$$\begin{cases} y_k = x_k + a_k(x_k - x_{k-1}), \\ \bar{x}_k = P_{C_{[k]_1}}(y_k - \gamma_k A^*(I - P_{Q_{[k]_2}})Ay_k), \\ C_k = \{u \in G : \|\bar{x}_k - u\|^2 \le \|y_k - u\|^2\}, \\ Q_k = \{u \in G : \langle (x_k - u, x_1 - x_k\rangle \ge 0\}, \\ x_{k+1} = P_{C_k\cap Q_k}x_1, \end{cases}$$

where $a_k \subset [0,+\infty)$ with $\limsup_{k\to\infty} a_k \le a < +\infty$ and γ_k is chosen by (3.26).

(ii) When $a_k \equiv 0$, algorithm (4.19) becomes the following self-adaptive primal-dual cyclic iterative algorithm for solving the MSFP :

$$\begin{cases} \bar{\omega}_k = (I - P_{C_{[k]_1}})(x_k - \gamma_k A^*(I - P_{Q_{[k]_2}})Ax_k) + (1-\lambda)\omega_k), \\ \bar{x}_k = x_k - \gamma_k A^*(I - P_{Q_{[k]_2}})Ax_k) - \lambda \bar{\omega}_k, \\ C_k = \{(u,v) \in G : \|\bar{x}_k - u\|^2 + \lambda \|\bar{\omega}_k - v\|^2 \le \|x_k - u\|^2 + \lambda \|\omega_k - v\|^2\}, \\ Q_k = \{(u,v) \in G : \langle (x_k, \omega_k) - (u,v), (x_1, \omega_1) - (x_k, \omega_k) \rangle \ge 0\}, \\ (x_{k+1}, \omega_{k+1}) = P_{C_k \cap Q_k}(x_1, \omega_1), \end{cases}$$

where γ_k is chosen by (3.29).

(iii) When $\lambda = 1$ and $a_k \equiv 0$, algorithm (4.19) becomes the following self-adaptive cyclic iterative algorithm for solving the MSFP:

$$\begin{cases} \bar{x}_k = P_{C_{[k]_1}}(x_k - \gamma_k A^*(I - P_{Q_{[k]_2}})Ax_k), \\ C_k = \{u \in G : \|\bar{x}_k - u\|^2 \le \|x_k - u\|^2\}, \\ Q_k = \{u \in G : \langle (x_k - u, x_1 - x_k \rangle \ge 0\}, \\ x_{k+1} = P_{C_k \cap Q_k} x_1, \end{cases}$$

where γ_k is chosen by (3.29).

5. Numerical Experiments

In this section, we demonstrate the performance of the proposed Algorithm 3.1 by using it to solve the MSFP. All the codes are written in MATLAB and are performed on a personal Lenovo computer with Intel(R) Core(TM) i5-7200U CPU @ 2.50GHz 2.70 GHz and RAM 4.00GB. For sake of convenience, we denote $e_0 = (0, 0, \cdots, 0)^T$ and $e_1 = (1, 1, \cdots, 1)^T$. In all tables, 'Iter' denotes the number of iteration and 'CPU' denotes the time of iteration.

Example 5.1. Let $A = (a_{ij})_{N \times M}$ be a random matrix, where $a_{ij} \in [100, 200]$ and N, M are two positive integers. Take $C_i = \{x \in \mathbb{R}^M | \sum_{l=1}^M x_l^2 \le r_i^2\}$, and $Q_j = \{x \in \mathbb{R}^N | x \le b_j\}$, where $1 \le i \le p$, $1 \le j \le r$ and $p = r$. For $1 \le i \le p$, given a random M-dimensional negative vector (each component is negative) z_i, $r_i = \|z_i\|$. Then, for $1 \le j \le r$, take $b_j = Az_j$. Find $x \in \bigcap_{i=1}^p C_i$ such that $Ax \in \bigcap_{j=1}^r Q_j$. We take experiment paramaters $p = r = 10$, $\eta = 0.9$, $\lambda = 0.5$, $\varepsilon_k = \frac{1}{k^2}$, and $\rho_k = 1$ for all $k \ge 1$. We define the function $p(x)$ by

