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SOME NEW q-HERMITE-HADAMARD-MERCER INEQUALITIES AND RELATED ESTIMATES IN QUANTUM CALCULUS

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Abstract. In this paper, we establish a quantum version of the Hermite-Hadamard-Mercer inequalities using the well-known Jensen-Mercer inequality. Moreover, we derive some new q-midpoint and q-trapezoidal type inequalities for differentiable functions. The newly developed inequalities are also shown to be the extensions of preexisting inequalities in the literature.

Keywords. Convex functions; Hermite–Hadamard inequality; Jensen-Mercer inequality; q-calculus.

1. Introduction

The Hermite-Hadamard inequality, named after Charles Hermite and Jacques Hadamard, is commonly known as Hadamard's inequality, that is, if a function $f:[a,b]\to\mathbb{R}$ is convex, the following double inequality holds:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\,dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

If f is a concave mapping, the above inequality holds in the opposite direction. Inequality (1.1) can be proved by using the Jensen inequality. There are numerous researches in the direction of Hermite-Hadamard for different kinds of convexities. For example, in [1, 2], the authors established some inequalities linked with midpoint and trapezoid formulas of numerical integration for convex functions.

In 2003, Mercer [3] proved another version of Jensen inequality, which is called Jensen-Mercer inequality and stated as follows.

Theorem 1.1. *For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequality holds for each $x_j\in[a,b]$:*

$$f\left(a+b-\sum_{j=i}^{n}u_jx_j\right)\leq f(a)+f(b)-\sum_{j=1}^{n}u_jf\left(x_j\right),$$

where $u_j\in[0,1]$ and $\sum_{j=1}^{n}u_j=1$.

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After that, in 2013, Kian et al. [4] used this new Jensen inequality and established the following new versions of Hermite-Hadamard inequality:

Theorem 1.2. *For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequalities hold for all $x,y\in[a,b]$ and $x<y$:*

$$f\left(a+b-\frac{x+y}{2}\right)\leq f(a)+f(b)-\frac{1}{y-x}\int_x^y f(u)\,du\leq f(a)+f(b)-f\left(\frac{x+y}{2}\right) \tag{1.2}$$

and

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq \frac{1}{y-x}\int_{a+b-y}^{a+b-x} f(u)\,du \\ &\leq \frac{f(a+b-x)+f(a+b-y)}{2} \\ &\leq f(a)+f(b)-\frac{f(x)+f(y)}{2}. \end{aligned} \tag{1.3}$$

Remark 1.1. It is easy to see that inequality (1.3) becomes the traditional Hermite-Hadamard inequality (1.1) for convex functions by setting $a=x$ and $b=y$.

For more recent inequalities related to (1.2) and (1.3), one can consult [5, 6, 7, 8, 9, 10].

On the other hand, quantum calculus is an important branch of calculus and it has a wide range of applications in the fields of mathematics and physics. Because of the numerous applications of quantum calculus (shortly, q-calculus) without limit calculus, many researchers began working on it and applying its concepts in various areas, such as differential equations, integral equalitions, mathematical modeling, and integral inequalities.

In [11, 12], two different versions of q-Hermite-Hadamard inequalities and some estimates were obtained based on q-derivatives and integrals (defined in Section 2) . The q-Hermite-Hadamard inequalities are described as:

Theorem 1.3. [11, 12] *For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequalities hold:*

$$f\left(\frac{qa+b}{[2]_q}\right) \leq \frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx \leq \frac{qf(a)+f(b)}{[2]_q}, \tag{1.4}$$

$$f\left(\frac{a+qb}{[2]_q}\right) \leq \frac{1}{b-a}\int_a^b f(x)\ {}^bd_qx \leq \frac{f(a)+qf(b)}{[2]_q}. \tag{1.5}$$

Remark 1.2. It is easy to observe that by adding (1.4) and (1.5), we have following q-Hermite-Hadamard inequality (see [12]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2(b-a)}\left[\int_a^b f(x)\ {}_ad_qx+\int_a^b f(x)\ {}^bd_qx\right]\leq\frac{f(a)+f(b)}{2}. \tag{1.6}$$

Recently, Ali et al. [13] and Sitthiwirattham et al. [14] used new techniques to prove the following two different and new versions of Hermite-Hadamard type inequalities:

Theorem 1.4. [13, 14] *For a convex mapping $f:[a,b]\to\mathbb{R}$, the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\ {}^{\frac{a+b}{2}}d_qx+\int_{\frac{a+b}{2}}^b f(x)\ {}_{\frac{a+b}{2}}d_qx\right]\leq\frac{f(a)+f(b)}{2}, \quad (1.7)$$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(x)\ {}_ad_qx+\int_{\frac{a+b}{2}}^b f(x)\ {}^bd_qx\right]\leq\frac{f(a)+f(b)}{2}. \quad (1.8)$$

Remark 1.3. By setting the limit as $q\to 1^-$ in (1.4)-(1.8), we recapture the traditional Hermite-Hadamard inequality (1.1).

There has been much research done in the direction of q-integral inequalities for different kind of convexities. For example, in [15], some new midpoint and trapezoidal type inequalities for q-integrals and q-differentiable convex functions were established. The authors of [16, 17, 18, 19] used q-integral and established Simpson's type inequalities for q-differentiable convex and general convex functions. For more recent inequalities in q-calculus, one can consult [20, 21, 22, 23, 24, 25] and the references therein.

