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CONVEX MINIMIZATION PROBLEMS BASED ON AN ACCELERATED FIXED POINT ALGORITHM WITH APPLICATIONS TO IMAGE RESTORATION PROBLEMS

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Abstract. In this paper, a new accelerated fixed point algorithm for finding a common fixed point of two countable families of nonexpansive mappings in a Hilbert space is introduced. The weak convergence of the proposed algorithm is analyzed under some suitable conditions. We apply our proposed algorithm to solve the convex optimization problems in the form of the sum of two lower semi-continuous and convex functions. Some numerical experiments are conducted to demonstrate the efficiency of the proposed algorithm.

Keywords. Common fixed points; Inertial forward-backward splitting; Images restoration problems; Nonexpansive mappings; Weak convergence.

1. INTRODUCTION

Let H be a real Hilbert space, and let C be a nonempty, closed, and convex subset of H with $\|\cdot\|$. In the past decades, many researchers paid attention to the convex minimization problem in the form of the sum of two functions. A general natural formulation by estimating the minimizer of the sum of two functions is defined as follows:

$$\min_{x\in H}\{f(x)+g(x)\}, \tag{1.1}$$

where f and g are proper, lower semi-continuous, and convex functions. The set of all solutions to problems (1.1) is denoted by $\operatorname{argmin}(f+g)$. If f is differentiable on H, then (1.1) can be described by the fixed point equation

$$x=\operatorname{prox}_{\mu g}(x-\mu\nabla f(x)), \tag{1.2}$$

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where $\mu > 0$ and $\text{prox}_{\mu g}$ is the proximity operator of g defined by $\text{prox}_{\mu g} = (I - \mu\partial g)^{-1}$, where I is the identity operator in H, and ∂g is the subdifferential of g. In [1–3], the authors provided important properties of proximal operators. For example, $\text{prox}_{\mu g}$ is well-defined with full domain, single valued, and nonexpansive.

Equation (1.2) leads to the classical forward-backward splitting algorithm (FBS) [4] which is generated by $x_1 \in \mathbb{R}^n$ and $x_{n+1} = \text{prox}_{\mu_n g}(I - \mu_n \nabla f)(x_n)$, $n \geq 0$, where $\mu_n \in (0, \frac{2}{L})$ is a step size, and L is a Lipschitz constant of ∇f. Several authors utilized the concept of the classical forward-backward splitting algorithm to solve problem (1.1); see, e.g., [5–9] and the references therein.

Fixed point methods are efficient to solve the convex minimization problem in the form of the sum of two functions. In the real word, fixed point theory have been applied to many branches of applied analysis; see, e.g., [10–12]. In addition, fixed point methods also play an important role in solving various problems in data science, economics, medicine, and engineering; see, e.g., [6, 13–17] and the references therein. In order to find a fixed point of a nonlinear operator $T : C \to C$, many researchers introduced various methods. One of the famous iterative methods is the Picard iteration process, defined by $x_{n+1} = Tx_n$. In 1953, Mann [18] proposed an iterative method known as the Mann iteration process in Hilbert spaces as follows: $x_1 \in C$, $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$. He also proved a weak convergence theorem of this iteration under some conditions on $\{\alpha_n\}$. In 1967, Halpern [19] introduced the following iterative scheme for a fixed point of T: $x_{n+1} = (1 - \alpha_n)x + \alpha_n Tx_n$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $x \in C$. This is now called Halpern iteration. He proved the strong convergence of the iterative sequence in a Hilbert space. Mann iteration may fail to provide a convergence result when T is a pseudo-contractive mapping. In order to overcome this problem, Ishikawa [20] defined an iterative process, called Ishikawa iterative process, as follows: $x_1 \in C$,

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n Ty_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Notice that if $\alpha_n = 1$ for all $n \in \mathbb{N}$, then the Ishikawa iterative process reduces to the Mann iteration process.

Recently, Agarwal et al. [21] modified the Ishikawa iteration to introduce a new iteration process, called the S-iteration process, $x_1 \in C$,

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n, \\ x_{n+1} = (1 - \beta_n)Tx_n + \beta_n Ty_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. The authors proved that the convergence behavior of the S-iteration is better than that of Mann and of Ishikawa iterations.

For common fixed points of two mappings, Das and Debata [22] and Takahashi and Tamura [23] generalized the Ishikawa iteration for two mappings S and T as follows: $x_1 \in C$,

$$\begin{cases} x_{n+1} = (1 - \beta_n)x_n + \beta_n Sy_n, \\ y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. Note that when $S = T$, the above generalized Ishikawa iterative process reduces to the Ishikawa iterative process. It is worth noting that the approximation of common fixed points of two mappings case can be linked with minimization problems directly, see, e.g., [24]. In 2006, Nakajo et al. [25] constructed a sequence by an hybrid method in mathematical programming for finding a common fixed point of family of nonexpansive mappings in Hilbert spaces. Consequently, several modifications of the above hybrid method was constructed for finding a common fixed point of a countable family of nonexpansive mappings; see, e.g., see [26–28]. In 2007, Aoyama et al. [29] improved the Halpern iterative sequence by considering $x_{n+1} = (1-\alpha_n)x + \alpha_n T_n x_n$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $x_1,\ x \in C$. They proved the strong convergence of the Halpern type algorithm under some suitable conditions. In 2008, Takahashi et al. [26] defined a useful condition so-called the NST-condition (I) for proving that the sequence generated by the hybrid method converges strongly to a common fixed point of a countable family of nonexpansive mappings in Hilbert spaces. Nakajo et al. [27] proposed the condition, called NST*-condition, which is weaker than that of the NST-condition (I) and proved the strong convergence of the hybrid method to a common fixed point of a countable family of nonexpansive mappings.

