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OPTIMALITY CONDITIONS OF QUASI (α,ε)-SOLUTIONS AND APPROXIMATE MIXED TYPE DUALITY FOR DC COMPOSITE OPTIMIZATION PROBLEMS

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Abstract. This paper is devoted to the approximate optimality condition and mixed type duality for DC composite optimization problems in locally convex Hausdorff topological vector spaces. By using the properties of the Fréchet subdifferential, a new constraint qualification is introduced. Under this constraint qualification, some approximate optimality conditions of the quasi (α,ε)-optimal solution for DC compose optimization problem and associated mixed type duality theorems are established, which extend and improve the corresponding results in the previous papers.

Keywords. Approximate optimality condition; Constraint qualification; DC composite optimization problem; Mixed type duality.

1. Introduction

Let X,Y, and Z be real locally convex Hausdorff topological vector spaces with dual spaces, X^*,Y^*, and Z^*, endowed with the weak*-topology $w^*(X^*,X)$, $w^*(Y^*,Y)$, and $w^*(Z^*,Z)$, respectively. Let Y and Z be partially ordered by closed convex cones $K\subseteq Y$ and $S\subseteq Z$, respectively. Denote $Y^\bullet = Y\cup\{\infty_Y\}$ and $Z^\bullet = Z\cup\{\infty_Z\}$, where ∞_Y and ∞_Z are the greatest elements with respect to the partial orders $\leq_K$ and $\leq_S$, respectively. Let $C\subseteq X$ be a nonempty convex subset, T be an arbitrary (possibly infinite) index set, $f_1 : Y\to\overline{\mathbb{R}} := \mathbb{R}\cup\{+\infty\}$ be a proper convex K-increasing function, $f_2 : X\to Y^\bullet$ be a proper K-convex function, $g_1 : Z\to\overline{\mathbb{R}}$ be a proper convex S-increasing function, $g_2 : X\to Z^\bullet$ be a proper S-convex function, and $h_t : X\to\overline{\mathbb{R}}$, $t\in T$ be a proper convex function.

Consider the following DC composite optimization problem

$$(P)\qquad \begin{aligned}&\inf\{(f_1\circ f_2)(x)-(g_1\circ g_2)(x)\}\\ &\text{s.t. } h_t(x)\leq 0, t\in T,\\ &\qquad x\in C.\end{aligned}$$

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This problem was studied extensively and numerous problems in optimization and approximation theory, such as the classical convex optimization problems, convex composite optimization problems, DC optimization problems, and the best approximation with restricted ranges can be recast into the form of (P); see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9] and the references therein.

Recently, a great deal of attention has been focused on the optimality conditions for the DC composite optimization problem. But, as one knows, it may not be always possible to find the point of minimizers in optimization problems or it is computationally expensive from a computational point of view. In these situations, we have to find an approximate solution for these optimization problems. Due to this, the study of approximate solutions becomes an important and interesting area. Numerous interesting results on the characterizations of approximate solutions to various types of optimization problems were obtained; see, e.g., ε-optimal solutions for convex programming or robust convex programming in [10, 11, 12, 13] and for composite convex optimization problems in [1, 14], quasi ε-optimal solutions for robust convex programming in [15] and for DC programming in [1], and quasi (α,ε)-optimal solutions for robust convex programming in [16] and for composite convex optimization problem in [17]. Note that, the above approximate optimality conditions are mainly focused on robust convex programming and composite convex optimization problems. To the best of our knowledge, not many results are known for DC composite optimization problem.

Motivated and inspired by the works in [1, 16, 17], we continue to study DC composite optimization problem (P) and devote this paper to some new characterizations of approximate optimality conditions and mixed type duality theorems. Our main aim in this paper is to give some constraint qualifications by using the properties of ε-subdifferential, and then establish some new characterizations for the quasi (α,ε)-optimal solutions to problem (P). Based on the approximate optimality conditions, we propose a mixed type approximate dual problem of (P) and then provide some mixed type duality theorems between problem (P) and its mixed type approximate dual problem.

The paper is organized as follows. In Section 2, we recall some necessary notations and preliminary results. In Section 3, some new regularity conditions are provided and several relationships among them are given. Under the new regularity conditions, quasi (α,ε)-optimality conditions for DC composite optimization problems are established. Approximate mixed type duality theorems are established in the last section, Section 4.

2. Notations and Preliminary Results

The notations used in the present paper are standard (see [18]). In particular, we assume throughout the whole paper that X, Y, and Z are real locally convex Hausdorff topological vector spaces with their dual spaces X^*, Y^*, and Z^*, endowed with the weak*-topology $w^*(X^*,X)$, $w^*(Y^*,Y)$, and $w^*(Z^*,Z)$, respectively. By $\langle x^*,x\rangle$, we denote the value of the functional $x^*\in X^*$ at $x\in X$, that is, $\langle x^*,x\rangle = x^*(x)$. We endow $X^*\times\mathbb{R}$ with the product topology of $w^*(X^*,X)$ and the usual Euclidean topology. The symbol $\mathbb{B}^*$ stands for the closed unit ball in X^*. The norm of $\xi\in X$ is denoted by $\|\xi\|$, i.e.,

$$\|\xi\| := \sup\{\langle \xi,d\rangle | d\in X, \|d\|\leq 1\}.$$

Let Y and Z be partially ordered by closed convex cones $K\subseteq Y$ and $S\subseteq Z$, respectively. Denote $Y^\bullet = Y\cup\{\infty_Y\}$ and $Z^\bullet = Z\cup\{\infty_Z\}$, where ∞_Y and ∞_Z are the greatest elements with respect to the partial orders $\leq_K$ and $\leq_S$, respectively. The following operations are defined on $Y^\bullet$ (resp. $Z^\bullet$): for any $y \in Y$ (resp. $z \in Z$), $y + \infty_Y = \infty_Y + y = \infty_Y$ and $t\infty_Y = \infty_Y$ (resp. $z + \infty_Z = \infty_Z + z = \infty_Z$ and $t\infty_Z = \infty_Z$) for any $t > 0$. Recall that a function $\psi : Y \to \overline{\mathbb{R}}$ is said to be K-increasing if, for any $x, y \in Y$ such that $y \leq_K x$, $\psi(y) \leq \psi(x)$, and $\varphi : X \to Y$ is said to be K-convex, if for any $x, y \in \operatorname{dom}\varphi := \{x \in X : \varphi(x) \in Y\}$ and every $t \in [0,1]$,

$$\varphi(tx + (1-t)y) \leq_K t\varphi(x) + (1-t)\varphi(y).$$

Let C be a nonempty subset in X. The closure of C is denoted by $\operatorname{cl} C$. The dual cone C^* and the indicator function δ_C of C are defined, respectively, by

$$C^* := \{x^* \in X^* : \langle x^*, x\rangle \geq 0 \quad \text{for each } x \in C\},$$

and

$$\delta_C(x) := \begin{cases} 0, & x \in C, \\ +\infty, & \text{otherwise.} \end{cases}$$

Moreover, we use $\mathbb{R}^{(T)}$ to denote the space of real tuples $\lambda = (\lambda_t)_{t \in T}$ with only finitely many $\lambda_t \neq 0$, and let $\mathbb{R}^{(T)}_+$ denote the nonnegative cone in $\mathbb{R}^{(T)}_+$, that is,

$$\mathbb{R}^{(T)}_+ := \left\{(\lambda_t)_{t\in T} \in \mathbb{R}^{(T)} : \lambda_t \geq 0 \quad \text{for each} \quad t \in T\right\}.$$

Let $f : X \to \overline{\mathbb{R}}$ be a proper convex function. The effective domain, conjugate function, and epigraph of f are denoted by $\operatorname{dom} f$, f^*, and $\operatorname{epi} f$, respectively, and they are defined by $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$, $f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in X\}$ for each $x^* \in X^*$, and $\operatorname{epi} f := \{(x,r) \in X \times \mathbb{R} : f(x) \leq r\}$. Then f is called proper if $\operatorname{dom} f \neq \emptyset$. It can easily seen that the following Young-Fenchel inequality holds:

$$f(x) + f^*(x^*) \geq \langle x, x^*\rangle \quad \text{for each pair } (x,x^*) \in X \times X^*. \tag{2.1}$$

