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MULTIPLICITY OF SOLUTIONS TO LINEARLY COUPLED HARTREE SYSTEMS WITH CRITICAL EXPONENT

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Abstract. We consider the existence multiple solutions to the linearly coupled elliptic system

$$\begin{cases} -\Delta u+\lambda_1 u=f(x)\left(\displaystyle\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}dy\right)|u|^{2^*_\mu-2}u+\beta v, & in\ \Omega,\\ -\Delta v+\lambda_2 v=g(x)\left(\displaystyle\int_\Omega \frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu}dy\right)|v|^{2^*_\mu-2}v+\beta u, & in\ \Omega,\\ u,v\geq 0, & in\ \Omega,\\ u,v=0, & on\ \partial\Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N\geq 3)$, $0<\mu<\min\{4,N\}$, $\lambda_1,\lambda_2>-\lambda_1(\Omega)$ are constants, $\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta,H_0^1(\Omega))$, $\beta\in\mathbb{R}$ is a coupling parameter, $f,\ g\in L^\infty(\Omega)$ are nonnegative, and $2^*_\mu=\frac{2N-\mu}{N-2}$ is the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality. We prove that the system has a positive ground state solution by mountain pass theorem for small $\beta>0$. By a perturbation argument, when $\lambda_1,\lambda_2\in(-\lambda_1(\Omega),0)$, comparing with the mountain pass type solution, another positive higher energy solution is obtained when $|\beta|$ is small. In addition, the asymptotic behaviours of these solutions are analyzed as $\beta\to 0$.
Keywords. Coupled Hartree systems; Critical exponent; Positive ground state solution; Variational method.

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1. INTRODUCTION

In this paper, we study the existence and multiplicity of solutions to the following coupled elliptic system

$$\begin{cases} -\Delta u+\lambda_1 u=f(x)\left(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}dy\right)|u|^{2^*_\mu-2}u+\beta v, & in\ \Omega,\\ -\Delta v+\lambda_2 v=g(x)\left(\int_\Omega \frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu}dy\right)|v|^{2^*_\mu-2}v+\beta u, & in\ \Omega,\\ u,v\geq 0, & in\ \Omega,\\ u,v=0, & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N\geq 3)$, $0<\mu<\min\{4,N\}$, $\lambda_1,\lambda_2>-\lambda_1(\Omega)$ are constants, $\lambda_1(\Omega)$ is the first eigenvalue of $(-\Delta,H_0^1(\Omega))$, $\beta\in\mathbb{R}$ is a coupling parameter, $f,\ g\in L^\infty(\Omega)$ are nonnegative, and $2^*_\mu=\frac{2N-\mu}{N-2}$ is the upper critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality.

This system concentrates on considering standing wave (pulse-like) solutions to the time-dependent 2-coupled Hartree systems of the from

$$\begin{cases} -i\frac{\partial}{\partial t}\Phi_1=\Delta\Phi_1-W_1(x)\Phi_1+\mu_1(\Psi(x)*|\Phi_1|^2)\Phi_1+\beta\Phi_2, & x\in\Omega,\ t>0,\\ -i\frac{\partial}{\partial t}\Phi_2=\Delta\Phi_2-W_2(x)\Phi_2+\mu_2(\Psi(x)*|\Phi_2|^2)\Phi_2+\beta\Phi_1, & x\in\Omega,\ t>0,\\ \Phi_j=\Phi_j(x,t)\in\mathbb{C},\ j=1,2,\\ \Phi_j(x,t)=0,\ x\in\partial\Omega,\ t>0,\ j=1,2, \end{cases} \tag{1.2}$$

which is interested in studying the nonlinear wave propagation in various physical situations, such as nonlinear optics and quantum physics. It is well known that the propagation of optical pulses in a nonlinear 2-core directional coupler can be described by two linearly coupled nonlinear Hartree equations. Here, $\Phi_j:\mathbb{R}^+\times\mathbb{R}^N\to\mathbb{C}$ $(i=1,2)$ are radially symmetric two-body potential functions, W_i $(i=1,2)$ are the external potentials, Ψ is a nonnegative response function which possesses information about the self-interaction between the particles, and μ_i measures the strength of the self-interactions in each component, $\mu_i>0$ corresponds to the attractive (focusing) and $\mu_i<0$ to the repulsive (defocusing), and the coupling constant $\beta>0$ corresponds to the attraction (cooperation) and $\beta<0$ to the repulsion (competition) between the two components in the system. Nonlocal nonlinearities have attracted considerable interest as a means of eliminating collapse and stabilizing multidimensional solitary waves in [1]. A type of basic external potential is the most classic Coulomb function $\Psi(x)=|x|^{-1}$. There are some remarkable achievements; see, e.g., [2, 3, 4, 5, 6, 7, 8].

A standing wave solutions to (1.2) is a solution of type

$$(\Phi_1(x,t),\Phi_2(x,t))=\left(e^{-iE_1t}u(x),e^{-iE_2t}v(x)\right),\ x\in\Omega, \tag{1.3}$$

which yields that (1.2) becomes a linearly coupled nonlinear systems

$$\begin{cases} -\Delta u+\lambda_1(x)u=\mu_1(\Psi(x)*u^2)u+\beta v, & in\ \Omega,\\ -\Delta v+\lambda_2(x)v=\mu_2(\Psi(x)*v^2)v+\beta u, & in\ \Omega,\\ u,v\geq 0, & in\ \Omega,\\ u,v=0, & on\ \partial\Omega,\\ \lambda_i(x)=W_i(x)-E_i,\ i=1,2. \end{cases}$$

When $W_i=0$, the above system reduces to the following nonlinear Hartree system

$$\begin{cases} -\Delta u+\lambda_1 u=\mu_1(\Psi(x)*u^2)u+\beta v, & in\ \Omega,\\ -\Delta v+\lambda_2 v=\mu_2(\Psi(x)*v^2)v+\beta u, & in\ \Omega,\\ u,v\geq 0, & in\ \Omega,\\ u,v=0, & on\ \partial\Omega. \end{cases} \tag{1.4}$$

This indicates that λ_i $(i=1,2)$ must have an effect on the number of the solutions. With a large number of new experimental advances in multi-component Bose-Einstein condensates, the systems of coupled nonlinear Hartree equations have been focused on theoretical studies and numerical astrophysics (we refer the readers to [7, 8, 9, 10] and the references therein for a more exhaustive discussion). In recent years, there has been increasing attention to systems like (1.4) on the existence and multiplicity of positive solutions, ground states solutions, radial and non-radial solutions and semiclassical states solutions for its profound physics backgrounds. Now let us recall some related works for (1.4) with subcritical or critical exponential growth. For the subcritical cases, Chen and Liu [11] used the Nehari manifold method to study the existence of a positive radial ground state solution with $\lambda_1=\lambda_2=1,\beta\in(0,1)$. For critical cases, we would like to mention a recent interesting paper [12] on system (1.4) with a parameter β in front of the linear term. Yang et al. [12] obtained positive radial ground state solutions on the Nehari manifold for $\lambda_1=\lambda_2=1,\beta\in(0,1)$. For more results on the subcritical or critical exponent problems, readers are referred to [13, 14, 15, 16, 17] for single equations and to [9, 10] for systems.

If system (1.2) involves a Dirac-delta function, i.e., $\Psi(x)=\delta(x)$, in light of (1.3) and $\lambda_i(x)=W_i(x)-E_i$ $(i=1,2)$, when $W_i=0$, (1.2) turns to be a coupled elliptic system of Schrödinger system given by

$$\begin{cases} -\Delta u+\lambda_1 u=\mu_1 u^3+\beta v, & in\ \Omega,\\ -\Delta v+\lambda_2 v=\mu_2 v^3+\beta u, & in\ \Omega,\\ u,v\geq 0, & in\ \Omega,\\ u,v=0, & on\ \partial\Omega. \end{cases}$$

If $N\geq 3,\lambda_1,\lambda_2>0,\beta\in(0,\sqrt{\lambda_1\lambda_2})$, and $\mu_1=\mu_2=1$, Chen and Zou [18] studied the following system with one critical exponent

$$\begin{cases} -\Delta u+\lambda_1 u=u^p+\beta v, & x\in\mathbb{R}^N,\\ -\Delta v+\lambda_2 v=v^{2^*-1}+\beta u, & x\in\mathbb{R}^N,\\ u,v\in\ H^1(\mathbb{R}^N). \end{cases}$$

The authors obtained a positive solution for $\lambda_1,\lambda_2>0$, and $\beta>0$ by Nehari manifold approach and blowup analysis, However, for $\lambda_1,\lambda_2>0$, and $0<\beta<\sqrt{\lambda_1\lambda_2}$, they proved the nonexistence of the nontrivial solutions with double critical exponents. Inspired by [18], more recently, Peng, Shuai and Wang [19] considered the linearly coupled system with double critical exponents

$$\begin{cases}-\Delta u+\lambda_1 u=u^{2^*-2}u+\beta v, & in\ \Omega,\\ -\Delta v+\lambda_2 v=v^{2^*-2}v+\beta u, & in\ \Omega,\\ u,v=0, & on\ \partial\Omega.\end{cases}$$

They proved that the system has a positive ground state solution for $\beta>0$, and they admitted the system possesses a positive higher energy solution when $|\beta|$ is small enough. Moreover, the asymptotic behaviour of these solutions were analyzed as $\beta\to 0$. In the case of $\lambda_1=\lambda_2=0$, Clapp and Pistoia [20] considered the functional constrained on a subset of the Nehari manifold consisting of functions invariant with respect to a subgroup of $O(N+1)$, and they proved that the system has a positive fully Γ-invariant solution.

