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A DOUBLE PROJECTION ALGORITHM WITH INERTIAL EFFECTS FOR SOLVING SPLIT FEASIBILITY PROBLEMS AND APPLICATIONS TO IMAGE RESTORATION

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Abstract. In this paper, we derive a new projection algorithm by incorporating inertial effects for solving a split feasibility problem in real Hilbert spaces. We then establish a weak convergence theorem under some suitable conditions. As an application, we apply our result to image restoration.

Keywords. Inertial technique; Projection method; Split feasibility problem; Weak convergence.

1. Introduction

The main objective of this paper is to present a new projection algorithm for solving the split feasibility problem (SFP), which was introduced by Censor and Elfving [1] and modeled as seeking a point x^* such that $x^* \in C$ and $Ax^* \in Q$, where C and Q are nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and A is a bounded linear operator from H_1 to H_2. The split feasibility problem appears in various fields of science and technology, such as signal processing, image reconstruction, and intensity-modulated radiation therapy; see, e.g., [2, 3, 4, 5] and the references therein.

For solving the SFP, many methods were developed, such as the CQ algorithm proposed by Byrne [2], the relaxed CQ algorithm proposed by Yang [6], the half space relaxation projection method proposed by Qu and Xiu [7]. One refers to [8, 9, 10, 11, 12] for recent various methods. Set $F(x) = A^*(I - P_Q)Ax$, $x \in H_1$, where A^* is an adjoint operator of A. In 2002, Byrne [2] introduced the following iterative procedure for the SFP:

$$x_{n+1} = P_C(x_n + \beta_n F(x_n)),$$

where $\beta_n \in (0, 2/L)$, L is the largest eigenvalue of the matrix A^*A, and P_C and P_Q are the metric projections onto C and Q, respectively. Later, Yang [6] replaced these projections by the

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projections onto half spaces. Nevertheless, the stepsize of the proposed algorithms depend on the operator norm which is not simple to compute in general.

In 2012, Zhao et al. [13] introduced the modified projection method for the SFP. Let $x_1 \in H_1$, $\gamma_0 > 0$, $\ell \in (0,1)$, $\mu \in (0,1)$, $\rho \in (0,1)$, and let $y_n = P_C(x_n - \beta_n F(x_n))$, where β_n is chosen to be the largest $\beta \in \{\gamma_n, \gamma_n \ell, \gamma_n \ell^2, \dots\}$ satisfying $\beta \|F(x_n) - F(y_n)\| \le \mu \|x_n - y_n\|$. Define

$$x_{n+1} = P_C(y_n - \beta_n(F(y_n) - F(x_n))).$$

If $\beta_n \|F(x_{n+1}) - F(x_n)\| \le \rho \|x_{n+1} - x_n\|$, then set $\gamma_n = \gamma_0$. Otherwise, set $\gamma_n = \beta_n$.

Recently, Dong et al. [14] presented an optimal choice of the step length of the projection and contraction methods for solving the SFP. Choose $\gamma > 0$, $\ell \in (0,1)$, $t \in (0,2)$, and $\mu \in (0,1)$. Take $x_1 \in H_1$ and let $y_n = P_C(x_n - \beta_n F(x_n))$, where $\beta_n = \gamma \ell^{m_n}$ and m_n is the smallest nonnegative integer such that $\beta_n \|F(x_n) - F(y_n)\| \le \mu \|x_n - y_n\|$. Define

$$x_{n+1} = x_n - t\rho_n d(x_n, y_n),$$

where $d(x_n, y_n) = (x_n - y_n) - \beta_n(F(x_n) - F(y_n))$ and

$$\rho_n = \frac{\langle x_n - y_n, d(x_n, y_n)\rangle + \beta_n \|(I - P_Q)A(y_n)\|^2}{\|d(x_n, y_n)\|^2}.$$

In 2017, Dang et al. [15] introduced double projection algorithms for solving the SFP, which do not require the fixed stepsize and do not employ the same projection region at different projection steps. Let $x_1 \in H_1$ be arbitrarily and $\gamma > 0$, $\ell \in (0,1)$, $\lambda > 1$, and $t \in (0,2)$. Define

$$y_n = P_C(x_n - \beta_n F(x_n)),$$

where $\beta_n = \gamma \ell^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\langle F(x_n), x_n - y_n\rangle \ge \lambda \langle F(x_n) - F(y_n), x_n - y_n\rangle.$$

Compute

$$x_{n+1} = P_C(x_n - t\alpha_n F(y_n)),$$

where

$$\alpha_n = \frac{\langle F(y_n), x_n - y_n\rangle}{\|F(y_n)\|^2}.$$

In 1964, Polyak [16] introduced an inertial extrapolation for solving the smooth convex minimization problem. Later, Nesterov [17] introduced a method of solving a convex programming problem with convergence rate $O(1/k^2)$. Let $x_0, x_1 \in H$ be arbitrarily and $0 \le \theta_n < 1$. Define

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = y_n + \beta \nabla f(y_n), \end{cases}$$

where β is a positive constant and the term $\theta_n(x_n - x_{n-1})$ is called the inertial term. Since the inertial term in the method speeds up the convergence, many inertial methods were extensively studied; see, e.g., [18, 19, 20, 21, 22] and the references therein.