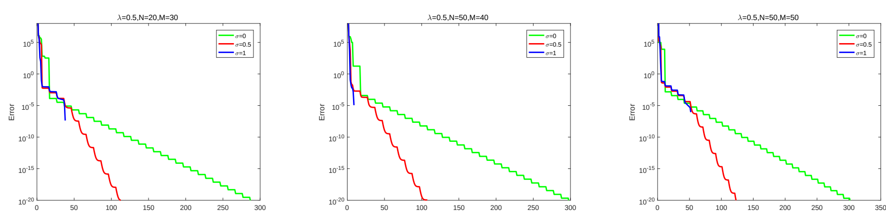
$$p(x) = \sum_{i=1}^p \frac{1}{p} \|x - P_{C_i}(x)\|^2 + \sum_{j=1}^r \frac{1}{r} \|Ax - P_{Q_j}(Ax)\|^2$$

and use the stopping rule $p(x) < \varepsilon = 10^{-20}$.

Applying algorithm (3.25) to solve Example 5.1, we can take inertial extrapolation factor $a_k \in [0, \bar{a}_k]$. Letting $a_k = \sigma \bar{a}_k$, we can choose different inertial extrapolation factors by adjusting parameter $\sigma \in [0, 1]$. When $\sigma = 0$, i.e., $a_k \equiv 0$, algorithm (3.25) becomes the primal-dual cyclic iterative algorithm (3.28) for solving the MSFP. Letting $x_0 = 5e_1$, $x_1 = 30e_1$, and $\omega_0 = -20e_1$, we make a comparison for different inertial extrapolation factors of the algorithm (3.25) with different dimension spaces. Table 1 demonstrates iteration numbers and CPU time of algorithm (3.25) with dimensions $(N, M) = (20, 30)$, $(50, 40)$, and $(50, 50)$. Further, Figure 1 presents error value versus the iteration numbers with dimensions $(N, M) = (20, 30)$, $(50, 40)$,

TABLE 1. Numerical results with different a_k, where $a_k = \sigma \bar{a}_k$.

Initial point	$x_0 = 5e_1$	$x_1 = 30e_1$	$\omega_0 = -20e_1$			
	N=20,M=30		N=50,M=40		N=50,M=50	
	Iter	CPU(s)	Iter	CPU(s)	Iter	CPU(s)
algo (3.28)	287	0.0161	294	0.0186	302	0.0214
$\sigma = 0.1$	249	0.0148	254	0.0204	272	0.0207
$\sigma = 0.2$	217	0.0151	214	0.0166	232	0.0144
$\sigma = 0.3$	177	0.0113	176	0.0125	193	0.0118
$\sigma = 0.4$	138	0.0109	137	0.0124	153	0.0129
$\sigma = 0.5$	90	0.0060	94	0.0061	112	0.0068
$\sigma = 0.6$	20	0.0010	19	0.0010	39	0.0023
$\sigma = 0.7$	19	0.0009	16	0.0009	34	0.0030
$\sigma = 0.8$	18	0.0009	15	0.0008	26	0.0029
$\sigma = 0.9$	18	0.0009	15	0.0008	24	0.0029
$\sigma = 1.0$	18	0.0009	15	0.0008	23	0.0029



(a) (b) (c)

FIGURE 1. Comparison of the iteration number of different inertial extrapolation factors of algorithm (3.25) with different N and M

and $(50,50)$. From Table 1 and Figure 1, we can see that algorithm (3.25) is more effective for solving Example 5.1 with different dimensions and different inertial extrapolation factors. The computation results demonstrate that algorithm (3.25) has better performance with adjusting parameter $\sigma = 1$ for different dimensions.

