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MULTIPLICITY OF POSITIVE RADIAL SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATION WITH LOCALLY CONCAVE-CONVEX VARIABLE EXPONENT

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Abstract. This paper is concerned with the following semilinear elliptic equation

$$\begin{cases} -\Delta u = u^{q(x)-1}, & \text{in } B, \\ u > 0, & \text{in } B, \\ u = 0, & \text{in } \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^N (N \geq 3)$, $q(x) = q(|x|)$ is a continuous radial function satifying $1 < \min_{x\in\overline{B}} q(x) = q_- < 2 < q_+ = \max_{x\in\overline{B}} q(x) < 2^* = \frac{2N}{N-2}$, and $q(0) > 2$. By means of variational methods and a priori estimate, we obtain that the problem above has at least two positive radial solutions.

Keywords. A priori estimate; Semilinear elliptic equation; Variable exponent; Variational methods.

1. Introduction and main result

In recent years, the following nonlinear elliptic equation

$$\begin{cases} -\Delta_{p(x)} u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

received considerable attention due to the fact that it can be applied to fluid mechanics and the field of image processing (see [1, 2, 3]), where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $p : \overline{\Omega} \to \mathbb{R}$ is a continuous function satisfying $1 < p_- = \min_{x\in\overline{\Omega}} p(x) \leq p(x) \leq \max_{x\in\overline{\Omega}} p(x) = p_+ < N$, $\Delta_{p(x)} u = div(|\nabla u|^{p(x)-2}\nabla u)$, and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a suitable function.

In 2003, Fan and Zhang in [4] gave several sufficient conditions for the existence and multiplicity of nontrivial solutions of problem (1.1). These conditions include either the sublinear growth condition $|f(x,t)| \leq C\left(1 + |t|^{p^-}\right)$ for $x \in \Omega$ and $t \in \mathbb{R}$ or Ambrosetti-Rabinowitz type growth condition ((AR)-condition, for short) $f(x,t)t \geq \theta F(x,t) > 0$ for all $x \in \Omega$ and $|t|$ sufficiently large, where $C > 0$, $\theta > p^+$, $F(x,t) = \int_0^t f(x,s)\,ds$, and $|f(x,t)t| \leq C(1 + |t|^{p^*(x)})$ with $p^* = \frac{Np(x)}{N-p(x)}$.

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Subsequently, the case $f(x,u) = \lambda|u|^{q(x)-2}u$ of problem (1.1) were considered by Fan, Zhang and Zhao in [5], and Mihăilescu and Rădulescu in [6]. More precisely, they studied the following nonhomogeneous eigenvalue problem

$$\begin{cases} -\Delta_{p(x)}u = \lambda|u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is a bounded domain with smooth boundary, $\lambda > 0$ is a real number, and p and q are continuous on $\overline{\Omega}$. For the case $p(x) = q(x)$, the authors in [5] established the existence of a sequence of eigenvalues of problem (1.2) by the Ljustemik-Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, they proved that $\sup\Lambda = +\infty$ and pointed out that $\inf\Lambda > 0$ only under additional assumptions. For the case $1 < \min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x) < \max_{x\in\overline{\Omega}} q(x)$, Mihăilescu and Rădulescu in [6] proved that any $\lambda > 0$ sufficiently small is an eigenvalue of problem (1.2) under the assumptions $\max_{x\in\overline{\Omega}} p(x) < N$ and $q(x) < \frac{Np(x)}{N-p(x)}$ for all $x \in \overline{\Omega}$. In fact, their method was Ekeland's variation principle, and the constraint $\lambda > 0$ sufficiently small played a major role. Moreover, they pointed out that the corresponding functional J_λ of problem (1.2) neither satisfies (AR)-condition nor is coercive. Therefore, they were not able to obtain a critical point of the functional J_λ by using the mountain pass theorem (see [7]) or a result as the Theorem 1.2 in Struwe [8].

Obviously, the assumption $1 < \min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x) < \max_{x\in\overline{\Omega}} q(x)$ does not exist in the elliptic equation with constant exponent. For fixed $\lambda > 0$, as mentioned above, it is still an interesting subject to study the solvability of problem (1.2). Unlike the concave-convex nonlinearities, the main difference of problem (1.2) is that the nonlinearity has both local superlinearity and local sublinearity. It is difficult to prove the boundedness of Palais-Smale sequence of the Euler-Lagrange functional. To the best of our knowledge, even for the case $p(x) = 2$, there are no results.

Let B be the unit ball in $\mathbb{R}^N (N \geq 3)$. Some scholars considered the following semilinear elliptic equation with variable exponent in [9] and [10]

$$\begin{cases} -\Delta u = u^{q(x)-1}, & \text{in } B, \\ u > 0, & \text{in } B, \\ u = 0, & \text{in } \partial B. \end{cases} \tag{1.3}$$

They obtained the existence or multiplicity of the nontrivial radial solutions of problem (1.3) with critical or supercritical exponent. In this paper, we consider multiple positive radial solutions of problem (1.3) involving local superlinearity and local sublinearity.

Denote $D_0 = \{x \in \overline{B} | q(x) = 2\}$, $D_- = \{x \in \overline{B} | q(x) < 2\}$, and $D_+ = \{x \in \overline{B} | q(x) > 2\}$, and assume

(Q_1) $q(x) = q(|x|) \in C(\overline{B})$;

(Q_2) $1 < \min_{x\in\overline{B}} q(x) < 2 < \max_{x\in\overline{B}} q(x) < 2^*$, $q(0) > 2$;

(Q_3) $S_N^{-1}|D_0|^{\frac{2}{N}} < 1$, with S_N is the Sobolev embedding constant.

The main results of this paper reals as follows.