In this paper, motivated and inspired by the previous works, we propose a new projection algorithm for solving the SFP and prove the weak convergence theorems under some suitable assumptions in real Hilbert spaces. As applications, we apply our main result to an image restoration.

2. Preliminaries

In this section, we collect some basic definitions and lemmas which are used in the sequel. Let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. In what follows, we use the following notations:

- the symbols $\rightharpoonup$ stands for the weak convergence.
- the symbols $\to$ stands for the strong convergence.

Recall that a mapping $T : H_1 \to H_1$ is said to be

(1) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in H_1$.

(2) firmly-nonexpansive if $\langle Tx - Ty, x - y \rangle \geq \|Tx - Ty\|^2$, $\forall x, y \in H_1$.

We note that if T is firmly-nonexpansive, then $I - T$ is also firmly-nonexpansive. In a real Hilbert space H_1, we know the following relations:

$$\langle x, y \rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x - y\|^2, \ \forall x, y \in H_1$$

and

$$\|\alpha x + (1-\alpha) y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2, \ \forall x, y \in H_1 \text{ and } \forall \alpha \in [0,1].$$

A differentiable function f is convex if and only if there holds the inequality: $f(z) \geq f(x) + \langle \nabla f(x), z - x \rangle$ for all $z \in H_1$. An element $g \in H_1$ is called a *subgradient* of $f : H_1 \to \mathbb{R}$ at x if $f(z) \geq f(x) + \langle g, z - x \rangle$ for all $z \in H_1$, which is called the *subdifferentiable inequality*. A function $f : H_1 \to \mathbb{R}$ is said to be *subdifferentiable* at x if it has at least one subgradient at x. The set of subgradients of f at the point x is called the *subdifferentiable* of f at x, which is denoted by $\partial f(x)$. A function f is said to be *subdifferentiable* if it is subdifferentiable at all $x \in H_1$. If a function f is differentiable and convex, then its gradient and subgradient coincide. A function $f : H_1 \to \mathbb{R}$ is said to be *weakly lower semi-continuous* (shortly, w-lsc) at x if $x_n \rightharpoonup x$ implies $f(x) \leq \liminf_{n\to\infty} f(x_n)$. We know that the orthogonal projection P_C from H_1 onto a nonempty closed convex subset $C \subset H_1$ is a typical example of a firmly nonexpansive mapping, which is defined by $P_C x = \arg\min_{y \in C} \|x - y\|^2$ for all $x \in H_1$.

Lemma 2.1. [23] *Let C be a closed, convex, and nonempty subset of a real Hilbert space H_1. Then, for any $x \in H_1$, the following assertions hold:*

(1) $\langle x - P_C x, z - P_C x \rangle \leq 0$ *for all* $z \in C$*;*

(2) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$ *for all* $x, y \in H_1$*;*

(3) $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2$ *for all* $z \in C$.

From Lemma 2.1, the operator $I - P_C$ is also firmly nonexpansive, where I denotes the identity operator, i.e., for any $x, y \in H_1$, $\langle (I - P_C)x - (I - P_C)y, x - y \rangle \geq \|(I - P_C)x - (I - P_C)y\|^2$.

Lemma 2.2. [24] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be positive real sequences such that $a_{n+1} \leq (1 + c_n)a_n + b_n$, $n \geq 1$. If $\Sigma_{n=1}^{\infty} c_n < +\infty$ and $\Sigma_{n=1}^{\infty} b_n < +\infty$, then $\lim_{n\to+\infty} a_n$ exists.*

Lemma 2.3. [25] *Let $\{a_n\}$ and $\{\theta_n\}$ be positive real sequences such that $a_{n+1} \leq (1 + \theta_n)a_n + \theta_n a_{n-1}$, $n \geq 1$. Then, $a_{n+1} \leq K \cdot \prod_{i=1}^{n}(1 + 2\theta_i)$, where $K = \max\{a_1, a_2\}$. Moreover, if $\sum_{n=1}^{\infty} \theta_n < +\infty$, then $\{a_n\}$ is bounded.*

Lemma 2.4. [26] *Let F be a mapping from a Hilbert space H_1 to H_1. For any $x \in H_1$ and $\alpha \geq 0$, define $x(\alpha) = P_C(x - \alpha F(x))$ and $e(x, \alpha) = x - x(\alpha)$. Then,* $\min\{1, \alpha\}\|e(x,1)\| \leq \|e(x,\alpha)\| \leq \max\{1,\alpha\}\|e(x,1)\|$.

