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STRONG CONVERGENT INERTIAL TSENG'S EXTRAGRADIENT METHOD FOR SOLVING NON-LIPSCHITZ QUASIMONOTONE VARIATIONAL INEQUALITIES IN BANACH SPACES

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Abstract. The class of quasimonotone variational inequalities is more general and applicable than the class of pseudomonotone and monotone variational inequalities. However, few results can be found in the literature on quasimonotone variational inequalities and currently results are mostly on weak convergent methods in the framework of Hilbert spaces. In this paper, we study the class of non-Lipschitz quasimonotone variational inequalities and the class of non-Lipschitz variational inequalities without monotonicity in the framework of Banach spaces. We propose a new inertial Tseng's extragradient method and obtain some strong convergence results for the proposed algorithm under some mild conditions on the control parameters. While the cost operator is non-Lipschitz, our proposed method does not require any linesearch procedure but employs a more efficient and simple self-adaptive step sizes with known parameters. Finally, we present several numerical experiments to demonstrate the implementability of our proposed method.
Keywords. Inertial technique; Quasimonotone variational inequalities; Non-Lipschitz operators; Tseng's extragradient method.

1. INTRODUCTION

Let C be a nonempty, convex, and closed subset of a real Banach space E with norm $||\cdot||$, and let E^* be the dual of E. For $x \in E$ and $f \in E^*$, let $\langle x, f\rangle$ be the value of f at x, and let $A : E \to E^*$ be a single-valued mapping. The variational inequality problem (VIP) is formulated as finding a point $x^* \in C$ such that

$$\langle x - x^*, Ax^*\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

We denote the solution set of the VIP (1.1) by $VI(C,A)$. Variational inequality theory finds numerous applications in diverse fields, and it can be viewed as a unified framework for several problems, such as necessary optimality conditions, complementarity problems, equilibrium problems, and systems of nonlinear equations; see, e.g., [1–5] and the references therein.

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Recall that the *dual variational inequality problem* (DVIP) of (1.1) is to find a point $x^* \in C$ such that

$$\langle Ax, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.2}$$

We denote the solution set of the DVIP (1.2) by $VI(C,A)^d$. Furthermore, we use $VI(C,A)^t$ and $VI(C,A)^n$ to denote the trivial solution set and the nontrivial solution set of VIP (1.1), respectively, that is,

$$VI(C,A)^t := \{x^* \in C | \langle Ax^*, y - x^* \rangle = 0, \text{ for all } y \in C\},$$
$$VI(C,A)^n := VI(C,A) \backslash VI(C,A)^t.$$

Recall that a mapping $A : H \to H$ is said to be

(i) α-*strongly monotone* if there exists a constant $\alpha > 0$ such that $\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2$, $\forall x, y \in H$.
(ii) *monotone* if $\langle Ax - Ay, x - y \rangle \geq 0$, $\forall x, y \in H$.
(iii) *pseudomonotone* if $\langle Ay, x - y \rangle \geq 0 \Rightarrow \langle Ax, x - y \rangle \geq 0$, $\forall x, y \in H$.
(iv) *quasimonotone* if $\langle Ay, x - y \rangle > 0 \Rightarrow \langle Ax, x - y \rangle \geq 0$, $\forall x, y \in H$.
(v) *Lipschitz-continuous* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \leq L\|x - y\|$, $\forall x, y \in H$. If $L \in [0,1)$, then A is said to be a *contraction mapping*.
(vi) *uniformly continuous* if, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that

$$\|Ax - Ay\| < \varepsilon \quad \text{whenever} \quad \|x - y\| < \delta, \quad \forall x, y \in H.$$

From the above definitions, we observe that (i)⇒(ii)⇒(iii)⇒(iv). However, the converses are not generally true. Also, it is known that the uniform continuity is a weaker notion than the Lipschitz continuity. Moreover, it is known that if D is a convex subset of E, then $A : D \to \text{range}(A)$ is uniformly continuous if and only if, for every $\varepsilon > 0$, there exists a constant $M < +\infty$ such that

$$\|Ax - Ay\| \leq M\|x - y\| + \varepsilon, \quad \forall x, y \in D. \tag{1.3}$$

In the last few decades, researchers developed various efficient iterative methods for solving the VIP. There are two common approaches to solving the VIP, namely, the regularised methods and the projection-based methods. In this study, our interest is in the projection methods. The earliest and simplest projection method for solving the VIP is the projected gradient method (PGM), which is presented as follows: $x_{n+1} = P_C(x_n - \lambda Ax_n)$, $n \geq 1$, where P_C denotes the metric projection map. We note that the PGM only requires computing one projection per iteration. However, the method only converges under very stringent conditions. If A is Lipschitz continuous with Lipschitz constant L and strongly monotone, then the sequence generated by the PGM converges to a solution of the VIP for $\lambda \in \left(0, \frac{2\alpha}{L^2}\right)$. In order to relax the hypotheses for the convergence of the PGM, Korpelevich [6] and Antipin [7] independently proposed the following extragradient method (EGM) in finite-dimensional Euclidean space: $x_1 \in C$,

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases}$$

where $\lambda \in (0, \frac{1}{L})$, and $A : \mathbb{R}^m \to \mathbb{R}^m$ is monotone and L- Lipschitz continuous. If the set $VI(C,A)$ is nonempty, the EGM converges to an element in $VI(C,A)$. Observe that the EGM requires two projections onto the feasible set C and two evaluations of the operator A per iteration. In general, computing projection onto an arbitrary closed and convex set C is complicated, and this can affect the efficiency of the EGM. In the recent years, the EGM has received great attention from many researchers, who improved it in various ways. These improvements focus on minimizing the number of projections onto the feasible set C and the number of evaluations of the operator A per iteration; see, e.g., [8–12] and the references therein. It is known that If A is continuous and C is convex, then $VI(C,A)^d \subseteq VI(C,A)$. If A is pseudomonotone and continuous, then $VI(C,A) = VI(C,A)^d$ (see [13]). However, the inclusion $VI(C,A) \subseteq VI(C,A)^d$ is not true when A is quasimonotone and continuous [14]. Also, it is known that if A is quasimonotone and continuous, then $VI(C,A)^n \subset VI(C,A)^d$ (see [14]).

Remark 1.1. We remark that the above results on VIP and most of the existing results in the literature were obtained under the common assumption that $VI(C,A) \subseteq VI(C,A)^d$, that is,

$$\text{for any } x^* \in VI(C,A), \quad \langle Ax, x - x^* \rangle \geq 0 \quad \text{for all } x \in C. \tag{1.4}$$

This assumption is a direct consequence of the pseudomonotonicity of operator A.

Recently, Liu and Yang [15] proposed the following iterative method for approximating the solution of a quasimonotone (or without monotonicity) variational inequality problem in infinite dimensional Hilbert spaces:

Algorithm 1.1.

Step 0. Take $\lambda_1 > 0$, $x_1 \in H$, and $0 < \mu < 1$. Choose a nonnegative real sequence $\{\phi_n\}$ such that $\sum_{n=1}^{\infty} \phi_n < +\infty$.

Step 1. Given the current iterate x_n, compute $y_n = P_C(x_n - \lambda_n A x_n)$. If $x_n = y_n$, (or $Ay_n = 0$), then stop and y_n is a solution of the VIP. Otherwise, go to **Step 2.**

Step 2. Compute $x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n)$, where

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \phi_n \right\} & \text{if} \quad Ax_n - Ay_n \neq 0, \\ \lambda_n + \phi_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go back to **Step 1**.

The authors obtained a weak convergence result for the proposed method under the following assumptions:

(D1) $VI(C,A)^d \neq \emptyset$.

(D2) The mapping A is Lipschitz-continuous with constant $L > 0$.

(D3) The mapping A is quasimonotone on H.

$(D3')$ If $x_n \rightharpoonup x^*$ and $\limsup_{n\to\infty} \langle Ax_n, x_n \rangle \leq \langle Ax^*, x^* \rangle$, then $\lim_{n\to\infty} \langle Ax_n, x_n \rangle = \langle Ax^*, x^* \rangle$.

(D4) The mapping A is sequentially weakly continuous on C.

(D5) The set $A = \{z \in C : Az = 0\} \setminus VI(C,A)^d$ is a finite set.

$(D5')$ The set $B = VI(C,A) \setminus VI(C,A)^d$ is a finite set.

It is known that $VI(C,A)^d \neq \emptyset \iff \exists x^* \in VI(C,A)$ such that $\langle Ax, x - x^* \rangle \geq 0$ for all $x \in C$. Therefore, assumption $VI(C,A)^d \neq \emptyset$ is weaker than assumption (1.4). Thus $VI(C,A) \neq \emptyset$ and pseudomonotonicity imply quasimonotonicity and $VI(C,A)^d \neq \emptyset$, but the converse implications are not true. For sufficient conditions for $VI(C,A)^d \neq \emptyset$, see below (Lemma 2.9). However, we note that the conditions (D4), (D5), and $(D5')$ of Algorithm 1.1 are too restrictive. Moreover, the authors were only able to obtain the weak convergence for the proposed algorithm. In solving optimization problems, strong convergence results are more desirable because they are more applicable in many practical applications. Thus it is important to develop algorithms that generate strong convergent sequences when solving optimization problems. More recently, Chinedu *et al.* [16] improved the result of Liu and Yang [15] by proposing the following iterative method for the solutions of quasimonotone VIP in the framework of Hilbert spaces:

Algorithm 1.2.

Step 0. Let $\lambda_0, \lambda_1 > 0$, and $\vartheta \in (\delta, \frac{1-2\delta}{2})$ with $\delta \in (0, \frac{1}{4})$. Choose a nonnegative real sequence $\{\phi_n\}$ such that $\sum_{n=1}^{\infty} \phi_n < +\infty$. Let $x_0, x_1 \in C$ be given starting points. Set n:=1.

Step 1. Compute $x_{n+1} = P_C(x_n - ((\lambda_n + \lambda_{n-1})Ax_n - \lambda_{n-1}Ax_{n-1}))$, $n \geq 1$, where

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\vartheta\|x_n - x_{n+1}\|}{\|Ax_n - Ax_{n+1}\|}, \lambda_n + \phi_n\}, & \text{if } Ax_n \neq Ax_{n+1}, \\ \lambda_n + \phi_n, & \text{otherwise.} \end{cases}$$

The authors only obtained the weak convergence for the proposed method under the following conditions:

(D1) $VI(C,A)^d \neq \emptyset$.

(D2) The mapping A is Lipschitz-continuous on C with constant $L > 0$.

(D3) The mapping A is quasimonotone on H.

(D4) A satisfies the condition: whenever $\{x_n\} \subset C$, $x_n \rightharpoonup z$, one has $\|Az\| \leq \liminf_{n\to\infty} \|Ax_n\|$.

(D5) $Ax \neq 0$ for all $x \in C$.

Next, we discuss the inertial extrapolation technique. Based on the heavy ball methods of a two-order time dynamical system, Polyak [17] first introduced an inertial extrapolation technique as an acceleration process to solve the smooth convex minimization problem. The inertial algorithm is a two-step iteration where the next iterate is defined by making use of the previous two iterates. In recent years, inertial technique as an acceleration method has attracted the attention of researchers in optimization community. Several authors have constructed some fast numerical algorithms by employing the inertial extrapolation technique for various many problems; see, e.g., [18–22] and the references therein.