Motivated by the works mentioned above, a nature questions whether system (1.1) with double upper critical exponents has nontrivial solutions arises. What is the existence of positive higher energy solutions to system (1.1)? What are the asymptotic behaviors of these solutions as $\beta\to 0$? They are surprisingly interesting and seminal problems. The present paper is devoted to these aspects and partially answers these questions.

It should be noted that (1.1) has no semi-trivial solutions (i.e. $(u,0)$ or $(0,v)$) provided $\beta\neq 0$ because of the linearly coupling terms. A solution $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$ to system (1.1) is called a nontrivial solution if $(u,v)\neq(0,0)$. A nontrivial solution is also called vector solution. A solution (u,v) with $u>0$ and $v>0$ is called a positive vector solution. A solution is called a ground state (or least energy or least action) solution if the nontrivial solution possesses the least energy among all nontrivial solutions to (1.1). System (1.1) is posed in the framework of the Sobolev space $H=H_0^1(\Omega)\times H_0^1(\Omega)$ with the norm

$$\|(u,v)\|=\left(\|u\|_{\lambda_1}^2+\|v\|_{\lambda_2}^2\right)^{\frac{1}{2}},$$

where $\|u\|_{\lambda_1}^2=\int_\Omega(|\nabla u|^2+\lambda_1 u^2)$ and $\|v\|_{\lambda_2}^2=\int_\Omega(|\nabla v|^2+\lambda_2 v^2)$. It is known that solutions to (1.1) correspond to the critical points of C^1 functional $I_\beta: H\to\mathbb{R}$ defined by

$$\begin{aligned}I_\beta(u,v)\ &=\tfrac{1}{2}\int_\Omega(|\nabla u|^2+\lambda_1 u^2+|\nabla v|^2+\lambda_2 v^2)-\beta\int_\Omega uv\\ &\quad-\tfrac{1}{22_\mu^*}\int_\Omega\int_\Omega\frac{f(x)|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}+g(x)|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^\mu}.\end{aligned}$$

Moreover, for every $(\varphi,\psi)\in H$, we have

$$\begin{aligned}&\langle I_\beta'(u,v),(\varphi,\psi)\rangle\\ &=\int_\Omega(\nabla u\nabla\varphi+\lambda_1 u\varphi)-\beta\int_\Omega v\varphi-\int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*-2}u(y)\varphi(y)}{|x-y|^\mu}\\ &\quad+\int_\Omega(\nabla v\nabla\psi+\lambda_2 v\psi)-\beta\int_\Omega u\psi-\int_\Omega\int_\Omega g(x)\frac{|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*-2}v(y)\psi(y)}{|x-y|^\mu}.\end{aligned}$$

Clearly, the critical points of I_β are the weak solutions to nonlocal problem (1.1). A necessary condition for $(u,v)\in H$ to be a critical point of I_β is that $\langle I'_\beta(u,v),(u,v)\rangle = 0$. This necessary condition defines the Nehari manifold

$$\begin{aligned}\mathcal{N} &= \Big\{(u,v)\in H\setminus\{(0,0)\} : \int_\Omega (|\nabla u|^2+\lambda_1 u^2+|\nabla v|^2+\lambda_2 v^2) \\ &= \int_\Omega\int_\Omega \frac{f(x)|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}+g(x)|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy+2\beta\int_\Omega uv.\Big\}\end{aligned}$$

Then every nontrivial solution to (1.1) belongs to $\mathcal{N}$. Taking $\varphi,\psi\in C_0^\infty(\Omega)$ with $\varphi,\psi\not\equiv 0$ and $supp(\varphi)\cap supp(\psi)=\emptyset$, we have that there exist $t_1,t_2>0$ such that $(t_1\varphi,t_2\psi)\in\mathcal{N}$, so $\mathcal{N}\neq\emptyset$. Thus, we define

$$\begin{aligned}c_\beta : &= \inf_{(u,v)\in\mathcal{N}} I_\beta(u,v) \\ &= \inf_{(u,v)\in\mathcal{N}} \tfrac{N-\mu+2}{4N-2\mu}\left(\int_\Omega(|\nabla u|^2+\lambda_1u^2+|\nabla v|^2+\lambda_2v^2)-2\beta\int_\Omega uv\right) \\ &= \inf_{(u,v)\in\mathcal{N}} \tfrac{N-\mu+2}{4N-2\mu}\int_\Omega\int_\Omega\frac{f(x)|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}+g(x)|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy.\end{aligned} \tag{1.5}$$

By the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^{2^*}(\Omega)$ and Hardy-Littlewood-Sobolev inequality, it is easy to see $c_\beta>0$.

Our first main result reads as follows.

Theorem 1.1. *Assume that $-\lambda_1(\Omega)<\lambda_1,\lambda_2<0$ for $N\geq 4$. Then system (1.1) has a positive ground state solution $(u_\beta,v_\beta)\in H$ with $I_\beta(u_\beta,v_\beta)=c_\beta$ for $0<\beta<\sqrt{(\lambda_1+\lambda_1(\Omega))(\lambda_2+\lambda_1(\Omega))}$. Let $\beta_n\in(0,\sqrt{(\lambda_1+\lambda_1(\Omega))(\lambda_2+\lambda_1(\Omega))}), n\in\mathbb{N}$ such that $\beta_n\to 0$ as $n\to\infty$. Then, passing to a subsequence, $(u_{\beta_n},v_{\beta_n})\to(\bar u,\bar v)$ in H as $n\to\infty$, and one of the following conclusion holds:*
(i) $\bar u\equiv 0$ and $\bar v$ is a positive ground state solution to

$$-\Delta v+\lambda_2 v = g(x)\Big(\int_\Omega\frac{|v(y)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|v|^{2^*_\mu-2}v,\ v\in H_0^1(\Omega);$$

(ii) $\bar v\equiv 0$ and $\bar u$ is a positive ground state solution to

$$-\Delta u+\lambda_1 u = f(x)\Big(\int_\Omega\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}dy\Big)|u|^{2^*_\mu-2}u,\ u\in H_0^1(\Omega).$$

In particular, if $\lambda_1>\lambda_2$, (i) occurs; if $\lambda_1<\lambda_2$, (ii) occurs.

Remark 1.1. If $\Omega=\mathbb{R}^N$ and $\lambda_1,\lambda_2>0$, and $0<\beta<\sqrt{\lambda_1\lambda_2}$, then by Pohožaev identity, we easily check the solutions (u,v) to (1.1) satisfy

$$
\begin{aligned}
0 &= \frac{1}{2}\int_{\partial\Omega}(|\nabla u|^2+|\nabla v|^2)x\nu ds\\
&= -\frac{N}{2}\int_{\Omega}\lambda_1 u^2+\frac{2N-\mu}{22^*_\mu}\int_{\Omega}\int_{\Omega}f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy+\frac{N}{2}\beta\int_{\Omega}uv\\
&\quad -\frac{N}{2}\int_{\Omega}\lambda_2 v^2+\frac{2N-\mu}{22^*_\mu}\int_{\Omega}\int_{\Omega}g(x)\frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy+\frac{N}{2}\beta\int_{\Omega}uv\\
&\quad -\frac{N-2}{2}\int_{\Omega}(|\nabla u|^2+|\nabla v|^2)\\
&= -\frac{N}{2}\int_{\Omega}\lambda_1 u^2+\frac{2N-\mu}{22^*_\mu}\int_{\Omega}\int_{\Omega}f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy+\frac{N}{2}\beta\int_{\Omega}uv\\
&\quad -\frac{N}{2}\int_{\Omega}\lambda_2 v^2+\frac{2N-\mu}{22^*_\mu}\int_{\Omega}\int_{\Omega}g(x)\frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy+\frac{N}{2}\beta\int_{\Omega}uv\\
&\quad -\frac{N-2}{2}\left[-\int_{\Omega}\lambda_1 u^2+\int_{\Omega}\int_{\Omega}f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy+\beta\int_{\Omega}uv\right]\\
&\quad -\frac{N-2}{2}\left[-\int_{\Omega}\lambda_2 v^2+\int_{\Omega}\int_{\Omega}g(x)\frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy+\beta\int_{\Omega}uv\right]\\
&= \lambda_1\int_{\Omega}|u|^2+\lambda_2\int_{\Omega}|v|^2-2\beta\int_{\Omega}uv,
\end{aligned}
$$

where $\frac{2N-\mu}{22^*_\mu}=\frac{N-2}{2}$. So $(u,v)=(0,0)$. Therefore, Ω is always supposed to be a bounded domain in $\mathbb{R}^N$ with smooth boundary. In addition, if Ω is a star shaped domain, $\lambda_1,\lambda_2>0$ and $0<\beta<\sqrt{\lambda_1\lambda_2}$, using Pohožaev type identity, we find that system (1.1) has no nontrivial solutions. However, for $\lambda_1\cdot\lambda_2<0$, we do not know whether there exist solutions or not. This is a more complicated and challenging problem, but we believe that there exist nontrivial solutions under some special situations. We would like to go further in this direction in the future.