Lemma 2.5. [27] *Let S be a nonempty subset of a real Hilbert space H_1, and let $\{x_n\}$ be a sequence in H_1 that satisfies the following properties:*
(i) $\lim_{n\to\infty}\|x_n - x\|$ *exists for each $x \in S$;*
(ii) every sequential weak limit point of $\{x_n\}$ is in S.
Then $\{x_n\}$ converges weakly to a point in S.

Lemma 2.6. [28] *Let C and Q be closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $f(x) = \frac{1}{2}\|(I-P_Q)Ax\|^2$, $x \in H_1$. Then ∇f is $\|A\|^2$-Lipschitz continuous.*

3. The Double Projection algorithm

In this section, we present our new algorithm with the aid of the inertial technique. Assume that the solution set of the split feasibility problem is nonempty and we denote by Γ. We define the function $F : H_1 \to H_1$ as $F(x) = A^*(I-P_Q)Ax$, $x \in H_1$.

Algorithm 3.1. Select a point $x_1 \in H_1$ arbitrarily, and choose $\gamma > 0$, $\ell \in (0,1)$, $\lambda > 1$, and $t \in (0,2)$. Let $\{\theta_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Define

$$\begin{aligned} w_n &= x_n + \theta_n(x_n - x_{n-1}), \\ y_n &= P_C(w_n - \beta_n F(w_n)), \end{aligned} \tag{3.1}$$

where $\beta_n = \gamma\ell^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\langle F(w_n), w_n - y_n\rangle \geq \lambda\langle F(w_n) - F(y_n), w_n - y_n\rangle. \tag{3.2}$$

Compute

$$x_{n+1} = P_C(w_n - t\alpha_n F(y_n)),$$

where

$$\alpha_n = \frac{\langle F(y_n), w_n - y_n\rangle}{\|F(y_n)\|^2}.$$

The following lemmas were proved in [15].

Lemma 3.1. *There exists a nonnegative number m_n satisfying (3.2) for all $n \geq 1$.*

Lemma 3.2. $\frac{\ell}{\lambda(\|A\|^2+1)} < \beta_n \leq \gamma$ *for all $n \geq 1$.*

Theorem 3.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. If $\sum_{n=1}^{\infty}\theta_n < \infty$, then $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a solution in Γ.*

Proof. Let $z \in \Gamma$. Then $z = P_C(z)$ and $Az = P_Q(Az)$. It follows that $F(z) = 0$. Using Lemma 2.1 (2), we see that

$$\begin{aligned} \langle F(y_n), y_n - z\rangle &= \langle F(y_n) - F(z), y_n - z\rangle \\ &= \langle A^*(I-P_Q)Ay_n - A^*(I-P_Q)Az, y_n - z\rangle \\ &\geq \|(I-P_Q)Ay_n\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} \langle F(y_n), w_n - z\rangle &= \langle F(y_n), w_n - y_n\rangle + \langle F(y_n), y_n - z\rangle \\ &\geq \langle F(y_n), w_n - y_n\rangle. \end{aligned} \tag{3.3}$$

By (3.2), we see that

$$\begin{aligned}\langle F(y_n), w_n - y_n\rangle &= -\langle F(w_n) - F(y_n), w_n - y_n\rangle + \langle F(w_n), w_n - y_n\rangle \\ &\geq -\frac{1}{\lambda}\langle F(w_n), w_n - y_n\rangle + \langle F(w_n), w_n - y_n\rangle \\ &= \left(1 - \frac{1}{\lambda}\right)\langle F(w_n), w_n - y_n\rangle. \end{aligned} \tag{3.4}$$

From definition of y_n and Lemma 2.1 (1), we have

$$\begin{aligned}\langle F(w_n), w_n - y_n\rangle &= \frac{1}{\beta_n}\langle \beta_n F(w_n) - w_n + y_n + w_n - y_n, w_n - y_n\rangle \\ &= \frac{1}{\beta_n}[\langle y_n - (w_n - \beta_n F(w_n)), w_n - y_n\rangle + \langle w_n - y_n, w_n - y_n\rangle] \\ &= \frac{1}{\beta_n}[\langle P_C(w_n - \beta_n F(w_n)) - (w_n - \beta_n F(w_n)), w_n - P_C(w_n - \beta_n F(w_n))\rangle \\ &\quad + \langle w_n - y_n, w_n - y_n\rangle] \\ &\geq \frac{1}{\beta_n}\|w_n - y_n\|^2. \end{aligned} \tag{3.5}$$