To accelerate convergence behavior of these algorithms, many researchers utilized an inertial technique, which was first introduced by Polyak [30] to solve smooth convex minimization problems. Recently, many inertial-type algorithms were proposed and studied; see, e.g., [7, 9, 31, 32] and the references therein. The inertial forward-backward splitting (IFBS) was presented by Moudafi and Oliny in [33] as follows:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = \operatorname{prox}_{\mu_n g}(y_n - \mu_n \nabla f(x_n)), \end{cases}$$

where $x_0,\ x_1 \in \mathbb{R}^n$, $\mu_n \in (0, \frac{2}{L})$, and θ_n is the inertial parameter which controls the momentum of $x_n - x_{n-1}$. It can be guaranteed the convergence of IFBS by proper choices of μ_n and θ_n.

The fast iterative shrinkage-thresholding algorithm (FISTA) was defined by

$$\begin{cases} y_n = x_n - \frac{1}{L}\nabla f(x_n), \\ t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}, \theta_n = \frac{t_n - 1}{t_{n+1}}, \\ x_{n+1} = y_n + \theta_n(y_n - y_{n-1}), \end{cases}$$

where $n \in \mathbb{N}$, $x_1 = y_0 \in \mathbb{R}^n$, and $t_1 = 1$. This notion was suggested by Beck and Teboulle [31]. The authors also analyzed the convergence rate of the FISTA and applied the FISTA to image restoration problems. It is pointed out from this work that the LASSO model is a suitable model for image restoration problems.

Recently, Verma and Shukla [7] proposed the new accelerated proximal gradient algorithm (NAGA) as follows:

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = (1-\beta_n)z_n + \beta_n \operatorname{prox}_{\mu_n g}(z_n - \mu_n \nabla f(z_n)), \\ x_{n+1} = \operatorname{prox}_{\mu_n g}(y_n - \mu_n \nabla f(y_n)), \end{cases}$$

where $n \in \mathbb{N},\ x_0,\ x_1 \in \mathbb{R}^n,\ \beta_n \in (0,1),\ \mu_n \in (0, \frac{2}{L})$, and $\theta_n \in (0,1)$ is the inertial parameter which controls the momentum of $x_n - x_{n-1}$. The authors proved NAGA's convergence theorem under the condition $\frac{\|x_n - x_{n-1}\|_2}{\theta_n} \to 0$ and applied their result to the convex minimization problem for a multitask learning framework using sparsity inducing regularizes.

Motivated and inspired by the results mentioned above, in this paper, we aim to introduce a new accelerated algorithm by using the inertial technique to solve a common fixed point problem of two countable families of nonexpansive operators. We prove a convergence theorem for common fixed points of two countable families of nonexpansive mappings in a real Hilbert space under some control conditions and also apply the algorithm to some convex minimization problems. This paper is organized as follows. Section 2 provides some preliminary results that will be utilized throughout the paper. In Section 3, we introduce our new accelerated algorithm via the inertial techniques and prove a weak convergence theorem. We also apply our main results to convex minimization problems and image restoration problems. Some numerical experiments of the proposed methods are given in Section 4. Finally, we give a brief conclusion of our work in Section 5.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ and $\mathbb{R}$ be the set of positive integers and real numbers, respectively. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, induced by the inner product. For a sequence $\{x_n\}$ in H, we denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

A mapping $T : C \to C$ is called L-Lipschitz if there exists $L > 0$ such that $\|Tx - Ty\| \leq L\|x-y\|$, $\forall x, y \in C$. If $L = 1$, then T is called nonexpansive. We denoted by $F(T)$ the set of all fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. Let $\{T_n\}$ and $\Im$ be families of nonexpansive operators from C into itself such that $\emptyset \neq F(\Im) \subset \bigcap_{n=1}^{\infty} F(T_n)$, where $F(\Im)$ is the set of all common fixed points of $\Im$ and $F(T_n)$ is the set of all fixed point of T_n. From [28], we see that $\{T_n\}$ satisfies the NST-condition (I) with $\Im$ if, for every bounded sequence $\{x_n\}$ in C,

$$\lim_{n\to+\infty} \|x_n - T_n x_n\| = 0 \Longrightarrow \lim_{n\to+\infty} \|x_n - Tx_n\| = 0, \ \forall T \in \Im.$$

In particular, if $\Im = \{T\}$, then $\{T_n\}$ is said to satisfy NST-condition (I) with T. After that, the concept of NST*-condition which is weaker than that of NST-condition (I) was introduced by Nakajo et al. [27] and they provided some of mappings that satisfying the NST*-condition.