Remark 1.2. If $\lambda_1=\lambda_2$, $\lambda_1-\beta\in(-\lambda_1(\Omega),0)$ and $f(x)=g(x)$, we obtain from [21] that system (1.1) has a positive solution (u_1,u_1), where u_1 is a positive least energy solution to

$$-\Delta u+\lambda_1 u=f(x)\left(\int_{\Omega}\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}dy\right)|u|^{2^*_\mu-2}u+\beta u,\ u\in H_0^1(\Omega).$$

On the other hand, if $\lambda_1=\lambda_2$, $\lambda_1+\beta\in(-\lambda_1(\Omega),0)$, and $f(x)=g(x)$, then system (1.1) has a nontrivial solution $(u_2,-u_2)$, where u_2 is a positive least energy solution to

$$-\Delta u+\lambda_1 u=f(x)\left(\int_{\Omega}\frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu}dy\right)|u|^{2^*_\mu-2}u-\beta u,\ u\in H_0^1(\Omega).$$

It is obvious that the solutions (u_β,v_β) obtained in Theorem 1.1 depend on β. Clearly, fix $\beta>0$ small enough, and then the solutions (u_1,u_1) and $(u_2,-u_2)$ are different from the ground state solution obtained in Theorem 1.1.

Analyzing under what conditions we have the ground state solutions, we also found in surprise that there are higher energy type positive solutions. In fact, as a by-product of our analysis, we have the following asymptotic behavior to these positive solutions. The result in this aspect can be stated as following.

Theorem 1.2. *Assume that λ_1 and λ_2 are in $(-\lambda_1(\Omega),0)$ for $N\geq 4$. Then there exists $\beta_0\in(0,\sqrt{(\lambda_1+\lambda(\Omega))(\lambda_2+\lambda_1(\Omega))})$ such that, for $|\beta|<\beta_0$, system (1.1) has a positive higher energy solution $(\tilde{u}_\beta,\tilde{v}_\beta)$ with $I_\beta(\tilde{u},\tilde{v})>c_\beta$. Let β_n, $n\in\mathbb{N}$, be a sequence with $|\beta_n|<\beta_0$ and $\beta_n\to 0$ as $n\to+\infty$. Then, passing to a subsequence, $(\tilde{u}_{\beta_n},\tilde{v}_{\beta_n})\to(\tilde{u},\tilde{v})$ in $H_0^1(\Omega)\times H_0^1(\Omega)$ as $n\to+\infty$, where $\tilde{u}$ is a positive ground state solution to*

$$-\Delta u+\lambda_1 u=f(x)\left(\int_\Omega\frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|u|^{2_\mu^*-2}u,\ u\in H_0^1(\Omega),$$

and $\tilde{v}$ is a positive ground state solution to

$$-\Delta v+\lambda_2 v=g(x)\left(\int_\Omega\frac{|v(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|v|^{2_\mu^*-2}v,\ v\in H_0^1(\Omega).$$

This theorem is an extension and complement of the corresponding result in [18, 19] and gives new insight into the higher energy solutions to (1.1). Obviously, for $\beta>0$ small enough, the higher energy solutions in Theorem 1.2 are different from the ground state solutions in Theorem 1.1. That is, system (1.1) has at least two positive solutions for $\lambda_1,\lambda_2<0$ and $\beta>0$ sufficiently small.

Our argument can also be used to study the following $k-$linaedry coupled system

$$\begin{cases}-\Delta u_i+\lambda_i u_i=\alpha_i\left(\int_\Omega\frac{|u_i(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|u_i|^{2_\mu^*-2}u_i+\sum\limits_{j\neq i}^{k}\beta_{ij}u_j, & \text{in } \Omega,\\ u_i=0, & \text{on } \partial\Omega\ i=1,2,3,\cdots,n.\end{cases}$$

Now, we follow the variational approach to prove our results. Since we are concerned with the system (1.1) involving two nonlocal convolution terms with the upper critical exponents, i.e., $\int_\Omega(\frac{1}{|x|^\mu}*|u|^{2_\mu^*})|u|^{2_\mu^*}$ and $\int_\Omega(\frac{1}{|x|^\mu}*|v|^{2_\mu^*})|v|^{2_\mu^*}$, the problem becomes more complicated in applying variational methods. Firstly, we cannot use the usual arguments (such as Ekeland variation principle) to seek a *PS* sequence for minimizing problem (1.5). Secondly, the approaches adopted in [11, 12, 20] do not work for this paper when the work place satisfies neither radially symmetric settings nor Γ-invariant bounded smooth domain. The above facts bring about two obstacles to the standard mountain pass argument both in checking the mountain pass geometrical construct in the corresponding energy functional and in proving the boundedness of corresponding *PS* sequences. In order to overcome these difficulties, we adopt an idea, due to [19, 22] together with new anlaysis techniques to prove the multiplicity of solutions to (1.1), where the Hardy-Littlewood-Sobolev inequality play an important role in proving the convergence of *PS* sequence. This idea is novel and effective to obtain a positive ground state solutions and higher energy solutions to linearly coupled Hartree system (1.1) with upper critical exponents.

To prove Theorem (1.1), we combine constraint minimization methods and the classical mountain pass theorem to obtain a positive ground state solution. Precisely, we perform a careful analysis of the behavior of *PS* sequences to study the possible reason of lack of compactness and to pick out the ranges of energy levels where the PS condition holds and compactness can be recovered. In fact, we work out a threshold value c_β of energy under which a PS sequence is compact, where all paths are required to be uniformly bounded in H with respect to the parameter β. That pulls the energy level down below critical level to recover PS sequence compactness. This indicates that c_β is less strict than the critical energy value

$$\frac{N-\mu+2}{4N-2\mu}\min\{S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}|f|_\infty^{-\frac{N-2}{N-\mu+2}},S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}|g|_\infty^{-\frac{N-2}{N-\mu+2}}\}.$$

Compared with the single Choquard problem [21] and Schrödinger systems like [19, 22], the minimum threshold level c_β cannot be obtained easily, and more careful analysis and new tricks are needed in this circumstance. At the same time, we will give an exact analysis to the positive solution with the desired asymptotic property.

To prove Theorem 1.2, we also use a perturbation approach to study the existence of a positive higher energy solution to (1.1). We mainly employ some ideas from Chen and Zou [22] and this argument was later generalized in [19]. However, this approach cannot be used directly, and we need some crucial modifications for our proof. In fact, it is based on a quantitative deformation lemma to construct a special PS sequence to nonlinear variational problems. To this end, some discussions are much more involved here. The paper is organized as follows. In Section 2, we first present some preliminary results for (1.1). Section 3 is devoted to the proof of Theorem 1.1. In Section 4, the last section, we prove Theorem 1.2.

2. Notations and Preliminaries

In this section, we begin with variational framework and some lemmas, which are important for us to prove our mains results.