Theorem 1.1. *Let (Q_1), (Q_2), and (Q_3) hold. Then there exists a constant $\Lambda_0 > 0$ such that problem (1.2) has at least two positive radial symmetric solutions with $|D_-| < \Lambda_0$.*

Remark 1.1. Conditions (Q_1) and (Q_2) mean that D_- and D_+ are positive measurable sets. Moreover, the conditions of Theorem 1.1 are different from those of other elliptic equations involving local superlinearity and sublinearity (see [11, 12, 13, 14]).

To end this section, we describe the basic ideas in the proof of Theorem 1.1. Note that $q(x) \leq 2$ for some $x \in B$. Inspired by [15], we first modify the nonlinear term to guarantee the boundedness of Palais-Smale sequence of the corresponding functional. Subsequently, we use the Moser iteration to prove that the positive solution of auxiliary problem is indeed a positive solution of original problem (1.3).

Throughout this paper, we use $\|\cdot\|$ to denote the usual norms of $H_0^1(B)$ and set $H_{0,r}^1(B) = \left\{u \in H_0^1(B) \mid u(x) = u(|x|)\right\}$. The letter C stands for positive constant which may take different values at different places.

2. The Modified Problem

According to $q_- < 2$, it seems to be difficult to confirm whether the energy function I corresponding to (1.3) satisfies the Palais-Smale condition or not. To apply the mountain pass theorem, the first step in proving Theorem 1.1 is to modify the nonlinear term. Since $q(x)$ is a continuous function, $1 < q_- < 2 < q_+ < 2^*$, and $q(0) > 2$, we see that there exist $\delta_0 \in (0, \frac{1}{4})$ and $r > 0$ such that

$$q(x) \geq 2 + r, \quad x \in B_{2\delta_0}; \qquad q_+ + r < 2^*, \quad x \in B. \tag{2.1}$$

Let $\psi(t) \in C_0^\infty(\mathbb{R}, [0,1])$ be a smooth even function with the following properties: $\psi(t) = 1$ for $|t| \leq 1$, $\psi(t) = 0$ for $|t| \geq 2$, and $\psi(t)$ is monotonically decreasing on the interval $(0, +\infty)$. Define $b_\mu(t) = \psi(\mu t)$ and $m_\mu(t) = \displaystyle\int_0^t b_\mu(\tau) d\tau$ for $\mu \in (0,1]$, and set

$$Q(x) = \begin{cases} 1, & \text{if } x \in D_0 \bigcup D_-, \\ 0, & \text{if } x \in D_+, \end{cases} \quad P(x) = 1 - Q(x).$$

Consider the perturbated problem

$$\begin{cases} -\Delta u = Q(x) u^{q(x)-1} + P(x) \left(\frac{u}{m_\mu(u)}\right)^r u^{q(x)-1}, & \text{in } B, \\ u > 0, & \text{in } B, \\ u = 0, & \text{in } \partial B. \end{cases} \tag{2.2}$$

The corresponding functional is

$$I_\mu(u) = \frac{1}{2} \int_B |\nabla u|^2 dx - \int_B \frac{Q(x)}{q(x)} (u^+)^{q(x)} dx - \int_B P(x) K_\mu(x, u^+) dx,$$

where $u \in H_0^1(B)$, $u^+ = \max\{u, 0\}$, $k_\mu(x,t) = (\frac{t}{m_\mu(t)})^r t^{q(x)-1}$ for $t > 0$, $k_\mu(x,t) = 0$ for $t = 0$, and $K_\mu(x,t) = \int_0^t k_\mu(x,\tau) d\tau$.

Theorem 2.1. *Let (Q_1), (Q_2), and (Q_3) hold. Then, for any $\mu \in (0,1]$, there exists $L > 0$ independent of μ and $\Lambda_0 > 0$ such that problem (2.2) has at least two positive radial symmetric solutions u_μ' and u_μ'' satisfying $I_\mu(u_\mu') < 0 < I_\mu(u_\mu'') < L$ for $|D_-| < \Lambda_0$.*

Lemma 2.1. *The function $K_\mu(x,t)$ defined above satisfies the following inequalities:*

$$K_\mu(x,t) \le \frac{1}{q(x)} t k_\mu(x,t), \qquad K_\mu(x,t) \le \frac{1}{q(x)+r} t k_\mu(x,t) + C_\mu,$$

for $t > 0$, where $C_\mu > 0$ is a positive constant.

Proof. Since $b_\mu(t)$ is a monotonically decreasing on the interval $(0,+\infty)$, we have

$$\frac{d}{dt}\left(\frac{t}{m_\mu(t)}\right) = \frac{m_\mu(t) - t b_\mu(t)}{m_\mu^2(t)} = \frac{t(b_\mu(\xi) - b_\mu(t))}{m_\mu^2(t)} \ge 0,$$

for $t > 0$, where $\xi \in (0,t)$. Therefore, $\frac{t}{m_\mu(t)}$ is monotonically increasing on the interval $(0,+\infty)$. Hence, $\frac{k_\mu(x,t)}{t^{q(x)-1}} = \left(\frac{t}{m_\mu(t)}\right)^r$ is also monotonically increasing on $(0,+\infty)$. It follows that

$$K_\mu(x,t) = \int_0^t k_\mu(x,\tau) d\tau \le \int_0^t \frac{k_\mu(x,t)}{t^{q(x)-1}} \tau^{q(x)-1} d\tau = \frac{1}{q(x)} t k_\mu(x,t),$$

for $t > 0$. By definition of the function m_μ, we have $m_\mu(t) = \frac{A}{\mu}$ for $t \ge \frac{2}{\mu}$, where $A = 1 + \int_1^2 \psi(\tau) d\tau$. For $t > \frac{2}{\mu}$, one has

$$\begin{aligned}
K_\mu(x,t) &= \int_0^{\frac{2}{\mu}} k_\mu(x,\tau)\, d\tau + \int_{\frac{2}{\mu}}^t \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\, d\tau \\
&= \int_0^{\frac{2}{\mu}} \left(k_\mu(x,\tau) - \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\right) d\tau + \int_0^t \left(\frac{\mu}{A}\right)^r \tau^{q(x)+r-1}\, d\tau \\
&\le C_\mu + \frac{t k_\mu(x,t)}{q(x)+r}.
\end{aligned} \tag{2.3}$$

