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FRACTIONAL KIRCHHOFF-CHOQUARD EQUATIONS INVOLVING UPPER CRITICAL EXPONENT AND GENERAL NONLINEARITY

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Abstract. In this paper, we mainly investigate the Kirchhoff-Choquard problem with upper critical exponent and general nonlinearity. Some appropriate restrictions are imposed on parameters. Furthermore, when the nonlinearity satisfies subcritical growth conditions, the existence of global minimizers and mountain pass type solutions of the problem are established by using the variational properties and fibbing maps.
Keywords. Choquard equation; Kirchhoff type problems; Nonlinearity; Upper critical exponent; Variational methods.

1. Introduction and Main Results

In this paper, we consider the following critical Kirchhoff-Choquard problem

$$\begin{cases} \left(a+b\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u = \left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dy\right)|u|^{2^*_{\mu,s}-2}u+\lambda f(x,u) \ in\ \Omega, \\ u=0 \qquad in\ \mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain, a and b are positive real numbers, $s\in(0,1)$, $N>4s$, $\mu>4s$, $2^*_{\mu,s}=\frac{2N-\mu}{N-2s}$ is the upper critical exponent in the Hardy-Littlewood-Sobolev inequality, λ is a positive parameter, f is a subcritical Carathéodory function, and $(-\Delta)^s$ is the fractional Laplacian operator.

The fractional Laplacian in (1.1) is defined as

$$(-\Delta)^s u(x) := K_{N,s}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(0)}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy,$$

where

$$\frac{1}{K_{N,s}} := \int_{\mathbb{R}^N}\frac{1-\cos\zeta_1}{|\zeta|^{N+2s}}d\zeta.$$

In recent years, the research on the Kirchhoff-Choquard equation become significant due to its application in physics; see, e.g., [1, 2, 3, 4, 5] for more details. In the local case, Goel and

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Sreenadh [6] studied the following equation

$$\begin{cases} \mathscr{K}u=\lambda f(x)|u|^{q-2}u+\left(\int_{\Omega}\frac{|u(y)|^{2_\mu^*}}{|x-y|^{\mu}}dy\right)|u|^{2_\mu^*-2}u & in\ \ \Omega, \\ u=0 & on\ \ \partial\Omega, \end{cases} \tag{1.2}$$

where $\mathscr{K}u=-\left(a+\varepsilon^p\left(\int_\Omega|\nabla u|^2dx\right)^{\theta-1}\right)\Delta u$ with $a>0$, $p>N-2$ $(N\geq 3)$ and $\theta\in[1,2_\mu^*)$. Here $0<\mu<N$, $1<q\leq 2$, and λ is a positive parameter. The function $f(x)$ is a continuous sign changing function. The authors established the existence of two positive solutions for the problem (1.2). When $1<q<2$, the authors applied minimization argument on the Nehari submanifolds to obtain the first solution. When $q=2$, they used the mountain pass lemma to obtain the second solution. Further, Luo et al. [7] studied the problem (1.2) with $p=1$ and $\theta=2$. The multiplicity and nonexistence of solutions were established for the above problem. Similarly, Liang et al. [8] used the Kajikiya new version of the symmetric mountain pass theorem to investigate the following Kirchhoff-Choquard type equation

$$-(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx)\Delta u=ak(x)|u|^{q-2}u+\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{2_\mu^*}}{|x-y|^{\mu}}dy\right)|u|^{2_\mu^*-2}u,\quad x\in\mathbb{R}^N, \tag{1.3}$$

where $a>0$, $b\geq 0$, $0<\mu<N$, $N\geq 3$, α and β are two positive real parameters, $2_\mu^*=\frac{2N-\mu}{N-2}$, $k\in L^r(\mathbb{R}^N)$ with $r=\frac{2^*}{2^*-q}$ if $1<q<2^*$, and $r=\infty$ if $q>2^*$. The authors proved the multiplicity of solutions for problem (1.3). More results on the Kirchhoff-Choquard problems can been found in [9, 10, 11, 12].

In the nonlocal case, fractional Kirchhoff-Choquard equation is also a hot topic of research. Applying Krasnoselskii's genus theory, Wang and Xiang [13] studied the following Kirchhoff-Choquard equation

$$(a+b[u]_{s,p}^p)(-\Delta)_p^s u=\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{p_{\mu,s}^*}}{|x-y|^{\mu}}dy\right)|u|^{p_{\mu,s}^*-2}u+\lambda h(x)|u|^{q-2}u\quad in\ \ \mathbb{R}^N, \tag{1.4}$$

where $[u]_{s,p}^p=\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{\frac{1}{p}}$, $a\geq 0$, $b>0$, $s\in\left(0,\min\{1,\frac{N}{2p}\}\right)$, $2sp\leq\mu<N$, $\lambda>0$, $p_{\mu,s}^*=\frac{(N-\frac{\mu}{2})p}{N-sp}$, $1<q<p_s^*=\frac{Np}{N-sp}$, and $h\in L^{\frac{p_s^*}{p_s^*-q}}(\mathbb{R}^N)$. The authors showed the multiplicity of nontrivial solutions for problem (1.4). Meanwhile, by using the concentration-compactness principle, Chen et al. [14] obtained the existence of a positive weak solution for the nonlocal fractional Kirchhoff-Choquard type equation. For some related results, we refer the readers to [15, 16, 17, 18, 19, 20]. For the general nonlinearity, Liang and Rădulescu [21] considered the following Kirchhoff-type Schrödinger-Choquard equation

$$M(\|u\|_s^p)[(-\Delta)_p^s u+V(x)|u|^{p-2}u]=\lambda f(x,u)+(|x|^{-\mu}*|u|^{p_{\mu,s}^*})|u|^{p_{\mu,s}^*-2}u\quad in\ \ \mathbb{R}^N, \tag{1.5}$$

where $\|u\|_s=\left(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{\frac{1}{p}}$, $p_{\mu,s}^*=\frac{(N-\frac{\mu}{2})p}{N-sp}$, and $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. The authors obtained the existence of infinitely many solutions for problem (1.5) by using variational methods. For more work on the nonlocal fractional equation involving general nonlinearity, we refer to [22, 23, 24, 25, 26] and the references therein.

Recently, Appolloin et al. [27] obtained some new results for the following problem

$$\begin{cases} \left(a+b\int_{\mathscr{Q}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u=|u|^{2_s^*-2}u+\lambda g(x,u) & in \ \ \Omega, \\ u=0 & in \ \ \mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.6}$$

where $(-\Delta)^s$ is the fractional Laplacian operator, $s\in(0,1)$, $N>4s$, $a>0$, $b>0$, $2_s^*=\frac{2N}{N-2s}$, λ is a parameter, and g is a Carathéodory function. By using variational and topological nature, the authors proved the weak lower semicontinuity, Palais-Smale condition, the convexity and existence of global minimizers, local minimizers, and mountain pass type solutions to (1.6). In [28], Faraci and Silva studied problem (1.6) with $s=1$ and they proved some similar conclusions.

Motivated by above discussions, we extend problem (1.6) to the fractional Kirchhoff-Choquard problem. In the present paper, coupled with the Hardy-Littlewood-Sobolev critical exponent and general nonlinearity, problem (1.1) is studied. The presence of the upper critical exponent will prevent us from using the variational methods in a standard way. Therefore, the key point is to overcome the lack of compactness, the main tool adopted in our proof is concentration-compactness principle of a nonlocal fractional problem with Choquard type term. Moreover, some new estimates on cut-off function will be reestablished. In fact, our general nonlinearity includes and generalizes subcritical terms in problems (1.2), (1.3), and (1.4).

Let

$$\mathscr{M}_{N,s,\mu}:=\left(\frac{\mu-4s}{2N+4s-2\mu}\right)^{\frac{4s-\mu}{\mu-2s-N}}\left(\frac{2N-\mu}{2N+4s-2\mu}\right)^{\frac{N-2s}{\mu-2s-N}}S_{H,L}^{\frac{2N-\mu}{\mu-2s-N}},$$

$$\mathscr{N}_{N,s,\mu}:=\left(\frac{\mu-4s}{N+2s-\mu}\right)^{\frac{4s-\mu}{\mu-2s-N}}\left(\frac{N-2s}{N+2s-\mu}\right)^{\frac{N-2s}{\mu-2s-N}}S_{H,L}^{\frac{2N-\mu}{\mu-2s-N}},$$

and

$$\mathscr{Q}_{N,s,\mu}:=\left(\frac{\mu-4s}{N+2s-\mu}\right)^{\frac{4s-\mu}{\mu-2s-N}}\left(\frac{(N-2s)^2}{(N+2s-\mu)(2s+3N-2\mu)}\right)^{\frac{N-2s}{\mu-2s-N}}S_{H,L}^{\frac{2N-\mu}{\mu-2s-N}},$$

where $S_{H,L}$ is defined by (2.1).

Our results are the following theorems.

Theorem 1.1. *The following results hold:*

(i) If $a^{\frac{4s-\mu}{\mu-2s-N}}b\geq\mathscr{M}_{N,s,\mu}$, then he energy functional $\mathscr{I}_{a,b}$ is sequentially weakly lower semicontinuous on $X_0^s(\Omega)$;

(ii) If $a^{\frac{4s-\mu}{\mu-2s-N}}b\geq\mathscr{N}_{N,s,\mu}$, then he energy functional $\mathscr{I}_{a,b}$ satisfies the compactness Palais-Smale condition at level $c\in\mathbb{R}$;

(iii) If $a^{\frac{4s-\mu}{\mu-2s-N}}b\geq\mathscr{Q}_{N,s,\mu}$, then the energy functional $\mathscr{I}_{a,b}$ is convex on $X_0^s(\Omega)$.