The subdifferential of f at $x \in \operatorname{dom} f$ is defined by

$$\partial f(x) := \{x^* \in X^* : f(x) + \langle x^*, y - x\rangle \leq f(y) \quad \text{for all} \quad y \in X\},$$

and for any $\varepsilon \geq 0$, the ε-subdifferential of f at $x \in \operatorname{dom} f$ is defined by

$$\partial_\varepsilon f(x) := \{x^* \in X^* : f(x) + \langle x^*, y - x\rangle \leq f(y) + \varepsilon \quad \text{for all} \quad y \in X\}. \tag{2.2}$$

Then, for each $\varepsilon \geq 0$ and $x \in \operatorname{dom} f$,

$$x^* \in \partial_\varepsilon f(x) \Leftrightarrow f(x) + f^*(x^*) \leq \langle x^*, x\rangle + \varepsilon. \tag{2.3}$$

In particular, if $\varepsilon = 0$, the set $\partial f(x) = \partial_0 f(x)$ is the classical subdifferential of convex analysis. By definition, the following Young's equality holds:

$$x^* \in \partial f(x) \Leftrightarrow f(x) + f^*(x^*) = \langle x^*, x\rangle. \tag{2.4}$$

If $0 \leq \varepsilon_1 \leq \varepsilon_2$, then $\partial_{\varepsilon_1} f(x) \subseteq \partial_{\varepsilon_2} f(x)$ for each $x \in \operatorname{dom} f$. If f is lsc, then $f^{**} = f$. Furthermore, the normal cone $N(x_0; C)$ and the ε-normal cone $N_\varepsilon(x_0; C)$ of a convex set $C \subseteq X$ at the point $x_0 \in C$ are defined, respectively, by

$$N(x_0; C) := \partial \delta_C(x_0) = \{x^* \in X^* : \langle x^*, x - x_0\rangle \leq 0 \quad \text{for all} \quad x \in C\},$$

and

$$N_\varepsilon(x_0; C) := \partial_\varepsilon \delta_C(x_0) = \{x^* \in X^* : \langle x^*, x - x_0\rangle \leq \varepsilon \quad \text{for all} \quad x \in C\}.$$

For a function $h : X \to Z^\bullet$, the S-epigraph of h is defined by

$$\mathrm{epi}_S h := \{(x,y) \in X \times Y : y \in h(x) + S\}.$$

Then h is called S-epi-closed if $\mathrm{epi}_S h$ is closed, and h is star S-lower semi-continuou (S-lsc in brief) if λh is lsc for each $\lambda \in S^*$. Note that if h is star S-lsc, then it is S-epi-closed.

Note that an element $p \in X^*$ can be naturally regarded as a function on X in such a way that $p(x) := \langle p, x\rangle$ for each $x \in X$. Thus the following facts are clear for any $r \in \mathbb{R}$ and any function $f : X \to \overline{\mathbb{R}}$:

$$(f + p + r)^*(x^*) = f^*(x^* - p) - r \quad \text{for each} \quad x^* \in X^*$$

and

$$\mathrm{epi}\,(f + p + r)^* = \mathrm{epi}\, f^* + (p, -r).$$

Below we drop the convexity assumption and consider the generalized differentials for arbitrary proper extend real value functions. Let $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ be an extended real valued function, and let $x_0 \in \mathrm{dom}\varphi$ with $|\varphi(x_0)| < \infty$. Following [19], one defines the analytic ε-subdifferential of φ at x_0 by

$$\hat{\partial}_\varepsilon \varphi(x_0) := \left\{ x^* \in X^* : \liminf_{x \to x_0} \frac{\varphi(x) - \varphi(x_0) - \langle x^*, x - x_0\rangle}{\|x - x_0\|} \geq -\varepsilon \right\}, \varepsilon \geq 0.$$

If $\varepsilon = 0$, then $\hat{\partial}\varphi(x_0) := \hat{\partial}_\varepsilon \varphi(x_0)$ is known as the Fréchet subdifferential of φ at x_0 and reduces in the convex case to the classical subdifferential of convex analysis. Moreover, if φ_1, φ_2 are finite at x_0 and $\hat{\partial}\varphi_2(x_0) \neq \emptyset$, then it follows from [19, Theorem 3.1] that

$$\hat{\partial}(\varphi_1 - \varphi_2)(x_0) \subseteq \bigcap_{x^* \in \hat{\partial}\varphi_2(x_0)} [\hat{\partial}\varphi_1(x_0) - x^*].$$

Furthermore, by the definition of Fréchet subdifferential, we can obtain the following result.

Proposition 2.1. *Let $\varepsilon_1, \varepsilon_2 \geq 0$. Assume that $\varphi_1, \varphi_2 : X \to \overline{\mathbb{R}}$ are finite at x_0 and $\hat{\partial}_{\varepsilon_2}\varphi_2(x_0) \neq \emptyset$. Then the following assertion holds:*

$$\hat{\partial}_{\varepsilon_1}(\varphi_1 - \varphi_2)(x_0) \subseteq \bigcap_{x^* \in \hat{\partial}_{\varepsilon_2}\varphi_2(x_0)} [\hat{\partial}_{\varepsilon_1+\varepsilon_2}\varphi_1(x_0) - x^*]. \tag{2.5}$$

Proof. Take $u^* \in \hat{\partial}_{\varepsilon_1}(\varphi_1 - \varphi_2)(x_0)$ and $x^* \in \hat{\partial}_{\varepsilon_2}\varphi_2(x_0)$. Let $\gamma > 0$ and $\eta > 0$. Then, by [20, Proposition 1.84 (ii)], there exist neighborhoods U_1 and U_2 of x_0 such that

$$(\varphi_1 - \varphi_2)(x) - (\varphi_1 - \varphi_2)(x_0) - \langle u^*, x - x_0\rangle + (\varepsilon_1 + \gamma)\|x - x_0\| \geq 0 \quad \text{for each} \quad x \in U_1$$

and

$$\varphi_2(x) - \varphi_2(x_0) - \langle x^*, x - x_0\rangle + (\varepsilon_2 + \eta)\|x - x_0\| \geq 0 \text{ for all } x \in U_2.$$

Adding the above inequalities, we have that, for each $x \in U_1 \cap U_2$,

$$\varphi_1(x) - \varphi_1(x_0) - \langle x^* + u^*, x - x_0\rangle + (\varepsilon_1 + \varepsilon_2 + \gamma + \eta)\|x - x_0\| \geq 0.$$

Let $\xi = \eta + \gamma > 0$. Then

$$\varphi_1(x) - \varphi_1(x_0) - \langle x^* + u^*, x - x_0\rangle + (\varepsilon_1 + \varepsilon_2 + \xi)\|x - x_0\| \geq 0 \text{ for all } x \in U_1 \cap U_2,$$

which implies that $x^* + u^* \in \hat{\partial}_{\varepsilon_1+\varepsilon_2}\varphi_1(x_0)$, that is, $u^* \in \hat{\partial}_{\varepsilon_1+\varepsilon_2}\varphi_1(x_0) - x^*$. Since $x^* \in \hat{\partial}_{\varepsilon_2}\varphi_2(x_0)$ is arbitrarily, it follows that (2.5) holds. The proof is complete. □

Lemma 2.1. [14] *Let* $f: Y^{\bullet} \to \overline{\mathbb{R}}$ *and* $h: X \to Y^{\bullet}$ *be proper functions. Assume that* $h^{-1}(\mathrm{dom} f) \neq \emptyset$. *Then, for any* $x^* \in X^*$ *and* $\xi \in \mathrm{dom} f^*$, $(f \circ h)^*(x^*) \leq f^*(\xi) + (\xi h)^*(x^*)$.