Next, we use Algorithm 3.1 to solve the following example by regarding projection operators as quasi-nonexpansive operators.

Example 5.2. For $1 \le i \le p$ and $1 \le j \le r$, we choose $C_i \subset \mathbb{R}^M$ and $Q_j \subset \mathbb{R}^N$, which are defined by $C_i := \{x \in \mathbb{R}^M | \langle a_i^C, x\rangle \le b_i^C\}$ and $Q_j := \{x \in \mathbb{R}^N | \langle a_j^Q, x\rangle \le b_j^Q\}$, where $a_i^C \in \mathbb{R}^M$, $a_j^Q \in \mathbb{R}^N$, and $b_i^C, b_j^Q \in \mathbb{R}$. For $1 \le i \le p$ and $1 \le j \le r$, the elements of a_i^C, a_j^Q are randomly generated in the closed interval $[1,3]$ and b_i^C, $b_j^Q \in \mathbb{R}$ are randomly generated in the closed interval $[2,4]$, and $A = (a_{ij})_{N\times M}$ is a bounded linear operator, where a_{ij} is randomly generated in the closed interval $[20,120]$. Further, we define the function $p(x)$ by

$$p(x) = \sum_{i=1}^{p} \frac{1}{p}\|x - P_{C_i}(x)\|^2 + \sum_{j=1}^{r} \frac{1}{r}\|Ax - P_{Q_j}(Ax)\|^2,$$

and use the stopping rule $p(x) < \varepsilon = 10^{-20}$. Set $p = r = 10$, $\eta = 0.9$, $\alpha_k = \beta_k \equiv \frac{1}{2}$, $\varepsilon_k = \frac{1}{k^2}$, and $\rho_k = 1.95$ for all $k \ge 1$.

Applying Algorithm 3.1 to solve Example 5.2, we can take inertial extrapolation factor $a_k \in [0, \bar{a}_k]$. Letting $a_k = \sigma\bar{a}_k$, we can choose different inertial extrapolation factors by adjusting parameter $\sigma \in [0,1]$. When $\sigma = 0$, i.e., $a_k \equiv 0$, Algorithm 3.1 becomes the primal-dual cyclic iterative algorithm (3.22) for solving the MSCFP. When $\lambda = 1$, Algorithm 3.1 becomes cyclic iterative algorithm (3.20) with only inertial technique for solving the MSCFP. When $\sigma = 0$ and $\lambda = 1$, Algorithm 3.1 becomes cyclic iterative algorithm (3.24) for solving the MSCFP. Letting

TABLE 2. Numerical results with different a_k and λ, where $a_k = \sigma\bar{a}_k$.

$N = 10$ $M = 15$

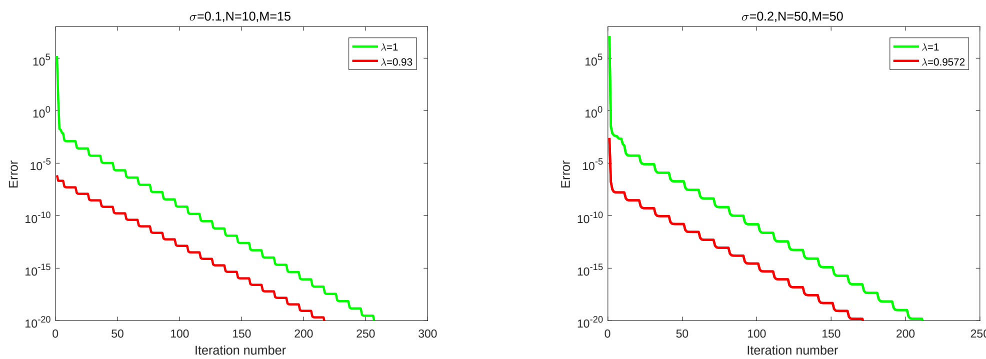
		$\sigma = 0$	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 1$
$\lambda = 1$	Iter	286	256	216	176	127	6
algo (3.20)	CPU(s)	0.0737	0.0658	0.0483	0.0368	0.0283	0.0015
$\lambda = 0.93$	Iter	246	216	186	147	116	3
Alg 3.1	CPU(s)	0.0714	0.0577	0.0480	0.0311	0.0237	0.0005