Inspired by these ongoing studies, we consider Jensen-Mercer inequality and establish q-Hermite-Hadamard inequalities using the left and right q-integrals. Moreover, we derive some midpoint and trapezoidal type inequalities using the Jensen-Mercer inequality. It is also shown that the newly established inequalities are the extension of some already existing inequalities.

2. Preliminaries of q-Calculus and Some Inequalities

We recall some basics of quantum calculus in this section. For the sake of brevity, let $q\in(0,1)$, and we use the following notation (see [26]):

$$[n]_q=\frac{1-q^n}{1-q}=1+q+q^2+\ldots+q^{n-1}.$$

Definition 2.1. [25] The left or q_a-derivative of $f:[a,b]\to\mathbb{R}$ at $x\in[a,b]$ is expressed as:

$${}_aD_qf(x) = \frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)},\ x\neq a. \quad (2.1)$$

Definition 2.2. [12] The right or q^b-derivative of $f:[a,b]\to\mathbb{R}$ at $x\in[a,b]$ is expressed as:

$${}^bD_qf(x) = \frac{f(qx+(1-q)b)-f(x)}{(1-q)(b-x)},\ x\neq b.$$

Definition 2.3. [25] The left or q_a-integral of $f:[a,b]\to\mathbb{R}$ at $x\in[a,b]$ is defined as:

$$\int_a^x f(t)\ {}_ad_qt = (1-q)(x-a)\sum_{n=0}^{\infty}q^nf(q^nx+(1-q^n)a).$$

Definition 2.4. [11] The right or q^b-integral of $f:[a,b]\to\mathbb{R}$ at $x\in[a,b]$ is defined as:

$$\int_x^b f(t)\ {}^bd_qt = (1-q)(b-x)\sum_{n=0}^{\infty}q^nf(q^nx+(1-q^n)b).$$

Lemma 2.1. [18] *For continuous functions $f,g:[a,b]\to\mathbb{R}$, the following equality is true:*

$$\begin{aligned}&\int_0^c g(t)\ {}^bD_qf(ta+(1-t)b)\,d_qt\\ =\ &\frac{1}{b-a}\int_0^c D_qg(t)\,f(qta+(1-qt)b)\,d_qt-\left.\frac{g(t)\,f(ta+(1-t)b)}{b-a}\right|_0^c.\end{aligned}$$

Lemma 2.2. [19] *For continuous functions $f,g:[a,b]\to\mathbb{R}$, the following equality is true:*

$$\begin{aligned}&\int_0^c g(t)\ {}_aD_qf(tb+(1-t)a)\,d_qt\\ =\ &\left.\frac{g(t)\,f(tb+(1-t)a)}{b-a}\right|_0^c-\frac{1}{b-a}\int_0^c D_qg(t)\,f(qtb+(1-qt)a)\,d_qt.\end{aligned}$$

3. q-HERMITE–HADAMARD-MERECER INEQUALITIES

In this section, we prove two new and different Hermite-Hadamard-Mercer type inequalities.

Theorem 3.1. *For a convex function $f:[a,b]\to\mathbb{R}$, the following inequalities hold:*

$$\begin{aligned}f\left(a+b-\frac{x+y}{2}\right)&\le f(a)+f(b)-\frac{1}{y-x}\left[\int_x^{\frac{x+y}{2}}f(u)\ {}_xd_qu+\int_{\frac{x+y}{2}}^y f(u)\ {}^yd_qu\right] &(3.1)\\ &\le f(a)+f(b)-f\left(\frac{x+y}{2}\right)\end{aligned}$$

and

$$\begin{aligned}f\left(a+b-\frac{x+y}{2}\right)&\le\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}}f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x}f(u)\ {}^{a+b-x}d_qu\right] &(3.2)\\ &\le\frac{f(a+b-x)+f(a+b-y)}{2}\\ &\le f(a)+f(b)-\frac{f(x)+f(y)}{2}\end{aligned}$$

for all $x,y\in[a,b]$ and $x<y$.

Proof. From the Jensen-Mercer inequality, we have

$$f\left(a+b-\frac{u+v}{2}\right)\le f(a)+f(b)-\frac{1}{2}\left[f(u)+f(v)\right]. \tag{3.3}$$

By setting $u=\frac{t}{2}x+\frac{2-t}{2}y$ and $v=\frac{t}{2}y+\frac{2-t}{2}x$, we have

$$f\left(a+b-\frac{x+y}{2}\right)\le f(a)+f(b)-\frac{1}{2}\left[f\left(\frac{t}{2}x+\frac{2-t}{2}y\right)+f\left(\frac{t}{2}y+\frac{2-t}{2}x\right)\right]. \tag{3.4}$$

q-Integrating (3.4) with respect to t over $[0,1]$ and using Definitions 2.3 and 2.4, we have

$$\begin{aligned} & f\left(a+b-\frac{x+y}{2}\right) \\ \leq\ & f(a)+f(b)-\frac{1}{2}\left[\int_0^1\left(f\left(\frac{t}{2}x+\frac{2-t}{2}y\right)+f\left(\frac{t}{2}y+\frac{2-t}{2}x\right)\right)d_qt\right] \\ =\ & f(a)+f(b)-\frac{1}{2}\left[\int_0^1\left(f\left(t\frac{x+y}{2}+(1-t)y\right)+f\left(t\frac{x+y}{2}+(1-t)x\right)\right)d_qt\right] \\ =\ & f(a)+f(b)-\frac{1}{y-x}\left[\int_x^{\frac{x+y}{2}} f(u)\ {}_xd_qu+\int_{\frac{x+y}{2}}^{y} f(u)\ {}^yd_qu\right]. \end{aligned}$$

Thus the first inequality in (3.1) is proved. To prove the second inequality, we use the inequality (1.8) to obtain

$$\frac{1}{y-x}\left[\int_x^{\frac{x+y}{2}} f(u)\ {}_xd_qu+\int_{\frac{x+y}{2}}^{y} f(u)\ {}^yd_qu\right]\geq f\left(\frac{x+y}{2}\right),$$

which implies that

$$f(a)+f(b)-\frac{1}{y-x}\left[\int_x^{\frac{x+y}{2}} f(u)\ {}_xd_qu+\int_{\frac{x+y}{2}}^{y} f(u)\ {}^yd_qu\right]\leq f(a)+f(b)-f\left(\frac{x+y}{2}\right).$$

Thus the proof of inequality (3.1) is completed.