Lemma 2.2. [5] *Let* $f, g: X \to \overline{\mathbb{R}}$ *be proper functions. Suppose that* g *is a lsc convex function. Then, for each* $p \in X^*$,

$$(f-g)^*(p) = \sup_{u^* \in \mathrm{dom} g^*} \{f^*(p+u^*) - g^*(u^*)\}.$$

Consequently,

$$\mathrm{epi}(f-g)^* = \bigcap_{u^* \in \mathrm{dom} g^*} \{\mathrm{epi} f^* - (u^*, g^*(u^*))\}.$$

Lemma 2.3. [18] *Let* $f: X \to \overline{\mathbb{R}}$ *be a proper convex function. If* $x \in \mathrm{dom} f$, *then*

$$\mathrm{epi} f^* = \bigcup_{\varepsilon \geq 0} \{(v, \langle v, x\rangle + \varepsilon - f(x)) | v \in \partial_{\varepsilon} f(x)\}.$$

3. Approximate Optimality Conditions

Throughout this paper, unless explicitly stated otherwise, we always assume that $C \subseteq X$ is a nonempty convex subset, T is an arbitrary (possibly infinite) index set, $f_1: Y^{\bullet} \to \overline{\mathbb{R}}$ is a proper convex K-increasing function, $f_2: X \to Y^{\bullet}$ is a proper K-convex function, $g_1: Z^{\bullet} \to \overline{\mathbb{R}}$ is a proper convex S-increasing function, $g_2: X \to Z^{\bullet}$ is a proper S-convex function and $h_t: X \to \overline{\mathbb{R}}$ is a proper convex function for each $t \in T$.

Let $A := \{x \in C : h_t(x) \leq 0, t \in T\}$ be the feasible set of problem (P). To avoid triviality, we always assume that $A \cap \mathrm{dom}(f_1 \circ f_2 - g_1 \circ g_2) \neq \emptyset$. To establish the approximate optimality condition for problem (P), we first introduce the following new constraint qualification. For simpleness, let $\varepsilon \geq 0$, $x \in A$, and $\partial(g_1 \circ g_2)(x) \neq \emptyset$ and denote

$$\Omega(x;\varepsilon) := \bigcap_{\substack{\beta \in \partial g_1(g_2(x)) \\ x^* \in \partial(\beta g_2)(x)}} \left(\bigcup_{\substack{\lambda \in \mathbb{R}_+^{(T)}, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_t \geq 0, \xi \in \partial_{\varepsilon_3} f_1(f_2(x)) \\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \sum_{t \in T} \lambda_t \varepsilon_t = \varepsilon + \sum_{t \in T} \lambda_t h_t(x)}} \left\{ \partial_{\varepsilon_1}(\xi f_2)(x) + N_{\varepsilon_2}(x;C) + \sum_{t \in T} \lambda_t \partial_{\varepsilon_t} h_t(x) - x^* \right\} \right).$$

Definition 3.1. Let $\varepsilon \geq 0$ and $x \in A \cap \mathrm{dom}(f_1 \circ f_2 - g_1 \circ g_2)$. It is said that $\{f_1, f_2, g_1, g_2, \delta_C; h_t : t \in T\}$ satisfies the approximate basic constraint qualification with Fréchet subdifferential ((*F-ABCQ*)$_\varepsilon$ in brief) at x if $\hat{\partial}_{\varepsilon}(f_1 \circ f_2 - g_1 \circ g_2 + \delta_A)(x) \subseteq \Omega(x;\varepsilon)$. Moreover, we say that the family $\{f_1, f_2, g_1, g_2, \delta_C; h_t : t \in T\}$ satisfies condition (*F-ABCQ*)$_\varepsilon$ if it satisfies condition (*F-ABCQ*)$_\varepsilon$ at each point $x \in A$.

Recall that the authors in [17] introduced the following generalized regularity condition (GRC)

$$\mathrm{epi}(f_1 \circ f_2 + \delta_A)^* = \bigcup_{\xi \in \mathrm{dom} f_1^*} \mathrm{epi}(\xi f_2)^* + (0, f_1^*(\xi)) + \mathrm{epi}\delta_C^* + \mathrm{cone}(\bigcup_{t \in T} \mathrm{epi} h_t^*)$$

and the Approximate Moreau-Rockafellar formula $(AMRF)$

$$\begin{aligned}&\partial_\varepsilon(f_1\circ f_2+\delta_A)(x)\\ =&\bigcup_{\substack{\lambda\in\mathbb{R}_+^{(T)},\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_t\geq 0,\xi\in\partial_{\varepsilon_3}f_1(f_2(x))\\ \varepsilon_1+\varepsilon_2+\varepsilon_3+\sum_{t\in T}\lambda_t\varepsilon_t=\varepsilon+\sum_{t\in T}\lambda_t h_t(x)}}\left\{\partial_{\varepsilon_1}(\xi f_2)(x)+N_{\varepsilon_2}(x;C)+\sum_{t\in T}\lambda_t\partial_{\varepsilon_t}h_t(x)\right\},\end{aligned}$$

where $\varepsilon\geq 0$ and $x\in A$. Let $\varepsilon\geq 0$. By [17, Proposition 3.3], we see that the condition $(AMRF)$ is weaker than the condition (GRC). Moreover, inspired by [7], we can introduce the following regularity condition

$$(CC)\qquad \mathrm{epi}(f_1\circ f_2-g_1\circ g_2+\delta_A)^*=\Lambda,$$

where

$$\begin{aligned}\Lambda:=\bigcap_{(\beta,x^*)\in\mathrm{dom}g_1^*\times\mathrm{dom}g_2^*}\Bigg\{&\bigcup_{\xi\in\mathrm{dom}f_1^*}\mathrm{epi}(\xi f_2)^*+(0,f_1^*(\xi))+\mathrm{epi}\delta_C^*+\mathrm{cone}\Big(\bigcup_{t\in T}\mathrm{epi}h_t^*\Big)\\ &-(x^*,g_1^*(\beta)+(\beta g_2)^*(x^*))\Bigg\}.\end{aligned}$$

The following Propositions 3.1 and 3.2 establish the relationships among the constraint qualifications (CC), (GRC) and $(F\text{-}ABCQ)_\varepsilon$. Since the proof of Proposition 3.1 is similar to that of [7, Lemma 3.3], we omit it here.

Proposition 3.1. *Let $f_1\circ f_2$ be a proper convex function and $g_1\circ g_2$ a proper lsc convex function. Suppose that*

$$(g_1\circ g_2)^*(x^*)=\min_{\beta\in\mathrm{dom}g_1^*}\{g_1^*(\beta)+(\beta g_2)^*(x^*)\}\quad \textit{for each}\quad x^*\in X^*. \tag{3.1}$$

Then, the following implication holds:

$$\textit{the condition } (GRC)\Rightarrow \textit{the condition } (CC).$$

Proposition 3.2. *The following implication holds:*

$$\textit{the condition } (CC)\Rightarrow \textit{the condition } (F\text{-}ABCQ)_\varepsilon.$$

Proof. Assume that condition (CC) holds. Let $x\in A\cap\mathrm{dom}(f_1\circ f_2-g_1\circ g_2)$ and $p\in\partial_\varepsilon(f_1\circ f_2-g_1\circ g_2+\delta_A)(x)$. By Lemma 2.3, we have

$$(p,\langle p,x\rangle+\varepsilon-(f_1\circ f_2-g_1\circ g_2+\delta_A)(x))\in\mathrm{epi}(f_1\circ f_2-g_1\circ g_2+\delta_A)^*.$$

Let $(\beta,x^*)\in\partial g_1(g_2(x))\times\partial(\beta g_2)(x)$. It follows that $(\beta,x^*)\in\mathrm{dom}g_1^*\times\mathrm{dom}g_2^*$. By the condition (CC), we obtain

$$\begin{aligned}&(p,\langle p,x\rangle+\varepsilon-(f_1\circ f_2-g_1\circ g_2+\delta_A)(x))\\ \in&\bigcup_{\xi\in\mathrm{dom}f_1^*}\mathrm{epi}(\xi f_2)^*+(0,f_1^*(\xi))+\mathrm{epi}\delta_C^*+\mathrm{cone}\Big(\bigcup_{t\in T}\mathrm{epi}h_t^*\Big)-(x^*,g_1^*(\beta)+(\beta g_2)^*(x^*)).\end{aligned}$$