$x_0 = -5e_1$, $x_1 = 10e_1$, and $\omega_0 = 10e_1$, we make a comparison for different inertial extrapolation factors and different value of λ of Algorithm 3.1 with different dimension spaces. Table 2 and Table 3 demonstrate iteration numbers and CPU time of Algorithm 3.1 for $\sigma = 0, 0.1, 0.2, 0.3, 0.4$, and 1 when dimensions $(N,M) = (10,15)$ and $(N,M) = (50,50)$, respectively. Further, The behavior of error is depicted in Figure 2. Figure 2(a) reports the behavior of $\lambda = 1$ and $\lambda = 0.93$ for $\sigma = 0.1$ and $(N,M) = (10,15)$. And Figure 2(b) reports the behavior of $\lambda = 1$ and $\lambda = 0.9572$ for $\sigma = 0.2$ and $(N,M) = (50,50)$. From Table 2, Table 3 and Figure 2, it can be seen easily that Algorithm 3.1 is faster than algorithm (3.20), algorithm (3.22), and algorithm (3.24) in the speed of convergence for different dimensions. The computation results show that Algorithm 3.1 has better performance with adjusting parameter $\sigma = 1$ for different dimensions and different value of λ.

TABLE 3. Numerical results with different a_k and λ, where $a_k = \sigma \bar{a}_k$.

$N = 50$ $M = 50$

		$\sigma = 0$	$\sigma = 0.1$	$\sigma = 0.2$	$\sigma = 0.3$	$\sigma = 0.4$	$\sigma = 1$
$\lambda = 1$	Iter	290	260	220	180	140	8
algo (3.20)	CPU(s)	0.0968	0.0634	0.0480	0.0372	0.0300	0.0017
$\lambda = 0.9572$	Iter	230	200	170	132	100	3
Algo 3.1	CPU(s)	0.0767	0.0407	0.0364	0.0321	0.0193	0.0005



(a) (b)

FIGURE 2. Comparison of the iteration number of different value of λ of Algorithm 3.1 with different N and M

Acknowledgments

This paper was supported by Scientific Research Project of Tianjin Municipal Education Commission (No. 2020ZD02).

REFERENCES

[1] Y. Censor, T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numer. Algorithms 8 (1994), 221-239.

[2] C. Byrne, Iterative oblique projection onto convex subsets and the split feasibility problem, Inverse Probl. 18 (2002), 441-453.

[3] Y. Censor, T. Bortfeld, B. Martin, A. Trofimov, A unified approach for inversion problems in intensity modulated radiation therapy, Phys. Med. Biol. 51 (2006), 2353-2365.

[4] Y. Censor, A.B. Israel, Y. Xiao, J.M. Galvin, On linear infeasibility arising in intensity-modulated radiation therapy inverse planning, Linear Algebra Appl. 428 (2008), 1406-1420.

[5] Y. Censor, A. Segal, The split common fixed point problem for directed operators, J. Convex Anal. 16 (2009), 587-600.

[6] Y. Censor, T. Elfving, N. Kopf, T. Bortfeld, The multiple-set split feasibility problem and its applications for inverse problems, Inverse Probl. 21 (2005), 2071-2084.

[7] G. Lopez, V. Martin, F.H. Wang, H.K. Xu, Solving the split feasibility problem without prior knowledge of matrix norms, Inverse Probl. 28 (2012), 085004.