From now on, we assume that $-\lambda_1(\Omega)<\lambda_1,\lambda_2<0$ for $N\geq 4$. Recall that

$$D^{1,2}(\mathbb{R}^N)=\Big\{u:u\in L^{2^*}(\mathbb{R}^N),\nabla u\in L^2(\mathbb{R}^N)\Big\}$$

with the standard norm $\|u\|_{D^{1,2}}^2:=\int_{\mathbb{R}^N}|\nabla u|^2$. Consider the following Brézis-Nirenberg problems with Hartree term for $\lambda_i\in(-\lambda_1(\Omega),0)$ $(i=1,2)$

$$-\Delta u+\lambda_1 u=f(x)\left(\int_\Omega\frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|u|^{2_\mu^*-2}u,\ u\in\ H_0^1(\Omega),\tag{2.1}$$

$$-\Delta v+\lambda_2 v=g(x)\left(\int_\Omega\frac{|v(y)|^{2_\mu^*}}{|x-y|^\mu}dy\right)|v|^{2_\mu^*-2}v,\ v\in\ H_0^1(\Omega),$$

and the corresponding energy functionals defined by

$$J_f(u)=\tfrac{1}{2}\int_\Omega(|\nabla u|^2+\lambda_1u^2)-\frac{1}{22_\mu^*}\int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy,$$
$$J_g(v)=\tfrac{1}{2}\int_\Omega(|\nabla v|^2+\lambda_2v^2)-\frac{1}{22_\mu^*}\int_\Omega\int_\Omega g(x)\frac{|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy,$$

and

$$\begin{aligned} B_f &:= \inf_{u\in\mathcal{N}_f} J_f(u) \le \tfrac{N+2-\mu}{4N-2\mu}|f|_\infty^{-\frac{N-2}{N+2-\mu}} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}, \\ B_g &:= \inf_{u\in\mathcal{N}_g} J_g(u) \le \tfrac{N+2-\mu}{4N-2\mu}|g|_\infty^{-\frac{N-2}{N+2-\mu}} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}, \end{aligned} \tag{2.2}$$

where

$$\mathcal{N}_f = \left\{ u \in H_0^1(\Omega)\setminus\{0\} : \int_\Omega (|\nabla u|^2 + \lambda u^2) = \int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \right\},$$

and

$$\mathcal{N}_g = \left\{ u \in H_0^1(\Omega)\setminus\{0\} : \int_\Omega (|\nabla v|^2 + \lambda v^2) = \int_\Omega\int_\Omega g(x)\frac{|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \right\}.$$

By [21], (2.1) has a positive least energy solution w with energy

$$\begin{aligned} \tfrac{N+2-\mu}{4N-2\mu}|f|_\infty^{-\frac{N-2}{N+2-\mu}} \left(S_{H,L}\tfrac{\lambda_1(\Omega)+\lambda}{\lambda_1(\Omega)}\right)^{\frac{2N-\mu}{N+2-\mu}} \le B_f &= \tfrac{N+2-\mu}{4N-2\mu}\int_\Omega (|\nabla w|^2 + \lambda w^2) \\ &= \tfrac{N+2-\mu}{4N-2\mu}\int_\Omega\int_\Omega f(x)\frac{|w(x)|^{2_\mu^*}|w(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\ &< \tfrac{N+2-\mu}{4N-2\mu}|f|_\infty^{-\frac{N-2}{N+2-\mu}} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}. \end{aligned}$$

Moreover

$$\|u\|_{\lambda_1}^2 = \int_\Omega (|\nabla u|^2 + \lambda_1 u^2) \ \ge \left(\tfrac{4N-2\mu}{N+2-\mu}B_f\right)^{\frac{N+2-\mu}{2N-\mu}} \left(\int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\right)^{\frac{1}{2_\mu^*}}.$$

Similarly,

$$\|v\|_{\lambda_2}^2 = \int_\Omega (|\nabla v|^2 + \lambda_2 v^2) \ \ge \left(\tfrac{4N-2\mu}{N+2-\mu}B_g\right)^{\frac{N+2-\mu}{2N-\mu}} \left(\int_\Omega\int_\Omega g(x)\frac{|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\right)^{\frac{1}{2_\mu^*}}.$$

Let $S_i = \{u \in H_0^1(\Omega), J_i'(u) = 0,\ B_i = J_i(u),\ i = f,g\}$. Obviously, the set is nonempty if $\lambda_i \in (-\lambda_1(\Omega), 0)$ $i = 1,2$ for $N \ge 4$. The Hardy-Littlewood-Sobolev inequality (see [23]) is important to our proof. Let $f = h = |u|^q$. Then, by Hardy-Littlewood-Sobolev inequality, the following integral

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}dxdy$$

is well defined if $|u|^q \in L^t(\mathbb{R}^N)$ for some $t > 1$ satisfying $\frac{2}{t} + \frac{\mu}{N} = 2$. Thus, for $u \in H^1(\mathbb{R}^N)$, by the Sobolev embedding theorem, we know $2 \le tq \le \frac{2N}{N-2}$, which implies that

$$\frac{2N-\mu}{N} \le q \le \frac{2N-\mu}{N-2}.$$

Hence, $\frac{2N-\mu}{N}$ (or $2_\mu^* = \frac{2N-\mu}{N-2}$) is called the *lower* (or *upper*) critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality. For $u \in D^{1,2}(\mathbb{R}^N)$, by Hardy-Littlewood-Sobolev inequality, one has

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy\right)^{\frac{1}{2^*_\mu}} \leq C(N,\mu)^{\frac{1}{2^*_\mu}}|u|^2_{2^*}.$$

Let $S_{H,L}$ be the best constant given by

$$S_{H,L} := \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N}|\nabla u|^2}{\left(\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy\right)^{\frac{1}{2^*_\mu}}}.$$

The constant $S_{H,L}$ is achieved if and only if

$$u = C\left(\frac{b}{b^2+|x-a|^2}\right)^{\frac{N-2}{2}},$$

where $C>0$, $a\in\mathbb{R}^N$, and $b\in(0,\infty)$.

Lemma 2.1. [21] *Let $N\geq 3$. For all open subset Ω of $\mathbb{R}^N$,*

$$S_{H,L}(\Omega) := \inf_{u\in D_0^{1,2}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_{\Omega}|\nabla u|^2}{\left(\displaystyle\int_{\Omega}\int_{\Omega}\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy\right)^{\frac{1}{2^*_\mu}}} = S_{H,L}, \tag{2.3}$$

where $S_{H,L}(\Omega)$ is never achieved except when $\Omega=\mathbb{R}^N$.

Lemma 2.2. *If the set S_i $(i=f,g)$ is nonempty, then S_i is compact in $H_0^1(\Omega)$.*

Proof. Without loss of generality, we suppose that S_i is nonempty and there exists a sequence $\{u_n\}\subset S_f$. Then, $\{u_n\}$ is a bounded $(PS)_{B_f}$ sequence of J_f and

$$\int_\Omega(|\nabla u_n|^2+\lambda_1 u_n^2) = \int_\Omega\int_\Omega f(x)\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy.$$

So, up to a subsequence, $u_n \rightharpoonup u_\infty$ in $H_0^1(\Omega)$ and $I'(u_\infty)=0$ in $H^{-1}(\Omega)$. Taking into account that

$$B_f < \frac{N-\mu+2}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}|f|_\infty^{-\frac{N-2}{N-\mu+2}}$$

and J_f satisfies the $(PS)_{B_f}$ condition, one has $u_n\to u_\infty$ in $H_0^1(\Omega)$ and $u_\infty\in S_f$. □

Lemma 2.3. *The functional I_β satisfies the mountain pass geometry, i.e.,*
(i) there exist $\alpha,\rho>0$ such that $I_\beta(u,v)>\alpha$ for all $\|(u,v)\|=\rho$;
(ii) there exists $(u_0,v_0)\in H$ such that $\|(u_0,v_0)\|>\rho$ and $I_\beta(u_0,v_0)<0$.

Proof. Since $\beta\in(0,\sqrt{(\lambda_1+\lambda_1(\Omega))(\lambda_2+\lambda_1(\Omega))})$, we choose β_1 and β_2 with $0<\beta_1<\lambda_1+\lambda_1(\Omega)$ and $0<\beta_2<\lambda_2+\lambda_1(\Omega)$ such that $\beta<\sqrt{\beta_1\beta_2}$. Note that $\lambda_1,\lambda_2>-\lambda_1(\Omega)$. From (2.3), we have