A sequence $\{T_n\}$ is said to satisfy the NST*-condition if, for every bounded sequence $\{x_n\}$ in C,

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0 = \lim_{n\to\infty} \|x_n - x_{n+1}\| \Longrightarrow \omega_w(x_n) \subset \bigcap_{n=1}^{\infty} F(T_n),$$

where $\omega_w(x_n)$ is the set of all weak-cluster points of $\{x_n\}$.

Note that the NST*-condition is more general than that of NST-condition (I). It is easy to see that if $\{T_n\}$ satisfies the NST-condition (I), then $\{T_n\}$ satisfies the NST*-condition.

We now recall the definition of forward-backward operator of lower semi-continuous and convex functions of $f : \mathbb{R}^n \to (-\infty, +\infty)$ and $g : \mathbb{R}^n \to (-\infty, +\infty)$ as: In 1962, Moreau [34] defined the proximity operator with respect to λ and g, denoted as $\text{prox}_{\lambda g}$. A forward-backward operator T is defined by $T := \text{prox}_{\lambda g}(I - \lambda\nabla f)$ for $\lambda > 0$, where ∇f is the gradient operator of function f and $\text{prox}_{\lambda g}(x) := \arg\min_{y\in H}\{g(y) + \frac{1}{2\lambda}\|y-x\|^2\}$ (see [3, 9]). Moreover, T is a nonexpansive mapping whenever $\lambda \in (0, \frac{2}{L})$, where L is a Lipschitz constant of ∇f.

We end this section with the following lemmas which will be used to prove our main result in next section.

Lemma 2.1 ([35]). *Let $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ be sequences of nonnegative real numbers such that $\alpha_{n+1} \leq (1+\gamma_n)\alpha_n + \beta_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \gamma_n < +\infty$ and $\sum_{n=1}^{\infty} \beta_n < +\infty$, then $\lim_{n\to\infty} \alpha_n$ exists.*

Lemma 2.2 ([33]). *Let H be a Hilbert space, and let $\{x_n\}$ be a sequence in H such that there exists a nonempty subset F of H satisfying*

(i) *for every $p \in F, \lim_{n\to+\infty} \|x_n - p\|$ exists;*
(ii) *each weak-cluster point of $\{x_n\}$ is in F.*

Then, there exists $x \in F$ such that $x_n \rightharpoonup x$.

Lemma 2.3 ([32]). *Let $\{a_n\}$ and $\{\theta_n\}$ be sequences of nonnegative real numbers such that $a_{n+1} \leq (1+\theta_n)a_n + \theta_n a_{n-1}$ for all $n \in \mathbb{N}$. Then the following holds $a_{n+1} \leq K \cdot \prod_{j=1}^{n}(1+2\theta_j)$, where $K = \max\{a_1, a_2\}$. Moreover, if $\sum_{n=1}^{\infty} \theta_n < +\infty$, then $\{a_n\}$ is bounded.*

Lemma 2.4 ([9]). *For a real Hilbert space H, let $g : H \to \mathbb{R} \cup \{+\infty\}$ be proper convex and lower semi-continuous function, and let $f : H \to \mathbb{R}$ be convex differentiable with gradient ∇f being $L-$Lipschitz constant for some $L > 0$. If $\{T_n\}$ is the forward-backward operator of f and g with respect to $c_n \in \left(0, \frac{2}{L}\right)$ such that $\{c_n\}$ converges to c, then $\{T_n\}$ satisfies the NST-condition (I) with T, where T is the forward-backward operator of f and g with respect to $c \in \left(0, \frac{2}{L}\right)$.*

3. Main Results

In this section, we first introduce the new algorithm for finding a common fixed point of two countable families of nonexpansive mappings in real Hilbert spaces. Let $\{S_n\}$ and $\{T_n\}$ be families of nonexpansive mappings on H into itself with $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

Algorithm 1: (IMIA): Inertial Modified Ishikawa algorithm.

Initial step. Take $x_0,\ x_1 \in H$ arbitrarily and set $n = 1$. Let $\{\theta_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \theta_n < +\infty$.

Iterative step. Compute $z_n,\ y_n$, and x_{n+1} via

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \beta_n z_n + (1-\beta_n)T_n z_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)S_n y_n. \end{cases}$$

We now prove our main result.

Theorem 3.1. *Let $\{S_n\}$ and $\{T_n\}$ be two countable families of nonexpansive mappings on a real Hilbert space H with $\bigcap_{n=1}^{\infty} F(S_n) \bigcap \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by Algorithm 1, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions: (i) $0 < \liminf_{n\to\infty} \beta_n(1-\beta_n)$ and (ii) $0 < \liminf_{n\to\infty} \alpha_n(1-\alpha_n)$. If $\{S_n\}$ and $\{T_n\}$ satisfy the NST*-condition, then $x_n \rightharpoonup x \in \Omega := \bigcap_{n=1}^{\infty} F(S_n) \bigcap \bigcap_{n=1}^{\infty} F(T_n)$.*