Now we prove the inequality (3.2). From the convexity, we have

$$\begin{aligned} f\left(a+b-\frac{u+v}{2}\right) &= f\left(\frac{a+b-u+a+b-v}{2}\right) \\ &\leq \frac{1}{2}\left[f(a+b-u)+f(a+b-v)\right]. \end{aligned}$$

By setting $u=\frac{t}{2}(a+b-y)+\frac{2-t}{2}(a+b-x)$ and $v=\frac{t}{2}(a+b-x)+\frac{2-t}{2}(a+b-y)$, we have

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) \leq\ & \frac{1}{2}\left[f\left(\frac{t}{2}(a+b-y)+\frac{2-t}{2}(a+b-x)\right)\right. \\ & \left.+f\left(\frac{t}{2}(a+b-x)+\frac{2-t}{2}(a+b-y)\right)\right]. \end{aligned} \tag{3.5}$$

q-Integrating (3.5) with respect to t over $[0,1]$ and Definitions 2.3 and 2.4, we have

$$f\left(a+b-\frac{x+y}{2}\right)\leq\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right].$$

Thus the first inequality in (3.2) is proved. We re-use the convexity to prove the second inequality in (3.2) as follows:

$$f\left(\frac{t}{2}(a+b-y)+\frac{2-t}{2}(a+b-x)\right)\leq\frac{t}{2}f(a+b-y)+\frac{2-t}{2}f(a+b-x) \tag{3.6}$$

and

$$f\left(\frac{t}{2}(a+b-x)+\frac{2-t}{2}(a+b-y)\right)\leq\frac{t}{2}f(a+b-x)+\frac{2-t}{2}f(a+b-y). \tag{3.7}$$

By applying convexity after adding (3.6) and (3.7), we have

$$\begin{aligned} & f\left(\frac{t}{2}(a+b-y)+\frac{2-t}{2}(a+b-x)\right)+f\left(\frac{t}{2}(a+b-x)+\frac{2-t}{2}(a+b-y)\right) \quad (3.8)\\ \leq\ & f(a+b-x)+f(a+b-y)\\ \leq\ & 2[f(a)+f(b)]-[f(x)+f(y)]. \end{aligned}$$

Thus we obtain the required inequality by q-integrating (3.8) with respect to t over $[0,1]$ and from Definitions 2.3 and 2.4. □

Remark 3.1. In Theorem 3.1, if we set the limit as $q\to 1^{-}$, then we recapture inequalities (1.2) and (1.3).

Remark 3.2. In Theorem 3.1, if we set $x=a$ and $y=b$, then inequality (3.2) becomes inequality (1.8).

4. Midpoint Inequalities

In this section, we establish some new midpoint type inequalities for differentiable functions that satisfy to Jensen-Mercer inequality:

Let us start with the following lemma.

Lemma 4.1. *Let $f:[a,b]\to\mathbb{R}$ be a q-differentiable mapping. If ${}_aD_qf$, ${}^bD_qf$ are q-integrable and continuous, then the following equality holds, for all $x,y\in[a,b]$ and $x<y$,*

$$\begin{aligned} & \frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right]-f\left(a+b-\frac{x+y}{2}\right)\\ =\ & \frac{y-x}{4}\left[\int_0^1 qt\ {}^bD_qf\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)d_qt\right.\\ & \left.-\int_0^1 qt\ {}_aD_qf\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)d_qt\right]. \end{aligned}$$

Proof. From Lemma 2.1 and Definition 2.4, we have

$$\begin{aligned} & \int_0^1 qt\ {}^bD_qf\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)d_qt\\ =\ & \int_0^1 qt\ {}^bD_qf\left(t\left(a+b-\frac{x+y}{2}\right)+(1-t)(a+b-x)\right)d_qt\\ =\ & -\frac{2q}{y-x}f\left(a+b-\frac{x+y}{2}\right)+\frac{2q}{y-x}\int_0^1 f\left(qt\left(a+b-\frac{x+y}{2}\right)+(1-qt)(a+b-x)\right)d_qt\\ =\ & -\frac{2q}{y-x}f\left(a+b-\frac{x+y}{2}\right)+\frac{2}{y-x}\left[(1-q)\sum_{n=0}^{\infty}q^nf\left(q^n\left(a+b-\frac{x+y}{2}\right)\right.\right.\\ & \left.\left.+(1-q^n)(a+b-x)\right)-(1-q)f\left(a+b-\frac{x+y}{2}\right)\right]\\ =\ & \frac{4}{(y-x)^2}\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu-\frac{2}{y-x}f\left(a+b-\frac{x+y}{2}\right). \end{aligned} \quad (4.1)$$