This implies that there exist $\xi\in\mathrm{dom}f_1^*$ and $\lambda\in\mathbb{R}_+^{(T)}$ such that

$$\begin{aligned}&(p,\langle p,x\rangle+\varepsilon-(f_1\circ f_2-g_1\circ g_2+\delta_A)(x))\\ \in&\mathrm{epi}(\xi f_2)^*+(0,f_1^*(\xi))+\mathrm{epi}\delta_C^*+\sum_{t\in T}\lambda_t\mathrm{epi}h_t^*-(x^*,g_1^*(\beta)+(\beta g_2)^*(x^*)).\end{aligned}$$

Thus there exist $(x_1^*,r_1)\in \text{epi}(\xi f_2)^*$, $(x_2^*,r_2)\in \text{epi}\delta_C^*$, $(x_t^*,r_t)\in \text{epi}h_t^*$, $t\in T$, such that

$$p+x^*=x_1^*+x_2^*+\sum_{t\in T}\lambda_t x_t^*, \tag{3.2}$$

and

$$\langle p,x\rangle+\varepsilon-(f_1\circ f_2-g_1\circ g_2+\delta_A)(x)=r_1+r_2+\sum_{t\in T}\lambda_t r_t+f_1^*(\xi)-g_1^*(\beta)-(\beta g_2)^*(x^*). \tag{3.3}$$

While, by (2.4), we have $g_1(g_2(x))+g_1^*(\beta)=\langle\beta,g_2(x)\rangle$, and $(\beta g_2)(x)+(\beta g_2)^*(x^*)=\langle x,x^*\rangle$. Therefore, $g_1(g_2(x))-\langle x,x^*\rangle=-g_1^*(\beta)-(\beta g_2)^*(x^*)$, which together with (3.3) and the fact $\delta_A(x)=0$ implies that

$$\langle p+x^*,x\rangle+\varepsilon-f_1(f_2(x))=r_1+r_2+\sum_{t\in T}\lambda_t r_t+f_1^*(\xi). \tag{3.4}$$

Moreover, by Lemma 2.3, there exist $\varepsilon_1,\varepsilon_2,\varepsilon_t\geq 0, t\in T$, such that

$$x_1^*\in\partial_{\varepsilon_1}(\xi f_2)(x), r_1=\varepsilon_1+\langle x_1^*,x\rangle-(\xi f_2)(x),$$

$$x_2^*\in N_{\varepsilon_2}(x;C), r_2=\varepsilon_2+\langle x_2^*,x\rangle,$$

and

$$x_t^*\in\partial_{\varepsilon_t}h_t(x), r_t=\varepsilon_t+\langle x_t^*,x\rangle-h_t(x) \text{ for each } t\in T.$$

Combining this with (3.2) and (3.4), we arrive at

$$p+x^*\in\partial_{\varepsilon_1}(\xi f_2)(x)+N_{\varepsilon_2}(x;C)+\sum_{t\in T}\lambda_t\partial_{\varepsilon_t}h_t(x),$$

and

$$\varepsilon=\varepsilon_1+\varepsilon_2+\sum_{t\in T}\lambda_t\varepsilon_t-\sum_{t\in T}\lambda_t h_t(x)+f_1(f_2(x))+f_1^*(\xi)-(\xi f_2)(x). \tag{3.5}$$

Let $\varepsilon_3:=f_1(f_2(x))+f_1^*(\xi)-(\xi f_2)(x)$. Then, by the Young-Fenchel inequality (2.1), we have that $\varepsilon_3\geq 0$ and $\xi\in\partial_{\varepsilon_3}f_1(f_2(x))$. Moreover, by (3.5), $\varepsilon_1+\varepsilon_2+\varepsilon_3+\sum_{t\in T}\lambda_t\varepsilon_t=\varepsilon+\sum_{t\in T}\lambda_t h_t(x)$. Therefore, $p\in\Omega(x;\varepsilon)$. Consequently, the condition $(F\text{-}ABCQ)_\varepsilon$ holds. The proof is complete. □

To characterize the approximate optimal solution to problem (P), we introduce the following definition.

Definition 3.2. Let $\alpha,\varepsilon\geq 0$. A point $x_0\in A$ is said to be a quasi (α,ε)-optimal solution of (P) if $(f_1\circ f_2)(x_0)-(g_1\circ g_2)(x_0)\leq(f_1\circ f_2)(x)-(g_1\circ g_2)(x)+\alpha\|x-x_0\|+\varepsilon$, $\forall x\in A$.

Theorem 3.1. *Let $\alpha,\varepsilon\geq 0$ and $x_0\in A\cap\text{dom}(f_1\circ f_2-g_1\circ g_2)$.*

(i) *Assume that $\{f_1,f_2,g_1,g_2,\delta_C;h_t:t\in T\}$ satisfies the condition $(F\text{-}ABCQ)_\varepsilon$ at x_0. If x_0 is a quasi (α,ε)-optimal solution to (P), then, for each $\beta\in\partial g_1(g_2(x_0)), x^*\in\partial(\beta g_2)(x_0)$, there exist $\bar\lambda\in\mathbb{R}_+^{(T)},\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_b,\varepsilon_t\geq 0,\bar\xi\in Y^*,u,w,v_t\in X^*,t\in T$ and $b\in\mathbb{B}^*$ such that*

(a) $0\leq(\bar\xi f_2)^*(u)+(\bar\xi f_2)(x_0)-\langle u,x_0\rangle\leq\varepsilon_1$;

(b) $0\leq\delta_C^*(w)+\delta_C(x_0)-\langle w,x_0\rangle\leq\varepsilon_2$;

(c) $0\leq f_1^*(\bar\xi)+f_1(f_2(x_0))-\langle\bar\xi,f_2(x_0)\rangle\leq\varepsilon_3$;

(d) $0\leq h_t^*(v_t)+h_t(x_0)-\langle v_t,x_0\rangle\leq\varepsilon_t, t\in T$;

(e) $\varepsilon_1+\varepsilon_2+\varepsilon_3+\sum_{t\in T}\bar\lambda_t\varepsilon_t+\varepsilon_b=\varepsilon+\sum_{t\in T}\bar\lambda_t h_t(x_0)$;

(f) $x^*=u+w+\sum_{t\in T}\bar\lambda_t v_t+\alpha b$.

(ii) *Assume that $g_1 \circ g_2$ is lsc and the equation (3.1) holds. If, for each $(\beta, x^*) \in \mathrm{dom} g_1^* \times X^*$, there exist $\bar{\lambda} \in \mathbb{R}_+^{(T)}, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_b, \varepsilon_t \geq 0, \bar{\xi} \in Y^*, u, w, v_t \in X^*, t \in T$ and $b \in \mathbb{B}^*$ such that assertions (a)-(f) hold at x_0, then x_0 is a quasi (α, ε)-optimal solution of (P).*

Proof. (i) Suppose that x_0 is a quasi (α, ε)-optimal solution of the problem (P). Then, by the definition of ε-subdifferential, we have $0 \in \hat{\partial}_\varepsilon (f_1 \circ f_2 - g_1 \circ g_2 + \alpha \| \cdot - x_0 \| + \delta_A)(x_0)$. This together with the condition $(F\text{-}ABCQ)_\varepsilon$ at x_0 implies that, for each $\beta \in \partial g_1(g_2(x_0)), x^* \in \partial(\beta g_2)(x_0)$, there exist $\bar{\lambda} \in \mathbb{R}_+^{(T)}$, $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_b, \varepsilon_t \geq 0, t \in T$ and $\bar{\xi} \in \partial_{\varepsilon_3} f_1(f_2(x_0))$ such that

$$x^* \in \partial_{\varepsilon_1}(\bar{\xi} f_2)(x_0) + N_{\varepsilon_2}(x_0; C) + \sum_{t \in T} \bar{\lambda}_t \partial_{\varepsilon_t} h_t(x_0) + \partial_{\varepsilon_b}(\alpha \| \cdot - x_0 \|)(x_0), \tag{3.6}$$

and

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \sum_{t \in T} \bar{\lambda}_t \varepsilon_t + \varepsilon_b = \varepsilon + \sum_{t \in T} \bar{\lambda}_t h_t(x_0).$$