Proof. Fix $u \in \Omega$. It follows from the definition of z_n that

$$\begin{aligned} \|z_n - u\| &= \|x_n + \theta_n(x_n - x_{n-1}) - u\| \\ &\leq \|x_n - u\| + \theta_n \|x_n - x_{n-1}\|. \end{aligned} \tag{3.1}$$

By the nonexpansiveness of T_n, we have

$$\begin{aligned}\|y_n-u\|^2 &= \|\beta_n(z_n-u)+(1-\beta_n)(T_nz_n-u)\|^2\\ &= \beta_n\|z_n-u\|^2+(1-\beta_n)\|T_nz_n-u\|^2-\beta_n(1-\beta_n)\|z_n-T_nz_n\|^2 \qquad (3.2)\\ &\le \|z_n-u\|^2, \qquad (3.3)\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-u\|^2 &= \|\alpha_n(y_n-u)+(1-\alpha_n)(S_ny_n-u)\|^2\\ &= \alpha_n\|y_n-u\|^2+(1-\alpha_n)\|S_ny_n-u\|^2-\alpha_n(1-\alpha_n)\|y_n-S_ny_n\|^2 \qquad (3.4)\\ &\le \alpha_n\|y_n-u\|^2+(1-\alpha_n)\|S_ny_n-u\|^2\\ &\le \|y_n-u\|^2. \qquad (3.5)\end{aligned}$$

From (3.1), (3.3), and (3.5), we have

$$\begin{aligned}\|x_{n+1}-u\| &\le \|z_n-u\|\\ &\le \|x_n-u\|+\theta_n\|x_n-x_{n-1}\|\\ &\le (1+\theta_n)\|x_n-u\|+\theta_n\|x_{n-1}-u\|. \qquad (3.6)\end{aligned}$$

Using Lemma 2.3, we obtain

$$\|x_{n+1}-u\| \le K\cdot\prod_{j=1}^{n}(1+2\theta_j),$$

where $K=\max\{\|x_1-u\|,\|x_2-u\|\}$. Since $\sum_{n=1}^{\infty}\theta_n<+\infty$, we see that $\{x_n\}$ is a bounded. It follows that $\sum_{n=1}^{\infty}\theta_n\|x_n-x_{n-1}\|<+\infty$. Using (3.6) and Lemma 2.1, we obtain $\lim_{n\to\infty}\|x_n-u\|$ exists for all $u\in\Omega$. By (3.2), (3.5), and nonexpansiveness of T_n, we have

$$\begin{aligned}\|x_{n+1}-u\|^2 &\le \|y_n-u\|^2\\ &= \beta_n\|z_n-u\|^2+(1-\beta_n)\|T_nz_n-u\|^2-\beta_n(1-\beta_n)\|z_n-T_nz_n\|^2\\ &= \beta_n\|z_n-u\|^2+(1-\beta_n)\|z_n-u\|^2-\beta_n(1-\beta_n)\|z_n-T_nz_n\|^2\\ &= \|z_n-u\|^2-\beta_n(1-\beta_n)\|z_n-T_nz_n\|^2\\ &= \|(x_n-u)+\theta_n(x_n-x_{n-1})\|^2-\beta_n(1-\beta_n)\|z_n-T_nz_n\|^2\\ &\le \|x_n-u\|^2+2\theta_n\|x_n-u\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2-\beta_n(1-\beta_n)\|z_n-T_nz_n\|^2,\end{aligned}$$

which implies

$$\beta_n(1-\beta_n)\|z_n-T_nz_n\|^2 \le \|x_n-u\|^2-\|x_{n+1}-u\|^2+2\theta_n\|x_n-u\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2.$$

This together with assumption (i), $\lim_{n\to\infty}\|x_n-u\|$ exists and $\sum_{n=1}^{\infty}\theta_n\|x_n-x_{n-1}\|<+\infty$, we obtain

$$\lim_{n\to\infty}\|z_n-T_nz_n\|=0. \qquad (3.7)$$

It follows from (3.3), (3.4) and nonexpansiveness of S_n that

$$\begin{aligned}
&\alpha_n(1-\alpha_n)\|y_n - S_n y_n\|^2 \\
&= \alpha_n\|y_n - u\|^2 + (1-\alpha_n)\|S_n y_n - u\|^2 - \|x_{n+1} - u\|^2 \\
&\le \|y_n - u\|^2 - \|x_{n+1} - u\|^2 \\
&\le \|z_n - u\|^2 - \|x_{n+1} - u\|^2 \\
&= \|(x_n - u) + \theta_n(x_n - x_{n-1})\|^2 - \|x_{n+1} - u\|^2 \\
&\le \|x_n - u\|^2 + 2\theta_n\|x_n - u\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 - \|x_{n+1} - u\|^2.
\end{aligned}$$

By condition (ii) and the facts that $\lim_{n\to\infty}\|x_n - u\|$ exists and $\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\| < +\infty$, we conclude that

$$\lim_{n\to\infty}\|y_n - S_n y_n\| = 0. \tag{3.8}$$

Since $\|z_n - x_n\| = \theta_n\|x_n - x_{n-1}\|$ and $\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\| < +\infty$, we arrive at