Recently, Alakoya *et al.* [23] proposed two inertial Tseng's extragradient method with the viscosity technique for approximating the solutions of the quasimonotone VIP in the framework of Hilbert spaces. The proposed methods are presented as follows:

Algorithm 1.3.

Step 1: Select initial point $x_0, x_1 \in H_1$. Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$, where

$$\hat{\theta}_n := \begin{cases} \min\left\{\theta, \frac{\xi_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2: Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$.

Step 3: Compute $y_n = P_C(w_n - \lambda_n A w_n)$. If $w_n = y_n$ (or $Ay_n = 0$), then stop: w_n is a solution to the VIP. Otherwise, go to **Step 4.**

Step 4: Compute $z_n = y_n + \lambda_n(Ay_n - Aw_n)$.

Step 5 Compute $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)z_n$, where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n + \phi_n\right\} & \text{if} \quad Aw_n - Ay_n \neq 0, \\ \lambda_n + \phi_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go back to **Step 1**.

Algorithm 1.4.

Step 1: Select initial point $x_0, x_1 \in H_1$. Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose θ_n such that $0 \leq \theta_n \leq \hat{\theta}_n$, where

$$\hat{\theta}_n := \begin{cases} \min\left\{\theta, \frac{\xi_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2: Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$.

Step 3: Compute $y_n = P_C(w_n - \lambda_n A w_n)$. If $w_n = y_n$ (or $Ay_n = 0$), then stop: w_n is a solution to the VIP. Otherwise, go to **Step 4.**

Step 4: Compute $z_n = y_n + \lambda_n(Ay_n - Aw_n)$.

Step 5 Compute $x_{n+1} = \alpha_n f(w_n) + (1-\alpha_n) z_n$, where

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n + \phi_n\right\} & \text{if } \quad Aw_n - Ay_n \neq 0, \\ \lambda_n + \phi_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go back to **Step 1**.

The authors obtained the strong convergence of the proposed methods under the following conditions:

(D1) $VI(C,A)^d \neq \emptyset$.
(D2) The mapping A is Lipschitz-continuous with constant $L > 0$.
(D3) The mapping A is quasimonotone on H.
$(D3')$ If $x_n \rightharpoonup x^*$ and $\limsup_{n\to\infty}\langle Ax_n, x_n\rangle \leq \langle Ax^*, x^*\rangle$, then $\lim_{n\to\infty}\langle Ax_n, x_n\rangle = \langle Ax^*, x^*\rangle$.
(D4) The mapping A is sequentially weakly continuous on C.
(D5) $Ax \neq 0$ for all $x \in C$.

Remark 1.2. At this point, we remark that all the above results for solving the quasimonotone VIP are not applicable when cost operator A is non-Lipschitz. Up to our knowledge, there are no existing results in the literature for solving the quasimonotone VIP when the cost operator is non-Lipschitz. Moreover, all the above results and the existing results in the literature are confined to the framework of Hilbert spaces. However, many important problems related to practical problems are generally defined in Banach spaces. For instance, Zhang et al. [24] remarked that in machine learning, Banach spaces possess much richer geometric structures, which are potentially useful for developing learning algorithms. This is due to the fact that any two Hilbert spaces over $\mathbb{C}$ of the same dimension are isometrically isomorphic. Also, Der and Lee [25] pointed out that most data in machine learning do not come with any natural notion distance that can be induced from an inner-product. Zhang et al. [24] further buttressed this statement by pointing out that the data come with intrinsic structures that make them impossible to be embedded into a Hilbert space. Therefore, it is more desirable and applicable to develop iterative methods for approximating solutions of the quasimonotone VIP in Banach spaces.

In the light of the above discourse, it is natural to ask the following research question: *Question. Can one develop an iterative method for solving non-Lipschitz quasimonotone (or without monotonicity) variational inequalities in the framework of Banach spaces and establish the strong convergence of the proposed method?*

In this study, we provide an affirmative answer to the above question. More precisely, we introduce a new inertial Tseng's extragradient method with self-adaptive step sizes for solving non-Lipschitz quasimonotone (or without monotonicity) variational inequalities in the framework of Banach spaces. Moreover, we establish the strong convergence of the proposed algorithm under some mild conditions on the control parameters. We point out that while the cost operator is non-Lipschitz, our proposed method does not involve any linesearch procedure but employs a more efficient self-adaptive step size technique with known parameters. Finally, we provide several numerical experiments to demonstrate the efficiency and applicability of our proposed algorithm. The outline of the paper is as follows. In Section 2, we recall some definitions and lemmas employed in the convergence analysis. In Section 3, we present the proposed algorithm and highlight some of its features while in Section 4, we analyze the convergence of the proposed method. In Section 5, we provide several numerical experiments to demonstrate the implementability and efficiency of our method. Finally, in Section 6 we give a concluding remark.

2. PRELIMINARIES

In this section, we recall some definitions and state some useful results needed in our convergence analysis. In the sequel, we assume that E is a real Banach space with dual space E^*. For a sequence $\{x_n\}$ in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. An element $z \in E$ is called a weak cluster point of $\{x_n\}$ if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to z. We write $w_\omega(x_n)$ to indicate the set of all weak cluster points of $\{x_n\}$.

Let $S_E : \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Găteaux differentiable* if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x,y \in S_E$. E is said to be *strictly convex* if $\|x+y\| < 2$ whenever $x,y \in S_E$ and $x \neq y$. E is said to be *uniformly convex* if, for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that, for all $x,y \in S_E$, $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \varepsilon$ imply $\|x+y\| < 2-\delta$. It is a known fact that a uniformly convex Banach space is strictly convex and reflexive. The *modulus of convexity* of E is defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x,y \in E, ||x|| = ||y|| = 1, ||x-y|| \geq \varepsilon\right\}.$$

E is *uniformly convex* if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. In particular, let H be a real Hilbert space, then $\delta_H(\varepsilon) = 1 - \sqrt{1-(\varepsilon/2)^2}$. E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta_E(\varepsilon) > c\varepsilon^p$ for all $\varepsilon \in (0,2]$ with $p \geq 2$. It is easy to see that a p-uniformly convex Banach space is uniformly convex.

A Banach space E is said to be *smooth* if its norm is *Găteaux differentiable* for all x,y. Moreover, if the limit in (2.1) is attained uniformly for $x,y \in S_E$, then E is said to be *uniformly smooth*. It is known that a uniformly smooth space is smooth. Let $1 < q \leq 2$. The Banach space E is said to be *q-uniformly smooth* if there exists a constant $\kappa > 0$ such that $\rho_E(t) \leq \kappa t^q$ for all $t > 0$, where ρ_E is the *modulus of smoothness* of E defined by

$$\rho_E(t) = \sup\left\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x,y \in S_E\right\}$$

for all $t \geq 0$. Let $1 < q \leq 2 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. It is known that every q-uniformly smooth (p-uniformly convex) space is uniformly smooth (uniformly convex). Moreover, it is well known that E is q-uniformly smooth (p-uniformly convex) if and only if its dual E^* is p-uniformly convex (q-uniformly smooth). If E is uniformly smooth then E is reflexive and smooth. Furthermore, it is known that for every $p > 1, L_p$ and ℓ_p are $\min\{p,2\}$-uniformly smooth and $\max\{p,2\}$-uniformly convex. In particular, Hilbert spaces are 2-uniformly smooth and 2-uniformly convex. Moreover, all the Lebesgue spaces L_p are uniformly smooth and 2-uniformly convex whenever $1 < p \leq 2$. For further details on the geometry of Banach spaces, one refers to [26, 27].

For $p > 1$, the generalized duality mapping $J_p : E \to 2^{E^*}$ is defined by

$$J_p x = \{f \in E^* : \langle x, f \rangle = ||x||^p, ||f|| = ||x||^{p-1},\ x \in E\}.$$

In particular, $J = J_2$ is called the normalized duality mapping. If $E = H$, where H is a real Hilbert space, then $J = I$ (see [26, 27]).

Remark 2.1. The following basic properties for a Banach space E and for the normalized duality mapping J can be found in Cioranescu [28].

(1) If E is an arbitrary Banach space, then J is monotone and bounded;
(2) If E is strictly convex, then J is one-to-one and strictly monotone;
(3) If E is reflexive, then J is onto;
(4) If E is a smooth Banach space, then J is single-valued, and hemi-continuous, i.e., J is continuous from the strong topology of E to the weak star topology of $E*$;
(5) If E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*, JJ^* = I_{E^*}$, and $J^*J = I_E$;
(6) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;
(7) A Banach space E is uniformly smooth if and only if E^* is uniformly convex. If E is uniformly smooth, then it is smooth and reflexive.

Let E be a smooth Banach space. The Lyapunov functional $\phi : E \times E \to \mathbb{R}$ [29] is defined by $\phi(x,y) = ||x||^2 - 2\langle x, Jy \rangle + ||y||^2, \forall x,y \in E$. From the definition, we observe that $\phi(x,x) = 0$ for every $x \in E$. If E is strictly convex, then $\phi(x,y) = 0 \Longleftrightarrow x = y$. If E is a Hilbert space, then $\phi(x,y) = ||x-y||^2$ for all $x,y \in E$. Furthermore, for every $x,y,z \in E$ and $\alpha \in (0,1)$, the Lyapunov functional ϕ satisfies the following properties:

(P1) $0 \leq (||x|| - ||y||)^2 \leq \phi(x,y) \leq (||x|| + ||y||)^2$;
(P2) $\phi(x, J^{-1}(\alpha Jz + (1-\alpha)Jy)) \leq \alpha\phi(x,z) + (1-\alpha)\phi(x,y)$;
(P3) $\phi(x,y) = \phi(x,z) - \phi(y,z) + 2\langle y - x, Jy - Jz \rangle$;
(P4) $\phi(x,y) \leq 2\langle y - x, Jy - Jx \rangle$;
(P5) $\phi(x,y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq ||x||||Jx - Jy|| + ||y - x||||y||$.

Also, the functional $V : E \times E^* \to [0, +\infty)$ studied in [29] is defined by $V(x, x^*) = ||x||_E^2 - 2\langle x, x^* \rangle + ||x^*||_{E^*}^2$, $\forall x \in E, x^* \in E^*$. It can be deduced that V is non-negative and $V(x,x^*) = \phi(x, J^{-1}(x^*))$.

The following result describes one of the properties of the functional $V(\cdot,\cdot)$.

Lemma 2.1. *[30] Let E be a reflexive strictly convex and smooth Banach space with dual E^*. Then, $V(x,x^*)+2\langle J^{-1}x^*-x,y^*\rangle \leq V(x,x^*+y^*)$, for all $x\in E$ and $x^*,y^*\in E^*$.*

Lemma 2.2. *[31] Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n,y_n)\to 0$ as $n\to\infty$, then $||x_n-y_n||\to 0$ as $n\to\infty$.*

Remark 2.2. From property (P4) of the Lyapunov functional, it follows that the converse of Lemma 2.2 also holds if the sequences $\{x_n\}$ and $\{y_n\}$ are bounded (see [32])

Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E. By Alber [29], for each $x\in E$, there exists a unique element $x_0\in C$ (denoted by $\Pi_C(x)$) such that $\phi(x_0,x)=\min_{y\in C}\phi(y,x)$. The mapping $\Pi_C : E\to C$, defined by $\Pi_C(x)=x_0$, is called the generalized projection from E onto C. Moreover, x_0 is called the generalized projection of x. If E is a real Hilbert space, then Π_C coincides with the metric projection operator P_C ([33]). We have the following well known results.