Theorem 1.2. *Let $a,b\in\mathbb{R}^+$ such that $a^{\frac{4s-\mu}{\mu-2s-N}}b\geq\mathscr{M}_{N,s,\mu}$, and set*

$$\mathscr{R}_\lambda^s:=\inf\left\{\mathscr{I}_{a,b}^\lambda(u)\mid u\in X_0^s(\Omega)\backslash\{0\}\right\},\ \ for\ any\ \ \lambda>0.$$

Then there exists $\overline{\lambda}_0^s \geq 0$ *such that, for any* $\lambda > \overline{\lambda}_0^s$*, we have* $u_\lambda^s \in X_0^s(\Omega)\setminus\{0\}$ *such that* $\mathscr{I}_{a,b}^{\lambda}(u_\lambda^s) = \mathscr{R}_\lambda^s < 0$*.*

Theorem 1.3. *Let* $\lambda = \overline{\lambda}_0^s$*. Then the following results hold:*

(i) *If* $a^{\frac{4s-\mu}{\mu-2s-N}} b > \mathscr{M}_{N,s,\mu}$*, then there exists* $u_{\overline{\lambda}_0^s}^s \in X_0^s(\Omega)\setminus\{0\}$ *such that* $\mathscr{I}_{a,b}^{\overline{\lambda}_0^s} = \mathscr{R}_{\overline{\lambda}_0^s}^s = 0$*;*

(ii) *If* $a^{\frac{4s-\mu}{\mu-2s-N}} b = \mathscr{M}_{N,s,\mu}$*, then* $u = 0$ *in the only minimizer for* $\mathscr{R}_{\overline{\lambda}_0^s}^s$*.*

Theorem 1.4. *Let* $((a_k)_k, (b_k)_k)$ *be a sequence and* $(a_k)_k > 0$*,* $(b_k)_k > 0$ *such that* $a_k \to a$*,* $b_k \to b$ *and* $a_k^{\frac{4s-\mu}{\mu-2s-N}} b_k \searrow \mathscr{M}_{N,s,\mu}$*. Setting* $\lambda_k := \overline{\lambda}_0^s(a_k, b_k)$*, then* $\lambda_k \to 0$ *as* $k \to \infty$*. And if* $(u_k)_k \subset X_0^s(\Omega)\setminus\{0\}$ *such that* $\lambda_k := \lambda_0^s(u_k)$*, then* $u_k \rightharpoonup 0$*, and*

$$\frac{\|u_k\|^2}{\left(\int_\Omega\int_\Omega \frac{|u_k(x)^{2^*_{\mu,s}}||u_k(y)^{2^*_{\mu,s}}|}{|x-y|^\mu}dxdy\right)^{\frac{1}{2^*_{\mu,s}}}} \to S_{H,L}.$$

Theorem 1.5. *If* $\lambda \geq \overline{\lambda}_0^s$*, then there exists a* $v_\lambda^s \in X_0^s(\Omega)\setminus\{0\}$ *such that* $\mathscr{I}_{a,b}^{\lambda}(v_\lambda^s) = c_\lambda^s$ *and* $\left(\mathscr{I}_{a,b}^{\lambda}\right)'(v_\lambda^s) = 0$*, where* $c_\lambda^s := \inf_{g\in\Gamma_\lambda^s}\max_{\zeta\in[0,1]} \mathscr{I}_{a,b}^{\lambda}(g(\zeta))$*, and*

$$\Gamma_\lambda^s := \left\{ g \in C([0,1],\, X_0^s(\Omega)) \mid g(0) = 0,\ g(1) = u_{\overline{\lambda}_0^s}^s \right\}.$$

Theorem 1.6. *Let* $\widehat{\mathscr{R}}_\lambda^s := \inf\left\{ \mathscr{I}_{a,b}^{\lambda}(u) \mid u \in X_0^s(\Omega),\ \|u\| \geq r \right\}$*, for any* $r > 0$*. Then there exist* δ*,* $r > 0$ *such that, for each* $\overline{\lambda}_0^s - \delta < \lambda < \overline{\lambda}_0^s$*, the value* $\widehat{\mathscr{R}}_\lambda^s$ *is achieved at a function* $w_\lambda^s \in X_0^s(\Omega)$ *satisfying* $\|w_\lambda^s\| > r$*.*

Theorem 1.7. *For each* $\overline{\lambda}_0^s - \delta < \lambda < \overline{\lambda}_0^s$*, there exists a* $v_\lambda^s \in X_0^s(\Omega)\setminus\{0\}$ *such that* $\mathscr{I}_{a,b}^{\lambda}(v_\lambda^s) = c_\lambda^s$ *and* $(\mathscr{I}_{a,b}^{\lambda})'(v_\lambda^s) = 0$*, where* $c_\lambda^s := \inf_{g\in\Gamma_\lambda^s}\max_{\zeta\in[0,1]} \mathscr{I}_{a,b}^{\lambda}(g(\zeta))$*, and*

$$\Gamma_\lambda^s := \left\{ g \in C([0,1],\, X_0^s(\Omega)) \mid g(0) = 0,\ g(1) = w_\lambda^s \right\}.$$

Theorem 1.8. *Assume that* (H_1)*-*(H_5) *hold. Then there exists* $\overline{\lambda}^s := \overline{\lambda}^s(a,b) \in (0, \overline{\lambda}_0^s)$ *such that if* $\lambda \in (0, \overline{\lambda}^s)$*, then (1.1) has no non-trivial solutions.*

Our article is organized as follows. In Section 2, in the case of $f(x,u) = 0$, we prove the weak lower semicontinuity, the validity of the Palais-Smale condition, and the convexity under appropriate restrictions on the parameters a and b. In Section 3, under the perturbation of function $f(x,u)$, we prove the existence of global minimizers and mountain pass type solutions with $\lambda \geq \overline{\lambda}_0^s$, and local minimizers and mountain pass type solutions with $\lambda < \overline{\lambda}_0^s$. And, when the nonlinear term f is strengthened, we prove the nonexistence result. In Appendix A, we obtain some more accurate estimates about the cut-off function.

2. Preliminaries and the Proof of Theorem 1.1

In this section, we state some preliminaries on fractional Sobolev spaces and Choquard equations, which can be found in [16, 20, 29, 30].

Let

$$X := \left\{ u \mid u : \mathbb{R}^N \to \mathbb{R} \ \ is \ \ measurable : u \mid_{\Omega} \in L^2(\Omega) \ \ and \ \ \frac{u(x)-u(y)}{|x-y|^{\frac{N}{2}+s}} \in L^2(Q) \right\},$$

where $Q = \mathbb{R}^{2N} \backslash (\mathscr{C}\Omega \times \mathscr{C}\Omega)$ and $\mathscr{C}\Omega = \mathbb{R}^N \backslash \Omega$. The space X is endowed with the norm

$$\|u\|_X = \|u\|_{L^2(\Omega)} + \left(\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy \right)^{\frac{1}{2}}.$$

Define $X_0^s(\Omega) := \{u \in X : u = 0 \ \ a.e. \ \ in \ \ \mathbb{R}^N \backslash \Omega\}$ and the best Sobolev constant as

$$S_{N,s} := \inf_{u \in X_0^s(\Omega) \backslash \{0\}} \frac{\|u\|^2}{\|u\|_{2_s^*}^2},$$

where

$$\|u\|^2 := \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy.$$

The norm introduced in the previous equation is derived form the following formula

$$\langle u, v \rangle_{X_0^s(\Omega)} := \int_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} dxdy, \ \ for\, any \ \ u, v \in X_0^s(\Omega).$$

The key point to apply variational approaches to problem (1.1) is the following well-known Hardy-Littlewood-Sobolev inequality; see [29, 30]. We have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_{\mu,s}^*} |u(y)|^{2_{\mu,s}^*}}{|x-y|^{\mu}} dxdy \le C(N,\mu) |u|_{2_s^*}^{2 \cdot 2_{\mu,s}^*},$$

where $C(N,\mu)$ is a suitable constant. We define

$$S_{H,L} := \inf_{u \in X_0^s(\Omega) \backslash \{0\}} \frac{\|u\|^2}{\left(\int_\Omega \left(\int_\Omega \frac{|u(y)|^{2_{\mu,s}^*}}{|x-y|^{\mu}} dy \right) |u(x)|^{2_{\mu,s}^*} dx \right)^{\frac{1}{2_{\mu,s}^*}}} \tag{2.1}$$

as the best constant which is achieved if and only if u is of the form

$$C_0 \left(\frac{t}{t^2 + |x-x_0|^2} \right)^{\frac{N-2s}{2}}, \ \ for \ \ all \ \ x \in \Omega,$$

for some $x_0 \in \mathbb{R}^N$, $C_0 > 0$ and $t > 0$. Also it satisfies

$$(-\Delta)^s u = \left(\int_\Omega \frac{|u(y)|^{2_{\mu,s}^*}}{|x-y|^{\mu}} dy \right) |u|^{2_{\mu,s}^* - 2} u \ \ in \ \ \Omega.$$

Moreover,

$$S_{H,L} = \frac{S_{N,s}}{C(N,\mu)^{\frac{1}{2_{\mu,s}^*}}}. \tag{2.2}$$

Denote by $\mathscr{I}_{a,b}^{\lambda} : X_0^s(\Omega) \to \mathbb{R}$ the energy functional associated to (1.1),

$$\mathscr{I}_{a,b}^{\lambda}(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2 \cdot 2_{\mu,s}^*} \int_\Omega \int_\Omega \frac{|u(x)|^{2_{\mu,s}^*} |u(y)|^{2_{\mu,s}^*}}{|x-y|^{\mu}} dxdy - \lambda \int_\Omega F(x,u) dx,$$

where $F(x,v)=\int_0^v f(x,w)dw$. One has

$$\left(\mathscr{I}_{a,b}^{\lambda}\right)'(u)[\varphi]=(a+b\|u\|^2)\langle u,\varphi\rangle_{X_0^s(\Omega)}-\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}-2}u(y)\varphi(y)}{|x-y|^{\mu}}dxdy$$
$$-\lambda\int_\Omega f(x,u)\varphi dx, \tag{2.3}$$

for all $u,\varphi\in X_0^s(\Omega)$. When $f(x,u)=0$, we will use the notation

$$\mathscr{I}_{a,b}(u):=\frac{a}{2}\|u\|^2+\frac{b}{4}\|u\|^4-\frac{1}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}dxdy,$$

and we point out the $\mathscr{I}_{a,b}$ is a C^2-functional.