Then (e) holds. Note that $\partial_{\varepsilon_b}(\alpha \| \cdot - x_0 \|)(x_0) = \alpha \mathbb{B}^*$. This together with (3.6) implies that

$$x^* \in \partial_{\varepsilon_1}(\bar{\xi} f_2)(x_0) + N_{\varepsilon_2}(x_0; C) + \sum_{t \in T} \bar{\lambda}_t \partial_{\varepsilon_t} h_t(x_0) + \alpha \mathbb{B}^*. \tag{3.7}$$

Moreover, since $\bar{\xi} \in \partial_{\varepsilon_3} f_1(f_2(x_0))$, it follows from (2.1) and (2.3) that (a) holds. By (3.7), there exist $u \in \partial_{\varepsilon_1}(\bar{\xi} f_2)(x_0), w \in N_{\varepsilon_2}(x_0; C), v_t \in \partial_{\varepsilon_t} h_t(x_0), t \in T$ and $b \in \mathbb{B}^*$ such that $x^* = u + w + \sum_{t \in T} \bar{\lambda}_t v_t + \alpha b$, that is, (f) holds. Again by (2.1) and (2.3), we can conclude that assertions (b)-(d) hold.

(ii) Suppose that, for each $(\beta, x^*) \in \mathrm{dom} g_1^* \times X^*$, there exist $\bar{\lambda} \in \mathbb{R}_+^{(T)}, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_b, \varepsilon_t \geq 0, \bar{\xi} \in Y^*, u, w, v_t \in X^*, t \in T$ and $b \in \mathbb{B}^*$ such that (a)-(f) hold. Then by (2.3), we have that $u \in \partial_{\varepsilon_1}(\bar{\xi} f_2)(x_0)$, $w \in \partial_{\varepsilon_2} \delta_C(x_0)$, $\bar{\xi} \in \partial_{\varepsilon_3} f_1(f_2(x_0))$ and $v_t \in \partial_{\varepsilon_t} h_t(x_0)$ for each $t \in T$. Moreover, by the definition of ε-subdifferential, it follows that, for each $x \in X$,

$$(\bar{\xi} f_2)(x) - (\bar{\xi} f_2)(x_0) \geq \langle u, x - x_0 \rangle - \varepsilon_1,$$

$$\delta_C(x) - \delta_C(x_0) \geq \langle w, x - x_0 \rangle - \varepsilon_2,$$

$$f_1(f_2(x)) - f_1(f_2(x_0)) \geq \langle \bar{\xi}, f_2(x) - f_2(x_0) \rangle - \varepsilon_3,$$

and

$$\sum_{t \in T} \bar{\lambda}_t h_t(x) - \sum_{t \in T} \bar{\lambda}_t h_t(x_0) \geq \sum_{t \in T} \bar{\lambda}_t \langle v_t, x - x_0 \rangle - \sum_{t \in T} \bar{\lambda}_t \varepsilon_t.$$

Thus, by assertions (e) and (f) , we see that, for each $x \in A$,

$$\begin{aligned}
& f_1(f_2(x)) - f_1(f_2(x_0)) \\
\geq & \langle u + w + \sum_{t \in T} \bar{\lambda}_t v_t, x - x_0 \rangle - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \sum_{t \in T} \bar{\lambda}_t \varepsilon_t - \sum_{t \in T} \bar{\lambda}_t (h_t(x) - h_t(x_0)) \\
= & \langle x^* - \alpha b, x - x_0 \rangle + \varepsilon_b - \varepsilon - \sum_{t \in T} \bar{\lambda}_t h_t(x) \\
\geq & \langle x^*, x - x_0 \rangle - \alpha \| x - x_0 \| + \varepsilon_b - \varepsilon - \sum_{t \in T} \bar{\lambda}_t h_t(x) \\
\geq & \langle x^*, x - x_0 \rangle - \alpha \| x - x_0 \| - \varepsilon,
\end{aligned}$$

where the last inequality holds by $\varepsilon_b \geq 0$ and $\bar{\lambda}_t h_t(x) \leq 0$ for each $t \in T$. This implies that, for each $(\beta, x^*) \in \mathrm{dom} g_1^* \times X^*$ and $x \in A$,

$$\begin{aligned}&(f_1 \circ f_2)(x_0) - \langle x^*, x_0\rangle + g_1^*(\beta) + (\beta g_2)^*(x^*)\\ \leq&(f_1 \circ f_2)(x) - \langle x^*, x\rangle + g_1^*(\beta) + (\beta g_2)^*(x^*) + \alpha\|x - x_0\| + \varepsilon.\end{aligned}$$

Since $(\beta, x^*) \in \mathrm{dom} g_1^* \times X^*$ is arbitrarily, we have that, for each $x \in A$,

$$\begin{aligned}&\inf_{(\beta,x^*)\in\mathrm{dom} g_1^*\times X^*}\left\{(f_1 \circ f_2)(x_0) - \langle x^*, x_0\rangle + g_1^*(\beta) + (\beta g_2)^*(x^*)\right\}\\ \leq&\inf_{(\beta,x^*)\in\mathrm{dom} g_1^*\times X^*}\left\{(f_1 \circ f_2)(x) - \langle x^*, x\rangle + g_1^*(\beta) + (\beta g_2)^*(x^*) + \alpha\|x - x_0\| + \varepsilon\right\}.\end{aligned}$$

This implies from (3.1) that

$$\begin{aligned}&\inf_{x^*\in X^*}\left\{(f_1 \circ f_2)(x_0) - \langle x^*, x_0\rangle + (g_1 \circ g_2)^*(x^*)\right\}\\ \leq&\inf_{x^*\in X^*}\left\{(f_1 \circ f_2)(x) - \langle x^*, x\rangle + (g_1 \circ g_2)^*(x^*) + \alpha\|x - x_0\| + \varepsilon\right\}.\end{aligned}$$

Again by definition of the conjugate function and the fact that $g_1 \circ g_2$ is a lsc function, we see that

$$(f_1 \circ f_2)(x_0) - (g_1 \circ g_2)(x_0) \leq (f_1 \circ f_2)(x) - (g_1 \circ g_2)(x) + \alpha\|x - x_0\| + \varepsilon \quad \text{for each} \quad x \in A.$$

Therefore, x_0 is a quasi (α, ε)-optimal solution to problem (P). The proof is complete. □

In the case that $g_1 = g_2 = 0$, optimization problem (P) becomes

$$(\mathscr{P}) \qquad \begin{aligned}&\inf\ (f_1 \circ f_2)(x)\\ &\text{s.t. } h_t(x) \leq 0, t \in T,\\ &\quad\ x \in C,\end{aligned}$$

and condition $(F\text{-}ABCQ)_\varepsilon$ for $\{f_1, f_2, g_1, g_2, \delta_C; h_t : t \in T\}$ reduces to condition $(AMRF)$ for the family $\{f_1, f_2, \delta_C; h_t : t \in T\}$. Therefore, it follows from Theorem 3.1 that we have the following corollary directly, which was given in [17, Theorem 3.6].