Lemma 2.3. *[34] Let C be a nonempty, closed, and convex subset of a reflexive, strictly convex, and smooth Banach space E. Given $x\in E$ and $z\in C$, $z=\Pi_C x$ implies $\phi(y,z)+\phi(z,x)\leq\phi(y,x)$, $\forall y\in C$.*

Lemma 2.4. *[34] Let C be a nonempty, closed, and convex subset of a smooth Banach space E and $x\in E$. Then, $x_0=\Pi_C x$ if and only if $\langle x_0-y,Jx-Jx_0\rangle\geq 0$, $\forall y\in C$.*

Lemma 2.5. *[35] Let E be a 2-uniformly convex Banach space. Then, for all $x,y\in E$, there exists $c\geq 1$ such that $\phi(x,y)\geq\frac{1}{c}||x-y||^2$, where c is the 2-uniform convexity constant of E. If E is a Hilbert space, then $c=1$.*

Lemma 2.6. *[36] Let $\frac{1}{p}+\frac{1}{q}=1, p,q>1$. The space E is q-uniformly smooth if and only if its dual E^* is p-uniformly convex.*

Lemma 2.7. *[37] Let E be a real Banach space. The following statements are equivalent:*

1. *E is 2-uniformly smooth;*
2. *There exists a constant $k>0$ such that $\forall x,y,\in E$, $\|x+y\|^2\leq\|x\|^2+2\langle y,Jx\rangle+2k^2\|y\|^2$, where k is the 2-uniform smoothness constant. If E is a Hilbert space, then $k=\frac{1}{\sqrt{2}}$.*

Lemma 2.8. *[37] Let E be a uniformly convex Banach space. Then, for any given real number $r>0$, there exists a continuous strictly increasing function $g:[0,\infty)\to[0,\infty)$ such that $g(0)=0$ and $\|\alpha x+(1-\alpha)y\|^2\leq\alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)g(\|x-y\|)$ for all $x,y\in E$ with $\|x\|\leq r$ and $\|y\|\leq r, \alpha\in[0,1]$.*

Lemma 2.9. *[14] If either*

(i) *A is pseudomonotone on C and $S\neq\emptyset$;*
(ii) *A is the gradient of G, where G is a differentiable quasiconvex function on an open set $K\supset C$ and attains its global minimum on C;*
(iii) *A is quasimonotone on $C, A\neq 0$ on C and C is bounded;*
(iv) *A is quasimonotone on $C, A\neq 0$ on C and there exists a positive number r such that, for every $x\in C$ with $\|x\|\geq r$, there exists $y\in C$ such that $\|y\|\leq r$ and $\langle Ax,y-x\rangle\leq 0$;*
(v) *A is quasimonotone on C, $intC\neq\emptyset$ and there exists $x^*\in S$ such that $Ax^*\neq 0$,*

then S_D is nonempty.

Lemma 2.10. *[38] Let $\{a_n\}$ be a sequence of nonnegative real numbers. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^{\infty}\alpha_n=\infty$, and let $\{b_n\}$ be a sequence of real numbers. Assume that $a_{n+1}\le(1-\alpha_n)a_n+\alpha_n b_n$ for all $n\ge 1$. If $\limsup\limits_{k\to\infty} b_{n_k}\le 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying $\liminf\limits_{k\to\infty}(a_{n_{k+1}}-a_{n_k})\ge 0$, then $\lim\limits_{n\to\infty} a_n=0$.*

Lemma 2.11. *[39] Suppose that $\{\lambda_n\}$ and $\{\theta_n\}$ are two nonnegative real sequences such that $\lambda_{n+1}\le\lambda_n+\phi_n$, $\forall n\ge 1$. If $\sum_{n=1}^{\infty}\phi_n<+\infty$, then $\lim\limits_{n\to\infty}\lambda_n$ exists.*

3. Main Results

In this section, we present our proposed algorithm for solving non-Lipschitz quasimonotone (or without monotonicity) variational inequalities in the framework of Banach spaces. We establish our strong convergence results under the following conditions:

Assumption A:

- (A1) E is a real 2-uniformly convex and uniformly smooth Banach space with dual E^*, c is the 2-uniform convexity constant of E, and k is the 2-uniform smoothness constant of E^*.
- (A2) The set $VI(C,A)^d\neq\emptyset$.
- (A3) $A:E\to E^*$ is uniformly continuous.
- (A4) $A:E\to E^*$ is quasimonotone.
- (A5) $A:E\to E^*$ satisfies the following property whenever $\{x_n\}\subset C$, $x_n\rightharpoonup z$, one has $\|Az\|\le\liminf_{n\to\infty}\|Ax_n\|$.
- (A4$'$) If $\{x_n\}\rightharpoonup x^*$ and $\limsup_{n\to\infty}\langle Ax_n,\, x_n\rangle\le\langle Ax^*,\, x^*\rangle$, then $\lim_{n\to\infty}\langle Ax_n,\, x_n\rangle=\langle Ax^*,\, x^*\rangle$.
- (A5$'$) A is sequentially weakly continuous.

Assumption B:

- (B1) Let $\{\alpha_n\}\subset(0,1)$ such that $\lim\limits_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=+\infty$.
- (B2) Let $\lambda_1>0,\theta>0$, and $0<\mu<\mu'<\frac{1}{k\sqrt{2c}}$. Let $\{\xi_n\}$ be a positive sequence such that $\lim_{n\to\infty}\frac{\xi_n}{\alpha_n}=0$.
- (B3) Let $\{\mu_n\}$ and $\{\phi_n\}$ be nonnegative sequences such that $\lim_{n\to\infty}\mu_n=0$ and $\sum_{n=1}^{\infty}\phi_n<+\infty$.
- (B4) $\{q_n\}\subset E$ such that $\lim_{n\to\infty}q_n=q\in E$.

Now, we present ou algorithm as follows.

Algorithm 3.1.

Step 0. Let $x_0,x_1\in E$ be two arbitrary initial points and set $n=1$.

Step 1. Given the $(n-1)th$ and nth iterates, choose θ_n such that $0\le\theta_n\le\hat{\theta}_n$ with $\hat{\theta}_n$ defined by

$$\hat{\theta}_n=\begin{cases}\min\left\{\theta,\ \frac{\xi_n}{\|x_n-x_{n-1}\|}\right\}, & \text{if } x_n\neq x_{n-1},\\ \theta, & \text{otherwise.}\end{cases}\tag{3.1}$$

Step 2. Compute $w_n=J^{-1}(Jx_n+\theta_n(Jx_{n-1}-Jx_n))$.

Step 3. Compute $y_n=\Pi_C J^{-1}(Jw_n-\lambda_n Aw_n)$. If $y_n=w_n$ or $Ay_n=0$, then stop and y_n is a solution of the VIP. Else go to **Step 4.**

Step 4. Compute $u_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Aw_n))$.
Step 5. Compute

$$x_{n+1} = J^{-1}(\alpha_n Jq_n + (1-\alpha_n)Ju_n). \tag{3.2}$$

Step 6. Compute

$$\lambda_{n+1} = \begin{cases} \min\{\frac{(\mu+\mu_n)\|w_n-y_n\|}{\|Aw_n-Ay_n\|},\ \lambda_n+\phi_n\}, & if\ Aw_n - Ay_n \neq 0, \\ \lambda_n + \phi_n, & \text{otherwise.} \end{cases} \tag{3.3}$$

Set $n := n+1$ and return to **Step 1**.

Remark 3.1.

- We remark that in the literature on quasimonotone VIP (in Hilbert spaces), the cost operator A is required to be Lipschitz continuous. However, our method only requires it to be uniformly continuous which is a weaker notion than the Lipschitz continuity. Moreover, while the cost operator is non-Lipschitz, our algorithm does not require any linesearch procedure which could be computationally expensive to implement. We employ a more efficient and simple step size rule in (3.3), which generates a non-monotonic sequence of step sizes. The step size is constructed such that it reduces the dependence of the algorithm on the initial step size λ_1. In addition, we introduce the sequence $\{\mu_n\}$ to relax parameter μ thereby enlarging the value range of the corresponding step sizes.
- We point out that condition (A5) is strictly weaker than the sequentially weakly continuity condition, which was commonly used in the literature when solving pseudomonotone and quasimonotone VIPs. For our first strong convergence theorem with the quasimonotonicity assumption, we do not require condition (A5$'$). We only require this condition when proving our second strong convergence theorem without the quasimonotonicity assumption.

4. Convergence Analysis

First, we establish some lemmas required to prove the strong convergence theorems for the proposed algorithm.

Lemma 4.1. *Let $\{\lambda_n\}$ be a sequence generated by Algorithm 3.1. Then,* $\lim_{n\to\infty}\lambda_n = \lambda$, *where* $\lambda \in \Big[\min\{\frac{\mu}{K},\lambda_1\},\lambda_1+\Phi\Big]$ *for some $K>0$ and* $\Phi = \sum_{n=1}^{\infty}\phi_n$.

Proof. Since A is uniformly continuous, then it follows from (1.3) that, for any given $\varepsilon > 0$, there exists $M < +\infty$ such that $\|Aw_n - Ay_n\| \leq M\|w_n - y_n\| + \varepsilon$. If $Aw_n - Ay_n \neq 0$ for all $n \geq 1$, then

$$\frac{(\mu+\mu_n)\|w_n-y_n\|}{\|Aw_n-Ay_n\|} \geq \frac{(\mu+\mu_n)\|w_n-y_n\|}{M\|w_n-y_n\|+\varepsilon} = \frac{(\mu+\mu_n)\|w_n-y_n\|}{(M+\varepsilon_1)\|w_n-y_n\|} = \frac{(\mu+\mu_n)}{K} \geq \frac{\mu}{K},$$

where $\varepsilon = \varepsilon_1\|w_n - y_n\|$ for some $\varepsilon_1 \in (0,1)$ and $K = M+\varepsilon_1$. Therefore, by the definition of λ_{n+1}, the sequence $\{\lambda_n\}$ has lower bound $\min\{\frac{\mu}{K},\lambda_1\}$ and upper bound $\lambda_1+\Phi$. By Lemma 2.11, it follows that $\lim_{n\to\infty}\lambda_n$ exists and denoted by $\lambda = \lim_{n\to\infty}\lambda_n$. It is obvious that $\lambda \in \big[\min\{\frac{\mu}{K},\lambda_1\},\lambda_1+\Phi\big]$. ∎

Lemma 4.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that conditions (A1)-(A5) and Assumption B hold. Then, the following inequality holds for all $p \in VI(C,A)^d$ and $n \in \mathbb{N}$:*

$$\phi(p,u_n) \leq (1-\theta_n)\phi(p,x_n) + \theta_n\phi(p,x_{n-1}) - \Big(1 - 2k^2(\mu+\mu_n)^2 c\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,w_n). \tag{4.1}$$

Moreover, there exists $N_0 > 0$ such that, for all $n > N_0$, $\phi(p,u_n) \leq (1-\theta_n)\phi(p,x_n) + \theta_n\phi(p,x_{n-1})$.