In the following, we completely describe the range of parameters a and b for which the functional $\mathscr{I}_{a,b}$ associated to the problem

$$\begin{cases}\left(a+b\int_Q\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u=\left(\int_\Omega\frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}dy\right)|u|^{2^*_{\mu,s}-2}u & in\ \ \Omega,\\ u=0 & in\ \ \mathbb{R}^N\setminus\Omega.\end{cases}$$

Proof of Theorem 1.1. (i) For $a^{\frac{4s-\mu}{\mu-2s-N}}b\geq\mathscr{M}_{N,s,\mu}$, there exists a sequence $(u_n)_n\subset X_0^s(\Omega)$ such that $u_n\rightharpoonup u$. Since the embedding $X_0^s(\Omega)\hookrightarrow L^p(\Omega)$ is compact (see Lemma 9 of [31]), $u_n\to u$ in $L^p(\Omega)$ for any $p\in[1,2^*_s)$. We derive that

$$\|u_n\|^2-\|u\|^2=\|u_n-u\|^2+2\langle u_n-u,u\rangle_{X_0^s(\Omega)}=\|u_n-u\|^2+o(1), \tag{2.4}$$

as $n\to\infty$ and

$$\|u_n\|^4-\|u\|^4=(\|u_n-u\|^2+o(1))(\|u_n-u\|^2+2\|u\|^2+o(1)). \tag{2.5}$$

Finally, from Lemma 3.3 of [29], we have

$$\int_\Omega\int_\Omega\frac{|u_n(x)|^{2^*_{\mu,s}}|u_n(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}dxdy-\int_\Omega\int_\Omega\frac{|(u_n-u)(x)|^{2^*_{\mu,s}}|(u_n-u)(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}dxdy$$
$$\to\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^{\mu}}dxdy, \tag{2.6}$$

as $n \to \infty$. Combining (2.4), (2.5), (2.6), and the Sobolev inequality (2.1), we obtain

$$\begin{aligned}
&\mathscr{I}_{a,b}(u_n) - \mathscr{I}_{a,b}(u) \\
&= \frac{a}{2}(\|u_n\|^2 - \|u\|^2) + \frac{b}{4}(\|u_n\|^4 - \|u\|^4) \\
&\quad - \frac{1}{2\cdot 2^*_{\mu,s}}\left(\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_{\mu,s}}|u_n(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy - \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy\right) \\
&= \frac{a}{2}(\|u_n\|^2 - \|u\|^2) + \frac{b}{4}(\|u_n\|^4 - \|u\|^4) \qquad (2.7)\\
&\quad - \frac{1}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega \frac{|(u_n-u)(x)|^{2^*_{\mu,s}}|(u_n-u)(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy + o(1) \\
&\geq \|u_n-u\|^2\left[\frac{a}{2} + \frac{b}{4}\|u_n-u\|^2 - \frac{S_{H,L}^{-2^*_{\mu,s}}}{2\cdot 2^*_{\mu,s}}\|u_n-u\|^{2\cdot 2^*_{\mu,s}-2}\right] + o(1),
\end{aligned}$$

as $n \to \infty$. At this point, we introduce the auxiliary function

$$l_{N,s,\mu}(t) = \frac{a}{2} + \frac{b}{4}t^2 - \frac{S_{H,L}^{-2^*_{\mu,s}}}{2\cdot 2^*_{\mu,s}}t^{2\cdot 2^*_{\mu,s}-2}, \ \ for \ \ t \geq 0.$$

It is easy to obtain that the function $l_{N,s,\mu}$ attains its minimum at the point

$$d_{N,s,\mu} = \left(\frac{b\cdot 2^*_{\mu,s}}{2\cdot 2^*_{\mu,s}-2}S_{H,L}^{2^*_{\mu,s}}\right)^{\frac{1}{2\cdot(2^*_{\mu,s}-2)}},$$

and that

$$a^{\frac{4s-\mu}{\mu-2s-N}}b \geq \mathscr{M}_{N,s,\mu} \Leftrightarrow l_{N,s,\mu}(d_{N,s,\mu}) \geq 0. \qquad (2.8)$$

It follows from (2.7) and (2.8) that

$$\liminf_{n\to\infty}(\mathscr{I}_{a,b}(u_n) - \mathscr{I}_{a,b}(u)) \geq \liminf_{n\to\infty}\|u_n-u\|^2 l_{N,s,\mu}(\|u_n-u\|) \geq 0,$$

which concludes this part of the proof. □

Proof of Theorem 1.1. (ii) Let $\{u_n\}_n \subset X_0^s(\Omega)$ be a $(PS)_c$ sequence, i.e., $\mathscr{I}_{a,b}(u_n) \to c$ and $\mathscr{I}'_{a,b}(u_n) \to 0$ as $n \to \infty$. Recalling (2.1), we have

$$\mathscr{I}_{a,b}(u) = a\|u\|^2 + b\|u\|^4 - \int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy \geq a\|u\|^2 + b\|u\|^4 - S_{H,L}^{-2^*_{\mu,s}}\|u\|^{2\cdot 2^*_{\mu,s}}.$$

Since $2^*_{\mu,s} < 2$, we have that $\mathscr{I}_{a,b}$ is coercive. From Lemma 9 in [31], up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u & in \ \ X_0^s(\Omega), \\ u_n \to u & in \ \ L^p(\Omega), \ \ for \ \ all \ \ p \in [1, 2^*_s), \\ u_n \to u & a.e \ \ in \ \ \mathbb{R}^N. \end{cases}$$

Using the Hölder inequality, it is easy to see that $\{u_n\}_n$ is bounded. Thus there exist two finite measures μ and ν such that $|(-\Delta)^{\frac{s}{2}}u_n|^2 \rightharpoonup^* \mu$, and

$$\left(\int_\Omega \frac{|u_n(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dy\right)|u_n(x)|^{2^*_{\mu,s}} \rightharpoonup^* \nu.$$

From Lemma 2.3 of [8], it follows that either $u_n \to u$ in $L^{2^*_s}(\Omega)$ or there exist a set I at most countable, two real sequences $\{\mu_i\}_{i\in I}$, $\{\nu_i\}_{i\in I}$ and distinct points $\{x_i\}_{i\in I} \subset \mathbb{R}^N$, such that

$$\nu = \left(\int_\Omega \frac{|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dy\right)|u(x)|^{2^*_{\mu,s}} + \sum_{i\in I}\nu_i\delta_{x_i}, \tag{2.9}$$

and

$$\mu = |(-\Delta)^{\frac{s}{2}}u|^2 + \widetilde{\mu} + \sum_{i\in I}\mu_i\delta_{x_i}, \tag{2.10}$$

for some positive finite measure $\widetilde{\mu}$, where

$$\nu_i \le S_{H,L}^{-2^*_{\mu,s}}\mu_i^{2^*_{\mu,s}}. \tag{2.11}$$

Claim: The set I is empty.