Using (3.3) and Lemma 2.1 (3), we obtain

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \|w_n - z - t\alpha_n F(y_n)\|^2 - \|x_{n+1} - w_n + t\alpha_n F(y_n)\|^2 \\ &= \|w_n - z\|^2 - 2t\alpha_n\langle F(y_n), w_n - z\rangle + t^2\alpha_n^2\|F(y_n)\|^2 \\ &\quad - \|x_{n+1} - w_n + t\alpha_n F(y_n)\|^2 \\ &\leq \|w_n - z\|^2 - 2t\alpha_n\langle F(y_n), w_n - y_n\rangle + t^2\alpha_n^2\|F(y_n)\|^2 \\ &\quad - \|x_{n+1} - w_n + t\alpha_n F(y_n)\|^2. \end{aligned}$$

Note that

$$\langle x_n - z, x_n - x_{n-1}\rangle = \frac{1}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2 - \frac{1}{2}\|x_{n-1} - z\|^2.$$

Combining (3.4), (3.5), and Lemma 3.2, we obtain

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ \leq\ & \|w_n - z\|^2 - \frac{2t\langle F(y_n), w_n - y_n\rangle^2}{\|F(y_n)\|^2} + \frac{t^2\langle F(y_n), w_n - y_n\rangle^2}{\|F(y_n)\|^2} - \|x_{n+1} - w_n + t\alpha_n F(y_n)\|^2 \\ =\ & \|w_n - z\|^2 - t(2-t)\frac{\langle F(y_n), w_n - y_n\rangle^2}{\|F(y_n)\|^2} - \|x_{n+1} - w_n + t\alpha_n F(y_n)\|^2 \\ \leq\ & \|w_n - z\|^2 - t(2-t)\left(1 - \frac{1}{\lambda}\right)^2\frac{1}{\gamma^2}\frac{\|w_n - y_n\|^4}{\|F(y_n)\|^2} - \|x_{n+1} - w_n + t\alpha_n F(y_n)\|^2. \end{aligned} \tag{3.6}$$

It follows that $\|x_{n+1} - z\| \leq \|w_n - z\|$. On the other hand, we have

$$\begin{aligned}\|x_{n+1} - z\| &\leq \|w_n - z\| \\ &\leq \|x_n - z\| + \theta_n\|x_n - x_{n-1}\| \\ &\leq (1+\theta_n)\|x_n - z\| + \theta_n\|x_{n-1} - z\|. \end{aligned} \tag{3.7}$$

By Lemma 2.3, we obtain $\|x_{n+1}-z\| \leq \prod_{i=1}^{n}(1+2\theta_i)$, where $K=\max\{\|x_1-z\|,\|x_2-z\|\}$. Thus $\{x_n\}$ is bounded. Since $\sum_{n=1}^{\infty}\theta_n<+\infty$ and $\{x_n\}$ is bounded, we have $\sum_{n=1}^{\infty}\theta_n\|x_n-x_{n-1}\|<+\infty$. By Lemma 2.2 and (3.7), we obtain $\lim_{n\to\infty}\|x_n-z\|$ exists. From definition of w_n, we have

$$\begin{aligned}\|w_n-z\|^2 &= \|x_n-z\|^2+2\theta_n\langle x_n-z, x_n-x_{n-1}\rangle+\theta_n^2\|x_n-x_{n-1}\|^2\\ &\leq \|x_n-z\|^2+2\theta_n\|x_n-z\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2. \quad (3.8)\end{aligned}$$

From (3.6) and (3.8), we conclude

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq\ & \|x_n-z\|^2+2\theta_n\|x_n-z\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2 \quad (3.9)\\ & -t(2-t)\left(1-\frac{1}{\lambda}\right)^2\frac{1}{\gamma^2}\frac{\|w_n-y_n\|^4}{\|F(y_n)\|^2}-\|x_{n+1}-w_n+t\alpha_nF(y_n)\|^2.\end{aligned}$$

From (3.9), it follows that

$$\lim_{n\to\infty} t(2-t)\left(1-\frac{1}{\lambda}\right)^2\frac{1}{\gamma^2}\frac{\|w_n-y_n\|^4}{\|F(y_n)\|^2}=0.$$