$$\begin{aligned}
I_\beta(u,v) &= \tfrac{1}{2}\int_\Omega(|\nabla u|^2+\lambda_1u^2+|\nabla v|^2+\lambda_2v^2)-\beta\int_\Omega uv \\
&-\tfrac{1}{22^*_\mu}\int_\Omega\int_\Omega\left[f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy+g(x)\frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy\right] \\
&\geq \tfrac{1}{2}\int_\Omega|\nabla u|^2+(\lambda_1-\beta_1)u^2+|\nabla v|^2+(\lambda_2-\beta_2)v^2 \\
&-\tfrac{1}{22^*_\mu}\int_\Omega\int_\Omega\left[f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy+g(x)\frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy\right] \\
&\geq \tfrac{1}{2}\int_\Omega|\nabla u|^2+\frac{1}{2}\frac{\lambda_1-\beta_1}{\lambda_1(\Omega)}\int_\Omega|\nabla u|^2-\frac{1}{22^*_\mu}|f|_\infty\left(S^{-1}_{H,L}\frac{\lambda_1(\Omega)}{\lambda_1(\Omega)+\lambda_1}\right)^{2^*_\mu}\|u\|^{22^*_\mu}_{\lambda_1} \\
&+\tfrac{1}{2}\int_\Omega|\nabla v|^2+\frac{1}{2}\frac{\lambda_2-\beta_2}{\lambda_1(\Omega)}\int_\Omega|\nabla v|^2-\frac{1}{22^*_\mu}|g|_\infty\left(S^{-1}_{H,L}\frac{\lambda_1(\Omega)}{\lambda_1(\Omega)+\lambda_2}\right)^{2^*_\mu}\|v\|^{22^*_\mu}_{\lambda_2} \\
&= \tfrac{1}{2}(1+\tfrac{\lambda_1-\beta_1}{\lambda_1(\Omega)})\int_\Omega|\nabla u|^2-\frac{1}{22^*_\mu}|f|_\infty\left(S^{-1}_{H,L}\frac{\lambda_1(\Omega)}{\lambda_1(\Omega)+\lambda_1}\right)^{2^*_\mu}\|u\|^{22^*_\mu}_{\lambda_1} \\
&+\tfrac{1}{2}(1+\tfrac{\lambda_2-\beta_1}{\lambda_1(\Omega)})\int_\Omega|\nabla v|^2-\frac{1}{22^*_\mu}|g|_\infty\left(S^{-1}_{H,L}\frac{\lambda_1(\Omega)}{\lambda_1(\Omega)+\lambda_2}\right)^{2^*_\mu}\|v\|^{22^*_\mu}_{\lambda_2} \\
&\geq \tfrac{1}{2}(1+\tfrac{\lambda_1-\beta_1}{\lambda_1(\Omega)})\int_\Omega(|\nabla u|^2+\lambda_1u^2)-\frac{1}{22^*_\mu}|f|_\infty\left(S^{-1}_{H,L}\frac{\lambda_1(\Omega)}{\lambda_1(\Omega)+\lambda_1}\right)^{2^*_\mu}\|u\|^{22^*_\mu}_{\lambda_1} \\
&+\tfrac{1}{2}(1+\tfrac{\lambda_2-\beta_2}{\lambda_1(\Omega)})\int_\Omega(|\nabla v|^2+\lambda_2v^2)-\frac{1}{22^*_\mu}|g|_\infty\left(S^{-1}_{H,L}\frac{\lambda_1(\Omega)}{\lambda_1(\Omega)+\lambda_2}\right)^{2^*_\mu}\|v\|^{22^*_\mu}_{\lambda_2} \\
&= \tfrac{1}{2}\tfrac{\lambda_1(\Omega)+\lambda_1-\beta_1}{\lambda_1(\Omega)}\|u\|^2_{\lambda_1}-\tfrac{1}{22^*_\mu}|f|_\infty\left(S^{-1}_{H,L}\tfrac{\lambda_1(\Omega)}{\lambda_1(\Omega)+\lambda_1}\right)^{2^*_\mu}\|u\|^{22^*_\mu}_{\lambda_1} \\
&+\tfrac{1}{2}\tfrac{\lambda_1(\Omega)+\lambda_2-\beta_2}{\lambda_1(\Omega)}\|v\|^2_{\lambda_2}-\tfrac{1}{22^*_\mu}|g|_\infty\left(S^{-1}_{H,L}\tfrac{\lambda_1(\Omega)}{\lambda_1(\Omega)+\lambda_2}\right)^{2^*_\mu}\|v\|^{22^*_\mu}_{\lambda_2} \\
&\geq C_1\|(u,v)\|^2-C_2\|(u,v)\|^{22^*_\mu} \\
&= (C_1-C_2\|(u,v)\|^{22^*_\mu-2})\|(u,v)\|^2.
\end{aligned}$$

Since $2<22^*_\mu$, one chooses $\alpha,\rho>0$ such that $I_\beta(u,v)\geq\alpha$ for $\|(u,v)\|=\rho$.

(ii) For some $(u_1,v_1)\in H^1_0(\Omega)\setminus\{(0,0)\}$, one has

$$\begin{aligned}
I_\beta(u_1,v_1) &= \tfrac{t^2}{2}\int_\Omega(|\nabla u_1|^2+\lambda_1u_1^2)-\frac{t^{22^*_\mu}}{22^*_\mu}\int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}-\beta t^2\int_\Omega uv \\
&+\tfrac{t^2}{2}\int_\Omega(|\nabla v_1|^2+\lambda_2v_1^2)-\frac{t^{22^*_\mu}}{22^*_\mu}\int_\Omega\int_\Omega g(x)\frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}-\beta t^2\int_\Omega uv<0
\end{aligned}$$

for t large enough. So, there exist $t_1,(u_0,v_0)>0$ such that

$$\|(u_0,v_0)\|=\|(t_1u_1,t_1v_1)\|=t_1\|(u_1,v_1)\|>\rho$$

and $I_\beta(t_1u_1,t_1v_1)<0$. The proof of this lemma is complete. □

From Lemma 2.3 and the mountain pass theorem, one concludes that there exists a $(PS)_{\bar{c}_\beta}$ sequence $\{(u_n, v_n)\}$ such that

$$I_\beta(u_n, v_n) \to \bar{c}_\beta,\ I_\beta^{'}(u_n, v_n) \to 0,\ as\ n \to \infty \tag{2.4}$$

at the minimax level $0 < \bar{c}_\beta := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\beta(\gamma(t))$, where

$$\Gamma = \{\gamma \in C([0,1], H) : \gamma(0) = (0,0), \gamma(1) = (u_0, v_0)\}.$$

Lemma 2.4. $\bar{c}_\beta = \inf_{H\setminus\{(0,0)\}} \max_{t\in[0,1]} I_\beta(tu,tv) = c_\beta$.

Proof. For every $(u,v) \in H$ with $(u,v) \neq (0,0)$, there exists a unique $t_{\beta,u,v} > 0$ such that

$$\begin{aligned}
&\max_{t>0} I_\beta(tu,tv) = I_\beta(t_{\beta,u,v}u, t_{\beta,u,v}v)\\
= &\ \tfrac{t^2_{\beta,u,v}}{2}\int_\Omega(|\nabla u|^2+\lambda_1 u^2) - \frac{t^{22^*_\mu}_{\beta,u,v}}{22^*_\mu}\int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} - \beta t^2_{\beta,u,v}\int_\Omega uv\\
&+\tfrac{t^2_{\beta,u,v}}{2}\int_\Omega(|\nabla v|^2+\lambda_2 v^2) - \frac{t^{22^*_\mu}_{\beta,u,v}}{22^*_\mu}\int_\Omega\int_\Omega g(x)\frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu} - \beta t^2_{\beta,u,v}\int_\Omega uv\\
&-\tfrac{1}{22^*_\mu}\left[\int_\Omega(|\nabla(t_{\beta,u,v}u)|^2+\lambda_1 t^2_{\beta,u,v}u^2) - \int_\Omega\int_\Omega f(x)\frac{|t_{\beta,u,v}u(x)|^{2^*_\mu}|t_{\beta,u,v}u(y)|^{2^*_\mu}}{|x-y|^\mu} - \beta\int_\Omega t^2_{\beta,u,v}uv\right]\\
&-\tfrac{1}{22^*_\mu}\left[\int_\Omega(|\nabla(t_{\beta,u,v}v)|^2+\lambda_2 t^2_{\beta,u,v}v^2) - \int_\Omega\int_\Omega g(x)\frac{|t_\beta v(x)|^{2^*_\mu}|t_{\beta,u,v}v(y)|^{2^*_\mu}}{|x-y|^\mu} - \beta\int_\Omega t^2_{\beta,u,v}uv\right]\\
= &\ \tfrac{N-\mu+2}{4N-2\mu}t^2_{\beta,u,v}\left(\int_\Omega|\nabla u|^2+\lambda_1u^2+|\nabla v|^2+\lambda_2v^2-2\beta\int_\Omega uv\right)\\
= &\ \tfrac{N-\mu+2}{4N-2\mu}t^{22^*_\mu}_{\beta,u,v}\int_\Omega\int_\Omega\left(f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}+g(x)\frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}\right),
\end{aligned}$$

where $t_{\beta,u,v} > 0$ satisfies

$$t^{22^*_\mu-2}_{\beta,u,v} = \frac{\int_\Omega(|\nabla u|^2+\lambda_1u^2+|\nabla v|^2+\lambda_2v^2)-2\beta\int_\Omega uv}{\int_\Omega\int_\Omega\left(f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}+g(x)\frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}\right)}. \tag{2.5}$$

Since $(t_{\beta,u,v}u, t_{\beta,u,v}v) \in \mathscr{N}$, one has

$$\bar{c}_\beta = \inf_{H\setminus\{(0,0)\}}\max_{t>0} I_\beta(tu,tv) = c_\beta. \tag{2.6}$$

This completes the proof. ∎

Lemma 2.5. $c_\beta \le \min\{B_f, B_g\} < \frac{N-\mu+2}{4N-2\mu}\min\left\{S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}|f|_\infty^{-\frac{N-2}{N-\mu+2}}, S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}|g|_\infty^{-\frac{N-2}{N-\mu+2}}\right\}$.