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.9}$$

By the nonexpansiveness of T_n, we have

$$\begin{aligned}
\|x_n - T_n x_n\| &\le \|x_n - z_n\| + \|z_n - T_n z_n\| + \|T_n z_n - T_n x_n\| \\
&\le 2\|x_n - z_n\| + \|z_n - T_n z_n\|, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Combining (3.7) and (3.9), we obtain $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. Moreover, we have

$$\begin{aligned}
\|x_n - y_n\| &\le \beta_n\|x_n - z_n\| + (1-\beta_n)\|x_n - T_n z_n\| \\
&\le \beta_n\|x_n - z_n\| + (1-\beta_n)(\|x_n - z_n\| + \|z_n - T_n z_n\|) \\
&\le \|x_n - z_n\| + \|z_n - T_n z_n\|.
\end{aligned}$$

It follows from (3.7) and (3.9) that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.11}$$

In view of the nonexpansiveness of S_n, we have

$$\begin{aligned}
\|x_n - S_n x_n\| &\le \|x_n - y_n\| + \|y_n - S_n y_n\| + \|S_n y_n - S_n x_n\| \\
&\le 2\|x_n - y_n\| + \|y_n - S_n y_n\|, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

This implies by (3.8) and (3.11) that $\lim_{n\to\infty}\|x_n - S_n x_n\| = 0$. Note that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Indeed, from the definition of x_{n+1}, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \alpha_n\|y_n - x_n\| + (1-\alpha_n)\|S_n y_n - x_n\| \\
&\le \alpha_n\|y_n - x_n\| + (1-\alpha_n)[\|S_n y_n - y_n\| + \|y_n - x_n\|]. \\
&\le \|y_n - x_n\| + \|S_n y_n - y_n\|.
\end{aligned}$$

Using (3.8) and (3.11), we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $\{S_n\}$ and $\{T_n\}$ satisfy the NST*-condition, we have that all the weak-cluster points of the bounded sequence $\{x_n\}$ are contained in Ω. By Opial's Lemma (Lemma 2.2), we obtain that $x_n \rightharpoonup x$ for some $x \in \Omega$. □

Remark 3.1. We observe from the proof of Theorem 3.1 that condition $\sum_{n=1}^{\infty}\theta_n < +\infty$ was assumed to guarantee the boundedness and convergence of the sequence $\{x_n\}$ generated by Algorithm 1. From inequality (3.6), if we assume that $\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\| < +\infty$, then Lemma 2.1 indicates that $\lim_{n\to\infty}\|x_n - u\|$ exists. Thus we can follow the same proof line from this point to conclude that $x_n \rightharpoonup x \in \Omega$. Hence, we may assume a weaker condition on $\{\theta_n\}$ in Theorem 3.1 by assuming that $\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\| < +\infty$ instead of $\sum_{n=1}^{\infty}\theta_n < +\infty$. However, in practice, condition $\sum_{n=1}^{\infty}\theta_n < +\infty$ is more suitable and gives us more flexibility for implementations (see TABLE 1).

As a direct consequence of Theorem 3.1, we obtain the following result by setting $S_n = T_n$.

Corollary 3.1. *Let $\{T_n\}$ be a countable family of nonexpansive mappings on a real Hilbert space H with $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \beta_n z_n + (1-\beta_n)T_n z_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)T_n y_n, \end{cases}$$

where $\sum_{n=1}^{\infty}\theta_n < +\infty$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the assumptions as in Theorem 3.1. If $\{T_n\}$ satisfies the NST*-condition, then $\{x_n\}$ converges weakly to $x \in \Omega$.

4. APPLICATIONS TO CONVEX MINIMIZATION PROBLEMS

In this section, we focus on our proposed method for finding a common solution of the following convex minimization problems:

$$\min_{x\in H}(f_1(x) + g_1(x)) \quad \text{and} \quad \min_{x\in H}(f_2(x) + g_2(x)), \tag{4.1}$$

where $f_i, g_i : H \to (-\infty, \infty]$, $i = 1,2$, are proper lower semi-continuous functions such that f_1 and f_2 are differentiable. The set of all solutions of problem (4.1) is denoted by

$$\Omega := \arg\min(f_1 + g_1) \cap \arg\min(f_2 + g_2).$$

When $f_1 = f_2$ and $g_1 = g_2$, the problem (4.1) can be reduced to the convex minimization problem of the following form:

$$\min_{x\in H}(f_1(x) + g_1(x)). \tag{4.2}$$

One knows that a point z in H is a solution to problem (4.2) if and only if $z = Tz$, where $T = \text{prox}_{\mu g_1}(I - \mu\nabla f_1)$ and $\mu > 0$. We also see that a point z in Ω if and only if $z = Tz = Sz$, where $T = \text{prox}_{\mu g_1}(I - \mu\nabla f_1)$ and $S = \text{prox}_{\kappa g_2}(I - \kappa\nabla f_2)$ with $\mu, \kappa > 0$. It is also guaranteed that T and S are nonxpansive if $\mu \in (0, \frac{2}{\rho_1})$ and $\kappa \in (0, \frac{2}{\rho_2})$ when ρ_1 and ρ_2 are Lipschitz constants of ∇f_1 and ∇f_2, respectively. For more details, one refers to [1–3]. To solve convex minimization problem (4.1), we set $T_n = \text{prox}_{\mu_n g_1}(I - \mu_n\nabla f_1)$ and $S_n = \text{prox}_{\kappa_n g_2}(I - \kappa_n\nabla f_2)$ in Algorithm 1, where $\mu_n \in (0, \frac{2}{\rho_1})$ and $\kappa_n \in (0, \frac{2}{\rho_2})$. So, Algorithm 1 can be written as Algorithm 2:

As a consequence of Theorem 3.1, we obtain the weak convergence of the sequence generated by Algorithm 2 to a solution of problem (4.1).