Corollary 3.1. *Let $\alpha, \varepsilon \geq 0$ and $x_0 \in A \cap f_2^{-1}(\mathrm{dom} f_1)$. Assume that $\{f_1, f_2, \delta_C; h_t : t \in T\}$ satisfies the condition $(AMRF)$ at x_0. Then x_0 is a quasi (α, ε)-optimal solution of $(\mathscr{P})$ if and only if there exist $\bar{\lambda} \in \mathbb{R}_+^{(T)}, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_b, \varepsilon_t \geq 0, \bar{\xi} \in Y^*, u, w, v_t \in X^*, t \in T$ and $b \in \mathbb{B}^*$ such that the assertions (a)-(e) in Theorem 3.1 and the following assertion hold:*
(f_1) $u + w + \sum_{t\in T} \bar{\lambda}_t v_t + \alpha b = 0.$

In the case that $X = Y = Z$ and $f_2 = g_2 = \mathrm{Id}_X$, optimization problem (P) turns into

$$(\mathbb{P}) \qquad \begin{aligned}&\inf\ f_1(x) - g_1(x)\\ &\text{s.t. } h_t(x) \leq 0, t \in T,\\ &\quad\ x \in C.\end{aligned}$$

Note that, for each $\xi \in \mathrm{dom} f_1^*, \beta \in \mathrm{dom} g_1^*$, $(\xi f_2)^* = \delta_{\{\xi\}}$ and $(\beta g_2)^* = \delta_{\{\beta\}}$. Consequently,

$$\bigcup_{\xi\in\mathrm{epi} f_1^*}(\mathrm{epi}(\xi f_2)^* + (0, f_1^*(\xi))) = \mathrm{epi} f_1^*.$$

Moreover, let $\varepsilon \geq 0$, $x \in A \cap \operatorname{dom}(f_1 - g_1)$. Then, for any $\beta \in \partial g_1(g_2(x)) = \partial g_1(x), \xi \in \partial_\varepsilon f_1(f_2(x)) = \partial_\varepsilon f_1(x)$, one has that $\partial(\beta g_2)(x) = \{\beta\}$ and $\partial_\varepsilon(\xi f_2)(x) = \{\xi\}$. Hence, condition (CC) becomes

$$\overline{(CC)} \qquad \operatorname{epi}(f_1 - g_1 + \delta_A)^* = \bigcap_{\beta \in \operatorname{dom} g_1^*} \left(\operatorname{epi} f_1^* + \operatorname{epi} \delta_C^* + \operatorname{cone}\Big(\bigcup_{t\in T} \operatorname{epi} h_t^*\Big) - (\beta, g_1^*(\beta)) \right),$$

and condition$(F\text{-}ABCQ)_\varepsilon$ becomes $\overline{(F\text{-}ABCQ)_\varepsilon}$:

$$\partial_\varepsilon(f_1 - g_1 + \delta_A)(x) \subseteq \bigcap_{\beta \in \partial g_1(x)} \left(\bigcup_{\substack{\lambda \in \mathbb{R}_+^{(T)} \\ \varepsilon_1, \varepsilon_2, \varepsilon_t \geq 0 \\ \varepsilon_1 + \varepsilon_2 + \sum_{t\in T} \lambda_t \varepsilon_t = \varepsilon + \sum_{t\in T} \lambda_t h_t(x)}} \left\{ \partial_{\varepsilon_1} f(x) + N_{\varepsilon_2}(x; C) + \sum_{t\in T} \lambda_t \partial_{\varepsilon_t} h_t(x) - \beta \right\} \right) \quad \text{for each} \quad x \in A.$$

Therefore, by Theorem 3.1, we have the following corollary straightforwardly.

Corollary 3.2. *Let $\varepsilon \geq 0$ and $x_0 \in A \cap \operatorname{dom}(f_1 - g_1)$. Assume that $\{f_1, g_1, \delta_C; h_t : t \in T\}$ satisfies the condition $\overline{(F\text{-}ABCQ)_\varepsilon}$ at x_0. If x_0 is an ε-optimal solution to $(\mathbb{P})$, then, for each $\beta \in \partial g_1(x_0)$, there exist $\bar{\lambda} \in \mathbb{R}_+^{(T)}, \varepsilon_1, \varepsilon_2, \varepsilon_t \geq 0, t \in T$, such that*

$$\beta \in \partial_{\varepsilon_1} f_1(x_0) + N_{\varepsilon_2}(x_0; C) + \sum_{t\in T} \bar{\lambda}_t \partial_{\varepsilon_t} h_t(x_0),$$

and

$$\varepsilon_1 + \varepsilon_2 + \sum_{t\in T} \bar{\lambda}_t \varepsilon_t + \varepsilon_b = \varepsilon + \sum_{t\in T} \bar{\lambda}_t h_t(x_0).$$

Remark 3.1. Recall that the authors in [21, Theorem 3.3] obtained the similar result via condition $\overline{(CC)}$. While, by Proposition 3.2, we see that condition $\overline{(CC)}$ is stronger than condition $\overline{(F\text{-}ABCQ)_\varepsilon}$. Thus, Corollary 3.2 improves the result in [21, Theorem 3.3].

4. Mixed Type Approximate Duality Theorems

This section is devoted to the mixed type approximate duality for problem (P). In the case that g_1 is a S-increasing lsc function and g_2 is a star S-epi-closed function, the standard convexification technique can be applied. In fact, in this case, problem (P) can be reformulated as the following one:

$$(P) \qquad \inf_{\sigma \in H^*} \inf_{x\in A} \{f_1(f_2(x)) - \langle x^*, x\rangle + g_1^*(\beta) + (\beta g_2)^*(x^*)\},$$

where $\sigma := (\beta, x^*)$ and $H^* := \operatorname{dom} g_1^* \times X^*$. Now, we defined the Lagrange function $L_\sigma : C \times \mathbb{R}_+^{(T)} \times \operatorname{dom} f_1^* \to \overline{\mathbb{R}}$ by

$$L_\sigma(y, \lambda, \xi) = -f_1^*(\xi) + (\xi f_2)(y) - \langle x^*, y\rangle + g_1^*(\beta) + (\beta g_2)^*(x^*) + \sum_{t\in T} \lambda_t h_t(y)$$

for each $(y, \lambda, \xi) \in C \times \mathbb{R}_+^{(T)} \times \operatorname{dom} f_1^*$. Then, mixed type dual problem (D) for (P) can be defined as follows:

$$(D) \qquad \inf_{\sigma \in H^*} \max_{(y,\lambda,\mu,\xi)\in F_\sigma} L_\sigma(y, \lambda, \xi),$$

where

$$F_\sigma = \Big\{ (y,\lambda,\mu,\xi) \in C \times \mathbb{R}_+^{(T)} \times \mathbb{R}_+^{(T)} \times \operatorname{dom} f_1^* : x^* \in \partial_{\varepsilon_1}(\xi f_2)(y) + N_{\varepsilon_2}(y;C) + \sum_{t\in T}(\lambda_t+\mu_t)\partial_{\varepsilon_t} h_t(y) + \alpha\mathbb{B}^*, \mu_t h_t(y) \geq 0, , t\in T, \xi \in \partial_{\varepsilon_3} f_1(f_2(y)), \varepsilon_1+\varepsilon_2+\varepsilon_3+\sum_{t\in T}(\lambda_t+\mu_t)\varepsilon_t + \varepsilon_b \leq \varepsilon \Big\}.$$

Moreover, for each $\sigma \in H^*$, the subproblem of (D) is defined by

$$(D_\sigma) \qquad \max_{(y,\lambda,\mu,\xi)\in F_\sigma} L_\sigma(y,\lambda,\xi).$$

Definition 4.1. Let $\alpha,\varepsilon \geq 0$ and $\sigma \in H^*$. $(y_0,\bar{\lambda},\bar{\mu},\bar{\xi}) \in F_\sigma$ is said to be a quasi (α,ε)-optimal solution to (D_σ) if, for any $(y,\lambda,\mu,\xi) \in F_\sigma$, $L_\sigma(y_0,\bar{\lambda},\bar{\xi}) \geq L_\sigma(y,\lambda,\xi) - \alpha\|y_0-y\| - \varepsilon$.