Similarly, from Lemma 2.2 and Definition 2.3, we have

$$\int_0^1 qt\ {}_aD_qf\left(t\left(a+b-\frac{x+y}{2}\right)+(1-t)(a+b-y)\right)d_qt \tag{4.2}$$
$$= \frac{2}{y-x}f\left(a+b-\frac{x+y}{2}\right)-\frac{4}{(y-x)^2}\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu.$$

Thus we obtain the required equality by subtracting (4.2) from (4.1). □

Theorem 4.1. *If the conditions of Lemma 4.1 hold and* $\left|{}_aD_qf\right|$, $\left|{}^bD_qf\right|$ *are convex, then we have the following inequality:*

$$\left|\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right]-f\left(a+b-\frac{x+y}{2}\right)\right|$$
$$\leq \frac{y-x}{4}\left[\frac{q}{[2]_q}\left(\left|{}^bD_qf(a)\right|+\left|{}^bD_qf(b)\right|\right)-\left(\frac{q\left([3]_q+q^2\right)}{2[2]_q[3]_q}\left|{}^bD_qf(x)\right|+\frac{q}{2[3]_q}\left|{}^bD_qf(y)\right|\right)\right.$$
$$\left.+\frac{q}{[2]_q}\left(\left|{}_aD_qf(a)\right|+\left|{}_aD_qf(b)\right|\right)-\left(\frac{q\left([3]_q+q^2\right)}{2[2]_q[3]_q}\left|{}_aD_qf(y)\right|+\frac{q}{2[3]_q}\left|{}_aD_qf(x)\right|\right)\right].$$

Proof. From equality (4.1) and Jensen-Mercer inequality, we have

$$\left|\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right]-f\left(a+b-\frac{x+y}{2}\right)\right|$$
$$\leq \frac{y-x}{4}\left[\int_0^1 qt\left|{}^bD_qf\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|d_qt\right.$$
$$\left.+\int_0^1 qt\left|{}_aD_qf\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)\right|d_qt\right]$$
$$\leq \frac{y-x}{4}\left[\int_0^1 qt\left(\left|{}^bD_qf(a)\right|+\left|{}^bD_qf(b)\right|-\frac{2-t}{2}\left|{}^bD_qf(x)\right|-\frac{t}{2}\left|{}^bD_qf(y)\right|\right)\right.$$
$$\left.+\int_0^1 qt\left(\left|{}_aD_qf(a)\right|+\left|{}_aD_qf(b)\right|-\frac{2-t}{2}\left|{}_aD_qf(y)\right|-\frac{t}{2}\left|{}_aD_qf(x)\right|\right)\right]$$
$$= \frac{y-x}{4}\left[\frac{q}{[2]_q}\left(\left|{}^bD_qf(a)\right|+\left|{}^bD_qf(b)\right|\right)-\left(\frac{q\left([3]_q+q^2\right)}{2[2]_q[3]_q}\left|{}^bD_qf(x)\right|+\frac{q}{2[3]_q}\left|{}^bD_qf(y)\right|\right)\right.$$
$$\left.+\frac{q}{[2]_q}\left(\left|{}_aD_qf(a)\right|+\left|{}_aD_qf(b)\right|\right)-\left(\frac{q\left([3]_q+q^2\right)}{2[2]_q[3]_q}\left|{}_aD_qf(y)\right|+\frac{q}{2[3]_q}\left|{}_aD_qf(x)\right|\right)\right],$$

which completes the proof. □

Remark 4.1. In Theorem 4.1, if we set $x=a$ and $y=b$, then we have the following inequality

$$\left|\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(u)\ {}_a d_q u+\int_{\frac{a+b}{2}}^{b} f(u)\ {}^b d_q u\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{8}\left[\frac{q}{[3]_q}\left|{}^bD_q f(a)\right|+\frac{q\left([3]_q+q^2\right)}{[2]_q[3]_q}\left|{}^bD_q f(b)\right|\right.$$
$$\left.+\left(\frac{q\left([3]_q+q^2\right)}{[2]_q[3]_q}\left|{}_aD_q f(a)\right|+\frac{q}{[3]_q}\left|{}_aD_q f(b)\right|\right)\right].$$

This was established by Ali et al. in [13].

Remark 4.2. If we take the limit as $q\to 1^-$, $x=a$, and $y=b$ in Theorem 4.1, then Theorem 4.1 gives [1, Theorem 2.2].

Theorem 4.2. *If the conditions of Lemma 4.1 hold and* $\left|{}_aD_q f\right|^s$, $\left|{}^bD_q f\right|^s$, $s\geq 1$ *are convex, then we have the following inequality:*

$$\left|\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_q u+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_q u\right]-f\left(a+b-\frac{x+y}{2}\right)\right|$$
$$\leq \frac{y-x}{4}\left(\frac{q}{[2]_q}\right)^{1-\frac{1}{s}}\left[\left(\frac{q}{[2]_q}\left(\left|{}^bD_q f(a)\right|^s+\left|{}^bD_q f(b)\right|^s\right)\right.\right.$$
$$\left.-\frac{q\left([3]_q+q^2\right)}{2[2]_q[3]_q}\left|{}^bD_q f(x)\right|^s-\frac{q}{2[3]_q}\left|{}^bD_q f(y)\right|^s\right)^{\frac{1}{s}}$$
$$\left.+\left(\frac{q}{[2]_q}\left(\left|{}_aD_q f(a)\right|^s+\left|{}_aD_q f(b)\right|^s\right)-\frac{q\left([3]_q+q^2\right)}{2[2]_q[3]_q}\left|{}_aD_q f(y)\right|^s-\frac{q}{2[3]_q}\left|{}_aD_q f(x)\right|^s\right)^{\frac{1}{s}}\right]. \tag{4.3}$$