If not, then there exists an index i_0 such that $\nu_{i_0} \neq 0$ at x_{i_0}. For any $\varepsilon > 0$, consider a cut-off function ϑ_ε such that

$$\begin{cases} 0 \le \vartheta_\varepsilon \le 1 & in \quad \Omega, \\ \vartheta_\varepsilon = 1 & in \quad B(x_{i_0},\varepsilon), \\ \vartheta_\varepsilon = 0 & in \quad \Omega\backslash B(x_{i_0},2\varepsilon). \end{cases}$$

Since the sequence $\{u_n\vartheta_\varepsilon\}_n$ is bounded in $X_0^s(\Omega)$, we have $\lim_{n\to\infty}\mathscr{I}_{a,b}(u_n)[u_n\vartheta_\varepsilon] = 0$. Thus

$$\begin{aligned} o(1) &= \mathscr{I}'_{a,b}(u_n)[u_n\vartheta_\varepsilon] \\ &= (a+b\|u_n\|^2)\langle u_n, u_n\vartheta_\varepsilon\rangle_{X_0^s(\Omega)} - \int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_{\mu,s}}|u_n(y)|^{2^*_{\mu,s}}\vartheta_\varepsilon}{|x-y|^\mu}dxdy \\ &= \Bigg[(a+b\|u_n\|^2)\int_Q u_n(y)\frac{(u_n(x)-u_n(y))(\vartheta_\varepsilon(x)-\vartheta_\varepsilon(y))}{|x-y|^{N+2s}}\,dxdy \\ &\quad + \int_Q \vartheta_\varepsilon(x)\frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}}dxdy\Bigg] - \int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_{\mu,s}}|u_n(y)|^{2^*_{\mu,s}}\vartheta_\varepsilon}{|x-y|^\mu}dxdy, \end{aligned} \tag{2.12}$$

as $n \to \infty$. We estimate the first term related to (2.12) by using the Hölder inequality,

$$\begin{aligned} &(a+b\|u_n\|^2)\int_Q u_n(y)\frac{(u_n(x)-u_n(y))(\vartheta_\varepsilon(x)-\vartheta_\varepsilon(y))}{|x-y|^{N+2s}}dxdy \\ &\le \overline{C}\int_Q |u_n(y)|^2\frac{|\vartheta_\varepsilon(x)-\vartheta_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy, \end{aligned}$$

for some $\overline{C} > 0$. From Lemma 2.1 of [32], we have

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_Q |u_n(y)|^2\frac{|\vartheta_\varepsilon(x)-\vartheta_\varepsilon(y)|^2}{|x-y|^{N+2s}}dxdy = 0. \tag{2.13}$$

For the second term of (2.12), we conclude from (2.10) that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}(a+b\|u_n\|^2)\int_Q \vartheta_\varepsilon(x)\frac{(u_n(x)-u_n(y))^2}{|x-y|^{N+2s}}dxdy \geq a\mu_{i_0}+b\mu_{i_0}^2. \tag{2.14}$$

Finally, by (2.9), we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_{\mu,s}}|u_n(y)|^{2^*_{\mu,s}}\vartheta_\varepsilon}{|x-y|^\mu}dxdy = \lim_{\varepsilon\to 0}\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}\vartheta_\varepsilon}{|x-y|^\mu}dxdy+\nu_{i_0}=\nu_{i_0}. \tag{2.15}$$

Combining (2.13), (2.14), (2.15), and using (2.11), we have

$$0\geq a\mu_{i_0}+b\mu_{i_0}^2-\nu_{i_0}\geq a\mu_{i_0}+b\mu_{i_0}^2-S_{H,L}^{-2^*_{\mu,s}}\mu_{i_0}^{2^*_{\mu,s}}=\mu_{i_0}(a+b\mu_{i_0}-S_{H,L}^{-2^*_{\mu,s}}\mu_{i_0}^{2^*_{\mu,s}-1}).$$

We introduce the auxiliary function $\widetilde{l}_{N,s,\mu}(t)=a+bt-S_{H,L}^{-2^*_{\mu,s}}t^{2^*_{\mu,s}-1}$ for $t\geq 0$. At this point, we deduce $a^{\frac{4s-\mu}{\mu-2s-N}}b>\mathscr{N}_{N,s,\mu}\Leftrightarrow \widetilde{l}_{N,s,\mu}(t)>0$. So, we have $a+b\mu_{i_0}-S_{H,L}^{-2^*_{\mu,s}}\mu_{i_0}^{2^*_{\mu,s}-1}>0$. Hence, $\mu_{i_0}=0$ and by (2.11), $\nu_{i_0}=0$ as well. Thus we conclude that $I=\emptyset$. Using the Brezis-Lieb lemma, we can obtain

$$\lim_{n\to\infty}\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_{\mu,s}}|u_n(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy=\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy.$$

Hence $u_n\to u$ in $L^{2^*_s}(\Omega)$ and

$$\lim_{n\to\infty}\int_\Omega\int_\Omega \frac{|u_n(x)|^{2^*_{\mu,s}}|u_n(y)|^{2^*_{\mu,s}-2}u_n(y)}{|x-y|^\mu}(u-u_n)(y)dxdy=0. \tag{2.16}$$

Since $\mathscr{I}'_{a,b}(u_n)\to 0$ as $n\to\infty$, we have

$$0=\lim_{n\to\infty}\mathscr{I}'_{a,b}(u_n)[u_n-u]=\lim_{n\to\infty}(a+b\|u_n\|^2)\langle u_n,u_n-u\rangle_{X_0^s(\Omega)}.$$

From above equalities, we obtain $\lim_{n\to\infty}\langle u_n,u_n-u\rangle_{X_0^s(\Omega)}=0$, which together with the fact that $u_n\rightharpoonup u$ yields that $\|u_n-u\|^2=\langle u_n,u_n-u\rangle_{X_0^s(\Omega)}-\langle u,u_n-u\rangle_{X_0^s(\Omega)}\to 0$, as $n\to\infty$. This completes the proof of this part. □

Proof of Theorem 1.1. (iii) To prove the convexity, we show that $\mathscr{I}''_{a,b}(u)[\varphi,\varphi]\geq 0$ for all $u,\varphi\in X_0^s(\Omega)$. Differentiating (2.3), we have

$$\begin{aligned}
&\mathscr{I}''_{a,b}(u)[\varphi,\varphi]\\
&=a\|\varphi\|^2+b\|u\|^2\|\varphi\|^2-\Bigg[2^*_{\mu,s}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}-1}|u(y)|^{2^*_{\mu,s}-2}u(y)\varphi^2}{|x-y|^\mu}dxdy\\
&\quad+(2^*_{\mu,s}-1)\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}-2}\varphi^2}{|x-y|^\mu}dxdy\Bigg].
\end{aligned} \tag{2.17}$$

Using the Hölder inequality, the Sobolev inequality, and (2.2), we have

$$2^*_{\mu,s}\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}-1}|u(y)|^{2^*_{\mu,s}-2}u(y)\varphi^2}{|x-y|^\mu}dxdy+(2^*_{\mu,s}-1)\int_\Omega\int_\Omega \frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}-2}\varphi^2}{|x-y|^\mu}dxdy$$
$$\le (2\cdot 2^*_{\mu,s}-1)\|\varphi\|^2\|u\|^{2\cdot(2^*_{\mu,s}-1)}S_{H,L}^{-2^*_{\mu,s}}. \tag{2.18}$$

Combining (2.17) and (2.18), we have

$$\mathscr{I}''_{a,b}(u)[\varphi,\varphi]\ge \|\varphi\|^2[a+b\|u\|^2-(2\cdot 2^*_{\mu,s}-1)\|u\|^{2\cdot(2^*_{\mu,s}-1)}S_{H,L}^{-2^*_{\mu,s}}].$$

We introduce the auxiliary function $\widehat{l}_{N,s,\mu}(t)=a+bt^2-(2\cdot 2^*_{\mu,s}-1)t^{2\cdot(2^*_{\mu,s}-1)}S_{H,L}^{-2^*_{\mu,s}}$ for all $t\ge 0$. It is easy to verify that $\widehat{l}_{N,s,\mu}$ attains its global minimum at the point

$$\widehat{d}_{N,s,\mu}:=\left(\frac{2bS_{H,L}^{2^*_{\mu,s}}}{(2\cdot 2^*_{\mu,s}-1)(2\cdot 2^*_{\mu,s}-2)}\right)^{\frac{1}{2\cdot(2^*_{\mu,s}-2)}},$$

and that

$$a^{\frac{4s-\mu}{\mu-2s-N}}b\ge \mathscr{Q}_{N,s,\mu}\Leftrightarrow \widehat{l}_{N,s,\mu}(t)\ge 0,$$

for all $t\ge 0$. It is clear from the proof that $\mathscr{I}_{a,b}$ is strictly convex provided that $a^{\frac{4s-\mu}{\mu-2s-N}}b>\mathscr{Q}_{N,s,\mu}$. □

3. Existence and Non-existence Results: General Case

On the application of Theorem 1.1, we study the set of solutions of perturbed problem (1.1). As for f, we make the following assumptions:

(H_1) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying $f(x,0)=0$ a.e. in Ω;

(H_2) $f(x,v)>0$ for every $v>0$ and $f(x,v)<0$ for every $v<0$ a.e. in Ω. Further, there exists $\mu>0$ such that $f(x,v)\ge\mu>0$ a.e. in Ω and for any $v\in J$, where J is open interval of $(0,\infty)$;

(H_3) $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a function satisfying $f(x,v)\le c(1+|v|^{p-1})$ for all $x\in\Omega$ and $v\in\mathbb{R}$, where $c>0$, $2<p<2^*_s$;

(H_4) $\lim_{v\to 0}\frac{f(x,v)}{|v|}=0$ uniformly in $x\in\Omega$;

(H_5) For every $u\in X_0^s(\Omega)$, the function $t\mapsto\int_\Omega f(x,tu(x))dx$ is C^1 with respect to $t\in(0,\infty)$.

We first establish the following results which are the fundamental tools in proving our theorems.