It implies from (2.3) that

$$K_\mu(x,t) \le \frac{1}{q(x)+r} t k_\mu(x,t) + C_\mu,$$

for $t > 0$. This completes the proof. $\square$

Lemma 2.2. *Let (Q_1), (Q_2) and (Q_3) hold. Then, for any $\mu \in (0,1]$, I_μ satisfies the (PS) condition.*

Proof. Let $\{u_n\}$ be a (PS) sequence of I_μ in $H^1_{0,r}(B)$. This means that there exists $C > 0$ such that

$$|I_\mu(u_n)| \le C,\ I'_\mu(u_n) \to 0 \text{ as } n \to \infty. \tag{2.4}$$

From (2.1), Lemma 2.1, the Hölder inequality, and the Sobolev embedding theorem, we derive that

$$\begin{aligned}
& I_\mu(u_n) - \frac{1}{2+r}\langle I'_\mu(u_n), u_n\rangle \\
= & \left(\frac{1}{2} - \frac{1}{2+r}\right)\left(\int_B |\nabla u_n|^2 dx - \int_{D_0} u_n^2 dx\right) - \int_{D_-}\left(\frac{1}{q(x)} - \frac{1}{2+r}\right)(u_n^+)^{q(x)} dx \\
& + \int_{D_+}\left(\frac{1}{2+r} u_n^+ k_\mu(x,u_n^+) - K_\mu(x,u_n^+)\right) dx \\
\ge & \left(\frac{1}{2} - \frac{1}{2+r}\right)(1 - S_N^{-1}|D_0|^{\frac{2}{N}}) \int_B |\nabla u_n|^2 dx - C_\mu - \int_{D_-} (u_n^+)^{q(x)} dx.
\end{aligned} \tag{2.5}$$

For $\delta > 0$, denote $E_\delta = \{x \in \overline{B} | 2-\delta < q(x) < 2\}$, and choose δ small enough such that

$$2S_N^{-1}|E_\delta|^{\frac{2}{N}} \leq \sigma := \left(\frac{1}{2} - \frac{1}{2+r}\right)(1 - S_N^{-1}|D_0|^{\frac{2}{N}}). \tag{2.6}$$

It follows that

$$\begin{aligned} \int_{D_-} (u_n^+)^{q(x)} dx &\leq \int_{E_\delta} |u_n|^{q(x)} dx + \int_{D_-\setminus E_\delta} |u_n|^{q(x)} dx \\ &\leq \int_{E_\delta} (u_n^2 + |u_n|^{2-\delta}) dx + \int_{D_-\setminus E_\delta} (|u_n|^{2-\delta} + |u_n|^{q_-}) dx \\ &\leq S_N^{-1}|E_\delta|^{\frac{2}{N}} \|u_n\|^2 + C_1(\|u_n\|^{2-\delta} + \|u_n\|^{q_-}). \end{aligned} \tag{2.7}$$

By (2.4), (2.5), (2.6), and (2.7), we have

$$C + 1 \geq \frac{1}{2}\sigma \|u_n\|^2 - C_1(\|u_n\|^{2-\delta} + \|u_n\|^{q_-}) - C_\mu. \tag{2.8}$$

According to (2.8), we obtain $\{u_n\}$ is bounded in $H^1_{0,r}(B)$. Up to a subsequence, we may assume that

$$\begin{cases} u_n \rightharpoonup u, & \text{in } H^1_{0,r}(B), \\ u_n \to u, & \text{in } L^s(B), \ 1 \leq s < 2^*. \end{cases}$$

For any integer pair (i,j), one has

$$\begin{aligned} \|u_i - u_j\|^2 &= \langle I'_\mu(u_i) - I'_\mu(u_j), u_i - u_j \rangle + \int_B P(x)\left(k_\mu(x,u_i^+) - k_\mu(x,u_j^+)\right)(u_i - u_j) dx \\ &\quad + \int_B Q(x)\left((u_i^+)^{q(x)-1} - (u_j^+)^{q(x)-1}\right)(u_i - u_j) dx. \end{aligned}$$

It follows from (2.4) that

$$\langle I'_\mu(u_i) - I'_\mu(u_j), u_i - u_j \rangle \to 0, \quad \text{as } i, \ j \to +\infty. \tag{2.9}$$

According to the proof of Lemma 2.1, it is easy to see that

$$|k_\mu(x,t)| \leq |t|^{q(x)-1} + \left(\frac{\mu}{A}\right)^r |t|^{q(x)+r-1}.$$

By the Hölder inequality, we obtain

$$\begin{aligned} &\left|\int_B P(x)(k_\mu(x,u_i^+) - k_\mu(x,u_j^+))(u_i - u_j) dx\right| \\ = \ &\int_{D_+} (k_\mu(x,u_i^+) - k_\mu(x,u_j^+))(u_i - u_j) dx \\ \leq \ &C_2 \int_B \left(|u_i| + |u_j| + |u_i|^{q_+ + r - 1} + |u_j|^{q_+ + r - 1}\right)|u_i - u_j| dx \to 0, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\left|\int_B Q(x)((u_i^+)^{q(x)-1} - (u_j^+)^{q(x)-1})(u_i - u_j) dx\right| \\ = \ &\int_{D_0 \bigcup D_-} ((u_i^+)^{q(x)-1} - (u_j^+)^{q(x)-1})(u_i - u_j) dx \\ \leq \ &C_3 \int_B \left(2 + |u_i| + |u_j|\right)|u_i - u_j| dx \to 0 \end{aligned} \tag{2.11}$$

as i and j tend to $+\infty$. From (2.9)-(2.11), we have $\|u_i - u_j\| \to 0$ as $i, j \to +\infty$, which implies that $\{u_n\}$ contains a strongly convergent subsequence in $H^1_{0,r}(B)$. Hence I_μ satisfies the (PS) condition. This completes the proof. □

In the following lemma, we verify that I_μ possesses the mountain pass geometry.