Proof. From equality (4.1) and the power mean inequality, we have

$$\left|\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_q u+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_q u\right]-f\left(a+b-\frac{x+y}{2}\right)\right|$$
$$\leq \frac{y-x}{4}\left[\int_0^1 qt\left|{}^bD_q f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|d_q t\right.$$
$$\left.+\int_0^1 qt\left|{}_aD_q f\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)\right|d_q t\right]$$

$$\leq \frac{y-x}{4}\left(\int_0^1 qt d_qt\right)^{1-\frac{1}{s}}\left[\left(\int_0^1 qt\left|{}^bD_qf\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1 qt\left|{}_aD_qf\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right].$$

By the Jensen-Mercer inequality, we have

$$\left|\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right]-f\left(a+b-\frac{x+y}{2}\right)\right|$$
$$\leq \frac{y-x}{4}\left(\frac{q}{[2]_q}\right)^{1-\frac{1}{s}}\left[\left(\frac{q}{[2]_q}\left(\left|{}^bD_qf(a)\right|^s+\left|{}^bD_qf(b)\right|^s\right)\right.\right.$$
$$\left.-\frac{q\left([3]_q+q^2\right)}{2[2]_q[3]_q}\left|{}^bD_qf(x)\right|^s-\frac{q}{2[3]_q}\left|{}^bD_qf(y)\right|^s\right)^{\frac{1}{s}}$$
$$+\left(\frac{q}{[2]_q}\left(\left|{}_aD_qf(a)\right|^s+\left|{}_aD_qf(b)\right|^s\right)\right.$$
$$\left.\left.-\frac{q\left([3]_q+q^2\right)}{2[2]_q[3]_q}\left|{}_aD_qf(y)\right|^s-\frac{q}{2[3]_q}\left|{}_aD_qf(x)\right|^s\right)^{\frac{1}{s}}\right].$$

Hence, the proof is completed. □

Remark 4.3. In Theorem 4.2, if we set $x=a$ and $y=b$, then we have the following inequality

$$\left|\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(u)\ {}_ad_qu+\int_{\frac{a+b}{2}}^b f(u)\ {}^bd_qu\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{q(b-a)}{4[2]_q}\left[\left(\frac{[2]_q\left|{}^bD_qf(a)\right|^s+\left([3]_q+q^2\right)\left|{}^bD_qf(b)\right|^s}{2[3]_q}\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\frac{\left([3]_q+q^2\right)\left|{}_aD_qf(a)\right|^s+[2]_q\left|{}_aD_qf(b)\right|^s}{2[3]_q}\right)^{\frac{1}{s}}\right].$$

This was established by Ali et al. in [13].

Remark 4.4. In Theorem 4.3, if we set $q\to 1^-$, then Theorem 4.2 gives [5, Corollary 3.10].

Theorem 4.3. *If the conditions of Lemma 4.1 hold and* $\left|{}_aD_qf\right|^s$, $\left|{}^bD_qf\right|^s$, $s>1$ *are convex, then we have the following inequality:*

$$\left|\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right]-f\left(a+b-\frac{x+y}{2}\right)\right|$$

$$\leq \frac{q(y-x)}{4}\left(\frac{1}{[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\left|{}^bD_qf(a)\right|^s+\left|{}^bD_qf(b)\right|^s-\frac{\left([2]_q+q\right)\left|{}^bD_qf(x)\right|^s+\left|{}^bD_qf(y)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right.$$

$$\left.+\left(\left|{}_aD_qf(a)\right|^s+\left|{}_aD_qf(b)\right|^s-\frac{\left([2]_q+q\right)\left|{}_aD_qf(y)\right|^s+\left|{}_aD_qf(x)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right],$$

where $r^{-1}+s^{-1}=1$.

Proof. From equality (4.1) and the Hölder inequality, we have

$$\left|\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right]-f\left(a+b-\frac{x+y}{2}\right)\right|$$

$$\leq \frac{y-x}{4}\left[\int_0^1 qt\left|{}^bD_qf\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|d_qt\right.$$

$$\left.+\int_0^1 qt\left|{}_aD_qf\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)\right|d_qt\right]$$

$$\leq \frac{y-x}{4}\left(\int_0^1 (qt)^r d_qt\right)^{\frac{1}{r}}\left[\left(\int_0^1\left|{}^bD_qf\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right.$$

$$\left.+\left(\int_0^1\left|{}_aD_qf\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right].$$

By the Jensen-Mercer inequality, we have

$$\left|\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right]-f\left(a+b-\frac{x+y}{2}\right)\right|$$

$$\leq \frac{q(y-x)}{4}\left(\frac{1}{[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\left|{}^bD_qf(a)\right|^s+\left|{}^bD_qf(b)\right|^s\right.\right.$$

$$\left.-\frac{\left([2]_q+q\right)\left|{}^bD_qf(x)\right|^s+\left|{}^bD_qf(y)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}$$

$$\left.+\left(\left|{}_aD_qf(a)\right|^s+\left|{}_aD_qf(b)\right|^s-\frac{\left([2]_q+q\right)\left|{}_aD_qf(y)\right|^s+\left|{}_aD_qf(x)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right].$$

Hence, the proof is completed. □

Remark 4.5. In Theorem 4.2, if we set $x=a$ and $y=b$, then we have the following inequality

$$\begin{aligned}&\left|\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(u)\ {}_a d_q u+\int_{\frac{a+b}{2}}^{b} f(u)\ {}^b d_q u\right]-f\left(\frac{a+b}{2}\right)\right|\\ \leq\ & \frac{q(b-a)}{4\left([r+1]_q\right)^{\frac{1}{r}}}\left[\left(\frac{\left|{}^bD_q f(a)\right|^s+\left([2]_q+q\right)\left|{}^bD_q f(b)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right.\\ &\left.+\left(\frac{\left([2]_q+q\right)\left|{}_aD_q f(a)\right|^s+\left|{}_aD_q f(b)\right|^s}{2[2]_q}\right)^{\frac{1}{s}}\right].\end{aligned}$$

This was established by Ali et al. in [13].