Theorem 4.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 2. Under the same conditions and assumptions as in Theorem 3.1, we have $x_n \rightharpoonup x \in \Omega$.*

Algorithm 2: (FBMIA (I)): Forward-Backward Modified Ishikawa algorithm

Initial step. Take $x_0,\ x_1 \in H$ arbitrarily and $n = 1$. Let $\{\theta_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \theta_n < +\infty$

Iterative step. Compute $z_n,\ y_n$ and x_{n+1} using

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \beta_n z_n + (1-\beta_n)\text{prox}_{\mu_n g_1}(I - \mu_n \nabla f_1) z_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)\text{prox}_{\kappa_n g_2}(I - \kappa_n \nabla f_2) y_n. \end{cases}$$

Proof. Let $T_n = \text{prox}_{\mu_n g_1}(I - \mu_n \nabla f_1)$ and $S_n = \text{prox}_{\kappa_n g_2}(I - \kappa_n \nabla f_2)$, where $\mu_n \in \left(0, \frac{2}{\rho_1}\right)$ and $\kappa_n \in \left(0, \frac{2}{\rho_2}\right)$. Then T_n and S_n are nonexpansive operators for all n. Similarly, we set T and S to be forward-backward operators of f_1 and f_2 with respect to μ and κ, respectively, where $\mu \in \left(0, \frac{2}{\rho_1}\right)$ and $\kappa \in \left(0, \frac{2}{\rho_2}\right)$. Then $T = \text{prox}_{\mu g_1}(I - \mu \nabla f_1)$ and $S = \text{prox}_{\kappa g_2}(I - \kappa \nabla f_2)$ are nonexpansive operators. By [36, Proposition 26.1], we know that $\bigcap_{n=1}^{\infty} F(T_n) = \text{argmin}(f_1 + g_1)$ and $\bigcap_{n=1}^{\infty} F(S_n) = \text{argmin}(f_2 + g_2)$. It is derived from Lemma 2.4 that $\{T_n\}$ and $\{S_n\}$ satisfy the NST*-condition. Using Theorem 3.1, we can conclude that $\{x_n\}$ converges weakly to $x \in \Omega$. □

Moreover, if $f_2 = f_1,\ g_2 = g_1$, and $\kappa_n = \mu_n$ in Algorithm 2, then we can prove $\{x_n\}$ converges weakly to solution of problem (4.2) via the following algorithm.

Algorithm 3: (FBMIA (II)): Forward-Backward Modified Ishikawa Algorithm

Initial step. Take $x_0,\ x_1 \in H$ arbitrarily and $n = 1$. Let $\{\theta_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} \theta_n < +\infty$

Iterative step. Compute $z_n,\ y_n$ and x_{n+1} using

$$\begin{cases} z_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \beta_n z_n + (1-\beta_n)\text{prox}_{\mu_n g_1}(I - \mu_n \nabla f_1) z_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)\text{prox}_{\mu_n g_1}(I - \mu_n \nabla f_1) y_n. \end{cases}$$

The following result is a consequence of Corollary 3.1.

Theorem 4.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3. Under the same conditions and assumption as in Corollary 3.1, we have $x_n \rightharpoonup x \in argmin(f_1 + g_1)$.*

Proof. Let $T_n = \text{prox}_{\mu_n g_1}(I - \mu_n \nabla f_1)$ and $T = \text{prox}_{\mu g_1}(I - \mu \nabla f_1)$, where $\mu_n, \mu \in \left(0, \frac{2}{\rho_1}\right)$. Then T_n and T are nonexpansive for all n. By [36, Proposition 26.1], we obtain $\cap_{n=1}^{\infty} F(T_n) = \text{argmin}(f_1 + g_1)$. It is derived from Lemma 2.4 that $\{T_n\}$ satisfies the NST*-condition. Using Corollary 3.1, we obtain that $\{x_n\}$ converges weakly to $x \in \arg\min(f_1 + g_1)$. □

We next present some experiments on image restoration problems by using Algorithm 2 and Algorithm 3. The model of image restoration problem is a simple linear inverse problem as follows:

$$Ax = b + w \tag{4.3}$$

where $x \in \mathbb{R}^{n\times 1}$ is the original image, A is the blurring operator, b is the observed image, and w is an additive noise. To approximate the original image $x^\star \in \mathbb{R}^{n\times 1}$ which satisfies (4.3), we need to minimize the value of w by using the least squares (LS) problem:

$$\min_x \left\{ \|Ax-b\|_2^2 \right\}, \tag{4.4}$$

where $\|\cdot\|_2$ is an ℓ_2 norm defined by $\|x\|_2 = \sqrt{\sum_{k=1}^n |x_k|^2}$.