Using the assumptions, we have

$$\lim_{n\to\infty}\frac{\|w_n-y_n\|^4}{\|F(y_n)\|^2}=0.$$

Using Lemma 2.6, we know that $\{\|F(y_n)\|\}$ is bounded. Hence, it implies that

$$\lim_{n\to\infty}\|w_n-y_n\|=0. \quad (3.10)$$

Furthermore, from (3.9) we obtain

$$\lim_{n\to\infty}\|x_{n+1}-w_n+t\alpha_nF(y_n)\|=0 \quad (3.11)$$

and

$$\alpha_n\|F(y_n)\| = \frac{\langle F(y_n), w_n-y_n\rangle}{\|F(y_n)\|}\leq\|w_n-y_n\|\to 0 \text{ as } n\to\infty. \quad (3.12)$$

From (3.11) and (3.12), we obtain

$$\lim_{n\to\infty}\|x_{n+1}-w_n\|=0. \quad (3.13)$$

From (3.1), we have

$$\lim_{n\to\infty}\|w_n-x_n\|=0. \quad (3.14)$$

From (3.13) and (3.14), we have $\|x_{n+1}-x_n\|\leq\|x_{n+1}-w_n\|+\|w_n-x_n\|\to 0$ as $n\to\infty$. In view of (3.10) and (3.13), we conclude that $\|x_{n+1}-y_n\|\leq\|x_{n+1}-w_n\|+\|w_n-y_n\|\to 0$ as $n\to\infty$. Since $\{x_n\}$ is bounded, there is a point x^* of $\{x_n\}$ with a subsequence $\{x_{n_k}\}$ converging to x^*. It follows that $\{x_{n_k+1}\}$ also converges to x^*.

Now, we prove that x^* is in Γ. From (3.13) and (3.12), we have

$$\begin{aligned}\|x_{n_k+1}-P_C(x_{n_k+1})\| &= \|P_C(w_{n_k}-t_{n_k}\alpha_{n_k}F(y_{n_k}))-P_C(x_{n_k+1})\|\\ &\leq \|w_{n_k}-t_{n_k}\alpha_{n_k}F(y_{n_k})-x_{n_k+1}\|\\ &\leq \|w_{n_k}-x_{n_k+1}\|+t_{n_k}\alpha_{n_k}\|F(y_{n_k})\|\\ &\to 0 \text{ as } k\to\infty,\end{aligned}$$

which indicates that $x^*\in C$ by the demiclosedness of P_C.

Next, we prove that $Ax^* \in Q$. Define $e_{n_k}(w,\mu) = w - P_C(w_{n_k} - \mu F(w_{n_k}))$. From Lemma 2.4, the definition of β_n, and (3.14), we have

$$\begin{aligned}\lim_{k\to\infty}\|e_{n_k}(w_{n_k},1)\| &\leq \lim_{k\to\infty}\frac{\|w_{n_k}-y_{n_k}\|}{\min\{1,\beta_{n_k}\}}\\ &\leq \lim_{k\to\infty}\frac{\|w_{n_k}-y_{n_k}\|}{\min\{1,\underline{\beta}\}}=0, \end{aligned} \tag{3.15}$$

where $\underline{\beta} = \frac{1}{\lambda(\|A\|^2+1)}$. Using Lemma 2.1 (1) and noting that $x^* \in C$, we have, for all $k = 1,2,...,$

$$\langle w_{n_k} - F(w_{n_k}) - P_C(w_{n_k} - F(w_{n_k})), x^* - P_C(w_{n_k} - F(w_{n_k}))\rangle \leq 0.$$

Thus $\langle e_{n_k}(w_{n_k},1) - F(w_{n_k}), w_{n_k} - x^* - e_{n_k}(w_{n_k},1)\rangle \geq 0$, which implies from Lemma 2.1 (1) that, for all $k = 1,2,...,$

$$\begin{aligned} &\langle w_{n_k} - x^*, e_{n_k}(w_{n_k},1)\rangle\\ \geq\ & \|e_{n_k}(w_{n_k},1)\|^2 - \langle F(w_{n_k}), e_{n_k}(w_{n_k},1)\rangle + \langle F(w_{n_k}), w_{n_k} - x^*\rangle\\ =\ & \|e_{n_k}(w_{n_k},1)\|^2 - \langle F(w_{n_k}), e_{n_k}(w_{n_k},1)\rangle + \langle (I-P_Q)Aw_{n_k} - (I-P_Q)Ax^*, Aw_{n_k} - Ax^*\rangle\\ \geq\ & \|e_{n_k}(w_{n_k},1)\|^2 - \langle F(w_{n_k}), e_{n_k}(w_{n_k},1)\rangle + \|(I-P_Q)Aw_{n_k} - (I-P_Q)Ax^*\|^2\\ =\ & \|e_{n_k}(w_{n_k},1)\|^2 - \langle F(w_{n_k}), e_{n_k}(w_{n_k},1)\rangle + \|(I-P_Q)Aw_{n_k}\|^2. \end{aligned} \tag{3.16}$$