Proposition 3.1. *Let $u\in X_0^s(\Omega)\backslash\{0\}$. we have:*

(i) for each $t>0$, it holds

$$\frac{a}{2}\|u\|^2+\frac{b}{4}t^2\|u\|^4-\frac{t^{2\cdot 2^*_{\mu,s}-2}}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy>l_{N,s,\mu}(t\|u\|)\|u\|^2;$$

(ii) for each $t>0$, it holds

$$a\|u\|^2+bt^2\|u\|^4-t^{2\cdot 2^*_{\mu,s}-2}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy>\widetilde{l}_{N,s,\mu}(t\|u\|)\|u\|^2.$$

Proof. From the boundedness of Ω and the Sobolev inequality, we have

$$\begin{aligned}
&t^2\left[\frac{a}{2}\|u\|^2+\frac{b}{4}t^2\|u\|^4-\frac{t^{2\cdot 2^*_{\mu,s}-2}}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy\right]\\
&=\frac{a}{2}(t\|u\|)^2+\frac{b}{4}(t\|u\|)^4-\frac{(t\|u\|)^{2\cdot 2^*_{\mu,s}}}{2\cdot 2^*_{\mu,s}}\cdot\frac{\|u\|^{-2\cdot 2^*_{\mu,s}}}{\left(\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy\right)^{-1}}\\
&>\frac{a}{2}(t\|u\|)^2+\frac{b}{4}(t\|u\|)^4-\frac{(t\|u\|)^{2\cdot 2^*_{\mu,s}}}{2\cdot 2^*_{\mu,s}}\cdot S_{H,L}^{-2^*_{\mu,s}}.
\end{aligned}$$

Moreover, we can obtain (ii) similarly. The proof is complete. □

Lemma 3.1. *Let $a,b\in\mathbb{R}^+$, $(u_k)_k\subset X_0^s(\Omega)$, and $\lambda_k\to\lambda\geq 0$ as $k\to\infty$.*

(1) If $a^{\frac{4s-\mu}{\mu-2s-N}}b\geq\mathscr{M}_{N,s,\mu}$ and $u_k\rightharpoonup u$ in $X_0^s(\Omega)$, then $\mathscr{I}^\lambda_{a,b}(u)\leq\liminf_{k\to\infty}\mathscr{I}^\lambda_{a,b}(u_k)$;

(2) If $a^{\frac{4s-\mu}{\mu-2s-N}}b>\mathscr{N}_{N,s,\mu}$, $\mathscr{I}^\lambda_{a,b}(u_k)\to c$ and $\left(\mathscr{I}^\lambda_{a,b}\right)'(u_k)\to 0$, then $(u_k)_k$ is convergent to some u in $X_0^s(\Omega)$ up to subsequence.

Proof. The proof process is similar to that of Theorem 1.1 (i) and (ii). □

For every $t>0$, we introduce the fiber map

$$\psi^{\lambda,u}_{a,b}(t):=\mathscr{I}^\lambda_{a,b}(tu)=\frac{a}{2}t^2\|u\|^2+\frac{b}{4}t^4\|u\|^4-\frac{t^{2\cdot 2^*_{\mu,s}}}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy-\lambda\int_\Omega F(x,tu)dx,$$

for $\lambda\geq 0$ and $u\in X_0^s(\Omega)$.

Proposition 3.2. *There exists a neighbourhood V of 0 such that $\psi^{\lambda,u}_{a,b}(t)>0$ for all $t\in V\cap(0,\infty)$. We have $\psi^{\lambda,u}_{a,b}(t)\to\infty$ as $t\to\infty$ and we also have that $\psi^{\lambda,u}_{a,b}(t)$ is bounded from below.*

Proof. Fix $\varepsilon>0$. Due to (H_4), we have

$$\begin{aligned}
\psi^{\lambda,u}_{a,b}(t)&=t^2\left(\frac{a}{2}\|u\|^2+\frac{b}{4}t^2\|u\|^4-\frac{t^{2\cdot 2^*_{\mu,s}-2}}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy-\lambda\int_\Omega\frac{F(x,tu)}{t^2}dx\right)\\
&\geq t^2\left(\frac{a}{2}\|u\|^2+\frac{b}{4}t^2\|u\|^4-\frac{t^{2\cdot 2^*_{\mu,s}-2}}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy-\lambda\frac{\varepsilon}{2}\|u\|_2^2\right).
\end{aligned}$$

By using the Sobolev inequality and taking ε and t appropriately, we obtain the results of the first part. Since $1<2^*_{\mu,s}<2$, we conclude that $\psi^{\lambda,u}_{a,b}(t)\to\infty$ as $t\to\infty$. At the same time, we also have that $\psi^{\lambda,u}_{a,b}(t)$ is bounded from below. □

Now, we consider the system

$$\begin{cases}\psi^{\lambda,u}_{a,b}(t)=0,\\ \left(\psi^{\lambda,u}_{a,b}\right)'(t)=0,\\ \psi^{\lambda,u}_{a,b}(t)=\inf_{\alpha>0}\psi^{\lambda,u}_{a,b}(\alpha),\end{cases}\tag{3.1}$$

in the unknowns λ and t.

Proposition 3.3. *Let $a,b\in\mathbb{R}^+$ such that $a^{\frac{4s-\mu}{\mu-2s-N}}b\geq\mathscr{M}_{N,s,\mu}$ and choose $u\in X_0^s(\Omega)\backslash\{0\}$. Then there exists a unique $\lambda=\lambda_0^s(u)$ that solves (3.1).*

Proof. The proof is the similar to that of Proposition 4 of [27]. We omit the concrete details. □

Corollary 3.1. *Let $u\in X_0^s(\Omega)\backslash\{0\}$. Then $\lambda_0^s(u)$ is the unique parameter such that*

$$\inf_{t\in(0,\infty)}\psi_{a,b}^{\lambda_0^s(u),u}(t)=0.$$

Moreover,

$$\inf_{t\in(0,\infty)}\psi_{a,b}^{\lambda,u}(t)\begin{cases}<0 & if\ \ \lambda>\lambda_0^s(u),\\ =0 & if\ \ 0<\lambda\leq\lambda_0^s(u).\end{cases}$$

Now, we introduce the following extremal parameter. Set $\overline{\lambda}_0^s:=\inf_{u\in X_0^s(\Omega)\backslash\{0\}}\lambda_0^s(u)$.

The next proposition demonstrates how the parameter $\overline{\lambda}_0^s$ varies according to the choices made for a and b.

Proposition 3.4. *The following assertions hold:*

(i) If $a^{\frac{4s-\mu}{\mu-2s-N}}b>\mathscr{M}_{N,s,\mu}$, then $\overline{\lambda}_0^s>0$;

(ii) If $a^{\frac{4s-\mu}{\mu-2s-N}}b=\mathscr{M}_{N,s,\mu}$, then $\overline{\lambda}_0^s=0$. Moreover, if $(u_k)_k\subset X_0^s(\Omega)\backslash\{0\}$ is a sequence such that $\lambda_0^s(u_k)\to\overline{\lambda}_0^s$ as $k\to\infty$, then $u_k\rightharpoonup 0$ and

$$\frac{\|u_k\|^2}{\left(\int_\Omega\int_\Omega\frac{|u_k(x)^{2^*_{\mu,s}}||u_k(y)^{2^*_{\mu,s}}|}{|x-y|^\mu}dxdy\right)^{\frac{1}{2^*_{\mu,s}}}}\to S_{H,L}.$$

Proof. (i) Since the proof is quite similar to that of Proposition 5 in [27], we omit the details.

(ii) We can assume that $0\in\Omega$. Take a nonnegative cut-off function such that $\phi(x)=1$ in $B_R(0)$ for some $R>0$. Set $\varepsilon>0$ and consider $\upsilon_\varepsilon(x):=\frac{\phi(x)}{(\varepsilon+|x|^2)^{\frac{N-2s}{2}}}$. We set $u_\varepsilon:=\frac{\upsilon_\varepsilon}{\|\upsilon_\varepsilon\|}$. By Appendix A, we obtain that

$$\|u_\varepsilon\|=1,\quad\int_\Omega\int_\Omega\frac{|u_\varepsilon(x)|^{2^*_{\mu,s}}|u_\varepsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy\geq S_{H,L}^{-2^*_{\mu,s}}+O(\varepsilon^{2N-\mu}),\quad\|\upsilon_\varepsilon\|\leq\varepsilon^{-\frac{N-2s}{4}}C_1+O(1),$$

as $\varepsilon\to0$ for some $C_1>0$. Therefore,

$$\begin{aligned}\psi_{a,b}^{\lambda,u_\varepsilon}(t)&=\frac{a}{2}t^2+\frac{b}{4}t^4-\frac{t^{2\cdot2^*_{\mu,s}}}{2\cdot2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u_\varepsilon(x)|^{2^*_{\mu,s}}|u_\varepsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy-\lambda\int_\Omega F(x,tu_\varepsilon)dx\\&\leq t^2l_{N,s,\mu}(t)-\frac{t^{2\cdot2^*_{\mu,s}}}{2\cdot2^*_{\mu,s}}O(\varepsilon^{2N-\mu})-\lambda\int_\Omega F(x,tu_\varepsilon)dx.\end{aligned}$$

Letting $t=d_{N,s,\mu}$, we obtain

$$\psi_{a,b}^{\lambda,u_\varepsilon}(d_{N,s,\mu})=-\frac{{d_{N,s,\mu}}^{2\cdot2^*_{\mu,s}}}{2\cdot2^*_{\mu,s}}O(\varepsilon^{2N-\mu})-\lambda\int_\Omega F(x,d_{N,s,\mu}u_\varepsilon)dx.$$

Claim: There exists a constant $C_2>0$ such that $\int_\Omega F(x,d_{N,s,\mu}u_\varepsilon)dx\geq C_2\varepsilon^{\frac{N}{2}}$ as $\varepsilon\to0$.