There are numerous iterations for solving problem (4.4), such as the Richardson iteration (see [37]). However, the number of unknown variables is much more than the observations which cause (4.4) to be an ill-posed problem; see [38] and [39]. Therefore, in order to improve the ill-conditioned least squares problem, several regularization methods were introduced. One of the most popular regularization methods is the least absolute shrinkage and selection operator (LASSO) model introduced by Tibshirani [40] as the following form:

$$\min_x \left\{ \|Ax-b\|_2^2 + \beta\|x\|_1 \right\}, \tag{4.5}$$

where β is a positive regularization parameter, $\|x\|_1 = \sum_{k=1}^n |x_k|$, and $\|x\|_2 = \sqrt{\sum_{k=1}^n |x_k|^2}$. This model can be used to solve problem (4.2) by utilizing optimization methods; see, e.g., [31, 40]. For solving image restoration problems, in particular, the true RGB images, this model is highly cost to compute the multiplication Ax and $\|x\|_1$ because of the size of matrix A and x as well as their members. In order to overcome this problem, most of researchers in this area employed the 2-D fast Fourier transform for transformation the true RGB images as the following form:

$$\min_x \{ \|\mathscr{A}x - v\|_2^2 + \beta\|\mathscr{W}x\|_1 \},$$

where $\mathscr{A}$ is the blurring operator which is often chosen as $\mathscr{A} = \mathscr{B}\mathscr{W}$, $\mathscr{B}$ is the blurring matrix, $\mathscr{W}$ is the 2-D fast Fourier transform, $v \in \mathbb{R}^{m\times n}$ is the observed image of size $m \times n$, and β is a positive regularization parameter. In real situation, we may have more than one observed images by different blurring operators. Let v_1 and v_2 be observed images occurred by blurring operators $\mathscr{A}_1 = \mathscr{B}_1\mathscr{W}$ and $\mathscr{A}_2 = \mathscr{B}_2\mathscr{W}$, where $\mathscr{B}_1$ and $\mathscr{B}_2$ are the blurring matrices, respectively. Our model of image restoration problem is to find the original image x which satisfies the following LASSO model:

$$\min_x \{ \|\mathscr{A}_1 x - v_1\|_2^2 + \lambda_1\|\mathscr{W}x\|_1 \} \quad \text{and} \quad \min_x \{ \|\mathscr{A}_2 x - v_2\|_2^2 + \lambda_2\|\mathscr{W}x\|_1 \}.$$

Hence, it can be viewed as two convex minimization problem (4.1). So, FBMIA (I) can be applied to image restoration problem (4.3) by setting $f_1(x) = \|\mathscr{A}_1 x - v_1\|_2^2$, $g_1(x) = \lambda_1\|\mathscr{W}x\|_1$, and $f_2(x) = \|\mathscr{A}_2 x - v_2\|_2^2$, $g_2(x) = \lambda_2\|\mathscr{W}x\|_1$. Furthermore, FBMIA (II), FBS [4], IFBS [33], FISTA [31], and NAGA [7] can be applied to image restoration problem $Ax = b + w$ by setting $f(x) = \|\mathscr{A}x - b\|_2^2$ and $g(x) = \gamma\|\mathscr{W}x\|_1$, where γ is a regularization parameter. In the following experiment, we set the regularization parameter $\gamma = 5e^{-5}$ and look at the original image (Wat Lok Moli) of size 256×256 pixels. The blurred and noisy images are created by a Gaussian blur of size 9×9 with the standard deviation $\sigma = 5$ and the motion blur of angle $\theta = 10$. In Figure 2, we can see the original image 2 (a) and both two blurred images 2 (b) and 2 (c). We use the peak signal-to-noise ratio (PSNR) in decibel (dB) [41] to measure the performance of restorative image which is defined by

$$\mathrm{PSNR}(x_n) = 10\log\left(\frac{255^2}{\mathrm{MSE}}\right),$$

where $\text{MSE} = \frac{1}{K}\Vert x_n - x^\star\Vert_2^2$, K is the number of image samples, and $x^\star$ is the original image. It is observed that a higher value of PSNR shows a higher quality of deblurring image. In addition, we compute the Lipschitz constant L by using the maximum eigenvalue of the matrix $A^T A$ and set parameters for FBMIA (I), FBMIA (II), FISTA, NAGA, IFBS and FBS as in Table 1.