Since $\{w_{n_k}\}$ is bounded, one asserts that $\{\|F(w_{n_k})\|\}$ is also bounded. From (3.15) and (3.16), we obtain that $\lim_{k\to\infty}\|(I-P_Q)Aw_{n_k}\| = 0$. Hence, $Aw_{n_k} \rightharpoonup Ax^*$. Furthermore, we obtain $Ax^* \in Q$. Finally, we conclude that the sequence $\{x_n\}$ converges weakly to a point in Γ by Lemma 2.5. This completes the proof. □

Remark 3.1. From the viewpoint of convergence speed, our algorithm, which is based on the inertial technique, mainly improves those of Zhao et al. [13], Dong et al. [14], and Dang et al. [15].

4. Numerical Experiments

In this section, we present numerical experiments to an image debluring. Let $C = [0,255]^D$ such that $D = M \times N$, where M is the pixels of width and N is the pixels of height of color image. Consider the minimization problem: $\min_{x\in C}\|Ax - y\|_2$. This problem can be solved via the SFP when $Q = \{y\}$ and $C = [0,255]^D$. We compare the following methods with $x_0 = x_1 = (1,1,1,...,1) \in \mathbb{R}^N$.

Method 1: The algorithm of Zhao et al. [13] with $\gamma_0 = 1$, $\ell = 0.8$, $\mu = 0.1$, and $\rho = 0.5$;
Method 2: The algorithm of Dong et al. [14] with $\gamma = 1$, $\ell = 0.8$, $\mu = 0.1$, and $t = 1.9$;
Method 3: The algorithm of Dang et al. [15] with $\gamma = 1$, $\ell = 0.8$, $\lambda = 1.2$, and $t = 1.9$;
Method 4: The algorithm 3.1 with $\gamma = 1$, $\ell = 0.8$, $\lambda = 1.2$, and $t = 1.9$.

We consider three blur types with the images size 268×201 for RGB images as follows:
(i) Motion blur with motion length of 45 pixels and motion orientation 180°.
(ii) Gaussian blur of filter size 5×5 with standard deviation 5.
(iii) Out of focus with radius 7.

To measure the restored images, we use the Peak-signal-to-noise ratio (PSNR) [29] defined by

$$PSNR = 10\log_{10}\left(\frac{255^2}{MSE}\right),$$

where MSE$= \|x_n - x\|^2$ such that x is an original image. We also use Structural Similarity Index Measure (SSIM) [30] for measuring the similarity between two images. From the definitions, it is clear that the high PSNR and SSIM values show the quality of restored images. All codes are written in Matlab (version R2020b) on MacBook Pro M1 with ram 8 GB.

We obtain numerical results as follows:

TABLE 1. The comparison of PSNR and SSIM of the restored images

Methods	Motion blur		Gaussian blur		Out of focus	
	PSNR	SSIM	PSNR	SSIM	PSNR	SSIM
Method 1	30.4787	0.8688	39.6649	0.9722	33.0624	0.8948
Method 2	31.9010	0.9016	41.1588	0.9793	34.7088	0.9188
Method 3	25.7223	0.7487	36.4745	0.9467	29.8484	0.8234
Method 4	40.3784	0.9817	47.4718	0.9939	40.6514	0.9703

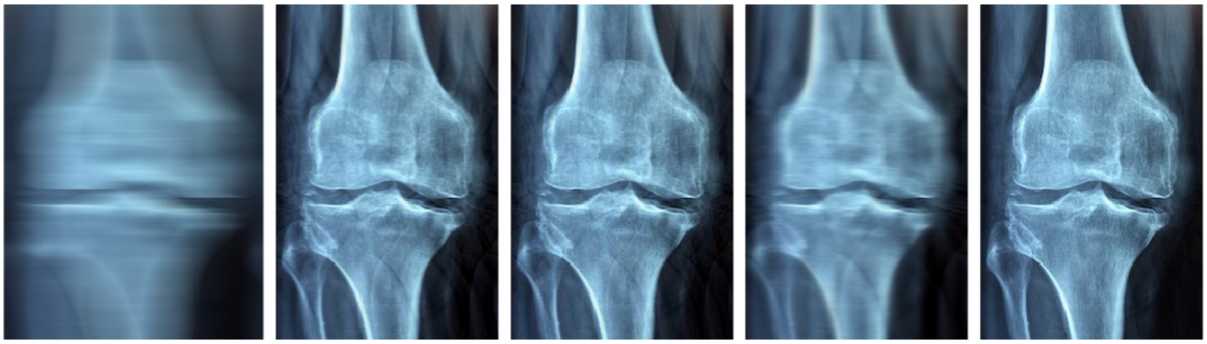
From Table 1, it appears that our algorithm (Method 4) is more efficient than the others (Methods 1, 2, 3) because the PSNR and SSIM values of our algorithm takes the highest number in the experiment for maximum 1,500 iterations. The following original RGB image was tested.