Proof. Without loss of generality, we assume that $B_f \le B_g$. By (2.2), we only need to prove $c_\beta \le B_f$. From Lemma 2.4, we have that there exists a unique $t_{\beta,u,0} > 0$ such that $(t_{\beta,u,0}u_f, 0) \in$

$\mathcal{N}$, where u_f is a positive ground state solution to (2.1). Moreover

$$c_\beta \le \max_{t>0} I_\beta(tu_f,0) = J_f(u_f) = \frac{N-\mu+2}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} |f|_\infty^{-\frac{N-2}{N-\mu+2}} = B_f.$$

The proof is completed. □

3. Existence of Positive Ground State Solutions

In this section, we give the proof, which is divided into two parts, to Theorem 1.1. In the first part, we prove system (1.1) has a nontrivial positive ground state solution. The asymptotic behavior of the ground state solution as $\beta \to 0^+$ will be given in the second part.

The proof of the first part of Theorem 1.1. By (2.4) and Lemma 2.5, there exists $\{(u_n,v_n)\} \subset H$ such that

$$\lim_{n\to+\infty} I_\beta(u_n,v_n) = c_\beta < \frac{N-\mu+2}{4N-2\mu} S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} \min\left\{ S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} |f|_\infty^{-\frac{N-2}{N-\mu+2}}, S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} |g|_\infty^{-\frac{N-2}{N-\mu+2}} \right\},$$

and $\lim_{n\to+\infty} I'_\beta(u_n,v_n) = 0$. We claim that $\{(u_n,v_n)\}$ is bounded in H. For n large enough, one has

$$\begin{aligned}
& c_\beta + 1 + \|(u_n,v_n)\| \\
& \ge I_\beta(u_n,v_n) - \frac{1}{22^*_\mu}\langle I'_\beta(u_n,v_n),(u_n,v_n)\rangle \\
& = \frac{1}{2}\int_\Omega (|\nabla u_n|^2 + \lambda_1 u_n^2) - \frac{1}{22^*_\mu}\int_\Omega\int_\Omega f(x)\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}}{|x-y|^\mu} - \frac{1}{2}\beta\int_\Omega u_n v_n \\
& + \frac{1}{2}\int_\Omega (|\nabla v_n|^2 + \lambda_2 v_n^2) - \frac{1}{22^*_\mu}\int_\Omega\int_\Omega g(x)\frac{|v_n(x)|^{2^*_\mu}|v_n(y)|^{2^*_\mu}}{|x-y|^\mu} - \frac{1}{2}\beta\int_\Omega u_n v_n \\
& - \frac{1}{22^*_\mu}\left[\int_\Omega (|\nabla u_n|^2 + \lambda_1 u_n^2) - \int_\Omega\int_\Omega f(x)\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}}{|x-y|^\mu} - \beta\int_\Omega uv\right] \\
& - \frac{1}{22^*_\mu}\left[\int_\Omega (|\nabla v_n|^2 + \lambda_2 v_n^2) - \int_\Omega\int_\Omega g(x)\frac{|v_n(x)|^{2^*_\mu}|v_n(y)|^{2^*_\mu}}{|x-y|^\mu} - \beta\int_\Omega u_n v_n\right] \\
& = \left(\frac{N-\mu+2}{4N-2\mu}\right)\left[\int_\Omega (|\nabla u_1|^2 + \lambda_1 u_n^2 + |\nabla v_n|^2 + \lambda_2 v_n^2) - 2\beta\int_\Omega u_n v_n\right] \\
& \ge \left(\frac{N-\mu+2}{4N-2\mu}\right)\int_\Omega (|\nabla u_n|^2 + (\lambda_1-\beta_1)u_n^2 + |\nabla v_n|^2 + (\lambda_2-\beta_2)u_n^2) \\
& \ge C\|(u_n,v_n)\|^2.
\end{aligned}$$

Hence, $\{(u_n,v_n)\}$ is bounded in H. Going if necessary to a subsequence $\{(u_n,v_n)\}$, we assume that there exists $(u,v) \in H$ such that

$$\begin{cases}
(u_n,v_n) \rightharpoonup (u,v), & in \ \ H_0^1(\Omega)\ , \\
(u_n,v_n) \longrightarrow (u,v), & in \ \ L^p(\Omega)\times L^p(\Omega), \quad 2 \le p < 2^*\ , \\
(u_n(x),v_n(x)) \longrightarrow (u(x),v(x)), & a.e. \ \ in \ \ \Omega.
\end{cases} \tag{3.1}$$

In fact, by Hardy-Littlewood-Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-\mu}}(\Omega)$ to $L^{\frac{2N}{\mu}}(\Omega)$. One has

$$f(x)\left(\frac{1}{|x|^{\mu}}*|u_n|^{2_\mu^*}\right)\rightharpoonup f(x)\left(\frac{1}{|x|^{\mu}}*|u|^{2_\mu^*}\right)\ \text{in}\ L^{\frac{2N}{\mu}}(\Omega),\ \text{as}\ n\to\infty$$

and

$$g(x)\left(\frac{1}{|x|^{\mu}}*|v_n|^{2_\mu^*}\right)\rightharpoonup g(x)\left(\frac{1}{|x|^{\mu}}*|v|^{2_\mu^*}\right)\ \text{in}\ L^{\frac{2N}{\mu}}(\Omega),\ \text{as}\ n\to\infty.$$

Since $|v_n|^{2_\mu^*}\rightharpoonup |v|^{2_\mu^*}$ in $L^{\frac{2N}{2N-\mu}}(\Omega)$ as $n\to\infty$, which together with the fact that $u_n^{2_\mu^*-1}\rightharpoonup u^{2_\mu^*-1}$ and $v_n^{2_\mu^*-1}\rightharpoonup v^{2_\mu^*-1}$ in $L^{\frac{2N}{N+2-\mu}}(\Omega)$ as $n\to\infty$, one has

$$f(x)\left(\frac{1}{|x|^{\mu}}*|u_n|^{2_\mu^*}\right)|u_n|^{2_\mu^*-1}\rightharpoonup f(x)\left(\frac{1}{|x|^{\mu}}*|u|^{2_\mu^*}\right)|u|^{2_\mu^*-1}\ \text{in}\ L^{\frac{2N}{N+2}}(\Omega),\ \text{as}\ n\to\infty$$

and

$$g(x)\left(\frac{1}{|x|^{\mu}}*|v_n|^{2_\mu^*}\right)|v_n|^{2_\mu^*-1}\rightharpoonup g(x)\left(\frac{1}{|x|^{\mu}}*|v|^{2_\mu^*}\right)|v|^{2_\mu^*-1}\ \text{in}\ L^{\frac{2N}{N+2}}(\Omega),\ \text{as}\ n\to\infty.$$

Moreover

$$\int_\Omega f(x)\left(\frac{1}{|x|^{\mu}}*|u_n|^{2_\mu^*}\right)|u_n|^{2_\mu^*}\to\int_\Omega f(x)\left(\frac{1}{|x|^{\mu}}*|u|^{2_\mu^*}\right)|u|^{2_\mu^*},\ \text{as}\ n\to\infty$$

and

$$\int_\Omega g(x)\left(\frac{1}{|x|^{\mu}}*|v_n|^{2_\mu^*}\right)|v_n|^{2_\mu^*}\to\int_\Omega g(x)\left(\frac{1}{|x|^{\mu}}*|v|^{2_\mu^*}\right)|v|^{2_\mu^*},\ \text{as}\ n\to\infty.$$

So $I_\beta'(u,v)=0$. By [21, Lemma 2.2], one has

$$\begin{aligned}
&\int_\Omega\int_\Omega\frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^{\mu}}dxdy-\int_\Omega\int_\Omega\frac{(|u_n(x)-u(x)|^{2_\mu^*}|u_n(y)-u(y)|^{2_\mu^*}}{|x-y|^{\mu}}dxdy\\
=\ &\int_\Omega\int_\Omega\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^{\mu}}dxdy+o(1),\\
&\int_\Omega\int_\Omega\frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^{\mu}}dxdy-\int_\Omega\int_\Omega\frac{(|v_n(x)-v(x)|^{2_\mu^*}|v_n(y)-v(y)|^{2_\mu^*}}{|x-y|^{\mu}}dxdy\\
=\ &\int_\Omega\int_\Omega\frac{|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^{\mu}}dxdy+o(1).
\end{aligned}\tag{3.2}$$