Remark 4.6. In Theorem 4.3, if we set the limit as $q\to 1^-$, then Theorem 4.3 gives [5, Corollary 3.13].

5. Trapezoidal Inequalities

In this section, we establish some new trapezoidal type inequalities for differentiable functions that satisfy to Jensen-Mercer inequality:

Let us start with the following lemma.

Lemma 5.1. *Let $f:[a,b]\to\mathbb{R}$ be a q-differentiable mapping. If ${}_aD_q f$, ${}^bD_q f$ are q-integrable and continuous, then the following equality holds for all $x,y\in[a,b]$ and $x<y$:*

$$\begin{aligned}&\frac{f(a+b-x)+f(a+b-y)}{2}-\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_q u+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_q u\right]\\ =\ & \frac{y-x}{4}\left[\int_0^1 (1-qt)\ {}^bD_q f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)d_q t\right.\\ &\left.+\int_0^1 (qt-1)\ {}_aD_q f\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)d_q t\right].\end{aligned}$$

Proof. From Lemma 2.1 and Defnition 2.4, we have

$$\begin{aligned}&\int_0^1 (1-qt)\ {}^bD_q f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)d_q t \qquad (5.1)\\ =\ & \int_0^1 {}^bD_q f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)d_q t-\int_0^1 qt\ {}^bD_q f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)d_q t\\ =\ & \frac{2}{y-x}f(a+b-x)-\frac{4}{(y-x)^2}\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-y}d_q u.\end{aligned}$$

Similarly, from Lemma 2.2 and Definition 2.3, we have

$$\begin{aligned}&\int_0^1 (qt-1)\ {}_aD_qf\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)d_qt \\ =\ & \frac{2}{y-x}f(a+b-y)-\frac{4}{(y-x)^2}\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu.\end{aligned} \tag{5.2}$$

Hence, we obtain the resultant equality by adding (5.1) and (5.2). □

Remark 5.1. In Lemma 5.1, if we set $x=a$ and $y=b$, then we have the following equality:

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{1}{y-x}\left[\int_a^{\frac{a+b}{2}} f(u)\ {}_ad_qu+\int_{\frac{a+b}{2}}^b f(u)\ {}^bd_qu\right] \\ =\ & \frac{b-a}{4}\left[\int_0^1 (1-qt)\ {}^bD_qf\left(\frac{t}{2}a+\frac{2-t}{2}b\right)d_qt+\int_0^1 (qt-1)\ {}_aD_qf\left(\frac{2-t}{2}a+\frac{t}{2}b\right)d_qt\right].\end{aligned}$$

This was established by Ali et al. in [13].

Theorem 5.1. *If the conditions of Lemma 5.1 hold and $\left|{}_aD_qf\right|$, $\left|{}^bD_qf\right|$ are convex, then we have the following inequality:*

$$\begin{aligned}&\left|\frac{f(a+b-x)+f(a+b-y)}{2}\right. \\ &\left.-\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right]\right| \\ \leq\ & \frac{y-x}{4}\left[\frac{1}{[2]_q}\left(\left|{}^bD_qf(a)\right|+\left|{}^bD_qf(b)\right|\right)\right. \\ &-\left(\frac{2[3]_q-q^3}{2[2]_q[3]_q}\left|{}^bD_qf(x)\right|+\frac{1}{2\left([4]_q+q[2]_q\right)}\left|{}^bD_qf(y)\right|\right) \\ &+\frac{1}{[2]_q}\left(\left|{}_aD_qf(a)\right|+\left|{}_aD_qf(b)\right|\right) \\ &\left.-\left(\frac{2[3]_q-q^3}{2[2]_q[3]_q}\left|{}_aD_qf(y)\right|+\frac{1}{2\left([4]_q+q[2]_q\right)}\left|{}_aD_qf(x)\right|\right)\right].\end{aligned}$$

Proof. From equality (5.1) and Jensen-Mercer inequality, we have

$$\begin{aligned}
&\left|\frac{f(a+b-x)+f(a+b-y)}{2}\right. \\
&\left.-\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_q u+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_q u\right]\right| \\
\leq\ & \frac{y-x}{4}\left[\int_0^1 (1-qt)\left|{}^{b}D_q f\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right| d_q t\right. \\
&\left.+\int_0^1 (1-qt)\left|{}_{a}D_q f\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)\right| d_q t\right] \\
\leq\ & \frac{y-x}{4}\left[\int_0^1 (1-qt)\left(\left|{}^{b}D_q f(a)\right|+\left|{}^{b}D_q f(b)\right|-\left(\frac{2-t}{2}\left|{}^{b}D_q f(x)\right|+\frac{t}{2}\left|{}^{b}D_q f(y)\right|\right)\right) d_q t\right. \\
&\left.+\int_0^1 (1-qt)\left(\left|{}_{a}D_q f(a)\right|+\left|{}_{a}D_q f(b)\right|-\left(\frac{2-t}{2}\left|{}_{a}D_q f(y)\right|+\frac{t}{2}\left|{}_{a}D_q f(x)\right|\right)\right) d_q t\right] \\
=\ & \frac{y-x}{4}\left[\frac{1}{[2]_q}\left(\left|{}^{b}D_q f(a)\right|+\left|{}^{b}D_q f(b)\right|\right)\right. \\
&-\left(\frac{2[3]_q-q^3}{2[2]_q[3]_q}\left|{}^{b}D_q f(x)\right|+\frac{1}{2\left([4]_q+q[2]_q\right)}\left|{}^{b}D_q f(y)\right|\right) \\
&+\frac{1}{[2]_q}\left(\left|{}_{a}D_q f(a)\right|+\left|{}_{a}D_q f(b)\right|\right) \\
&\left.-\left(\frac{2[3]_q-q^3}{2[2]_q[3]_q}\left|{}_{a}D_q f(y)\right|+\frac{1}{2\left([4]_q+q[2]_q\right)}\left|{}_{a}D_q f(x)\right|\right)\right].
\end{aligned}$$