Proof. Let $p \in VI(C,A)^d$. If $Aw - Ay_n \neq 0$, we have from the definition of λ_{n+1} that

$$\lambda_{n+1} = \min\Big\{\frac{(\mu+\mu_n)\|w_n - y_n\|}{\|Aw_n - Ay_n\|},\ \lambda_n + \phi_n\Big\} \leq \frac{(\mu+\mu_n)\|w_n - y_n\|}{\|Aw_n - Ay_n\|},$$

which implies that

$$\|Aw_n - Ay_n\| \leq \frac{(\mu+\mu_n)}{\lambda_{n+1}}\|w_n - y_n\|, \quad \forall n \geq 1. \tag{4.2}$$

Observe that (4.2) still holds when $Aw_n - Ay_n = 0$. Next, from the definition of w_n, we have

$$\phi(p,w_n) \leq (1-\theta_n)\phi(p,x_n) + \theta_n\phi(p,x_{n-1}). \tag{4.3}$$

From Lemma 2.6, E^* is 2-uniformly smooth. Thus, by applying Lemma 2.5, Lemma 2.7, property (P3) of the Lyapunov function, and (4.2), we have

$$\begin{aligned}
\phi(p,u_n) &= \phi(p, J^{-1}(Jy_n - \lambda_n(Ay_n - Aw_n))) \\
&= V(p, Jy_n - \lambda_n(Ay_n - Aw_n)) \\
&= \|p\|^2 - 2\langle p, Jy_n - \lambda_n(Ay_n - Aw_n)\rangle + \|Jy_n - \lambda_n(Ay_n - Aw_n)\|^2 \\
&\leq \|p\|^2 - 2\langle p, Jy_n\rangle + \|y_n\|^2 - 2\lambda_n\langle y_n - p, Ay_n - Aw_n\rangle + 2k^2\lambda_n^2\|Ay_n - Aw_n\|^2 \\
&= \phi(p,y_n) - 2\lambda_n\langle y_n - p, Ay_n - Aw_n\rangle + 2k^2\lambda_n^2\|Ay_n - Aw_n\|^2 \\
&\leq \phi(p,w_n) - \phi(y_n,w_n) + 2k^2(\mu+\mu_n)^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2 - 2\lambda_n\langle y_n - p, Ay_n\rangle \\
&\quad + 2\langle p - y_n, Jw_n - \lambda_n Aw_n - Jy_n\rangle \\
&\leq \phi(p,w_n) - \Big(1 - 2k^2(\mu+\mu_n)^2 c\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,w_n) - 2\lambda_n\langle y_n - p, Ay_n\rangle \\
&\quad + 2\langle p - y_n, Jw_n - \lambda_n Aw_n - Jy_n\rangle.
\end{aligned} \tag{4.4}$$

Since $y_n \in C$ and $p \in VI(C,A)^d$, we have $\langle y_n - p, Ay_n\rangle \geq 0$. By the property of the generalized projection operator and the definition of y_n, we have $\langle p - y_n, Jw_n - \lambda_n Aw_n - Jy_n\rangle \leq 0$. Now, by applying (4.3) and (4.4), we obtain

$$\begin{aligned}
\phi(p,u_n) &\leq \phi(p,w_n) - \Big(1 - 2k^2(\mu+\mu_n)^2 c\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,w_n) \\
&\leq (1-\theta_n)\phi(p,x_n) + \theta_n\phi(p,x_{n-1}) - \Big(1 - 2k^2(\mu+\mu_n)^2 c\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,w_n).
\end{aligned} \tag{4.5}$$

Consider the limit

$$\lim_{n\to\infty}\Big(1 - 2k^2(\mu+\mu_n)^2 c\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big) = \Big(1 - 2k^2\mu^2 c\Big) > 0, \qquad (0 < \mu < \mu' < \frac{1}{k\sqrt{2c}}). \tag{4.6}$$

Therefore, there exists $N_0 > 0$ such that for all $n > N_0$, we have $\left(1 - 2k^2(\mu + \mu_n)^2 c \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) > 0$. Consequently, it follows from (4.5) that, for all $n > N_0$,

$$\phi(p,u_n) \le (1-\theta_n)\phi(p,x_n) + \theta_n\phi(p,x_{n-1}), \tag{4.7}$$

which completes the proof. □

Lemma 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that conditions (A1)-(A5) and Assumption B hold. Then $\{x_n\}$ is bounded.*

Proof. Let $p \in VI(C,A)^d$. From the definition of x_{n+1} and (4.7), we have

$$\begin{aligned}
\phi(p,x_{n+1}) &\le \alpha_n\phi(p,q_n) + (1-\alpha_n)\phi(p,u_n) \\
&\le \alpha_n\phi(p,q_n) + (1-\alpha_n)\Big((1-\theta_n)\phi(p,x_n) + \theta_n\phi(p,x_{n-1})\Big) \\
&\le \max\left\{\phi(p,q_n),\ \phi(p,x_n),\ \phi(p,x_{n-1})\right\}.
\end{aligned}$$

Since $\{q_n\}$ is bounded and J is bounded on bounded subsets of E, we see that there exists a real number $L > 0$ such that $\phi(p,q_n) \le L$ for all $n \in \mathbb{N}$. Thus, by induction,

$$\begin{aligned}
\phi(p,x_{n+1}) &\le \max\left\{L, \phi(p,x_n), \phi(p,x_{n-1})\right\} \\
&\ \ \vdots \\
&\le \max\left\{L, \phi(p,x_{N_0}), \phi(p,x_{N_0-1})\right\}.
\end{aligned}$$

This implies that $\{\phi(p,x_n)\}$ is bounded. Consequently, $\{x_n\}$ is bounded. Moreover, from the construction of the algorithm, we have that $\{w_n\}, \{y_n\}$, and $\{u_n\}$ are all bounded. □

Lemma 4.4. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 and suppose $p \in VI(C,A)^d$. Then*

i. $\lim_{n\to\infty} \theta_n\Big(\phi(p,x_{n-1}) - \phi(p,x_n)\Big) = 0,$

ii. $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\Big(\phi(p,x_{n-1}) - \phi(p,x_n)\Big) = 0.$

Proof. i. Let $p \in VI(C,A)^d$. From (3.1), we have

$$\theta_n\|x_n - x_{n-1}\| \le \xi_n \quad \text{for each } n \ge 1. \tag{4.8}$$

Since $\lim_{n\to\infty}\frac{\xi_n}{\alpha_n} = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, it follows that $\lim_{n\to\infty}\xi_n = 0$. Consequently, we obtain

$$\lim_{n\to\infty}\theta_n\|x_n - x_{n-1}\| \le \lim_{n\to\infty}\xi_n = 0. \tag{4.9}$$

Since J is norm-to-norm continuous on subsets of E, it follows that

$$\lim_{n\to\infty}\theta_n\|Jx_n - Jx_{n-1}\| = 0. \tag{4.10}$$

Observe that

$$\begin{aligned}
\phi(p,x_{n-1}) - \phi(p,x_n) &= \|p\|^2 - 2\langle p, Jx_{n-1}\rangle + \|x_{n-1}\|^2 - \left(\|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2\right) \\
&= \|x_{n-1}\|^2 - \|x_n\|^2 + 2\langle p, Jx_n - Jx_{n-1}\rangle \\
&\le \|x_{n-1} - x_n\|\left(\|x_{n-1}\| + \|x_n\|\right) + 2\|p\|\|Jx_n - Jx_{n-1}\|.
\end{aligned} \tag{4.11}$$

By applying (4.9) and (4.10), we obtain from (4.11) that

$$\begin{aligned}&\lim_{n\to\infty}\theta_n\Big(\phi(p,x_{n-1})-\phi(p,x_n)\Big)\\&\le\lim_{n\to\infty}\Big(\theta_n\|x_{n-1}-x_n\|\big(\|x_{n-1}\|+\|x_n\|\big)+2\|p\|\theta_n\|Jx_n-Jx_{n-1}\|\Big)=0,\end{aligned}$$

as required.

ii. Since $\lim_{n\to\infty}\frac{\xi_n}{\alpha_n}=0$, then it follows from (4.8) that $\lim\limits_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\le\lim\limits_{n\to\infty}\frac{\xi_n}{\alpha_n}=0$. Since J is norm-to-norm continuous on subsets of E, we obtain $\lim\limits_{n\to\infty}\frac{\theta_n}{\alpha_n}\|Jx_n-Jx_{n-1}\|=0$. In view of (4.11), we conclude that

$$\begin{aligned}&\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\Big(\phi(p,x_{n-1})-\phi(p,x_n)\Big)\\&\le\lim_{n\to\infty}\Big(\frac{\theta_n}{\alpha_n}\|x_{n-1}-x_n\|\big(\|x_{n-1}\|+\|x_n\|\big)+2\|p\|\frac{\theta_n}{\alpha_n}\|Jx_n-Jx_{n-1}\|\Big)=0,\end{aligned}$$

which completes the proof. □

Lemma 4.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that conditions (A1)-(A5) and Assumption B hold. If $p\in VI(C,A)^d$, then the following inequality holds, for all $n\in\mathbb{N}$,*

$$\begin{aligned}&\phi(p,x_{n+1})\\&\le(1-\alpha_n)\phi(p,x_n)+\alpha_n\Big((1-\alpha_n)\frac{\theta_n}{\alpha_n}\big(\phi(p,x_{n-1})-\phi(p,x_n)\big)+2\langle x_{n+1}-p,Jq_n-Jp\rangle\Big)\\&-(1-\alpha_n)\Big(1-2k^2(\mu+\mu_n)^2c\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,w_n).\end{aligned}$$

Proof. Let $p\in VI(C,A)^d$. From (3.2), (4.1), and Lemma 2.1, we have

$$\begin{aligned}&\phi(p,x_{n+1})\\&=V\big(p,\alpha_nJq_n+(1-\alpha_n)Ju_n\big)\\&\le V\big(p,(1-\alpha_n)Ju_n+\alpha_nJq_n-\alpha_n(Jq_n-Jp)\big)+2\alpha_n\langle x_{n+1}-p,Jq_n-Jp\rangle\\&\le(1-\alpha_n)\phi(p,u_n)+\alpha_n\phi(p,p)+2\alpha_n\langle x_{n+1}-p,Jq_n-Jp\rangle\\&\le(1-\alpha_n)\Big((1-\theta_n)\phi(p,x_n)+\theta_n\phi(p,x_{n-1})-\Big(1-2k^2(\mu+\mu_n)^2c\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,w_n)\Big)\\&+2\alpha_n\langle x_{n+1}-p,Jq_n-Jp\rangle\\&=(1-\alpha_n)\phi(p,x_n)+(1-\alpha_n)\big(\theta_n\phi(p,x_{n-1})-\theta_n\phi(p,x_n)\big)\\&-(1-\alpha_n)\Big(1-2k^2(\mu+\mu_n)^2c\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,w_n)+2\alpha_n\langle x_{n+1}-p,Jq_n-Jp\rangle\\&=(1-\alpha_n)\phi(p,x_n)+\alpha_n\Big((1-\alpha_n)\frac{\theta_n}{\alpha_n}\big(\phi(p,x_{n-1})-\phi(p,x_n)\big)+2\langle x_{n+1}-p,Jq_n-Jp\rangle\Big)\\&-(1-\alpha_n)\Big(1-2k^2(\mu+\mu_n)^2c\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,w_n),\end{aligned}$$

which is the required inequality.