Since $\{(u_n,v_n)\}$ is bounded in H and $I_\beta^{'}(u_n,v_n)\to 0$ as $n\to\infty$, one obtains from (3.1) and (3.2) that

$$\begin{aligned}
&c_\beta \leftarrow I_\beta(u_n,v_n)\\
&=\frac{1}{2}\int_\Omega(|\nabla u_n|^2+\lambda_1 u_n^2)-\frac{1}{2}\beta\int_\Omega u_nv_n-\frac{1}{22_\mu^*}\int_\Omega\int_\Omega f(x)\frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\\
&+\frac{1}{2}\int_\Omega(|\nabla v_n|^2+\lambda_2 v_n^2)-\frac{1}{2}\beta\int_\Omega u_nv_n-\frac{1}{22_\mu^*}\int_\Omega\int_\Omega g(x)\frac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\\
&=\frac{1}{2}\int_\Omega[|\nabla(u_n-u)|^2+|\nabla u|^2+\lambda_1u^2-\beta uv]\\
&-\frac{1}{22_\mu^*}\int_\Omega\int_\Omega f(x)\frac{|u_n(x)-u(x)|^{2_\mu^*}|u_n(y)-u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\\
&-\frac{1}{22_\mu^*}\int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy-\frac{1}{2}\int_\Omega[|\nabla(v_n-v)|^2+|\nabla v|^2+\lambda_2v^2-\beta uv]\\
&-\frac{1}{22_\mu^*}\int_\Omega\int_\Omega g(x)\frac{|v_n(x)-v(x)|^{2_\mu^*}|v_n(y)-v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy-\frac{1}{22_\mu^*}\int_\Omega\int_\Omega g(x)\frac{|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\\
&=I_\beta(u,v)+\frac{1}{2}\int_\Omega|\nabla(u_n-u)|^2-\frac{1}{22_\mu^*}\int_\Omega\int_\Omega f(x)\frac{|u_n(x)-u(x)|^{2_\mu^*}|u_n(y)-u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\\
&+\frac{1}{2}\int_\Omega|\nabla(v_n-v)|^2-\frac{1}{22_\mu^*}\int_\Omega\int_\Omega g(x)\frac{|v_n(x)-v(x)|^{2_\mu^*}|v_n(y)-v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\\
&\geq\frac{1}{2}\int_\Omega|\nabla(u_n-u)|^2-\frac{1}{22_\mu^*}\int_\Omega\int_\Omega f(x)\frac{|u_n(x)-u(x)|^{2_\mu^*}|u_n(y)-u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\\
&+\frac{1}{2}\int_\Omega|\nabla(v_n-v)|^2-\frac{1}{22_\mu^*}\int_\Omega\int_\Omega g(x)\frac{|v_n(x)-v(x)|^{2_\mu^*}|v_n(y)-v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy.
\end{aligned}$$

Observe $I_\beta(u,v)\geq 0$ and $\int_\Omega u_n^2\to\int_\Omega u^2,\ \int_\Omega v_n^2\to\int_\Omega v^2$ as $n\to\infty$. Recalling that $\langle I_\beta^{'}(u,v),(u,v)\rangle=0$, we derive that

$$\begin{aligned}
&o(1)\\
&=\langle I_\beta^{'}(u_n,v_n),(u_n,v_n)\rangle\\
&=\int_\Omega(|\nabla u_n|^2+\lambda_1u_n^2-\beta u_nv_n)-\int_\Omega\int_\Omega f(x)\tfrac{|u_n|^{2_\mu^*}|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\\
&+\int_\Omega(|\nabla v_n|^2+\lambda_2v_n^2-\beta u_nv_n)-\int_\Omega\int_\Omega g(x)\tfrac{|v_n(x)|^{2_\mu^*}|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\\
&=\int_\Omega|\nabla(u_n-u)|^2+\int_\Omega|\nabla u|^2+\lambda_1\int_\Omega u^2-\beta\int_\Omega(u_n-u)(v_n-v)-\beta\int_\Omega uv\\
&-\int_\Omega\int_\Omega f(x)\tfrac{|u_n(x)-u(x)|^{2_\mu^*}|u_n(y)-u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy-\int_\Omega\int_\Omega f(x)\tfrac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\\
&+\int_\Omega|\nabla(v_n-v)|^2+\int_\Omega|\nabla v|^2+\lambda_2\int_\Omega v^2-\beta\int_\Omega(u_n-u)(v_n-v)-\beta\int_\Omega uv\\
&-\int_\Omega\int_\Omega g(x)\frac{|v_n(x)-v(x)|^{2_\mu^*}|v_n(y)-v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy-\int_\Omega\int_\Omega g(x)\frac{|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy+o(1)
\end{aligned}$$

$$
\begin{aligned}
&= \langle I_\beta'(u,v),(u,v)\rangle + \textstyle\int_\Omega |\nabla(u_n-u)|^2 - \int_\Omega\int_\Omega f(x)\frac{|u_n(x)-u(x)|^{2_\mu^*}|u_n(y)-u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\quad + \textstyle\int_\Omega |\nabla(v_n-v)|^2 - \int_\Omega\int_\Omega g(x)\frac{|v_n(x)-v(x)|^{2_\mu^*}|v_n(y)-v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + o(1) \\
&= \textstyle\int_\Omega |\nabla(u_n-u)|^2 - \int_\Omega\int_\Omega f(x)\frac{|u_n(x)-u(x)|^{2_\mu^*}|u_n(y)-u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\quad + \textstyle\int_\Omega |\nabla(v_n-v)|^2 - \int_\Omega\int_\Omega g(x)\frac{|v_n(x)-v(x)|^{2_\mu^*}|v_n(y)-v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy + o(1),
\end{aligned}
\tag{3.3}
$$

which implies that

$$
\begin{aligned}
&\langle I_\beta'(u_n,v_n),(u_n,0)\rangle \\
&= \int_\Omega |\nabla(u_n-u)|^2 - \int_\Omega\int_\Omega f(x)\frac{(|u_n(x)-u(x)|^{2_\mu^*}|u_n(y)-u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy = o(1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle I_\beta'(u_n,v_n),(0,v_n)\rangle \\
&= \int_\Omega |\nabla(v_n-v)|^2 - \int_\Omega\int_\Omega g(x)\frac{(|v_n(x)-v(x)|^{2_\mu^*}|v_n(y)-v(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy = o(1).
\end{aligned}
$$

So

$$
I_\beta(u_n,v_n) = I_\beta(u,v) + \tfrac{N-\mu+2}{4N-2\mu}\int_\Omega |\nabla(u_n-u)|^2 + \frac{N-\mu+2}{4N-2\mu}\int_\Omega |\nabla(v_n-v)|^2 + o(1). \tag{3.4}
$$

We claim that $(u_n,v_n)\to(u,v)$ in H. Suppose to the contradiction $u_n \not\to u$ in $H_0^1(\Omega)$. Passing to a subsequence, without loss of generality, we may assume there exists a constant $l>0$ such that

$$
\int_\Omega |\nabla(u_n-u)|^2 \to l \text{ and } \int_\Omega\int_\Omega f(x)\frac{|u_n(x)-u(x)|^{2_\mu^*}|u_n(y)-u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \to l,\ as\ n\to\infty.
$$

From (3.3) and (3.4), we have

$$
c_\beta \geq \tfrac{N+2-\mu}{4N-2\mu}l. \tag{3.5}
$$

Note that

$$
\begin{aligned}
l &= \int_\Omega\int_\Omega f(x)\frac{|u_n(x)-u(x)|^{2_\mu^*}|u_n(y)-u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\leq |f|_\infty \int_\Omega\int_\Omega \frac{|u_n(x)-u(x)|^{2_\mu^*}|u_n(y)-u(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy \\
&\leq |f|_\infty \left(S_{H,L}^{-1}\int_\Omega |\nabla(u_n-u)|^2\right)^{2_\mu^*},
\end{aligned}
$$

which implies $l \leq |f|_\infty (S_{H,L}^{-1}l)^{2_\mu^*}$, so $l=0$ or $l \geq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}|f|_\infty^{-\frac{N-2}{N-\mu+2}}$. If $l=0$, the proof is complete. If $l\neq 0$, then $l \geq S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}|f|_\infty^{-\frac{N-2}{N-\mu+2}}$. By (3.5), we obtain

$$
\frac{N+2-\mu}{4N-2\mu}S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}|f|_\infty^{-\frac{N-2}{N-\mu+2}} \leq \frac{N+2-\mu}{4N-2\mu}l \leq c_\beta,
$$

which contradicts

$$
c_\beta < \frac{N+2-\mu}{4N-2\mu}\min\left\{S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}|f|_\infty^{-\frac{N-2}{N-\mu+2}}, S_{H,L}^{\frac{2N-\mu}{N-\mu+2}}|g|_\infty^{-\frac{N-2}{N-\mu+2}}\right\}.
$$

Hence, $l=0$ and $\|u_n-u\|\to 0$ as $n\to\infty$. Moreover $I_\beta(u_n,v_n)=c_\beta$ and $I'_\beta(u_n,v_n)=0$, which imply that (u,v) is a nontrivial ground state solution to system (1.1). Clearly, $u\not\equiv 0, v\not\equiv 0$. This completes the proof.