TABLE 1. Algorihtms and their setting of parameters

Methods	**Setting of parameters**
FBMIA (I) FBMIA (II)	$\alpha_n = \frac{1}{100},\ \beta_n = 0.5,\ \mu_n, \kappa_n = \frac{n}{L(n+1)}$, and $\theta_n = \begin{cases} \frac{n}{n+1} & \text{if } 1 \leq n < N, \\ \frac{1}{2^n}, & \text{otherwise, where } N \text{ is a stop number of iteration.} \end{cases}$
FISTA	$\mu = \frac{1}{L},\ \theta_n = \frac{t_n - 1}{t_{n+1}}$, where $t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}$.
NAGA	$\beta_n = 0.5,\ \mu_n = \frac{n}{L(n+1)}$ and $\theta_n = \frac{t_n - 1}{t_{n+1}}$, where $t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}$.
IFBS	$\mu_n = \frac{n}{L(n+1)}$ and $\theta_n = \begin{cases} \frac{1}{n^2\Vert x_n - x_{n-1}\Vert_2^2} & \text{if } x_n \neq x_{n-1}, \\ 0, & \text{otherwise.} \end{cases}$
FBS	$\mu_n = \frac{n}{L(n+1)}$

As seen in Table 1, all parameters are selected to satisfy all the conditions for each algorithms. By Theorem 4.1 and Theorem 4.2, the sequence $\{x_n\}$ generated by FBMIA (I) and FBMIA (II) converges to the original image, respectively. All experiments are performed on Intel(R) core(TM) i7-9700CPU with 32.00 GB RAM, windows 10, under Matlab computing environment. All parameters in each algorithm are set as in Table 1, we obtain the results of deblurring image of Wat Lok Moli with 500 iterations in Table 2.

TABLE 2. The values of PSNR at $x_{10}, x_{50}, x_{100}, x_{200}, x_{300}, x_{400}, x_{500}$

Iteration No.	The peak signal-to-noise ratio (PSNR)					
	FBMIA (I)	FBMIA (II)	NAGA	FISTA	IFBS	FBS
10	23.3669	22.5153	22.0882	21.9384	21.3223	21.3223
50	29.4914	25.5639	25.5513	25.1685	22.5736	22.5736
100	32.8390	27.2957	26.8910	26.5220	23.4065	23.4065
200	35.9721	28.9271	22.4459	27.9742	24.2912	24.2912
300	37.7413	29.7341	29.3091	28.9318	24.7922	24.7922
400	38.7311	30.1881	29.9754	29.5728	25.1320	25.1320
500	39.4552	30.4147	30.3461	30.0153	25.3848	25.3848

From Table 2, comparing with FBMIA (II), NAGA, FISTA, IFBS and FBS, FBMIA (I) provides higher value of PSNR. This means that the performance of the image restoration of our proposed algorithm is the best. Moreover, we present the results of deblurring image of Wat Lok Moli at the 500^{th} iteration of such studied algorithms as in Figure 1. It is derived from the graph

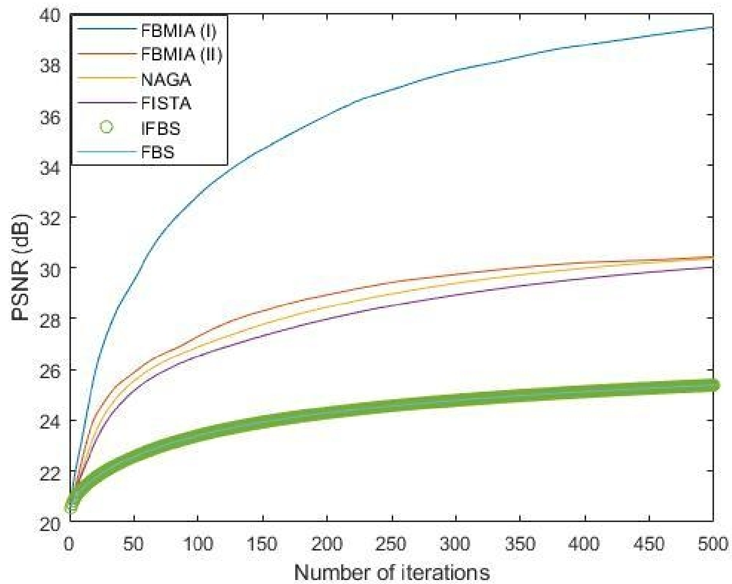

FIGURE 1. The graph of PSNR for Wat Lok Moli

of PSNR in Figure 1 that FBMIA (I) provides the higher value of PSNR than other algorithms. This implies that FBMIA (I) has a better performance than those of FBMIA (II), NAGA, FISTA, IFBS, and FBS. We can conclude from Figure 2 that Algorithm 2 provides the best results of deblurring.

5. CONCLUSIONS

In this paper, a new accelerated algorithm for solving a common fixed point of two countable families of nonexpansive operators was introduced. Under some conditions, a weak convergence theorem of this algorithm was proved. The main result of this paper was applied to a convex minimization problem in the form of the sum of two proper lower semi-continuous and convex functions. Image restoration problems were also considered as applications. some numerical experiments were provided to compare the performance of the studied algorithms with NAGA [7], FISTA [31], IFBS [33], and FBS [4] and the efficiency for images restoration of our proposed algorithm was obtained.

(A) Original image

(B) Gaussian Blurred

(C) Motion Blurred

(D) FBS

(E) IFBS

(F) FISTA

(G) NAGA

(H) FBMIA (II)

(I) FBMIA (I)

FIGURE 2. Results for image restoration at 500^{th} iteration.

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