Theorem 4.1. *Let $\alpha,\varepsilon \geq 0$ and $x_0 \in A\cap \operatorname{dom}(f_1\circ f_2 - g_1\circ g_2)$ be a quasi (α,ε)-optimal solution to (P). If, for each $\sigma \in H^*$, there exist $\bar{\lambda} \in \mathbb{R}_+^{(T)}$, $\bar{\xi} \in \operatorname{dom} f_1^*, \varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_b,\varepsilon_t \geq 0, u,w,x^*,v_t \in X^*, t\in T$ and $b \in \mathbb{B}^*$ such that $\bar{\lambda}_t h_t(x_0) = 0$ and assertions (a)-(f) in Theorem 3.1 hold, then $(x_0,\bar{\lambda},0,\bar{\xi})$ and $(x_0,0,\bar{\lambda},\bar{\xi})$ are quasi $(\alpha,2\varepsilon)$-optimal solutions to (D_σ).*

Proof. Let $\sigma \in H^*$, $\bar{\lambda} \in \mathbb{R}_+^{(T)}$, $\bar{\xi} \in \operatorname{dom} f_1^*, \varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_b,\varepsilon_t \geq 0, u,w,x^*,v_t \in X^*, t\in T$, $b \in \mathbb{B}^*$ be such that $\bar{\lambda}_t h_t(x_0) = 0$ and (a)-(f) in Theorem 3.1 hold. Obviously, $(x_0,\bar{\lambda},0,\bar{\xi})$ and $(x_0,0,\bar{\lambda},\bar{\xi})$ are feasible solutions of (D_σ). Then, by (2.1), (2.2) and Lemma 2.1, we have

$$\begin{aligned} L_\sigma(x_0,\bar{\lambda},\bar{\xi}) &= -f_1^*(\bar{\xi}) + (\bar{\xi} f_2)(x_0) - \langle x^*, x_0\rangle + g_1^*(\beta) + (\beta g_2)^*(x^*) \\ &\geq f_1(f_2(x_0)) - \varepsilon_3 - \langle x^*, x_0\rangle + g_1^*(\beta) + (\beta g_2)^*(x^*) \\ &\geq f_1(f_2(x_0)) - \langle x^*, x_0\rangle + g_1^*(\beta) + (\beta g_2)^*(x^*) - \varepsilon. \end{aligned} \tag{4.1}$$

Take $(y,\lambda,\mu,\xi) \in F_\sigma$. Then there exist $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_b,\varepsilon_t \geq 0, t\in T$ such that $\xi \in \partial_{\varepsilon_3} f_1(f_2(y))$,

$$x^* \in \partial_{\varepsilon_1}(\xi f_2)(y) + N_{\varepsilon_2}(y;C) + \sum_{t\in T}(\lambda_t+\mu_t)\partial_{\varepsilon_t} h_t(y) + \alpha\mathbb{B}^*,$$

$$\lambda_t \geq 0, \mu_t \geq 0, \mu_t h_t(y) \geq 0, t\in T,$$

and

$$\varepsilon_1+\varepsilon_2+\varepsilon_3+\sum_{t\in T}(\lambda_t+\mu_t)\varepsilon_t + \varepsilon_b \leq \varepsilon. \tag{4.2}$$

Therefore, there exist $u \in \partial_{\varepsilon_1}(\xi f_2)(y), w \in N_{\varepsilon_2}(y;C), v_t \in \partial_{\varepsilon_t} h_t(y)$ and $b \in \mathbb{B}^*$ such that

$$x^* = u + w + \sum_{t\in T}(\lambda_t+\mu_t)v_t + \alpha b. \tag{4.3}$$

By the definition of the ε-subdifferential, we see that

$$(\xi f_2)(x_0) - (\xi f_2)(y) \geq \langle u, x_0 - y\rangle - \varepsilon_1, \tag{4.4}$$

$$\delta_C(x_0) - \delta_C(y) \geq \langle w, x_0 - y\rangle - \varepsilon_2, \tag{4.5}$$

$$f_1(f_2(x_0)) - f_1(f_2(y)) \geq \langle \xi, f_2(x_0) - f_2(y)\rangle - \varepsilon_3, \tag{4.6}$$

and

$$\sum_{t\in T}(\lambda_t+\mu_t)h_t(x_0) - \sum_{t\in T}(\lambda_t+\mu_t)h_t(y) \geq \sum_{t\in T}(\lambda_t+\mu_t)\langle v_t, x_0-y\rangle - \sum_{t\in T}(\lambda_t+\mu_t)\varepsilon_t. \tag{4.7}$$

Summing up (4.4)-(4.7) and combining with (4.2)-(4.3), we obtain that, for each $x \in A$,

$$\begin{aligned} & f_1(f_2(x_0)) - f_1(f_2(y)) \\ & \geq \langle u + \sum_{t\in T}(\lambda_t+\mu_t)v_t + w, x_0 - y\rangle - \sum_{t\in T}(\lambda_t+\mu_t)(h_t(x_0) \\ & \quad - h_t(y)) - \sum_{t\in T}(\lambda_t+\mu_t)\varepsilon_t - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 \\ & \geq \langle x^* - \alpha b, x_0 - y\rangle + \sum_{t\in T}\lambda_t h_t(y) - \varepsilon + \varepsilon_b \\ & \geq \langle x^*, x_0 - y\rangle + \sum_{t\in T}\lambda_t h_t(y) - \alpha\|x_0 - y\| - \varepsilon, \end{aligned}$$

where the second inequality holds by (4.2), $\sum_{t\in T}(\lambda_t+\mu_t)h_t(x_0) \leq 0$, and $\mu_t h_t(y) \geq 0$ for each $t \in T$. Hence, by (2.1), we have that

$$\begin{aligned} & f_1(f_2(x_0)) - \langle x^*, x_0\rangle + g_1^*(\beta) + (\beta g_2)^*(x^*) \\ \geq & f_1(f_2(y)) - \langle x^*, y\rangle + g_1^*(\beta) + (\beta g_2)^*(x^*) + \sum_{t\in T}\lambda_t h_t(y) - \alpha\|x_0 - y\| - \varepsilon \\ \geq & - f_1^*(\xi) + (\xi f_2)(y) - \langle x^*, y\rangle + g_1^*(\beta) + (\beta g_2)^*(x^*) + \sum_{t\in T}\lambda_t h_t(y) - \alpha\|x_0 - y\| - \varepsilon, \end{aligned}$$

that is,

$$-L_\sigma(y,\lambda,\xi) \geq -f_1(f_2(x_0)) + \langle x^*, x_0\rangle - g_1^*(\beta) - (\beta g_2)^*(x^*) - \alpha\|x_0 - y\| - \varepsilon.$$

This together with (4.1) implies that

$$L_\sigma(x_0,\bar\lambda,\bar\xi) - L_\sigma(y,\lambda,\xi) \geq -\alpha\|x_0 - y\| - 2\varepsilon.$$

Note that $(y,\lambda,\mu,\xi) \in F_\sigma$ is arbitrary. It follows that $(x_0,\bar\lambda,0,\bar\xi)$ and $(x_0,0,\bar\lambda,\bar\xi)$ are quasi $(\alpha,2\varepsilon)$-optimal solutions to (D_σ). The proof is complete. □

Theorem 4.2. *Let $\alpha,\varepsilon \geq 0$. Suppose that $g_1 \circ g_2$ is lsc and equation (3.1) holds. If, for each $\sigma \in H^*$, there exist $x_0 \in A \cap \operatorname{dom}(f_1 \circ f_2 - g_1 \circ g_2)$, $\bar\lambda \in \mathbb{R}_+^{(T)}$, $\bar\xi \in \operatorname{dom} f_1^*$ such that $\bar\lambda_t h_t(x_0) = 0$ and $(x_0,\bar\lambda,0,\bar\xi) \in F_\sigma$ or $(x_0,0,\bar\lambda,\bar\xi) \in F_\sigma$, then x_0 is a quasi (α,ε)-optimal solution to problem (P).*