Lemma 2.3. *Let (Q_1), (Q_2) and (Q_3) hold. Then (1) there exist positive constants Λ_0, ρ, and m independent of μ such that $I_\mu(u) \geq m > 0$ with $\|u\| = \rho$ for $|D_-| < \Lambda_0$; (2) there exists $u_0 \in H^1_{0,r}(B)$ such that $\|u_0\| > \rho$ and $I_\mu(u_0) < 0$.*

Proof. According to Lemma 2.1, we have

$$\begin{aligned} \int_{D_+} K_\mu(x,u^+)dt &\leq \int_{D_+} u^+ k_\mu(x,u^+)dx \\ &\leq \int_{D_+} (1+|u|^r)|u|^{q(x)}dx \\ &\leq \int_{D_+} |u|^{q(x)}dx + C\int_{D_+} |u|^r(u^2+|u|^{q_+})dx \\ &\leq \int_{D_+} |u|^{q(x)}dx + C(\|u\|^{2+r} + \|u\|^{q_+ + r}). \end{aligned} \tag{2.12}$$

For $\delta > 0$, set $F_\delta = \{x \in \overline{B} | 2 < q(x) < 2+\delta\}$, and choose $\delta < r$ such that

$$\sigma_1 = 1 - S_N^{-1}(|D_0|^{\frac{2}{N}} + 2|F_\delta|^{\frac{2}{N}}) > 0. \tag{2.13}$$

Then

$$\begin{aligned} \int_{D_+} |u|^{q(x)}dx &= \int_{F_\delta} |u|^{q(x)}dx + \int_{D_+\setminus F_\delta} |u|^{q(x)}dx \\ &\leq \int_{F_\delta} (u^2+|u|^{2+\delta})dx + \int_{D_+\setminus F_\delta} (|u|^{2+\delta}+|u|^{q_+})dx \\ &\leq S_N^{-1}|F_\delta|^{\frac{2}{N}}\|u\|^2 + C(\|u\|^{2+\delta} + \|u\|^{q_+}). \end{aligned} \tag{2.14}$$

Define

$$J_\mu(u) = \frac{1}{2}\int_B |\nabla u|^2 dx - \frac{1}{2}\int_{D_0} u^2 dx - \int_{D_+} K_\mu(x,u^+)dx.$$

By (2.12), (2.13) and (2.14), we obtain

$$\begin{aligned} J_\mu(u) &\geq \frac{1}{2}\left(1 - S_N^{-1}(|D_0|^{\frac{2}{N}} + 2|F_\delta|^{\frac{2}{N}})\right)\int_B |\nabla u|^2 dx - C(\|u\|^{2+\delta} + \|u\|^{q_+ + r}) \\ &\geq \frac{1}{2}\sigma_1\|u\|^2 - C(\|u\|^{2+\delta} + \|u\|^{q_+ + r}). \end{aligned}$$

If we choose $\rho > 0$ such that $C(\rho^{2+\delta} + \rho^{q_+ + r}) \leq \frac{1}{4}\sigma_1\rho^2$, then

$$J_\mu(u) \geq \frac{1}{2}\sigma_1\rho^2 - C(\rho^{2+\delta} + \rho^{q_+ + r}) \geq \frac{1}{4}\sigma_1\rho^2 \tag{2.15}$$

for $\|u\| = \rho$. By the Hölder inequality and the Sobolev imbedding theorem, we have

$$\begin{aligned}\int_{D_-} \frac{1}{q(x)} |u^+|^{q(x)} dx &\leq \int_{D_-} (u^2 + |u|^{q_-}) dx \\ &\leq |D_-|^{\frac{2}{N}} |u|_{2^*}^2 + |D_-|^{\frac{2^*-q}{2^*}} |u|_{2^*}^q \\ &\leq C_4(\|u\|^2 |D_-|^{\frac{2}{N}} + \|u\|^q |D_-|^{\frac{2^*-q}{2^*}}), \end{aligned} \tag{2.16}$$

where $|\cdot|_{2^*} = \left(\int_\Omega |\cdot|^{2^*} dx\right)^{\frac{1}{2^*}}$. Choose $\Lambda_0 > 0$ such that $C_4(\rho^2 \Lambda_0^{\frac{2}{N}} + \rho^q \Lambda_0^{\frac{2^*-q}{2^*}}) \leq \frac{1}{8}\sigma_1 \rho^2$, which together with (2.15) and (2.16) implies that

$$I_\mu(u) = J_\mu(u) - \int_{D_-} \frac{1}{q(x)} (u^+)^{q(x)} dx \geq \frac{1}{8} \sigma_1 \rho^2 := m$$

with $\|u\| = \rho$ for $|D_-| < \Lambda_0$. By definition of the function k_μ, we obtain $k_\mu(x,t) \geq |t|^{q(x)-1}$. According to (Q_2), we know that there exists a positive measurable set $U \subset D_+$ such that $q(x) \geq \frac{2+q_+}{2}$ for any $x \in U$. Fix a nonnegative radial function $v_0 \in C_0^\infty(U) \backslash \{0\}$. Then, for $t > 0$ sufficiently large, we obtain

$$I_\mu(tv_0) \leq \frac{t^2}{2} \int_U |\nabla v_0|^2 dx - t^{\frac{2+q_+}{2}} \int_U \frac{|v_0|^{q(x)}}{q(x)} dx < 0. \tag{2.17}$$

Choosing $u_0 = t_0 v_0$ with $t_0 > 0$ large enough, we have $\|u_0\| > \rho$ and $I_\mu(u_0) < 0$. $\square$

Now we are in a position to prove the main result of this section.