FIGURE 1. The original of RGB image

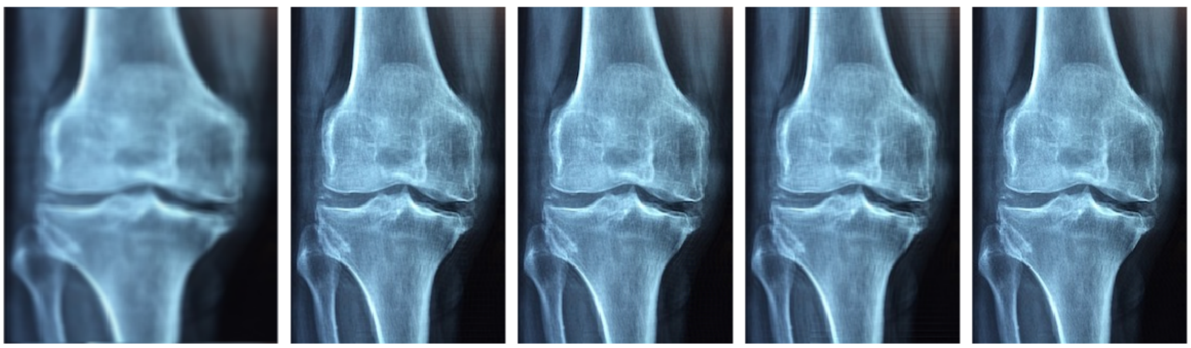
We next demonstrate the figures of blurred images and restored images.

5. CONCLUSIONS

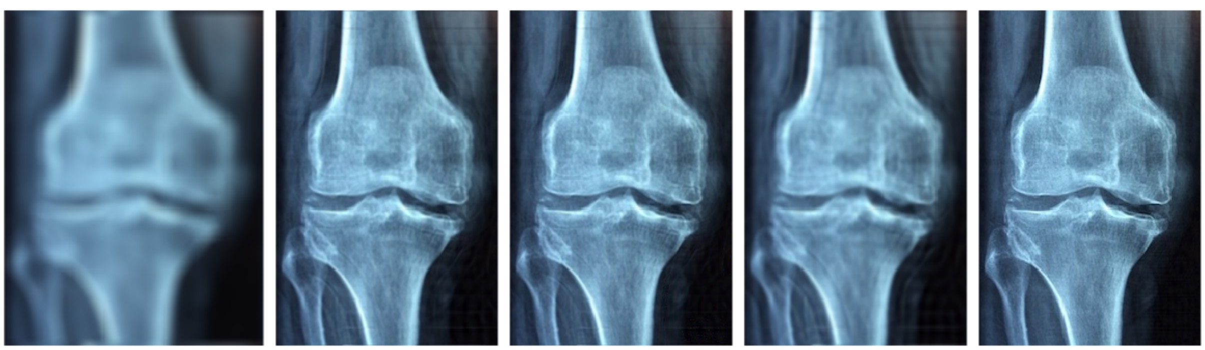
In this paper, we proposed a new double projection algorithm with the inertial effects. Under suitable conditions, a weak convergence theorem of the proposed algorithm was established. Numerical experiments in an image restoration demonstrated that our algorithm has a higher efficiency than other methods in terms PSNR and SSIM.



(a) Motion blurred image (b) PSNR=30.4787 (c) PSNR=31.9010 (d) PSNR=25.7223 (e) PSNR=40.3784

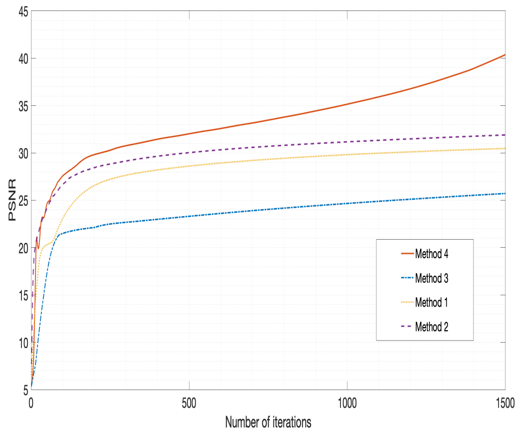


(f) Gaussian blurred image (g) PSNR=39.6649 (h) PSNR=41.1588 (i) PSNR=36.4745 (j) PSNR=47.4718