Thus the proof is completed. □

Remark 5.2. In Theorem 5.1, if we set $x=a$ and $y=b$, then we have the following inequality:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(u)\ {}_{a}d_q u+\int_{\frac{a+b}{2}}^{b} f(u)\ {}^{b}d_q u\right]\right| \\
\leq\ & \frac{(b-a)}{8[2]_q[3]_q}\left[\left|{}^{b}D_q f(a)\right|+\left([3]_q+q[2]_q\right)\left|{}^{b}D_q f(b)\right|\right. \\
&\left.+\left([3]_q+q[2]_q\right)\left|{}_{a}D_q f(a)\right|+\left|{}_{a}D_q f(b)\right|\right].
\end{aligned}$$

This was established by Ali et al. in [13].

Remark 5.3. If we take the limit as $q\to 1^-$, $x=a$ and $y=b$, then Theorem 5.1 gives [2, Theorem 2.2].

Theorem 5.2. *If the conditions of Lemma 5.1 hold and* $\left|{}_aD_qf\right|^s$, $\left|{}^bD_qf\right|^s$, $s\geq 1$ *are convex, then we have the following inequality:*

$$\begin{aligned}
&\left|\frac{f(a+b-x)+f(a+b-y)}{2}\right. \\
&\left.-\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right]\right| \\
\leq\ &\frac{y-x}{4}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{s}}\left[\left(\frac{1}{[2]_q}\left(\left|{}^bD_qf(a)\right|^s+\left|{}^bD_qf(b)\right|^s\right)\right.\right. \\
&\left.-\left(\frac{2[3]_q-q^3}{2[2]_q[3]_q}\left|{}^bD_qf(x)\right|^s+\frac{1}{2\left([4]_q+q[2]_q\right)}\left|{}^bD_qf(y)\right|^s\right)\right)^{\frac{1}{s}} \\
&+\left(\frac{1}{[2]_q}\left(\left|{}_aD_qf(a)\right|^s+\left|{}_aD_qf(b)\right|^s\right)\right. \\
&\left.\left.-\left(\frac{2[3]_q-q^3}{2[2]_q[3]_q}\left|{}_aD_qf(y)\right|^s+\frac{1}{2\left([4]_q+q[2]_q\right)}\left|{}_aD_qf(x)\right|^s\right)\right)^{\frac{1}{s}}\right].
\end{aligned}$$

Proof. From equality (5.1) and the power mean inequality, we have

$$\begin{aligned}
&\left|\frac{f(a+b-x)+f(a+b-y)}{2}\right. \\
&\left.-\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right]\right| \\
\leq\ &\frac{y-x}{4}\left[\int_0^1(1-qt)\left|{}^bD_qf\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|d_qt\right. \\
&\left.+\int_0^1(1-qt)\left|{}_aD_qf\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)\right|d_qt\right] \\
\leq\ &\frac{y-x}{4}\left(\int_0^1(1-qt)\,d_qt\right)^{1-\frac{1}{s}}\left[\left(\int_0^1(1-qt)\left|{}^bD_qf\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right. \\
&\left.+\left(\int_0^1(1-qt)\left|{}_aD_qf\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right].
\end{aligned}$$

From the Jensen-Mercer inequality, we have

$$\begin{aligned}
&\left|\frac{f(a+b-x)+f(a+b-y)}{2}\right. \\
&\left.-\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_q u+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_q u\right]\right| \\
\leq\ & \frac{y-x}{4}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{s}}\left[\left(\frac{1}{[2]_q}\left(\left|{}^bD_q f(a)\right|^s+\left|{}^bD_q f(b)\right|^s\right)\right.\right. \\
&\left.-\left(\frac{2[3]_q-q^3}{2[2]_q[3]_q}\left|{}^bD_q f(x)\right|^s+\frac{1}{2\left([4]_q+q[2]_q\right)}\left|{}^bD_q f(y)\right|^s\right)\right)^{\frac{1}{s}} \\
&+\left(\frac{1}{[2]_q}\left(\left|{}_aD_q f(a)\right|^s+\left|{}_aD_q f(b)\right|^s\right)\right. \\
&\left.\left.-\left(\frac{2[3]_q-q^3}{2[2]_q[3]_q}\left|{}_aD_q f(y)\right|^s+\frac{1}{2\left([4]_q+q[2]_q\right)}\left|{}_aD_q f(x)\right|^s\right)\right)^{\frac{1}{s}}\right].
\end{aligned}$$

Hence, the proof is completed. □

Remark 5.4. In Theorem 5.2, if we set $x=a$ and $y=b$, then we have the following inequality:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(u)\ {}_a d_q u+\int_{\frac{a+b}{2}}^b f(u)\ {}^b d_q u\right]\right| \\
\leq\ & \frac{(b-a)}{4[2]_q}\left[\left(\frac{\left|{}^bD_q f(a)\right|^s+\left([3]_q+q[2]_q\right)\left|{}^bD_q f(b)\right|^s}{2[3]_q}\right)^{\frac{1}{s}}\right. \\
&\left.+\left(\frac{\left([3]_q+q[2]_q\right)\left|{}_aD_q f(a)\right|^s+\left|{}_aD_q f(b)\right|^s}{2[3]_q}\right)^{\frac{1}{s}}\right].
\end{aligned}$$

This was established by Ali et al. in [13].