□

Lemma 4.6. *Assume that $\{w_n\}$ and $\{y_n\}$ are two sequences generated by Algorithm 3.1 such that conditions (A1)-(A5) and Assumption B hold. Suppose that there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\{w_{n_k}\}$ converges weakly to some $x^* \in E$ and $\|y_{n_k} - w_{n_k}\| \to 0$, as $k \to \infty$. Then, either $x^* \in VI(C,A)^d$ or $Ax^* = 0$.*

Proof. Since w_n is bounded, then $w_\omega(w_n)$ is nonempty. Let $x^* \in w_\omega(w_n)$ be an arbitrary element. Then, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \rightharpoonup x^*$ as $k \to \infty$. By the hypothesis of the lemma and the definition of y_n, we have that $y_{n_k} \rightharpoonup x^*$ as $k \to \infty$ and $x^* \in C$. We now divide the proof of this lemma into two cases.

CASE 1: If $\limsup_{k\to\infty} \|Ay_{n_k}\| = 0$, then $\lim_{k\to\infty} \|Ay_{n_k}\| = \liminf_{k\to\infty} \|Ay_{n_k}\| = 0$. Since A satisfies condition (A3) and $\{y_{n_k}\}$ converges weakly to $x^* \in C$, one obtains that $0 \leq \|Ax^*\| \leq \liminf_{k\to\infty} \|Ay_{n_k}\| = 0$. Thus $Ax^* = 0$.

CASE 2: If $\limsup_{k\to\infty} \|Ay_{n_k}\| > 0$, without loss of generality, we take $\lim_{k\to\infty} \|Ay_{n_k}\| = Z > 0$. Then, it follows that there exists a constant $K \in \mathbb{N}$ such that $\|Ay_{n_k}\| > \frac{Z}{2}$, $\forall k \geq K$. By Lemma 2.4, we have $\langle Jw_{n_k} - \lambda_{n_k} Aw_{n_k} - Jy_{n_k}, z - y_{n_k} \rangle \leq 0$, $\forall z \in C$, which implies that $\langle Jw_{n_k} - Jy_{n_k},\, z - y_{n_k} \rangle \leq \lambda_{n_k} \langle Aw_{n_k}, z - y_{n_k} \rangle$, $\forall z \in C$. Consequently, we have

$$\frac{1}{\lambda_{n_k}} \langle Jw_{n_k} - Jy_{n_k},\, z - y_{n_k} \rangle + \langle Aw_{n_k},\, y_{n_k} - w_{n_k} \rangle \leq \langle Aw_{n_k},\, z - w_{n_k} \rangle,\ \forall z \in C. \tag{4.12}$$

In view of $\lim\limits_{k\to\infty} \lambda_{n_k} = \lambda > 0$, $\|y_{n_k} - w_{n_k}\| \to 0$ as $k \to \infty$, the norm-to-norm continuity of J, and (4.12), we obtain

$$0 \leq \liminf_{k\to\infty} \langle Aw_{n_k},\, z - w_{n_k} \rangle \leq \limsup_{k\to\infty} \langle Aw_{n_k},\, z - w_{n_k} \rangle < +\infty. \tag{4.13}$$

Observe that

$$\langle Ay_{n_k},\, z - y_{n_k} \rangle = \langle Ay_{n_k} - Aw_{n_k},\, z - w_{n_k} \rangle + \langle Aw_{n_k},\, z - w_{n_k} \rangle + \langle Ay_{n_k},\, w_{n_k} - y_{n_k} \rangle. \tag{4.14}$$

Since A is uniformly continuous, it follows from the hypothesis of the lemma that $\lim_{k\to\infty} \|Aw_{n_k} - Ay_{n_k}\| = 0$. Consequently, from (4.13) and (4.14), we obtain

$$0 \leq \liminf_{k\to\infty} \langle Ay_{n_k},\, z - y_{n_k} \rangle \leq \limsup_{k\to\infty} \langle Ay_{n_k},\, z - y_{n_k} \rangle < +\infty,\ \forall\, z \in C. \tag{4.15}$$

If $\limsup_{k\to\infty} \langle Ay_{n_k}, z - y_{n_k} \rangle > 0$, then there exists a subsequence $\{y_{n_{k_j}}\}$ such that $\lim_{j\to\infty} \langle Ay_{n_{k_j}}, z - y_{n_{k_j}} \rangle > 0$. Hence, there exists $j_0 \in \mathbb{N}$ such that $\langle Ay_{n_{k_j}}, z - y_{n_{k_j}} \rangle > 0$, $\forall j \geq j_0$. By the quasimonotonicity of A, we have $\langle Az, z - y_{n_{k_j}} \rangle \geq 0$ for all $j \geq j_0$. Letting $j \to \infty$, we have $x^* \in VI(C,A)^d$.

If $\limsup_{k\to\infty} \langle Ay_{n_k}, z - y_{n_k} \rangle = 0$, it follows from (4.15) that

$$\lim_{k\to\infty} \langle Ay_{n_k},\, z - y_{n_k} \rangle = \limsup_{k\to\infty} \langle Ay_{n_k},\, z - y_{n_k} \rangle = \liminf_{k\to\infty} \langle Ay_{n_k},\, z - y_{n_k} \rangle = 0.$$

Let $\zeta_k = |\langle Ay_{n_k},\, z - y_{n_k} \rangle| + \frac{1}{k+1}$. It follows that $\langle Ay_{n_k},\, z - y_{n_k} \rangle + \zeta_k > 0$. For some $z_{n_k} \in E$ with $\lim_{k\to\infty} z_{n_k} = b \in E$ and $\langle Ay_{n_k}, z_{n_k} \rangle = 1$ for each $k \geq K$, we deduce that $\langle Ay_{n_k}, z + \zeta_k z_{n_k} - y_{n_k} \rangle > 0$. Since A is quasimonotone, we have, for all $k \geq K$, $\langle A(z + \zeta_k z_{n_k}),\, z + \zeta_k z_{n_k} - y_{n_k} \rangle \geq 0$, which together with (1.3) yields that, $\forall k \geq K$,

$$\begin{aligned}\langle Az,\, z+\zeta_k z_{n_k} - y_{n_k}\rangle &= \langle Az - A(z+\zeta_k z_{n_k}),\, z+\zeta_k z_{n_k} - y_{n_k}\rangle + \langle A(z+\zeta_k z_{n_k}),\, z+\zeta_k z_{n_k} - y_{n_k}\rangle \\ &\geq \langle Az - A(z+\zeta_k z_{n_k}),\, z+\zeta_k z_{n_k} - y_{n_k}\rangle \\ &\geq -\|Az - A(z+\zeta_k z_{n_k})\|\|z+\zeta_k z_{n_k} - y_{n_k}\| \\ &\geq -\zeta_k(M'+\varepsilon_1')\|z_{n_k}\|\|z+\zeta_k z_{n_k} - y_{n_k}\|, \end{aligned} \tag{4.16}$$

where $\varepsilon' = \varepsilon_1{}'\|\zeta_k z_{n_k}\|$ for some $\varepsilon_1{}' \in (0,1)$ and M' is some constant. Letting $k \to \infty$ in (4.16), and using the fact that $\lim_{k\to\infty} \zeta_k = 0$, the boundedness of $\{z_{n_k}\}$, and $\{\|z+\zeta_k z_{n_k} - y_{n_k}\|\}$, we have $\langle Az,\, z-x^*\rangle \geq 0$, $\forall z \in C$, which implies that $x^* \in VI(C,A)^d$. This completes the proof. ☐

Now, we are in a position to state and prove our first strong theorem for the proposed algorithm.

Theorem 4.1. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that conditions (A1)-(A5), Assumption B hold, and $Ax \neq 0$, $\forall x \in C$. Then $\{x_n\}$ converges strongly to $\hat{x} \in VI(C,A)^d \subset VI(C,A)$, where $\hat{x} = \Pi_{VI(C,A)^d}(q)$.*

Proof. Let $\hat{x} = \Pi_{VI(C,A)^d}(q)$. From Lemma 4.5, we obtain

$$\begin{aligned}\phi(\hat{x},x_{n+1}) &\leq (1-\alpha_n)\phi(\hat{x},x_n) + \alpha_n\Big((1-\alpha_n)\frac{\theta_n}{\alpha_n}\big(\phi(\hat{x},x_{n-1}) - \phi(\hat{x},x_n)\big) + 2\langle x_{n+1}-\hat{x}, Jq_n - J\hat{x}\rangle\Big) \\ &= (1-\alpha_n)\phi(\hat{x},x_n) + \alpha_n b_n, \end{aligned} \tag{4.17}$$

where

$$b_n = (1-\alpha_n)\frac{\theta_n}{\alpha_n}\big(\phi(\hat{x},x_{n-1}) - \phi(\hat{x},x_n)\big) + 2\langle x_{n+1}-\hat{x}, Jq_n - J\hat{x}\rangle.$$

Now, we prove that $\{\phi(\hat{x},x_n)\}$ converges to zero. To this end, in view of Lemma 2.10, it suffices to demonstrate that $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{\phi(\hat{x},x_{n_k})\}$ of $\{\phi(\hat{x},x_n)\}$ satisfying

$$\liminf_{k\to\infty}\Big(\phi(\hat{x},x_{n_k+1}) - \phi(\hat{x},x_{n_k})\Big) \geq 0. \tag{4.18}$$

Suppose that $\{\phi(\hat{x},x_{n_k})\}$ is a subsequence of $\{\phi(\hat{x},x_n)\}$ such that (4.18) holds. Again, from Lemma 4.5, we have

$$\begin{aligned}&(1-\alpha_{n_k})\Big(1-2k^2(\mu+\mu_{n_k})^2 c\frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2}\Big)\phi(y_{n_k},w_{n_k}) \\ &\leq (1-\alpha_{n_k})\phi(\hat{x},x_{n_k}) - \phi(\hat{x},x_{n_k+1}) + \alpha_{n_k}\Big((1-\alpha_{n_k})\frac{\theta_{n_k}}{\alpha_{n_k}}\big(\phi(\hat{x},x_{n_k-1}) \\ &\quad - \phi(\hat{x},x_{n_k})\big) + 2\langle x_{n_k+1}-\hat{x}, Jq_{n_k} - J\hat{x}\rangle\Big).\end{aligned}$$

By using Lemma 4.4(ii), (4.18), and the fact that $\lim_{k\to\infty}\alpha_{n_k} = 0$, we have

$$\lim_{k\to\infty}(1-\alpha_{n_k})\Big(1-2k^2(\mu+\mu_{n_k})^2 c\frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2}\Big)\phi(y_{n_k},w_{n_k}) = 0.$$