[8] J. Zhao, D. Hou, A self-adaptive iterative algorithm for the split common fixed point problems, Numer. Algorithms 82 (2019), 1047-1063.
[9] A. Cegielski, General method for solving the split common fixed point problem, J. Optim. Theory Appl. 165 (2015), 385-404.
[10] X. Qin, A. Petrusel, J.C. Yao, CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces, J. Nonlinear Convex Anal. 19 (2018), 157-165.
[11] M. Eslamian, G.Z. Eskandani, M. Raeisi, Split common null point and common fixed point problems between Banach spaces and Hilbert spaces, Mediterr. J. Math. 14 (2017), 119.
[12] S. Reich, T.M. Tuyen, Two projection algorithms for solving the split common fixed point problem, J. Optim. Theory Appl., 186 (2020), 148-168.
[13] J. Zhao, S. He, Viscosity approximation methods for split common fixed-point problem of directed operators, Numer. Funct. Anal. Optim. 36 (2018), 1199-1215.
[14] S. He, H. Tian, H.K. Xu, The selective projection method fpr convex feasibility and split feasibility problems, J. Nonlinear Convex Anal. 19 (2015), 528-547.
[15] J. Zhao, H. Zong, Iterative algorithm for solving the split feasibility problem in Hilbert space, J. Fixed Point Theory Appl. 20 (2018), 11.
[16] Q.L. Dong, et al., An alternated inertial general splitting method with linearization for the split feasibility problem, Optimmization, doi: 10.1080/02331934.2022.2069567.
[17] A. Moudafi, The split common fixed point problem for demicontractive mappings, Inverse Probl. 26 (2010), 055007.
[18] X. Qin, J.C. Yao, A viscosity iterative method for a split feasibility problem, J. Nonlinear Convex Anal. 20 (2019), 1497-1506.
[19] M. Wen, J.G. Peng, Y.C. Tang, A cyclic and simultaneous iterative method for solving the multiple-sets split feasibility problem, J. Optim. Theory Appl. 166 (2015), 844-860.
[20] F.H. Wang, H.K. Xu, Cyclic algorithms for split feasibility problems in Hilbert spaces, Nonlinear Anal. 74 (2011), 4105-4111.
[21] S. Reich, T.M. Tuyen, N.M. Trang, Parallel iterative methods for solving the split common fixed point problem in Hilbert spaces, Numer. Funct. Anal. Optim. 41 (2020), 778-805.
[22] Y.Z. Dang, J. Sun, H.L. Xu, Inertial accelerated algorithms for solving a split feasibility problem, J. Ind. Manag. Optim. 13 (2017), 1383-1394.
[23] D.V. Thong, D.V. Hieu, An inertial method for solving split common fixed point problems, J. Fixed Point Theory Appl. 19 (2017), 3029-3051.
[24] Y. Wang, X. Li, B. Jiang, Two new inertial relaxed gradient CQ algorithms on the split equality problem, J. Appl. Anal. Comput. 12 (2022), 436-454.
[25] H.H. Cui, H.X. Zhang, L.C. Ceng, An inertial Censor-Segal algorithm for split common fixed-point problems, Fixed Point Theory. 21 (2021), 93-103.
[26] Z. Zhou, B. Tan, S. Li, An accelerated hybrid projection method with a self-adaptive step-size sequence for solving split common fixed point problems, Math. Meth. Appl. Sci. 44 (2021), 7294-7303.
[27] A. Moudafi, A note on the split common fixed point problem for quasi-nonexpansive operators, Nonlinear Anal. 74 (2011), 4083-4087.
[28] P.E. Mainge, Convergence theorems for inertial KM-type algorithms, J. Comput. Appl. Math. 219 (2008), 223-236.
[29] H.K. Xu, Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces, Inverse Probl. 26 (2010), 105018.
[30] H.H. Bauschke, The approximation of fixed points of composition of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. 202 (1996), 150-159.
[31] E.H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory, Contributions to Nonlinear Functional Analysis, pp. 19-32, American Press, New York, 1971.