Proof of Theorem 2.1. According to the condition (Q_2), we know that there exist $\sigma_2 > 0$ and a positive measurable set $\Omega \subset D_-$, such that $q_- \leq q(x) \leq 2 - \sigma_2$ for $x \in \Omega$. Fix a radial function $\varphi \in C_0^\infty(\Omega) \backslash \{0\}$. For $s > 0$ sufficiently small, we have

$$\begin{aligned} I_\mu(s\varphi) &= \frac{1}{2} s^2 \int_B |\nabla \varphi|^2 dx - \int_B \frac{1}{q(x)} |s\varphi|^{q(x)} dx \\ &\leq \frac{1}{2} s^2 \int_\Omega |\nabla \varphi|^2 dx - \frac{s^{2-\sigma_2}}{2-\sigma_2} \int_\Omega |\varphi|^{q(x)} dx \\ &< 0. \end{aligned}$$

Thus, we deduce that

$$c_* = \inf_{u \in \overline{B_{\rho(0)}}} I_\mu(u) < 0 < \inf_{u \in \partial B_{\rho(0)}} I_\mu(u).$$

By app1ying the Ekeland's variational principle in $\overline{B_{\rho(0)}}$ (see [16]), we obtain that problem (2.2) has a solution $u_\mu^{'}$ satisfying $I_\mu(u_\mu^{'}) = c_* < 0$. From Lemmas 2.2 and 2.3 we see that the functional I_μ satisfies the (PS) condition and has the mountain pass geometry. Define

$$\Gamma = \{\gamma \in C([0,1], H_{0,r}^1(B)) |\ \gamma(0) = 0,\ \gamma(1) = u_0\}, \ \ c^* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\mu(\gamma(t)).$$

By the mountain pass theorem (see [7]), we obtain that problem (2.2) has a solution $u_\mu^{''}$ satisfying $I_\mu(u_\mu^{''}) = c^* > 0$. Let u_μ be a nontrivial critical of I_μ. After a direct calculation, we derive that $\|u_\mu^-\|^2 = \langle I_\mu'(u_\mu), u_\mu^- \rangle = 0$, which implies that $u_\mu^- = 0$. Hence, $u_\mu \geq 0$. Since $I_\mu(u_\mu) \neq 0 = I(0)$, we have $u_\mu \neq 0$. By the Strong Maximum Principle (see [17]), we obtain u_μ is a positive solution of problem (2.2). Since $I_\mu(u_\mu^{'}) < 0 < I_\mu(u_\mu^{''})$, we know that $u_\mu^{'}$ and $u_\mu^{''}$ are two positive solutions to problem (2.2). It follows from (2.17) that

$$c^* \le \max_{t\in[0,1]} I_\mu(tu_0) \le \max_{t\in[0,1]} \Big(\frac{t^2}{2}\int_B |\nabla u_0|^2 dx - \frac{t^{2+q_+}}{q_+}\int_U |u_0|^{q(x)}dx\Big) = L.$$

Therefore, c^* is uniformly bounded. That is, we have

$$I_\mu(u_\mu^{'}) < 0 < I_\mu(u_\mu^{''}) < L.$$

□

3. A Priori Estimate and the Proof of Theorem 1.1

In this section, we prove that radial symmetric solutions of auxiliary problem (2.2) sufficiently small μ are indeed solutions of original problem (1.3). For this purpose, we need the following uniform L^∞-estimate for critical points of the functional I_μ.

Lemma 3.1. *If v is a positive radial symmetric solution to problem (2.2), then $v(x)$ is decreasing about $|x|$.*

Proof. Set $\rho = |x|$. Since v is positive radially symmetric, one has

$$-\frac{1}{\rho^{N-1}}\frac{d}{d\rho}\Big(\rho^{N-1}\frac{dv}{d\rho}\Big) = (1-Q(\rho))\Big(\frac{v}{m_\mu(v)}\Big)^\rho v^{q(\rho)-1} + Q(\rho)v^{q(\rho)-1} \ge 0,$$

which implies that $\frac{d}{d\rho}\Big(\rho^{N-1}\frac{dv}{d\rho}\Big) \le 0$. In view of $\rho^{N-1}\frac{dv}{d\rho}|_{\rho=0} = 0$, we have $\rho^{N-1}\frac{dv}{d\rho} \le 0$. That is, $\frac{dv}{d\rho} \le 0$. Therefore, $v(x)$ is decreasing about $|x|$. □

Lemma 3.2. *Let (Q_1), (Q_2) and (Q_3) hold. Assume that $v \in H^1_{0,r}(B)$ is a positive radial symmetric solution to problem (2.2) satisfying $I_\mu(v) \leqslant L$. Then there exists a constant $M_0 > 0$ independent of μ such that $\int_B |\nabla v|^2 dx \le M_0$.*

Proof. Since $q(0) > 2$, we have $\rho > 0$ and $\sigma > 0$ such that

$$q(\bar{y}) \ge 2 + 2\sigma,\ for\ \bar{y} \in B_\rho(0). \tag{3.1}$$

Without loss of generality, we assume

$$4\rho \le R = d(0) = dist(0, D_0),$$

and

$$2 < q(y) \le 2 + \sigma,\ for\ y \in D_\rho = \{y \in D_+ | d(y) = dist(y, D_0) \le \rho\}. \tag{3.2}$$

Letting $x \in D_0 \bigcap \overline{D_\rho}$, $y \in B_\rho(x)$, $\bar{y} = y - x \in B_\rho(0)$, we have

$$|y| \ge |x| - |\bar{y}| \ge R - \rho \ge \rho \ge |\bar{y}|. \tag{3.3}$$

By (3.3) and Lemma 3.1, we have $v(y) \le v(\bar{y})$ for $y \in B_\rho(x)$ and $\bar{y} = y - x \in B_\rho(0)$. On the other hand, (3.1) and (3.2) yield that $q(y) \le 2 + \sigma \le 2 + 2\sigma \le q(\bar{y})$ for $y \in B_\rho(x)$, $\bar{y} \in B_\rho(0)$. Note that if $v(y) \le 1$, $\mu \in (0,1]$, then

$$v(y)k_\mu(y, v(y)) = \Big(\frac{v(y)}{m_\mu(v(y))}\Big)^r v(y)^{q(y)} = v(y)^{q(y)} \le 1. \tag{3.4}$$