As a consequence of the claim ,we obtain

$$\psi_{a,b}^{\lambda,u_\varepsilon}(d_{N,s,\mu}) \leq \varepsilon^{2N-\mu}\left(-\frac{d_{N,s,\mu}{}^{2\cdot 2^*_{\mu,s}}}{2\cdot 2^*_{\mu,s}}O(1)-\lambda C_2\varepsilon^{\mu-\frac{3N}{2}}\right)<0.$$

Hence, $\lambda_0^s(u_\varepsilon)<\lambda$. We obtain $\overline{\lambda}_0^s=0$ as $\lambda\to 0$. Now, we see the last part, We suppose that $(u_k)_k\subset X_0^s(\Omega)\backslash\{0\}$ satisfies $\lambda_k:=\lambda_0^s(u_k)\to\overline{\lambda}_0^s=0$. Let $\|u_k\|=1$ and $u_k\rightharpoonup u$, Then there exists $t_k>0$ such that

$$\frac{a}{2}+\frac{b}{4}t_k^2-\frac{t_k^{2\cdot 2^*_{\mu,s}-2}}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u_k(x)|^{2^*_{\mu,s}}|u_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy-\lambda_k\int_\Omega\frac{F(x,t_ku_k)}{t_k^2}dx=0. \tag{3.2}$$

Combining (H_3), (H_4), and (3.2), we can deduce that $t_k\to\bar{t}$ and $\int_\Omega\int_\Omega\frac{|u_k(x)|^{2^*_{\mu,s}}|u_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy\to\sigma$, as $k\to\infty$. Passing to the limit in (3.2), we arrive at

$$\frac{a}{2}+\frac{b}{4}\bar{t}^2-\frac{\bar{t}^{2\cdot 2^*_{\mu,s}-2}}{2\cdot 2^*_{\mu,s}}\sigma=0.$$

From $a^{\frac{4s-\mu}{\mu-2s-N}}b=\mathscr{M}_{N,s,\mu}$, it follows that $\sigma=S_{H,L}^{-2^*_{\mu,s}}$. Thus $(u_k)_k$ is a minimizing sequence for $S_{H,L}$. Now, if $u\neq 0$, by the lower semicontinuity of the norm, we have $\|u\|\leq 1$. Coupling this fact with $l_{N,s,\mu}$, we obtain

$$\begin{aligned}
0&\leq\frac{a}{2}+\frac{b}{4}\bar{t}^2-\frac{\bar{t}^{2\cdot 2^*_{\mu,s}-2}}{2\cdot 2^*_{\mu,s}}S_{H,L}^{-2^*_{\mu,s}}\|u\|^{2\cdot 2^*_{\mu,s}}\\
&\leq\frac{a}{2}+\frac{b}{4}\bar{t}^2-\frac{\bar{t}^{2\cdot 2^*_{\mu,s}-2}}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u(x)|^{2^*_{\mu,s}}|u(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy\\
&\leq\limsup_{k\to\infty}\left(\frac{a}{2}+\frac{b}{4}t_k^2-\frac{t_k^{2\cdot 2^*_{\mu,s}-2}}{2\cdot 2^*_{\mu,s}}\int_\Omega\int_\Omega\frac{|u_k(x)|^{2^*_{\mu,s}}|u_k(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy-\lambda_k\int_\Omega\frac{F(x,t_ku_k)}{t_k^2}dx\right)=0,
\end{aligned}$$

which cannot happen since Ω is bounded. □

The following proposition summarizes the condition of the infimum depending on the choice of the parameter λ for the functional $\psi_{a,b}^{\lambda,u}(t)$.

Proposition 3.5. *If $\lambda\leq\overline{\lambda}_0^s$, then $\inf_{t>0}\psi_{a,b}^{\lambda,u}(t)=0$ for any $u\in X_0^s(\Omega)\backslash\{0\}$. If $\lambda>\overline{\lambda}_0^s$, then $\inf_{t>0}\psi_{a,b}^{\lambda,u}(t)<0$ for any $u\in X_0^s(\Omega)\backslash\{0\}$.*

Proof. The proof is the similar to that of Proposition 6 of [27]. We omit the details here. □

After some preliminary results, we use the traditional variational methods to study the set of solutions of problem (1.1). The first step giving the proof for Theorems 1.2 and 1.3 provides the existence of global minimizers for $\lambda\geq\overline{\lambda}_0^s$.

Proof of Theorem 1.2. Combining (H_3) and (H_4), it is easy to verify that $\mathscr{I}_{a,b}^\lambda$ is coercive. Moreover, we also have the lower semicontinuity. Finally, we recall that Proposition 3.5 implies the existence of a function and the functional is proved to be negative. □

Proof of Theorem 1.3. (i) Take a sequence $\lambda_k \searrow \overline{\lambda}_0^s$. From Theorem 1.2, for each k, we can find $(u_k)_k \subset X_0^s(\Omega)\backslash\{0\}$ such that $\mathscr{R}_{\lambda_k}^s = \mathscr{I}_{a,b}^{\lambda_k}(u_k) < 0$. Since $\lambda_k \searrow \overline{\lambda}_0^s$, it follows that $(u_k)_k$ is bounded and we may suppose that $u_k \rightharpoonup u$ in $X_0^s(\Omega)$. According to Lemma 3.1 (1), we obtain $\mathscr{I}_{a,b}^{\overline{\lambda}_0^s}(u) \leq \liminf_{k\to\infty} \mathscr{I}_{a,b}^{\overline{\lambda}_0^s}(u_k) \leq 0$. Proposition 3.5 ensures that $\mathscr{I}_{a,b}^{\overline{\lambda}_0^s}(\omega) \geq 0$ for each $\omega \in X_0^s(\Omega)$. Thus $\mathscr{I}_{a,b}^{\overline{\lambda}_0^s}(u) = \mathscr{R}_{\overline{\lambda}_0^s}^s = 0$. To conclude the proof, we have to prove that $u \neq 0$. In fact

$$
\begin{aligned}
&\frac{a}{2}\|u_k\|^2 + \frac{b}{4}\|u_k\|^4 - \frac{S_{H,L}^{-2_{\mu,s}^*}}{2\cdot 2_{\mu,s}^*}\|u_k\|^{2\cdot 2_{\mu,s}^*} \\
&\leq \frac{a}{2}\|u_k\|^2 + \frac{b}{4}\|u_k\|^4 - \frac{1}{2\cdot 2_{\mu,s}^*}\int_\Omega\int_\Omega \frac{|u_k(x)|^{2_{\mu,s}^*}|u_k(y)|^{2_{\mu,s}^*}}{|x-y|^\mu}dxdy \\
&\leq \lambda_k \int_\Omega F(x,u_k)dx.
\end{aligned}
$$

Thus

$$
l_{N,s,\mu}(\|u_k\|) = \frac{a}{2} + \frac{b}{4}\|u_k\|^2 - \frac{S_{H,L}^{-2_{\mu,s}^*}}{2\cdot 2_{\mu,s}^*}\|u_k\|^{2\cdot 2_{\mu,s}^*-2} \leq \lambda_k \int_\Omega \frac{F(x,u_k)}{\|u_k\|^2}dx.
$$

If $u = 0$, by (H_3) and (H_4), we have $l_{N,s,\mu}(\|u_k\|) \to 0$ as $k \to \infty$. This fact is in contradiction with $l_{N,s,\mu}(\|u_k\|) \geq l_{N,s,\mu}(d_{N,s,\mu}) > 0$. Since $a^{\frac{4s-\mu}{\mu-2s-N}}b > \mathscr{M}_{N,s,\mu}$, then u must be different from zero.

(ii) From Proposition 3.4 (ii), we have $\overline{\lambda}_0^s$ and

$$
\mathscr{I}_{a,b}^{\overline{\lambda}_0^s}(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{2\cdot 2_{\mu,s}^*}\int_\Omega\int_\Omega \frac{|u(x)|^{2_{\mu,s}^*}|u(y)|^{2_{\mu,s}^*}}{|x-y|^\mu}dxdy.
$$

Furthermore, $\mathscr{I}_{a,b}^{\overline{\lambda}_0^s}(u) = \|u\|^2 l_{N,s,\mu}(\|u\|) > 0$ for any $u \in X_0^s(\Omega)\backslash\{0\}$. So $u = 0$ is the only minimizer for this functional. □

Corollary 3.2. *Let $a^{\frac{4s-\mu}{\mu-2s-N}}b > \mathscr{M}_{N,s,\mu}$ and $u \in X_0^s(\Omega)\backslash\{0\}$ such that $\mathscr{I}_{a,b}^{\overline{\lambda}_0^s}(u) = \mathscr{R}_{\overline{\lambda}_0^s}^s$. Then $\overline{\lambda}_0^s = \lambda_0^s(u)$.*

Proof. The pair $(\overline{\lambda}_0^s, u)$ solves system (3.1). □

Proof of Theorem 1.4. The proof process is similar to that of Proposition 3.4 (ii). We omit the proof here. □

Now we investigate the solutions of mountain pass type. As we can see, the situation changes if $\lambda \geq \overline{\lambda}_0^s$ or $\lambda < \overline{\lambda}_0^s$. Next we consider positive parameters $a, b \in \mathbb{R}$ such that $a^{\frac{4s-\mu}{\mu-2s-N}}b > \mathscr{M}_{N,s,\mu}$.