Proof. Take $\sigma \in H^*$. Let $x_0 \in A \cap \operatorname{dom}(f_1 \circ f_2 - g_1 \circ g_2)$, $\bar\lambda \in \mathbb{R}_+^{(T)}$, $\bar\xi \in \operatorname{dom} f_1^*$ be such that $\bar\lambda_t h_t(x_0) = 0$ and $(x_0,\bar\lambda,0,\bar\xi)$ or $(x_0,0,\bar\lambda,\bar\xi) \in F_\sigma$. Then $\bar\xi \in \partial_{\varepsilon_3} f_1(f_2(x_0))$, and there exist $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_b,\varepsilon_t \geq 0, u \in \partial_{\varepsilon_1}(\xi f_2)(x_0), w \in N_{\varepsilon_2}(x_0;C), v_t \in \partial_{\varepsilon_t} h_t(x_0), b \in \mathbb{B}^*, t \in T$ such that

$$x^* = u + w + \sum_{t\in T}\bar\lambda_t v_t + \alpha b, \tag{4.8}$$

and

$$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \sum_{t\in T}\bar\lambda_t\varepsilon_t + \varepsilon_b \leq \varepsilon. \tag{4.9}$$

If, on the contrary, x_0 is not a quasi (α,ε)-optimal solution to problem (P), then there exists $\bar x \in A$ such that

$$(f_1 \circ f_2)(\bar x) - (g_1 \circ g_2)(\bar x) + \alpha\|\bar x - x_0\| + \varepsilon < (f_1 \circ f_2)(x_0) - (g_1 \circ g_2)(x_0). \tag{4.10}$$

While, by the definition of ε-subdifferential, we have

$$(\bar\xi f_2)(\bar x) - (\bar\xi f_2)(x_0) \geq \langle u, \bar x - x_0\rangle - \varepsilon_1,$$

$$\delta_C(\bar{x})-\delta_C(x_0)\geq\langle w,\bar{x}-x_0\rangle-\varepsilon_2,$$
$$f_1(f_2(\bar{x}))-f_1(f_2(x_0))\geq\langle\bar{\xi},f_2(\bar{x})-f_2(x_0)\rangle-\varepsilon_3,$$

and

$$\sum_{t\in T}\bar{\lambda}_t h_t(\bar{x})-\sum_{t\in T}\bar{\lambda}_t h_t(x_0)\geq\sum_{t\in T}\bar{\lambda}_t\langle v_t,\bar{x}-x_0\rangle-\sum_{t\in T}\bar{\lambda}_t\varepsilon_t.$$

Adding the above inequalities and combining with (4.8)-(4.9), we arrive at

$$\begin{aligned}&f_1(f_2(\bar{x}))-f_1(f_2(x_0))\\&\geq\langle u+w+\sum_{t\in T}\bar{\lambda}_t v_t,\bar{x}-x_0\rangle-\sum_{t\in T}\bar{\lambda}_t(h_t(\bar{x})-h_t(x_0))-\varepsilon_1-\varepsilon_2-\varepsilon_3-\sum_{t\in T}\bar{\lambda}_t\varepsilon_t\\&\geq\langle x^*-\alpha b,\bar{x}-x_0\rangle+\varepsilon_b-\varepsilon\\&\geq\langle x^*,\bar{x}-x_0\rangle-\alpha\|\bar{x}-x_0\|-\varepsilon,\end{aligned}$$

where the second inequality holds by (4.9), $\bar{\lambda}_t h_t(\bar{x})\leq 0$ and $\bar{\lambda}_t h_t(x_0)=0$ for each $t\in T$. This means that

$$\begin{aligned}&(f_1\circ f_2)(x_0)-\langle x^*,x_0\rangle+g_1^*(\beta)+(\beta g_2)^*(x^*)\\&\leq(f_1\circ f_2)(\bar{x})-\langle x^*,\bar{x}\rangle+g_1^*(\beta)+(\beta g_2)^*(x^*)+\alpha\|\bar{x}-x_0\|+\varepsilon.\end{aligned}$$

Note that the above inequality holds for each $\sigma\in H^*$, it follows that

$$\begin{aligned}&\inf_{\sigma\in H^*}\{(f_1\circ f_2)(x_0)-\langle x^*,x_0\rangle+g_1^*(\beta)+(\beta g_2)^*(x^*)\}\\&\leq\inf_{\sigma\in H^*}\{(f_1\circ f_2)(x)-\langle x^*,x\rangle+g_1^*(\beta)+(\beta g_2)^*(x^*)+\alpha\|x-x_0\|+\varepsilon\}.\end{aligned}$$

Then, by (3.1) and the assumption that $g_1\circ g_2$ is a lsc function, we have

$$(f_1\circ f_2)(x_0)-(g_1\circ g_2)(x_0)\leq(f_1\circ f_2)(\bar{x})-(g_1\circ g_2)(\bar{x})+\alpha\|\bar{x}-x_0\|+\varepsilon,$$

which contradicts (4.10). Therefore, x_0 is a quasi (α,ε)-optimal solution to problem (P). The proof is complete. □

In the case that $g_1=g_2=0$, the corresponding Lagrange function and mixed type dual problem of problem (P) can be expressed respectively as

$$\mathscr{L}(y,\lambda,\xi):=-f_1^*(\xi)+(\xi f_2)(y)+\sum_{t\in T}\lambda_t h_t(y)$$

for each $(y,\lambda,\xi)\in C\times\mathbb{R}_+^{(T)}\times\operatorname{dom}f_1^*$ and

$$(\mathscr{D})\qquad\max_{(y,\lambda,\mu,\xi)\in\mathscr{F}}\mathscr{L}(y,\lambda,\xi),$$

where

$$\begin{aligned}\mathscr{F}=\Big\{&(y,\lambda,\mu,\xi)\in C\times\mathbb{R}_+^{(T)}\times\mathbb{R}_+^{(T)}\times\operatorname{dom}f_1^*:0\in\partial_{\varepsilon_1}(\xi f_2)(y)+N_{\varepsilon_2}(y;C)\\&+\sum_{t\in T}(\lambda_t+\mu_t)\partial_{\varepsilon_t}h_t(y)+\alpha\mathbb{B}^*,\xi\in\partial_{\varepsilon_3}f_1(f_2(y)),\mu_t h_t(y)\geq 0,t\in T,\\&\varepsilon_1+\varepsilon_2+\varepsilon_3+\sum_{t\in T}(\lambda_t+\mu_t)\varepsilon_t+\varepsilon_b\leq\varepsilon\Big\}.\end{aligned}$$

Then, by Theorems 4.1 and 4.2, we have the following corollaries straightforwardly, which were given in [17, Theorems 4.3-4.4].

Corollary 4.1. *Let $\alpha,\varepsilon \geq 0$ and $x_0 \in A\cap f_2^{-1}(\mathrm{dom} f_1)$ be a quasi (α,ε)-optimal solution to $(\mathscr{P})$. Suppose that there exist $\bar{\lambda} \in \mathbb{R}_+^{(T)}$, $\bar{\xi} \in \mathrm{dom} f_1^*, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_b, \varepsilon_t \geq 0, u, w, v_t \in X^*, t \in T$ and $b \in \mathbb{B}^*$ such that $\bar{\lambda}_t h_t(x_0) = 0$ and assertions (a)-(e) in Theorem 3.1 and (f_1) in Corollary 3.1 hold. Then $(x_0, \bar{\lambda}, 0, \bar{\xi})$ and $(x_0, 0, \bar{\lambda}, \bar{\xi})$ are quasi $(\alpha, 2\varepsilon)$-optimal solutions to $(\mathscr{D})$.*

Corollary 4.2. *Suppose that $(x_0, \bar{\lambda}, 0, \bar{\xi})$ or $(x_0, 0, \bar{\lambda}, \bar{\xi}) \in \mathscr{F}$ satisfies $\bar{\lambda}_t h_t(x_0) = 0$. If $x_0 \in A\cap f_2^{-1}(\mathrm{dom} f_1)$, then x_0 is a quasi (α,ε)-optimal solution to problem $(\mathscr{P})$.*

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