Note that $\left(\frac{t}{m_\mu(t)}\right)^r$ is monotonically increasing on the interval $(0,+\infty)$. If $v(y)\geq 1$, then

$$\begin{aligned} v(y)k_\mu(y,v(y)) &= \left(\frac{v(y)}{m_\mu(v(y))}\right)^r v(y)^{q(y)} \leq \left(\frac{v(y)}{m_\mu(v(y))}\right)^r v(y)^{q(\bar{y})} \\ &\leq \left(\frac{v(\bar{y})}{m_\mu(v(\bar{y}))}\right)^r v(\bar{y})^{q(\bar{y})} = v(\bar{y})k_\mu(\bar{y},v(\bar{y})). \end{aligned} \tag{3.5}$$

For $x\in D_0\bigcap\overline{D_\rho}$, one concludes from (3.4) and (3.5) that

$$\int_{B_\rho(x)} vk_\mu(y,v)dy \leq \int_{B_\rho(x)} dy + \int_{B_\rho(0)} vk_\mu(\bar{y},v)d\bar{y}. \tag{3.6}$$

By Lemma 2.1, we have

$$\begin{aligned} L &\geq I_\mu(v)-\frac{1}{2}\langle I_\mu^{'}(v),v\rangle \\ &= \int_{D_+}\left(\frac{1}{2}vk_\mu(x,v)-K_\mu(x,v)\right)dx-\int_{D_-}\left(\frac{1}{q(x)}-\frac{1}{2}\right)v^{q(x)}dx \\ &\geq \int_{D_+}\left(\frac{1}{2}-\frac{1}{q(x)}\right)vk_\mu(x,v)dx - C\int_{D_-} v^{q(x)}dx. \end{aligned} \tag{3.7}$$

By (3.7), we obtain

$$\int_{B_\rho(0)} vk_\mu(x,v)dx \leq C\left(1+\int_{D_-} v^{q(x)}dx\right). \tag{3.8}$$

Obviously, the family of open sets $\{B_\rho(x)|x\in D_0\bigcap\overline{D_\rho}\}$ is an open cover of the close set $\overline{D_{\frac{1}{2}\rho}}$. Therefore, there exists a finite number of sets $B_\rho(x_1)$,...,$B_\rho(x_k)$ such that $\overline{D_{\frac{1}{2}\rho}}\subset\bigcup_{i=1}^k B_\rho(x_i)$, where $x_i\in D_0\bigcap\overline{D_\rho}$, $i=1,2,\ldots,k$. By (3.6) and (3.8), we have

$$\begin{aligned} \int_{D_{\frac{1}{2}\rho}} vk_\mu(x,v)dx &\leq \sum_{i=1}^k\int_{B_\rho(x_i)} vk_\mu(x,v)dx \\ &\leq \sum_{i=1}^k\left(\int_{B_\rho(x_i)} dy+\int_{B_\rho(0)} vk_\mu(\bar{y},v)d\bar{y}\right) \\ &\leq C\left(1+\int_{D_-} v^{q(x)}dx\right). \end{aligned} \tag{3.9}$$

Since v is a positive solution to problem (2.2), we have

$$\int_B |\nabla v|^2dx = \int_{D_+} vk_\mu(x,v)dx + \int_{D_0} v^2dx + \int_{D_-} v^{q(x)}dx. \tag{3.10}$$

By (3.7) and (3.9), we deduce from (Q_1) that

$$\int_{D_+} vk_\mu(x,v)dx = \int_{D_{\frac{1}{2}\rho}} vk_\mu(x,v)dx + \int_{D_+\backslash D_{\frac{1}{2}\rho}} vk_\mu(x,v)dx \leq C\left(1+\int_{D_-} v^{q(x)}dx\right). \tag{3.11}$$

It follows from (3.10) and (3.11) that $\int_B|\nabla v|^2dx\leq C+\int_{D_0}v^2dx+C_5\int_{D_-}v^{q(x)}dx$. For $\delta>0$, denote E_δ=$\{x\in\overline{B}|\ 2-\delta<q(x)<2\}$. We can choose δ small enough such that

$$S_N^{-1}(|D_0|^{\frac{2}{N}}+C_5|E_\delta|^{\frac{2}{N}})<1. \tag{3.12}$$

Then

$$\begin{aligned}\int_{D_-} v^{q(x)}dx &= \int_{E_\delta} v^{q(x)}dx+\int_{D_-\setminus E_\delta} v^{q(x)}dx\\ &\le \int_{E_\delta}(v^2+v^{2-\delta})dx+\int_{D_-\setminus E_\delta}(v^{2-\delta}+v^{q_-})dx\\ &\le \int_{E_\delta} v^2dx+C(\|v\|^{2-\delta}+\|v\|^{q_-}).\end{aligned}\tag{3.13}$$

By (3.12) and (3.13), we have

$$\begin{aligned}\int_B|\nabla v|^2dx &\le C+\int_{D_0}v^2dx+C_5\int_{E_\delta}v^2dx+C_6(\|v\|^{2-\delta}+\|v\|^{q_-})\\ &\le (|D_0|^{\frac{2}{N}}+C_5|E_\delta|^{\frac{2}{N}})|v|_{2^*}^2+C_7(1+\|v\|^{2-\delta}+\|v\|^{q_-})\\ &\le S_N^{-1}(|D_0|^{\frac{2}{N}}+C_5|E_\delta|^{\frac{2}{N}})\|v\|^2+C_7(1+\|v\|^{2-\delta}+\|v\|^{q_-}).\end{aligned}$$