Proof of Theorem 1.5. Combining (H_3) and (H_4), and recalling $X_0^s(\Omega) \hookrightarrow L^q(\Omega)$ continuously for each $q \in [2, 2_s^*]$, there exists a positive constant C such that

$$\begin{aligned}\mathscr{I}_{a,b}^{\lambda} &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{C}{2\cdot 2_{\mu,s}^*}\int_{\Omega}\int_{\Omega}\frac{|u(x)|^{2_{\mu,s}^*}|u(y)|^{2_{\mu,s}^*}}{|x-y|^{\mu}}dxdy - \lambda C(\varepsilon\|u\|^2 + \|u\|^p)\\ &= (\frac{a}{2} - \lambda C\varepsilon)\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{C}{2\cdot 2_{\mu,s}^*}\int_{\Omega}\int_{\Omega}\frac{|u(x)|^{2_{\mu,s}^*}|u(y)|^{2_{\mu,s}^*}}{|x-y|^{\mu}}dxdy - \lambda C\|u\|^p.\end{aligned} \tag{3.3}$$

By selecting $\varepsilon < a/(2\lambda C)$, we see that there exists ι_λ^s such that $\inf_{\|u\|=\iota_\lambda^s}\mathscr{I}_{a,b}^{\lambda} > 0$. Hence, we have $\mathscr{I}_{a,b}^{\lambda}(0) = 0$ and $\mathscr{I}_{a,b}^{\lambda}(u_{\overline{\lambda}_0^s}^s) \leq 0$. In fact, $\mathscr{I}_{a,b}^{\lambda}(u_{\overline{\lambda}_0^s}^s) = 0$ if $\lambda = \overline{\lambda}_0^s$ while $\mathscr{I}_{a,b}^{\lambda}(u_{\overline{\lambda}_0^s}^s) < 0$ for $\lambda > \overline{\lambda}_0^s$ by Proposition 3.5. The functional possesses a mountain pass geometry. Moreover, recalling Lemma 3.1 (2), we have that $\mathscr{I}_{a,b}^{\lambda}$ satisfies the Palais-Smale condition. So we obtain the conclusion by the mountain pass theorem. □

After solving the case of $\lambda \geq \overline{\lambda}_0^s$, we now study the case of $\lambda < \overline{\lambda}_0^s$, that is, the existence of nontrivial solutions which are local minimizers or mountain pass type.

Proposition 3.6. *Let $\lambda \leq \overline{\lambda}_0^s$. Then there exist $r = r(s)$ and $M = M(s) > 0$ such that*

$$\inf\left\{\mathscr{I}_{a,b}^{\lambda}(u) : u \in X_0^s(\Omega),\ \ \|u\| = r\right\} \geq M. \tag{3.4}$$

Proof. Fix $\varepsilon > 0$. Since $\lambda \leq \overline{\lambda}_0^s$ and (3.3), we have

$$\mathscr{I}_{a,b}^{\lambda} \geq (\frac{a}{2} - \overline{\lambda}_0^s C\varepsilon)\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{C}{2\cdot 2_{\mu,s}^*}\int_{\Omega}\int_{\Omega}\frac{|u(x)|^{2_{\mu,s}^*}|u(y)|^{2_{\mu,s}^*}}{|x-y|^{\mu}}dxdy - \overline{\lambda}_0^s C\|u\|^p,$$

for any $u \in X_0^s(\Omega)$. If we take ε in such a way that $a/2 - \overline{\lambda}_0^s C\varepsilon > 0$, then the proof is complete. □

Now, we show that the infimum in Theorem 1.6. Set

$$\widehat{\mathscr{R}}_\lambda^s := \inf\left\{\mathscr{I}_{a,b}^{\lambda}(u) : u \in X_0^s(\Omega),\ \ \|u\| \geq r\right\},\ \ for\ \ any\ \ r > 0.$$

Remark 3.1. We can easily see that $\widehat{\mathscr{R}}_0^s \to 0$ as $\lambda \to \overline{\lambda}_0^s$. There exists a function $u \in X_0^s(\Omega)$ such that $\overline{\lambda}_0^s = \lambda_0^s(u)$ and we have $0 \leq \widehat{\mathscr{R}}_\lambda^s \leq \mathscr{I}_{a,b}^{\lambda}(u) \to 0$ as $\lambda \to \overline{\lambda}_0^s$.

Remark 3.2. w_λ^s is a local minimizer and a critical point for $\mathscr{I}_{a,b}^{\lambda}$.

Proof of Theorem 1.6. For $r, M > 0$, if $\overline{\lambda}_0^s - \delta < \lambda < \overline{\lambda}_0^s$, we have that $\widehat{\mathscr{R}}_\lambda^s < M$ for $\delta > 0$. As a result, if $(u_k)_k$ is a minimizing sequence, then there exist $\beta > 0$ and $k \gg 0$ such that $\|u_k\| \geq M + \beta$. we conclude the existence of a minimizing sequence and the convergence to a local minimizer $w_\lambda^s \in X_0^s(\Omega)$ such that $\|w_\lambda^s\| \geq M$ and $\widehat{\mathscr{R}}_\lambda^s = \mathscr{I}_{a,b}^{\lambda}(w_\lambda^s)$ is established. □

Next, we prove the existence of mountain pass solutions in Theorem 1.7 for $\lambda < \overline{\lambda}_0^s$ close enough to $\overline{\lambda}_0^s$.

Proof of Theorem 1.7. We obtain that $\min\{\mathscr{I}_{a,b}^{\lambda}(0),\ \ \mathscr{I}_{a,b}^{\lambda}(w_\lambda^s)\} < M$, recalling $\|w_\lambda^s\| > M$ and (3.4). Hence, we have a mountain pass geometry. Since the Palais-Smale condition is satisfied, we obtain the conclusion by the Mountain Pass Theorem. □

Finally, we introduce the nonexistence result of solutions for problem (1.1). For (H_5), the following system is well defined

$$\begin{cases} \left(\psi_{a,b}^{\lambda,u}\right)'(t) = 0, \\ \left(\psi_{a,b}^{\lambda,u}\right)''(t) = 0, \\ \left(\psi_{a,b}^{\lambda,u}\right)'(t) = \inf_{\alpha>0}\left(\psi_{a,b}^{\lambda,u}\right)'(\alpha). \end{cases} \tag{3.5}$$

Fixing $u \in X_0^s(\Omega)$, we see that there exists a unique $\lambda^s(u) > 0$ that solves (3.5).

Proposition 3.7. *For each $u \in X_0^s(\Omega)\backslash\{0\}$, the parameter $\lambda^s(u)$ is the unique $\lambda > 0$ for which the fiber map $\psi_{a,b}^{\lambda,u}(t)$ has a critical point where the second derivative is zero. Moreover, $\psi_{a,b}^{\lambda,u}(t)$ has no critical points if and only if $0 < \lambda < \lambda^s(u)$.*

Proof. If $0 < \lambda < \lambda^s(u)$ for each $t > 0$, then $\psi_{a,b}^{\lambda,u}(t) > \psi_{a,b}^{\lambda^s(u),u}(t) > 0$. □

Corollary 3.3. *For each $u \in X_0^s(\Omega)\backslash\{0\}$, then $\lambda_0^s(u) > \lambda^s(u)$.*

Proof. Indeed, we assume on the contrary that $\lambda_0^s(u) \le \lambda^s(u)$. From Proposition 3.2, we obtain that $\psi_{a,b}^{\lambda_0^s(u),u}(t)$ is increasing which contradicts with the existence of solutions for system (3.1). □

Define the extremal value $\overline{\lambda}^s := \inf_{u\in X_0^s(\Omega)\backslash\{0\}} \lambda^s(u)$.

Proposition 3.8. *Let $a^{\frac{4s-\mu}{\mu-2s-N}} b > \mathscr{N}_{N,s,\mu}$. Then $0 < \overline{\lambda}^s < \overline{\lambda}_0^s$.*

Proof. In fact, it follows from Theorem 1.3 and Corollary 3.2 that there exists a $u \in X_0^s(\Omega)\backslash\{0\}$ such that $\overline{\lambda}_0^s = \lambda_0^s(u)$. From Corollary 3.3, it follows that $\overline{\lambda}^s \le \lambda^s(u) < \lambda_0^s(u) = \overline{\lambda}_0^s$. □

Proposition 3.9. *For each $0 < \lambda < \overline{\lambda}^s$, the fiber map $\psi_{a,b}^{\lambda,u}(t)$ is increasing and has no critical points.*

Proof. This follows from the fact that $\lambda < \overline{\lambda}^s \le \lambda^s(u)$ for each $u \in X_0^s(\Omega)\backslash\{0\}$ and Proposition 3.7. □

Proof of Theorem 1.8. In fact, by Proposition 3.9, we have that $\left(\psi_{a,b}^{\lambda,u}\right)'(t) > 0$ for all $t > 0$ and $u \in X_0^s(\Omega)\backslash\{0\}$. So $u = 0$ is the only critical point. □

Appendix A. Estimation Bout the Cut-Off Function

In this section, we give some new estimates on the cut-off function which needs to be reestablished. It is also applied to our main theorems.