From the conditions on the control parameters, we obtain $\lim_{k\to\infty}\phi(y_{n_k},w_{n_k})=0$. Applying Lemma 2.5, we obtain

$$\lim_{k\to\infty}\|y_{n_k}-w_{n_k}\|=0. \tag{4.19}$$

By the continuity of A and J, we have $\lim_{k\to\infty}\|Ay_{n_k}-Aw_{n_k}\|=\lim_{k\to\infty}\|Jy_{n_k}-Jw_{n_k}\|=0$. Using the definition of u_n, we obtain

$$\lim_{k\to\infty}\|Ju_{n_k}-Jy_{n_k}\|=\lim_{k\to\infty}\|Jy_{n_k}-\lambda_{n_k}(Ay_{n_k}-Aw_{n_k})-Jy_{n_k}\|=\lim_{k\to\infty}\lambda_{n_k}\|Ay_{n_k}-Aw_{n_k}\|=0. \tag{4.20}$$

Since J is norm-to-norm continuous, we obtain from (4.19) and (4.20) that

$$\lim_{k\to\infty}\|u_{n_k}-y_{n_k}\|=\lim_{k\to\infty}\|u_{n_k}-w_{n_k}\|=0. \tag{4.21}$$

From the definition of w_n and (4.10), we have

$$\lim_{k\to\infty}\|Jw_{n_k}-Jx_{n_k}\|=\lim_{k\to\infty}\|Jx_{n_k}+\theta_{n_k}(Jx_{n_k-1}-Jx_{n_k})-Jx_{n_k}\|=\lim_{k\to\infty}\theta_{n_k}\|Jx_{n_k-1}-Jx_{n_k}\|=0. \tag{4.22}$$

Using (4.19)-(4.22), we obtain

$$\lim_{k\to\infty}\|x_{n_k}-w_{n_k}\|=\lim_{k\to\infty}\|x_{n_k}-u_{n_k}\|=\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\|=0. \tag{4.23}$$

Since J is norm-to-norm continuous, we have $\lim_{k\to\infty}\|Jx_{n_k}-Ju_{n_k}\|=0$, which together with the fact that $\lim\limits_{k\to\infty}\alpha_{n_k}=0$ yields that $\|Jx_{n_k+1}-Jx_{n_k}\|\le\alpha_{n_k}\|Jq_{n_k}-Jx_{n_k}\|+(1-\alpha_{n_k})\|Ju_{n_k}-Jx_{n_k}\|\to 0$ as $k\to\infty$. Consequently, we have

$$\lim_{k\to\infty}\|x_{n_k+1}-x_{n_k}\|=0. \tag{4.24}$$

Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $x^*\in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$ as $k\to\infty$. By (4.23), we have $y_{n_k}\rightharpoonup x^*$ as $k\to\infty$. Since C is weakly closed and $\{y_n\}\subset C$, we have that $x^*\in C$. Then by the assumption that $Ax\neq 0$, $\forall x\in C$, we have $Ax^*\neq 0$. Therefore, it follows from (4.19) and Lemma 4.6 that $x^*\in VI(C,A)^d$. Since $x^*\in w_\omega(x_n)$ was chosen arbitrarily, it follows that $w_\omega(x_n)\subset VI(C,A)^d$.

Next, by the boundedness of $\{x_{n_k}\}$, we see that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}}\rightharpoonup v$ and $\lim_{j\to\infty}\langle x_{n_{k_j}}-\hat{x},Jq_{n_{k_j}}-J\hat{x}\rangle=\limsup_{k\to\infty}\langle x_{n_k}-\hat{x},Jq_{n_k}-J\hat{x}\rangle$. Since $\hat{x}=\Pi_{VI(C,A)^d}(q)$, it follows from Lemma 2.4 that

$$\limsup_{k\to\infty}\langle x_{n_k}-\hat{x},Jq_{n_k}-J\hat{x}\rangle=\lim_{j\to\infty}\langle x_{n_{k_j}}-\hat{x},Jq_{n_{k_j}}-J\hat{x}\rangle=\langle v-\hat{x},Jq-J\hat{x}\rangle\le 0.$$

In view of (4.24), we have $\limsup_{k\to\infty}\langle x_{n_k+1}-\hat{x},Jq_{n_k}-J\hat{x}\rangle\le 0$. By applying Lemma 4.4(ii), we have $\limsup_{k\to\infty}b_{n_k}\le 0$. Now, invoking Lemma 2.10, it follows from (4.17) that $\{\phi(\hat{x},x_n)\}$ converges to zero, which implies that $\lim_{n\to\infty}x_n=\hat{x}$ as required. □

Remark 4.1. We observe that quasimonotonicity of the mapping A was only employed in **CASE 2** of Lemma 4.6. Now, we prove our second strong convergence theorem for the proposed Algorithm 3.1 without monotonicity by replacing conditions (A4) and (A5) with conditions (A4′) and (A5′), respectively.

Lemma 4.7. *Assume that* $\{w_n\}$ *and* $\{y_n\}$ *are sequences generated by Algorithm 3.1 such that conditions (A1)-(A3), (A4′)-(A5′) and Assumption B hold. Suppose that there exists a subsequence* $\{w_{n_k}\}$ *of* $\{w_n\}$ *such that* $\{w_{n_k}\}$ *converges weakly to some* $x^* \in E$ *and* $\|y_{n_k} - w_{n_k}\| \to 0$ *as* $k \to \infty$. *Then, either* $x^* \in VI(C,A)^d$ *or* $Ax^* = 0$.

Proof. Following the similar line of argument as in Lemma 4.6 and fixing $z \in C$, we obtain from (4.15) that $y_{n_k} \rightharpoonup x^* \in C$ and $\liminf_{k\to\infty}\langle Ay_{n_k},\, z - y_{n_k}\rangle \geq 0$.

Next, we choose a positive sequence $\{\zeta_k\}$ such that $\lim_{k\to\infty}\zeta_k = 0$ and $\langle Ay_{n_k}, z - y_{n_k}\rangle + \zeta_k > 0$, $\forall k \in \mathbb{N}$. From this, we obtain

$$\langle Ay_{n_k},\, z\rangle + \zeta_k > \langle Ay_{n_k},\, y_{n_k}\rangle,\ \forall k \in \mathbb{N}. \tag{4.25}$$

In particular, setting $z = x^*$ in (4.25), we have

$$\langle Ay_{n_k},\, x^*\rangle + \zeta_k > \langle Ay_{n_k}, y_{n_k}\rangle,$$

$\forall k \in \mathbb{N}$. Letting $k \to \infty$ in the last inequality, and applying the fact that $y_{n_k} \rightharpoonup x^*$ together with condition (A5′), we have $\langle Ax^*, x^*\rangle \geq \limsup_{k\to\infty}\langle Ay_{n_k}, y_{n_k}\rangle$. By (A4′), we have

$$\lim_{k\to\infty}\langle Ay_{n_k}, y_{n_k}\rangle = \langle Ax^*, x^*\rangle.$$

From (4.25) we obtain

$$\langle Ax^*,\, z\rangle = \lim_{k\to\infty}(\langle Ay_{n_k},\, z\rangle + \zeta_k) \geq \liminf_{k\to\infty}\langle Ay_{n_k},\, y_{n_k}\rangle = \lim_{k\to\infty}\langle Ay_{n_k},\, y_{n_k}\rangle = \langle Ax^*,\, x^*\rangle.$$

Hence, $\langle Ax^*, z - x^*\rangle \geq 0$ for all $z \in C$. This implies that $x^* \in VI(C,A)$. Therefore, we obtain that either $x^* \in VI(C,A)$ or $Ax^* = 0$ as required. □

Theorem 4.2. *Let* $\{x_n\}$ *be a sequence generated by Algorithm 3.1 such that conditions (A1)-(A3), (A4′)-(A5′) and Assumption B hold, and* $Ax \neq 0$, $\forall x \in C$. *Then* $\{x_n\}$ *converges strongly to* $\hat{x} \in VI(C,A)^d \subset VI(C,A)$, *where* $\hat{x} = \Pi_{VI(C,A)^d}(q)$.

Proof. By following similar argument as in Theorem 4.1 and applying Lemma 4.7, we have that $\lim_{n\to\infty} x_n = \hat{x}$ as required. □

Remark 4.2. Our result in this paper extends and complements the result of Liu and Yang [15], Chinedu *et al.* [16], Alakoya *et al.* [23], and Ogwo *et al.* [40] in the following senses:

- Algorithm 3.1 solves a larger class of quasimonotone VIP than the ones considered in [15, 16, 23, 40]. To the best of our knowledge, Algorithm 3.1 is the unique method which can solve quasimonotone VIP with non-Lipschitz operator currently.
- Our result extends the results in [15, 16, 23, 40] from Hilbert spaces to 2-uniformly convex uniformly smooth Banach spaces.
- Unlike the results in [15, 16], our proposed algorithm employs the inertial technique to speed up rate of convergence.
- In [15, 16, 40], only weak convergence results were obtained while our result in this paper are strongly convergent under more relaxed conditions.

5. NUMERICAL EXAMPLES

In this section, we present some numerical experiments to illustrate the performance of our method, Algorithm 3.1 in comparison with Algorithm 1.1, Algorithm 1.2, Algorithm 1.3, Algorithm 1.4 and Appendix 6.1. All numerical computations were carried out using Matlab version R2019(b).

We choose $\alpha_n = \frac{1}{n+1}, \xi_n = \frac{1}{(n+1)^3}, \theta = 1.5, \lambda_0 = \lambda_1 = 0.85, c = 1, k = \frac{1}{\sqrt{2}}, \mu = 0.97, \mu_n = \frac{50}{n^{0.0001}}, \phi_n = \frac{100}{n^2}, \psi = 0.7, \vartheta = \frac{3}{16}, \psi_n = \frac{n}{5n+1}, \tau_n = 0.76$, and $f(x) = \frac{2}{3}x$.

Example 5.1. Let $C := [-1,1]$ and

$$Ax = \begin{cases} 2x-1 & x > 1, \\ x^2 & x \in [-1,1], \\ -2x-1 & x < -1. \end{cases}$$

Then A is quasimonotone and Lipschitz continuous, and $VI(C,A)^d = \{-1\}$ and $VI(C,A) = \{-1,0\}$.

We use $|x_{n+1} - x_n| < 10^{-3}$ as the stopping criterion, take $q_n = \frac{n}{2n+1}x_0$, and choose different starting points as follows:
Case 1: $x_0 = 0.88,\ x_1 = 0.04$;
Case 2: $x_0 = 0.69,\ x_1 = 0.02$;
Case 3: $x_0 = 0.90,\ x_1 = 0.05$;
Case 4: $x_0 = 0.74,\ x_1 = 0.03$.

The numerical results are reported in Figures 1-4 and Table 1.