Since $S_N^{-1}(|D_0|^{\frac{2}{N}}+C_5|E_\delta|^{\frac{2}{N}})<1$, $q_-<2$, we obtain that there exists $M_0>0$ independent of μ such that $\int_B|\nabla v|^2dx\le M_0$. This completes the proof. □

Lemma 3.3. *Let (Q_1), (Q_2), and (Q_3) hold. If v is a positive radial symmetric critical point of I_μ with $I_\mu(v)\le L$, then there exists a positive constant M independent of μ such that $\|v\|_{L^\infty(B)}\le M$.*

Proof. Let $\alpha>2$, $B_{\frac{\delta}{2}}(0)\subset D_+$, and $\zeta\in C_0^\infty(B_{\frac{\delta}{2}}(0),\mathbb{R})$. By the Young inequality, we have

$$\begin{aligned}-\int_{B_{\frac{\delta}{2}}(0)}\zeta^2v^{\alpha-1}\Delta v\,dx &= (\alpha-1)\int_{B_{\frac{\delta}{2}}(0)}\zeta^2v^{\alpha-2}|\nabla v|^2dx+2\int_{B_{\frac{\delta}{2}}(0)}\zeta v^{\alpha-1}\nabla v\cdot\nabla\zeta dx\\ &= \frac{4(\alpha-1)}{\alpha^2}\int_{B_{\frac{\delta}{2}}(0)}\zeta^2|\nabla v^{\frac{\alpha}{2}}|^2dx+2\int_{B_{\frac{\delta}{2}}(0)}\zeta v^{\frac{\alpha}{2}}\nabla v^{\frac{\alpha}{2}}\cdot\nabla\zeta dx\\ &\ge \frac{2(\alpha-1)}{\alpha^2}\int_{B_{\frac{\delta}{2}}(0)}\zeta^2|\nabla v^{\frac{\alpha}{2}}|^2dx-\frac{\alpha^2}{2(\alpha-1)}\int_{B_{\frac{\delta}{2}}(0)}v^\alpha|\nabla\zeta|^2dx\\ &\ge \frac{1}{\alpha}\int_{B_{\frac{\delta}{2}}(0)}\zeta^2|\nabla v^{\frac{\alpha}{2}}|^2dx-\alpha\int_{B_{\frac{\delta}{2}}(0)}v^\alpha|\nabla\zeta|^2dx.\end{aligned}\tag{3.14}$$

On the other hand, one has

$$\begin{aligned}&\int_{B_{\frac{\delta}{2}}(0)}\left(P(x)\left(\frac{v}{m_\mu(v)}\right)^r v^{q(x)-1}+Q(x)v^{q(x)-1}\right)v^{\alpha-1}\zeta^2dx\\ &=\int_{B_{\frac{\delta}{2}}(0)}\left(\frac{v}{m_\mu(v)}\right)^r v^{q(x)+\alpha-2}\zeta^2dx\\ &\le\int_{B_{\frac{\delta}{2}}(0)}v^\alpha\zeta^2dx+\int_{B_{\frac{\delta}{2}}(0)}v^{q_++r+\alpha-2}\zeta^2dx.\end{aligned}\tag{3.15}$$

Combining (3.14) with (3.15), and noticing that v is a solution to problem (2.2), we obtain

$$
\begin{aligned}
&\int_{B_{\frac{\delta}{2}}(0)} \zeta^2 |\nabla v^{\frac{\alpha}{2}}|^2 dx \\
&\leq \alpha \left(\alpha \int_{B_{\frac{\delta}{2}}(0)} v^{\alpha} |\nabla \zeta|^2 dx + \int_{B_{\frac{\delta}{2}}(0)} v^{\alpha} \zeta^2 dx + \int_{B_{\frac{\delta}{2}}(0)} v^{q_+ + r + \alpha - 2} \zeta^2 dx \right). \qquad (3.16)
\end{aligned}
$$

Suppose that $\delta_k = \frac{\delta}{4}\left(1+\frac{1}{2^k}\right)$ and $\zeta_k \in C_0^{\infty}(B_{\delta_k}(0), \mathbb{R})$ has the following properties: $0 \leq \zeta_k \leq 1$, $\delta_k = 1$ for $x \in B_{\delta_{k+1}}(0)$, and $|\nabla \zeta_k| \leq \frac{1}{4(\delta_k - \delta_{k+1})} = \frac{2^{k+1}}{\delta}$. $B_{\frac{\delta}{2}}(0)$ and ζ are taken to be $B_{\delta_k}(0)$ and ζ_k in inequality (3.16), respectively. Using the Hölder inequality, we have

$$
\begin{aligned}
&\left(\int_{B_{\delta_{k+1}}(0)} v^{\frac{2^*\alpha}{2}} dx \right)^{\frac{2}{2^*}} \\
&\leq \left(\int_{B_{\delta_k}(0)} \left(\zeta_k v^{\frac{\alpha}{2}} \right)^{2^*} dx \right)^{\frac{2}{2^*}} \\
&\leq C \int_{B_{\delta_k}(0)} |\nabla (\zeta_k v^{\frac{\alpha}{2}})|^2 dx \\
&\leq C \left(\int_{B_{\delta_k}(0)} \zeta_k^2 |\nabla v^{\frac{\alpha}{2}}|^2 dx + \int_{B_{\delta_k}(0)} v^{\alpha} |\nabla \zeta_k|^2 dx \right) \\
&\leq C\alpha \left(\left(\alpha + \frac{1}{\alpha} \right) \int_{B_{\delta_k}(0)} v^{\alpha} |\nabla \zeta_k|^2 dx + \int_{B_{\delta_k}(0)} v^{\alpha} \zeta_k^2 dx + \int_{B_{\delta_k}(0)} v^{q_+ + r + \alpha - 2} \zeta_k^2 dx \right) \\
&\leq C\alpha \left(\left(\left(\alpha + \frac{1}{\alpha} \right) \frac{4^{k+1}}{\delta^2} + 1 \right) \int_{B_{\delta_k}(0)} v^{\alpha} dx + \int_{B_{\delta_k}(0)} v^{q_+ + r + \alpha - 2} dx \right) \\
&\leq C\alpha \left(\frac{\alpha 4^{k+2}}{\delta^2} |B_{\delta_k}(0)|^{\frac{q_+ + r - 2}{2^*}} + \left(\int_{B_{\delta_k}(0)} v^{2^*} dx \right)^{\frac{q_+ + r - 2}{2^*}} \right) \left(\int_{B_{\delta_k}(0)} v^{\frac{2^*\alpha}{2^* - q_+ - r + 2}} dx \right)^{\frac{2^* - q_+ - r + 2}{2^*}}.
\end{aligned}
$$