Lemma A.1. *Let $s \in (0,1)$, $N > 4s$, and $\mu > 4s$. Then, the following estimate holds*

$$\|\upsilon_\varepsilon\| \le \varepsilon^{-\frac{N-2s}{4}} C_1 + O(1).$$

Proof. The proof is based on the previous estimates. And it is complicated-definitely more difficult than the one for similar results in the case of the Laplacian

$$\|\upsilon_\varepsilon\|^2 \le \int_Q \frac{|\upsilon_\varepsilon(x) - \upsilon_\varepsilon(y)|^2}{|x-y|^{N+2s}} dxdy + O(1).$$

By Proposition 3.4 (ii), we can obtain

$$\|\upsilon_\varepsilon\|^2 \le \int_Q \frac{\left|\frac{1}{(\varepsilon+|x|^2)^{\frac{N-2s}{2}}} - \frac{1}{(\varepsilon+|y|^2)^{\frac{N-2s}{2}}}\right|^2}{|x-y|^{N+2s}} dxdy + O(1).$$

Note that $|x-y|^{N+2s} \ge \left||x|^{N+2s} - |y|^{N+2s}\right|$. Hence,

$$\begin{aligned}
\|\upsilon_\varepsilon\|^2 &\le \int_Q \frac{\left|\frac{1}{(\varepsilon+|x|^2)^{\frac{N-2s}{2}}} - \frac{1}{(\varepsilon+|y|^2)^{\frac{N-2s}{2}}}\right|^2}{|x-y|^{N+2s}} dxdy + O(1) \\
&\le \int_Q \frac{\left|\frac{1}{(\varepsilon+|x|^2)^{\frac{N-2s}{2}}} - \frac{1}{(\varepsilon+|y|^2)^{\frac{N-2s}{2}}}\right|^2}{||x|^{N+2s} - |y|^{N+2s}|} dxdy + O(1) \qquad \text{(A.1)} \\
&= \varepsilon^{-(N-2s)} \int_Q \frac{\left|\frac{1}{(1+\frac{|x|^2}{\varepsilon})^{\frac{N-2s}{2}}} - \frac{1}{(1+\frac{|y|^2}{\varepsilon})^{\frac{N-2s}{2}}}\right|^2}{||x|^{N+2s} - |y|^{N+2s}|} dxdy + O(1).
\end{aligned}$$

Let $\frac{|x|}{\sqrt{\varepsilon}} = r_1$ and $\frac{|y|}{\sqrt{\varepsilon}} = r_2$. We have $dx = (\sqrt{\varepsilon} r_1)^{N-1}\sqrt{\varepsilon} dr_1$ and $dy = (\sqrt{\varepsilon} r_2)^{N-1}\sqrt{\varepsilon} dr_2$. Substituting them into the formula above, we have

$$\begin{aligned}
\|\upsilon_\varepsilon\|^2 &\le \varepsilon^{-(N-2s)} \int_Q \frac{\left|\frac{1}{(1+\frac{|x|^2}{\varepsilon})^{\frac{N-2s}{2}}} - \frac{1}{(1+\frac{|y|^2}{\varepsilon})^{\frac{N-2s}{2}}}\right|^2}{||x|^{N+2s} - |y|^{N+2s}|} dxdy + O(1) \\
&= \varepsilon^{-(N-2s)} \cdot \frac{\varepsilon^N}{\varepsilon^{\frac{N+2s}{2}}} \int_Q \frac{\left|\frac{1}{(1+r_1^2)^{\frac{N-2s}{2}}} - \frac{1}{(1+r_2^2)^{\frac{N-2s}{2}}}\right|^2 r_1^{N-1} r_2^{N-1}}{||r_1|^{N+2s} - |r_2|^{N+2s}|} dr_1 dr_2 + O(1) \qquad \text{(A.2)} \\
&= \varepsilon^{-\frac{N-2s}{2}} \int_Q \frac{\left|\frac{1}{(1+r_1^2)^{\frac{N-2s}{2}}} - \frac{1}{(1+r_2^2)^{\frac{N-2s}{2}}}\right|^2 r_1^{N-1} r_2^{N-1}}{||r_1|^{N+2s} - |r_2|^{N+2s}|} dr_1 dr_2 + O(1).
\end{aligned}$$

Thus $\|\upsilon_\varepsilon\| \le \varepsilon^{-\frac{N-2s}{4}} C_1 + O(1)$ for some $C_1 > 0$. This completes the proof. □

Lemma A.2. *Let $s \in (0,1)$, $N > 4s$ and $\mu > 4s$. Then, the following estimate holds*

$$\int_\Omega \int_\Omega \frac{|u_\varepsilon(x)|^{2^*_{\mu,s}} |u_\varepsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu} dxdy \ge S_{H,L}^{-2^*_{\mu,s}} + O(\varepsilon^{2N-\mu}).$$

Proof. By Proposition 3.4 (ii) and Lemma A.1, we can obtain

$$\begin{aligned}\int_\Omega\int_\Omega \frac{|u_\varepsilon(x)|^{2^*_{\mu,s}}|u_\varepsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy &= \int_\Omega\int_\Omega \frac{\left|\frac{\upsilon_\varepsilon(x)}{\|\upsilon_\varepsilon\|}\right|^{2^*_{\mu,s}}\left|\frac{\upsilon_\varepsilon(y)}{\|\upsilon_\varepsilon\|}\right|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy\\ &= \frac{1}{\|\upsilon_\varepsilon\|^{2\cdot 2^*_{\mu,s}}}\int_\Omega\int_\Omega \frac{|\upsilon_\varepsilon(x)|^{2^*_{\mu,s}}|\upsilon_\varepsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy\\ &= \frac{1}{\|\upsilon_\varepsilon\|^{2\cdot 2^*_{\mu,s}}}\int_\Omega\int_\Omega \frac{\left|\frac{1}{(\varepsilon+|x|^2)^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}}\left|\frac{1}{(\varepsilon+|y|^2)^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy.\end{aligned} \tag{A.3}$$

We calculate the above formula

$$\begin{aligned}&\int_\Omega\int_\Omega \frac{\left|\frac{1}{(\varepsilon+|x|^2)^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}}\left|\frac{1}{(\varepsilon+|y|^2)^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy\\ &= \varepsilon^{-(2N-\mu)}\int_\Omega\int_\Omega \frac{\left|\frac{1}{(1+\frac{|x|^2}{\varepsilon})^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}}\left|\frac{1}{(1+\frac{|y|^2}{\varepsilon})^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy.\end{aligned}$$

Note that

$$S_{H,L} := \inf_{\upsilon_\varepsilon\in X_0^s(\Omega)\setminus\{0\}} \frac{\|\upsilon_\varepsilon\|^2}{\left(\int_\Omega\left(\int_\Omega \frac{|\upsilon_\varepsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dy\right)|\upsilon_\varepsilon(x)|^{2^*_{\mu,s}}dx\right)^{\frac{1}{2^*_{\mu,s}}}}. \tag{A.4}$$

Combining (A.2), (A.3), and (A.4), we have

$$S_{H,L} := \inf_{\upsilon_\varepsilon\in X_0^s(\Omega)\setminus\{0\}} \frac{\left(\varepsilon^{-\frac{N-2s}{4}}\int_Q \frac{\left|\frac{1}{(1+r_1^2)^{\frac{N-2s}{2}}}-\frac{1}{(1+r_2^2)^{\frac{N-2s}{2}}}\right|^2 r_1^{N-1}r_2^{N-1}}{||r_1|^{N+2s}-|r_2|^{N+2s}|}dr_1dr_2\right)^2}{\left(\varepsilon^{-(2N-\mu)}\int_\Omega\int_\Omega \frac{\left|\frac{1}{(1+\frac{|x|^2}{\varepsilon})^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}}\left|\frac{1}{(1+\frac{|y|^2}{\varepsilon})^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy\right)^{\frac{1}{2^*_{\mu,s}}}}. \tag{A.5}$$

Setting $\frac{|x|}{\sqrt{\varepsilon}} = r_1$ and $\frac{|y|}{\sqrt{\varepsilon}} = r_2$, we have $dx = (\sqrt{\varepsilon}r_1)^{N-1}\sqrt{\varepsilon}dr_1$ and $dy = (\sqrt{\varepsilon}r_2)^{N-1}\sqrt{\varepsilon}dr_2$. Substituting them into the formula above, we have

$$\begin{aligned}&\varepsilon^{-(2N-\mu)}\int_\Omega\int_\Omega \frac{\left|\frac{1}{(1+\frac{|x|^2}{\varepsilon})^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}}\left|\frac{1}{(1+\frac{|y|^2}{\varepsilon})^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy\\ &\leq \varepsilon^{-(N-\frac{\mu}{2})}\int_\Omega\int_\Omega \frac{\left|\frac{1}{(1+r_1^2)^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}}\left|\frac{1}{(1+r_2^2)^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}} r_1^{N-1}r_2^{N-1}}{|r_1-r_2|^\mu}dr_1dr_2.\end{aligned}$$

Hence, combining (A.3) and (A.5), we have

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|u_\varepsilon(x)|^{2^*_{\mu,s}}|u_\varepsilon(y)|^{2^*_{\mu,s}}}{|x-y|^\mu}dxdy \\
&\geq \frac{\varepsilon^{-(N-\frac{\mu}{2})}\int_\Omega\int_\Omega \frac{\left|\frac{1}{(1+r_1^2)^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}}\left|\frac{1}{(1+r_2^2)^{\frac{N-2s}{2}}}\right|^{2^*_{\mu,s}} r_1^{N-1}r_2^{N-1}}{|r_1-r_2|^\mu}dr_1dr_2}{\left(\varepsilon^{-\frac{N-2s}{4}}\int_Q \frac{\left|\frac{1}{(1+r_1^2)^{\frac{N-2s}{2}}}-\frac{1}{(1+r_2^2)^{\frac{N-2s}{2}}}\right|^2 r_1^{N-1}r_2^{N-1}}{||r_1|^{N+2s}-|r_2|^{N+2s}|}dr_1dr_2\right)^{2\cdot 2^*_{\mu,s}}} \\
&\geq S_{H,L}^{-2^*_{\mu,s}} + O(\varepsilon^{2N-\mu}).
\end{aligned}$$

This completes the proof. □

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