TABLE 1. Numerical Results for Example 5.1

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Liu & Yang Alg.	44	0.01486	40	0.0138	44	0.0138	41	0.0146
Izuchukwu *et al. Alg.*	39	0.0059	38	0.0063	39	0.0067	39	0.0066
Alakoya *et al. Alg.*	47	0.0077	42	0.0079	50	0.0078	45	0.0079
Alakoya *et al. Alg.*	37	0.0067	35	0.0075	38	0.0068	36	0.0073
Ogwo *et al. Alg.*	63	0.0108	72	0.0147	68	0.0111	73	0.0118
Proposed Alg. 3.1	25	0.0069	23	0.0073	25	0.0065	24	0.0063

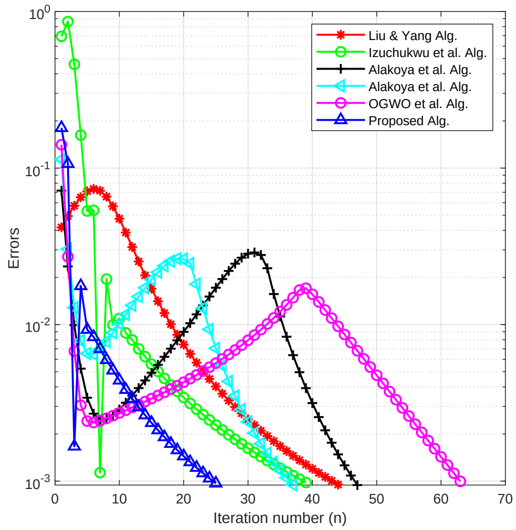

FIGURE 1. Example 5.1 Case 1

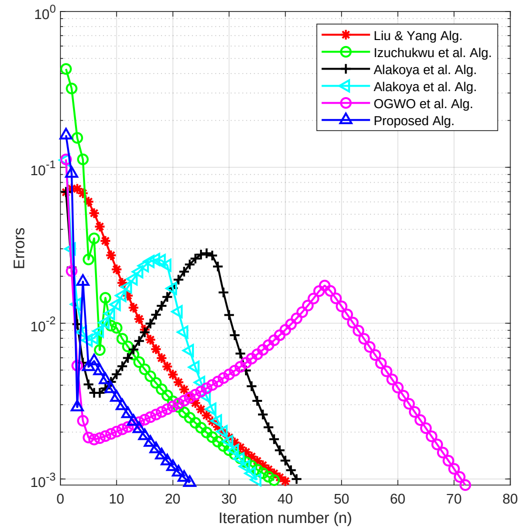

FIGURE 2. Example 5.1 Case 2

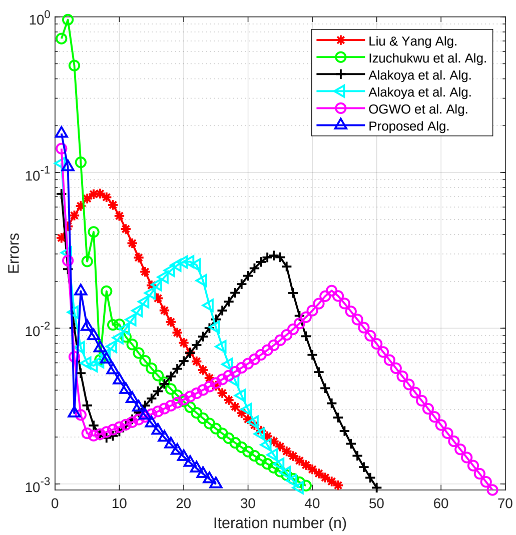

FIGURE 3. Example 5.1 Case 3

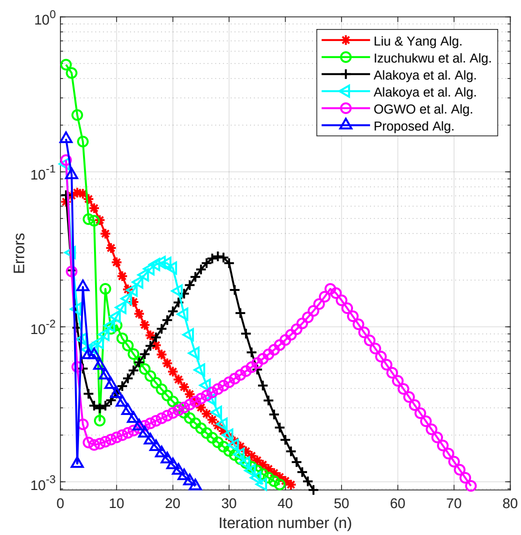

FIGURE 4. Example 5.1 Case 4

Example 5.2. Let $A:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $A(x_1,x_2)=(-x_1e^{x_2},x_2)$ and $C=\{x\in\mathbb{R}^2 : x_1^2+x_2^2\leq 1,\ 0\leq x_1\}$. Then, $(1,0)^T\in VI(C,A)^d$ and $VI(C,A)=\{(1,0)^T,(0,0)^T\}$. We use $\|x_{n+1}-x_n\|<10^{-3}$ as the stopping criterion, take $q_n=\frac{2n}{3n+1}x_1$, and choose different starting points as follows:

Case 1: $x_0=(0.1,0.2)^T,\ x_1=(0.2,0.4)^T$,

Case 2: $x_0=(0.3,0.5)^T,\ x_1=(0.2,0.2)^T$,

Case 3: $x_0 = (0.1, 0.4)^T$, $x_1 = (0.2, 0.5)^T$,
Case 4: $x_0 = (0.2, 0.3)^T$, $x_1 = (0.3, 0.5)^T$.

The numerical results are reported in Figures 5-8 and Table 2.

TABLE 2. Numerical Results for Example 5.2

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Liu & Yang Alg.	46	0.0147	58	0.0155	54	0.0149	54	0.0277
Izuchukwu *et al. Alg.*	132	0.0112	123	0.0116	129	0.0115	129	0.0159
Alakoya *et al. Alg.*	54	0.0097	56	1.0100	53	0.0099	53	0.0127
Alakoya *et al. Alg.*	54	0.0096	57	0.0097	53	0.0107	53	0.0116
Ogwo *et al. Alg.*	76	0.0138	51	0.0121	80	0.0150	80	0.0184
Proposed Alg. 3.1	26	0.0090	26	0.0091	25	0.0092	25	0.0123

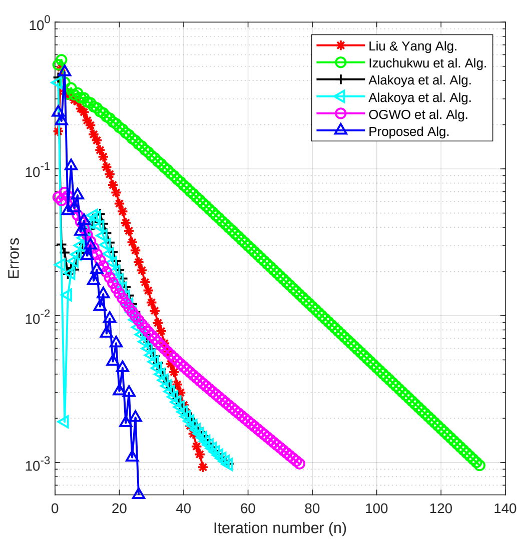

FIGURE 5. Example 5.2 Case 1

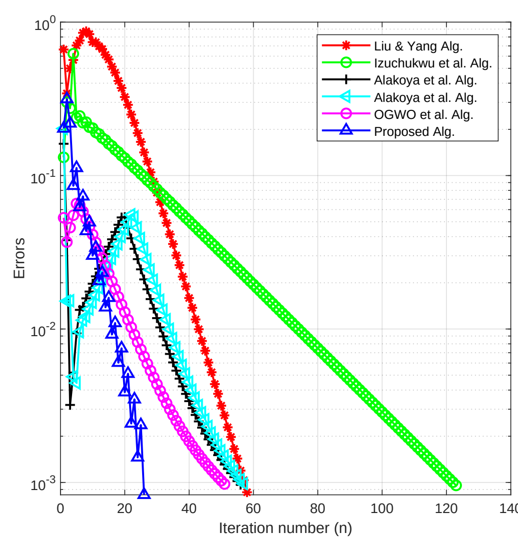

FIGURE 6. Example 5.2 Case 2

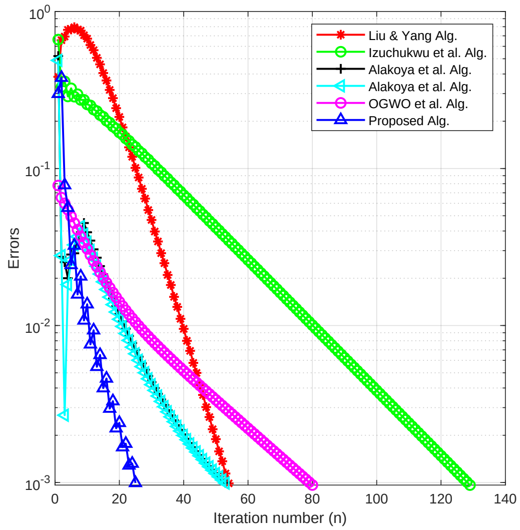

FIGURE 7. Example 5.3 Case 3

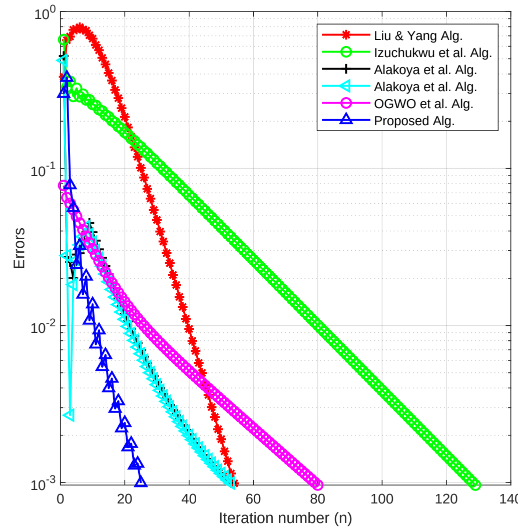

FIGURE 8. Example 5.4 Case 4

Example 5.3. Let $C := [0,1]^m$ and $Ax = (h_1x, h_2x, \cdots, h_mx)$, where $h_ix = x_{i-1}^2 + x_i^2 + x_{i-1}x_i + x_ix_{i+1} - 2x_{i-1} + 4x_i + x_{i+1} - 1$, i=1,2,...,m, $x_0 = x_{m+1} = 0$. We consider the cases $m = 10, m = 30, m = 60$, and $m = 120$ while the starting points x_0 and x_1 are generated randomly. We use $\|x_{n+1} - x_n\| < 10^{-3}$ as the stopping criterion. The numerical results are reported in Figures 9-12 and Table 3.