Theorem 5.3. *If the conditions of Lemma 5.1 hold and* $\left|{}_aD_qf\right|^s$, $\left|{}^bD_qf\right|^s$, $s>1$ *are convex, then we have the following inequality:*

$$\begin{aligned}
&\Bigg|\frac{f(a+b-x)+f(a+b-y)}{2}\\
&-\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right]\Bigg|\\
\leq\ &\frac{y-x}{4}\left(\int_0^1(1-qt)^r\,d_qt\right)^{\frac{1}{r}}\\
&\times\left[\left(\left|{}^bD_qf(a)\right|^s+\left|{}^bD_qf(b)\right|^s-\left(\frac{\left([2]_q+q\right)\left|{}^bD_qf(x)\right|^s+\left|{}^bD_qf(y)\right|^s}{2[2]_q}\right)\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\left|{}^bD_qf(a)\right|^s+\left|{}^bD_qf(b)\right|^s-\left(\frac{\left([2]_q+q\right)\left|{}_aD_qf(y)\right|^s+\left|{}_aD_qf(x)\right|^s}{2[2]_q}\right)\right)^{\frac{1}{s}}\right],
\end{aligned}$$

where $s^{-1}+r^{-1}=1$.

Proof. From equality (5.1) and the Hölder inequality, we have

$$\begin{aligned}
&\Bigg|\frac{f(a+b-x)+f(a+b-y)}{2}\\
&-\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_qu+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_qu\right]\Bigg|\\
\leq\ &\frac{y-x}{4}\left[\int_0^1(1-qt)\left|{}^bD_qf\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|d_qt\right.\\
&\left.+\int_0^1(1-qt)\left|{}_aD_qf\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)\right|d_qt\right]\\
\leq\ &\frac{y-x}{4}\left(\int_0^1(1-qt)^r\,d_qt\right)^{\frac{1}{r}}\left[\left(\int_0^1\left|{}^bD_qf\left(a+b-\left(\frac{2-t}{2}x+\frac{t}{2}y\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\int_0^1\left|{}_aD_qf\left(a+b-\left(\frac{2-t}{2}y+\frac{t}{2}x\right)\right)\right|^s d_qt\right)^{\frac{1}{s}}\right].
\end{aligned}$$

After applying the Jensen-Mercer inequality, we have

$$\begin{aligned}
&\left|\frac{f(a+b-x)+f(a+b-y)}{2}\right.\\
&\left.-\frac{1}{y-x}\left[\int_{a+b-y}^{a+b-\frac{x+y}{2}} f(u)\ {}_{a+b-y}d_q u+\int_{a+b-\frac{x+y}{2}}^{a+b-x} f(u)\ {}^{a+b-x}d_q u\right]\right|\\
\leq\ & \frac{y-x}{4}\left(\int_0^1 (1-qt)^r\, d_q t\right)^{\frac{1}{r}}\\
&\times\left[\left(\left|{}^bD_q f(a)\right|^s+\left|{}^bD_q f(b)\right|^s-\left(\frac{\left([2]_q+q\right)\left|{}^bD_q f(x)\right|^s+\left|{}^bD_q f(y)\right|^s}{2\,[2]_q}\right)\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\left|{}_aD_q f(a)\right|^s+\left|{}_aD_q f(b)\right|^s-\left(\frac{\left([2]_q+q\right)\left|{}_aD_q f(y)\right|^s+\left|{}_aD_q f(x)\right|^s}{2\,[2]_q}\right)\right)^{\frac{1}{s}}\right].
\end{aligned}$$

Thus the proof is completed. □

Remark 5.5. In Theorem 5.3, if we set $x=a$ and $y=b$, then we have the following inequality:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(u)\ {}_a d_q u+\int_{\frac{a+b}{2}}^{b} f(u)\ {}^b d_q u\right]\right|\\
\leq\ & \frac{b-a}{4}\left(\int_0^1 (1-qt)^r\, d_q t\right)^{\frac{1}{r}}\left[\left(\frac{\left|{}^bD_q f(a)\right|^s+\left([2]_q+q\right)\left|{}^bD_q f(b)\right|^s}{2\,[2]_q}\right)^{\frac{1}{s}}\right.\\
&\left.+\left(\frac{\left([2]_q+q\right)\left|{}_aD_q f(a)\right|^s+\left|{}_aD_q f(b)\right|^s}{2\,[2]_q}\right)^{\frac{1}{s}}\right].
\end{aligned}$$

This was established by Ali et al. in [13].

6. Conclusion

With the help of the well-known Jensen-Mercer inequality, we derived some new versions of the q-Hermite-Hadamard-Mercer inequalities. We also proved some q-midpoint and q-trapezoidal estimates for the differentiable functions to have the Jensen-Mercer inequality properties. It is an intriguing problem in which upcoming researchers can obtain similar inequalities for various types of convexities in their future work.

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