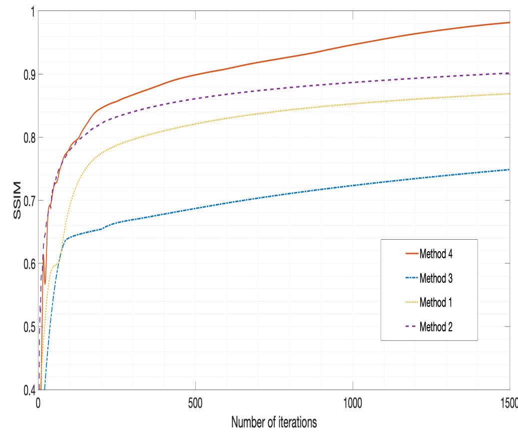


(k) Out of focus image (l) PSNR=33.0624 (m) PSNR=34.7088 (n) PSNR=29.8484 (o) PSNR=40.6514

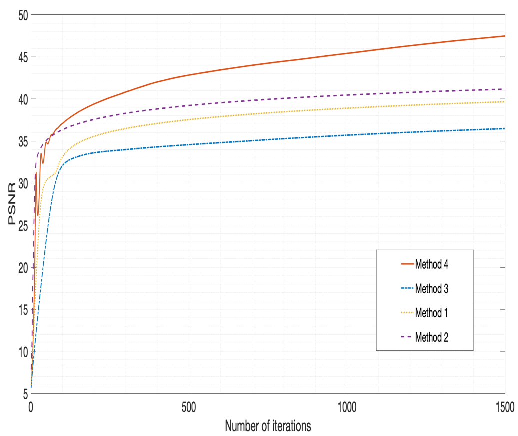
FIGURE 2. The restored images with PSNR values for each blurs using Methods 1,2,3, and 4 from left to right



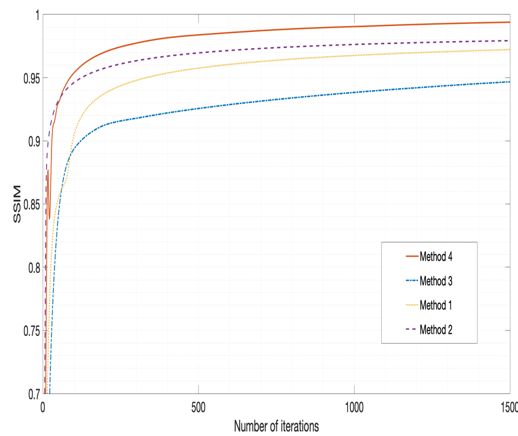
(a) PSNR plotting of Motion blur



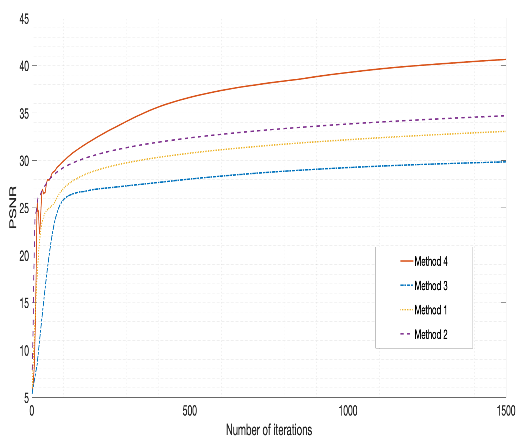
(b) SSIM plotting of Motion blur



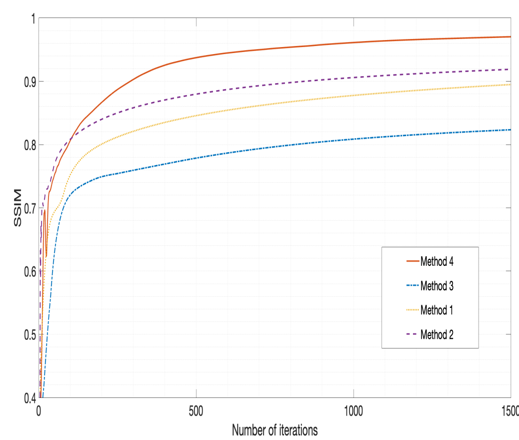
(c) PSNR plotting of Gaussian blur



(d) SSIM plotting of Gaussian blur



(e) PSNR plotting of Out of focus



(f) SSIM plotting of Out of focus

FIGURE 3. Graphs of PSNR and SSIM plotting for each blurs and methods

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