By the Sobolev embedding theorem and Lemma 3.2, we obtain

$$
\begin{aligned}
\left(\int_{B_{\delta_k}(0)} v^{2^*} dx \right)^{\frac{q_+ + r - 2}{2^*}} &\leq C \left(\int_{B_{\delta_k}(0)} (|\nabla v|^2 + v^2) dx \right)^{\frac{q_+ + r - 2}{2}} \\
&\leq C_8 \left(\int_B |\nabla v|^2 dx \right)^{\frac{q_+ + r - 2}{2}} \\
&\leq C_8 M_0^{\frac{q_+ + r - 2}{2}}.
\end{aligned}
$$

According to the above two inequalities, we have

$$\begin{aligned}
&\left(\int_{B_{\delta_{k+1}}(0)} v^{\frac{2^*\alpha}{2}}dx\right)^{\frac{2}{2^*}}\\
&\leq C\alpha\left(\frac{\alpha 4^{k+2}}{\delta^2}|B_{\delta_k}(0)|^{\frac{q_++r-2}{2^*}}+C_8M_0^{\frac{q_++r-2}{2}}\right)\left(\int_{B_{\delta_k}(0)} v^{\frac{2^*\alpha}{2^*-q_++r+2}}dx\right)^{\frac{2^*-q_++r+2}{2^*}}\\
&\leq C\alpha^2 4^{k+1}\left(\int_{B_{\delta_k}(0)} v^{\frac{2^*\alpha}{2^*-q_+-r+2}}dx\right)^{\frac{2^*-q_+-r+2}{2^*}}.
\end{aligned}$$

It implies that

$$\|v\|_{L^{\frac{2^*\alpha}{2}}(B_{\delta_{k+1}}(0))}\leq\left(C\alpha^2 4^{k+1}\right)^{\frac{1}{\alpha}}\|v\|_{L^{\frac{2^*\alpha}{2^*-q_+-r+2}}(B_{\delta_k}(0))}. \tag{3.17}$$

Now we carry out an iteration process. Set $\beta_k=2\left(\frac{2^*-q_+-r+2}{2}\right)^k\alpha$ for $k=0,1,\cdots$. Then $\frac{2^*}{2^*-q_+-r+2}\beta_{k+1}=\frac{2^*}{2}\beta_k$. By (3.17), we have

$$\|v\|_{L^{2^*\beta_{k+1}}(B_{\delta_{k+1}(0)})}\leq\left(C\beta_{k+1}^2 4^{k+2}\right)^{\frac{1}{2\beta_{k+1}}}\|v\|_{L^{2^*\beta_k}(B_{\delta_k(0)})}.$$

This implies that

$$\begin{aligned}
\|v\|_{L^{2^*\beta_k}(B_{\delta_k(0)})}&\leq C^{\sum_{j=1}^k\frac{1}{2\beta_j}}\cdot\Pi_{j=1}^k\beta_j^{\frac{1}{\beta_j}}\cdot 4^{\sum_{j=1}^k\frac{j+1}{\beta_j}}\|v\|_{L^{2^*}\left(B_{\frac{\delta}{2}(0)}\right)}\\
&\leq C^{\frac{1}{2}\sum_{j=1}^k\beta_1^{-j}}\cdot\beta_1^{\sum_{j=1}^k j\beta_1^{-j}}\cdot 4^{\sum_{j=1}^k(j+1)\beta_1^{-j}}\|v\|_{L^{2^*}\left(B_{\frac{\delta}{2}(0)}\right)}.
\end{aligned}$$

Since $\beta_1>2$, the series $\sum\limits_{j=1}^{\infty}\beta_1^{-j}$ and $\sum\limits_{j=1}^{\infty}j\beta_1^{-j}$ are convergent. Letting $k\to\infty$, we conclude from Lemma 3.2 that

$$\|v\|_{L^\infty\left(B_{\frac{\delta}{4}}(0)\right)}\leq C\|v\|_{L^{2^*}\left(B_{\frac{\delta}{2}}(0)\right)}\leq C\left(\int_{B_{\frac{\delta}{2}}(0)}(|\nabla v|^2+v^2)dx\right)^{\frac{1}{2}}\leq C_9\left(\int_B|\nabla v|^2dx\right)^{\frac{1}{2}}\leq M.$$

By Lemma 3.1, we obtain

$$\|v\|_{L^\infty(B)}\leq\|v\|_{L^\infty\left(B_{\frac{\delta}{4}}(0)\right)}\leq M.$$

The proof is complete. □

Proof of Theorem 1.1. By definition of m_μ, we have $m_\mu(t)=t$ for $t\leq\frac{1}{\mu}$. It is easy to see that problem (2.2) reduces to problem (1.3) for $|u|\leq\frac{1}{\mu}$. Let $\mu>\frac{1}{M}$. We see that two positive radial symmetric solutions $u_\mu^{'}$ and $u_\mu^{''}$ of problem (2.2) are indeed two positive radial symmetric solution to problem (1.3). □

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