TABLE 3. Numerical Results for Example 5.3

	$m = 10$		$m = 30$		$m = 60$		$m = 120$	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Liu & Yang Alg.	65	0.0190	68	0.0382	62	0.0351	60	0.0406
Izuchukwu *et al. Alg.*	40	0.0110	30	0.0220	31	0.0177	11	0.0137
Alakoya *et al. Alg.*	104	0.0148	145	0.0342	176	0.0322	212	0.0361
Alakoya *et al. Alg.*	104	0.0169	145	0.0358	176	0.0322	212	0.0435
Ogwo *et al. Alg.*	58	0.0157	58	0.0224	60	0.0228	61	0.0304
Proposed Alg. 3.1	22	0.0105	23	0.0226	24	0.0196	25	0.0294

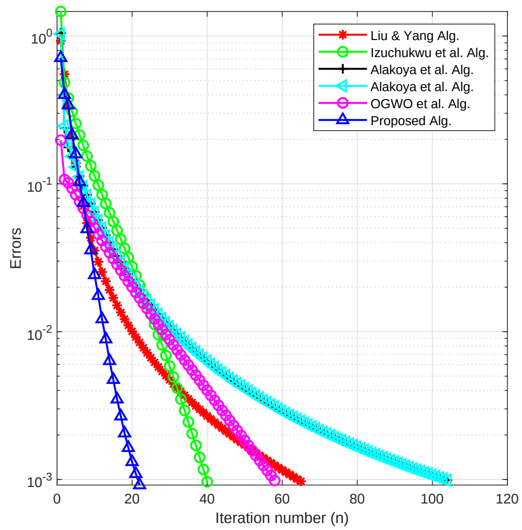

FIGURE 9. Example 5.3 with $m = 10$

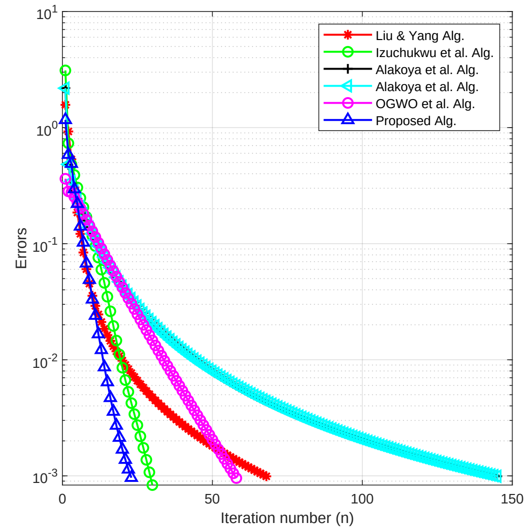

FIGURE 10. Example 5.3 with $m = 30$

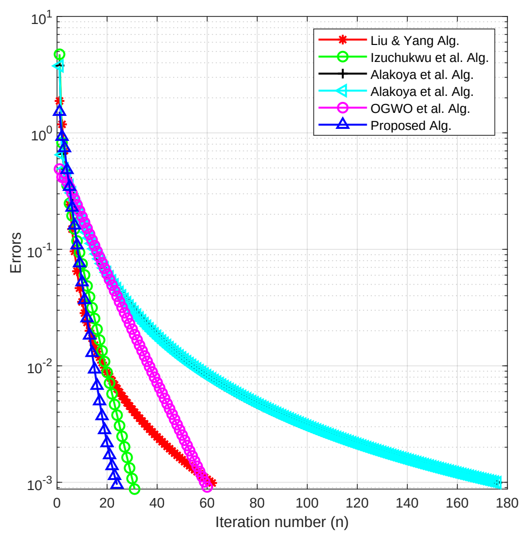

FIGURE 11. Example 5.3 with $m = 60$

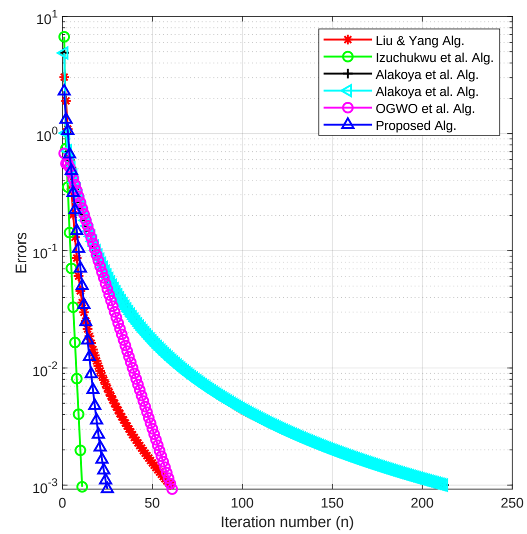

FIGURE 12. Example 5.3 with $m = 120$

Next, we present the last example in infinite dimensional Hilbert space and compare our proposed Algorithm 3.1 with Algorithm 1.3 and Algorithm 1.4, which are strongly convergent.

Example 5.4. Let $H = \ell_2 := \{x = (x_1, x_2, \ldots, x_i, \ldots) : \sum_{i=1}^{\infty} |x_i|^2 < +\infty\}$. Let $s, t \in \mathbb{R}$ be such that $s > t > \frac{s}{2} > 0$. Take $C_t = \{x \in H : \|x\| \leq t\}$ and $A_s(x) = (s - \|x\|)x$. Then A is quasimonotone

and Lipschitz continuous. Let $t = 3$ and $s = 4$. We use $\|x_{n+1} - x_n\| < 10^{-3}$ as the stopping criterion, take $q_n = \frac{3n}{5n+1}x_1$, and choose different starting points as follows:
Case 1: $x_0 = (0.1, -0.01, 0.001, \ldots)$, $x_1 = (0.4, -0.04, 0.004, \ldots)$,
Case 2: $x_0 = (0.2, -0.02, 0.002, \ldots)$, $x_1 = (0.3, 0.03, 0.003, \ldots)$,
Case 3: $x_0 = (0.1, -0.01, 0.001, \ldots)$, $x_1 = (0.3, 0.03, 0.003, \ldots)$,
Case 4: $x_0 = (-0.2, 0.02, -0.002, \ldots)$, $x_1 = (0.4, -0.04, 0.004, \ldots)$.

The numerical results are reported in Figures 13-16 and Table 4.

TABLE 4. Numerical Results for Example 5.4

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time	Iter.	CPU Time
Alakoya *et al. Alg.*	30	0.0298	28	0.0251	28	0.0232	30	0.0465
Alakoya *et al. Alg.*	33	0.0165	30	0.0142	31	0.0103	33	0.0139
Proposed Alg. 3.1	25	0.0175	25	0.0117	24	0.0125	25	0.0108

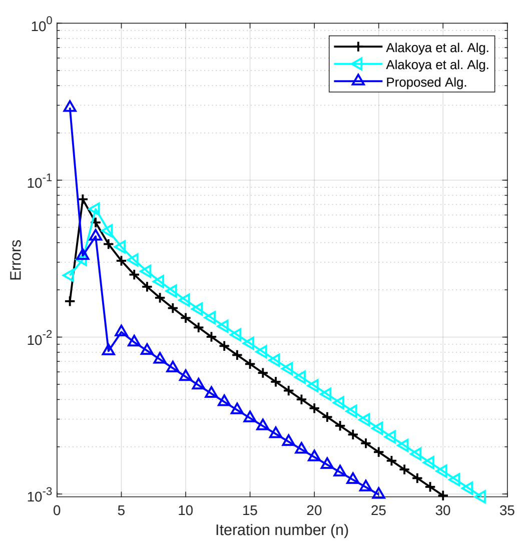

FIGURE 13. Example 5.4 Case 1

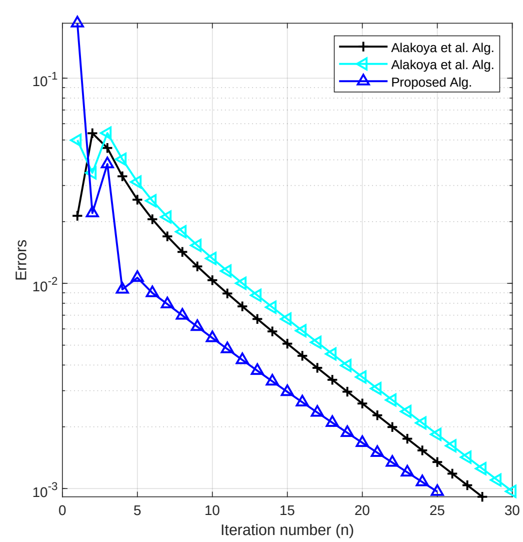

FIGURE 14. Example 5.4 Case 2

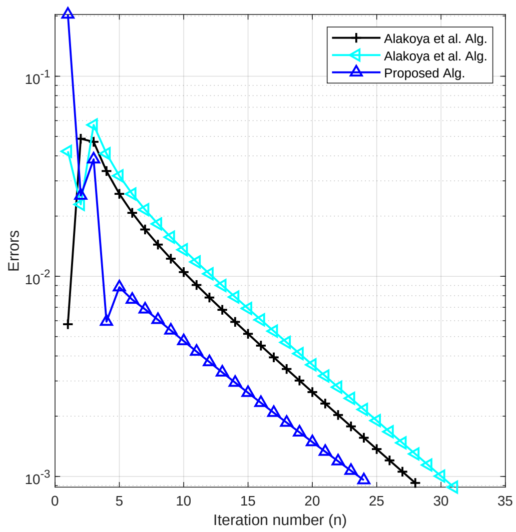

FIGURE 15. Example 5.4 Case 3

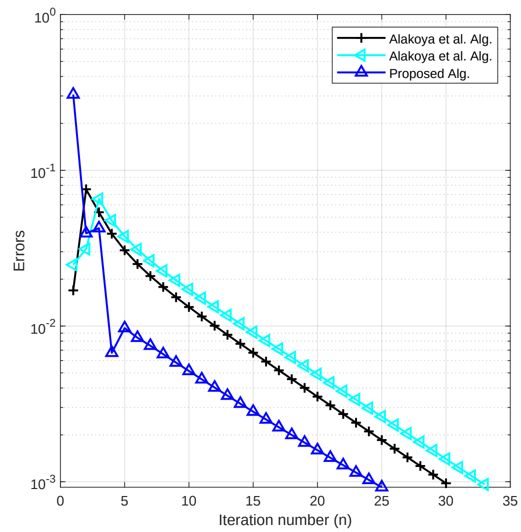

FIGURE 16. Example 5.4 Case 4

6. CONCLUSION

We studied the class of non-Lipschitz quasimonotone variational inequalities and the class of non-Lipschitz variational inequalities without monotonicity in the framework of Banach spaces. We proposed a new inertial Tseng's extragradient method for approximating the solution of the problem and obtained some strong convergence results for the proposed algorithm under some mild conditions on the control parameters. Our method does not involve any linesearch procedure but employs a more efficient and simple self-adaptive step size technique with known parameters. We presented several numerical experiments to demonstrate the applicability of our proposed method. Our result extends and generalizes the existing results in the literature in this direction.

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Appendix 6.1. Algorithm 3.4 of Ogwo et al. [40]

Step 0: Choose sequences $\{\psi_n\}$ and $\{\tau_n\}$ such that $\psi_n \in [0,1)$ and $\tau_n \in (0,1]$ for all $n \geq 1$. Let $\lambda_1 > 0$, $\mu \in (0,1)$ and $x_0, x_1 \in H$ be given arbitrarily. Choose a nonnegative real sequence $\{\rho_n\}$ such that $\sum_{n=1}^{\infty} \phi_n < +\infty$. Set $n := 1$.

Step 1: Given the current iterates x_{n-1} and x_n $(n \geq 1)$, compute

$$w_n = x_n + \psi_n(x_n - x_{n-1})$$

and

$$y_n = P_C(w_n - \lambda_n A w_n).$$

If $w_n = y_n$: STOP. Otherwise, go to **Step 2.**

Step 2: Construct the half-space

$$T_n = \{x \in H : \langle w_n - \lambda_n A w_n - y_n,\, x - y_n \rangle \leq 0\}.$$

Then, compute

$$z_n = P_{T_n}(w_n - \lambda_n A y_n)$$

and

$$x_{n+1} = (1 - \tau_n) w_n + \tau_n z_n,$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\mu\left(\|w_n - y_n\|^2 + \|z_n - y_n\|^2\right)}{2\langle A w_n - A y_n, z_n - y_n \rangle},\ \lambda_n + \phi_n \right\}, & \text{if } \langle A w_n - A y_n, z_n - y_n \rangle > 0, \\ \lambda_n + \phi_n, & \text{otherwise.} \end{cases} \tag{6.1}$$

Set $n := n + 1$ and return to **Step 1.**