In order to prove Theorem (1.1), we give the following lemmas.

Lemma 3.1. *Let $\beta\in(0,\sqrt{(\lambda_1+\lambda_1(\Omega))(\lambda_2+\lambda_1(\Omega))})$, and let $(u_\beta,v_\beta)\in H$ be any critical point of I_β with $I_\beta(u_\beta,v_\beta)=c_\beta$. Then either $u_\beta>0, v_\beta>0$ or $u_\beta<0, v_\beta<0$.*

Proof. Let $(u_\beta,v_\beta)\in H$ be any critical point of I_β with $I_\beta(u_\beta,v_\beta)=c_\beta$. Combining the definition of c_β with (2.6), it follows that $c_\beta=\inf_{H\setminus\{(0,0)\}}\max_{t>0}I_\beta(tu,tv)$. Note that $\beta>0$ and system (1.1) has no semi-trivial solutions. Then neither of u_β and v_β can be identically zero. Without loss of generality, we may assume that $(u_\beta^+,v_\beta^+)\neq(0,0)$. It follows from (2.5) that $t_{\beta,|u_\beta|,|v_\beta|}\le t_{\beta,u_\beta,v_\beta}=1$, and

$$c_\beta\le\max_{t>0}I_\beta(t|u_\beta|,t|v_\beta|)=t_{\beta,|u_\beta|,|v_\beta|}^{22_\mu^*}I_\beta(u_\beta,v_\beta)\le c_\beta.$$

So $t_{\beta,|u_\beta|,|v_\beta|}=1$ and $I_\beta(|u_\beta|,|v_\beta|)=I_\beta(u_\beta,v_\beta)=c_\beta$, i.e., $\int_\Omega|u_\beta||v_\beta|=\int_\Omega u_\beta v_\beta$. Moreover

$$\int_\Omega u_\beta^+v_\beta=\int_\Omega u_\beta^+v_\beta^+=\int_\Omega u_\beta v_\beta^+. \tag{3.6}$$

From $I'_\beta(u_\beta,v_\beta)(u_\beta^+,v_\beta^+)=0$ and (3.6), we conclude

$$\begin{aligned}&\|u_\beta^+\|_{\lambda_1}^2+\|v_\beta^+\|_{\lambda_2}^2\\&=\int_\Omega\int_\Omega\frac{f(x)|u_\beta^+(x)|^{2_\mu^*}|u_\beta^+(y)|^{2_\mu^*}+g(x)|v_\beta^+(x)|^{2_\mu^*}|v_\beta^+(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy-2\beta\int_\Omega u_\beta^+v_\beta^+.\end{aligned}$$

Hence $t_{\beta,u_\beta^+,v_\beta^+}=1$, and it follows from (3.6) that

$$c_\beta\le\max_{t>0}I_\beta(tu_\beta^+,tv_\beta^+)=I_\beta(u_\beta^+,v_\beta^+)\le I_\beta(u_\beta,v_\beta)\le c_\beta.$$

This means that the inequalities above are equalities. Thus

$$\begin{aligned}&\int_\Omega\int_\Omega f(x)\frac{|u_\beta^+(x)|^{2_\mu^*}|u_\beta^+(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy+\int_\Omega\int_\Omega g(x)\frac{|v_\beta^+(x)|^{2_\mu^*}|v_\beta^+(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy\\=&\int_\Omega\int_\Omega f(x)\frac{|u_\beta(x)|^{2_\mu^*}|u_\beta(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy+\int_\Omega\int_\Omega g(x)\frac{|v_\beta(x)|^{2_\mu^*}|v_\beta(y)|^{2_\mu^*}}{|x-y|^\mu}dxdy.\end{aligned}$$

Moreover $(u_\beta^+,v_\beta^+)=(u_\beta,v_\beta)$, that is, $u_\beta\ge 0, v_\beta\ge 0$. Applying the strong maximum principle, one has $u_\beta>0, v_\beta>0$. The proof of the lemma is completed. □

Lemma 3.2. *Let $\beta_n\in(0,\sqrt{(\lambda_1+\lambda_1(\Omega))(\lambda_2+\lambda_1(\Omega))}), n\in\mathbb{N}$, be a sequence with $\beta_n\to 0$ as $n\to\infty$. Then, passing to a subsequence, $(u_{\beta_n},v_{\beta_n})\to(\bar u,\bar v)$ in H as $n\to\infty$, and one of the following conclusion holds:*

(i) $\bar u\equiv 0$ and $\bar v$ is a positive ground state to $-\Delta v+\lambda_2 v=g(x)(\int_\Omega\frac{|v(y)|^{2_\mu^*}}{|x-y|^\mu}dy)|v|^{2_\mu^*-2}v$, $v\in H_0^1(\Omega)$;

(ii) $\bar v\equiv 0$ and $\bar u$ is a positive ground state to $-\Delta u+\lambda_1 u=f(x)(\int_\Omega\frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}dy)|u|^{2_\mu^*-2}u$, $u\in H_0^1(\Omega)$.

In particular, if $\lambda_2<\lambda_1$, then, (i) occurs; if $\lambda_1<\lambda_2$, (ii) occurs.

Proof. We claim that c_β is strictly decreasing for $\beta \in (0, \sqrt{(\lambda_1+\lambda_1(\Omega))(\lambda_2+\lambda_1(\Omega))})$. For every $0 \le \beta_1 \le \beta_2 < \sqrt{(\lambda_1+\lambda_1(\Omega))(\lambda_2+\lambda_1(\Omega))}$, one concludes from (2.5) that

$$\begin{aligned} t_{\beta_2,u_{\beta_1},v_{\beta_1}}^{22_\mu^*-2} &= \frac{\int_\Omega (|\nabla u_{\beta_1}|^2+\lambda_1 u_{\beta_1}^2+|\nabla v_{\beta_1}|^2+\lambda_2 v_{\beta_1}^2) - 2\beta_2 \int_\Omega u_{\beta_1}v_{\beta_1}}{\int_\Omega\int_\Omega \left[f(x)\frac{|u_{\beta_1}(x)|^{2_\mu^*}|u_{\beta_1}(y)|^{2_\mu^*}}{|x-y|^\mu} + g(x)\frac{|v_{\beta_1}(x)|^{2_\mu^*}|v_{\beta_1}(y)|^{2_\mu^*}}{|x-y|^\mu}\right] dxdy} \\ &< \frac{\int_\Omega (|\nabla u_{\beta_1}|^2+\lambda_1 u_{\beta_1}^2+|\nabla v_{\beta_1}|^2+\lambda_2 v_{\beta_1}^2) - 2\beta_1 \int_\Omega u_{\beta_1}v_{\beta_1}}{\int_\Omega\int_\Omega \left[f(x)\frac{|u_{\beta_1}(x)|^{2_\mu^*}|u_{\beta_1}(y)|^{2_\mu^*}}{|x-y|^\mu} + g(x)\frac{|v_{\beta_1}(x)|^{2_\mu^*}|v_{\beta_1}(y)|^{2_\mu^*}}{|x-y|^\mu}\right] dxdy} = 1, \end{aligned}$$

and

$$\begin{aligned} c_{\beta_2} &\le \max_{t>0} I_{\beta_2}(tu_{\beta_1}, tv_{\beta_1}) \\ &= \tfrac{N-\mu+2}{4N-2\mu} t_{\beta_2,u_{\beta_1},v_{\beta_1}}^{22_\mu^*} \int_\Omega\int_\Omega \left[f(x)\frac{|u_{\beta_1}(x)|^{2_\mu^*}|u_{\beta_1}(y)|^{2_\mu^*}}{|x-y|^\mu} + g(x)\frac{|v_{\beta_1}(x)|^{2_\mu^*}|v_{\beta_1}(y)|^{2_\mu^*}}{|x-y|^\mu}\right] dxdy \\ &= t_{\beta_2,u_{\beta_1},v_{\beta_1}}^{22_\mu^*} c_{\beta_1} < c_{\beta_1}. \end{aligned}$$

So, c_β is strictly decreasing for $\beta \in (0, \sqrt{(\lambda_1+\lambda_1(\Omega))(\lambda_2+\lambda_1(\Omega))})$.

Next, let $\beta_n \in (0, \sqrt{(\lambda_1+\lambda_1(\Omega))(\lambda_2+\lambda_1(\Omega))}), n \in \mathbb{N}$, be a sequence with $\beta_n \to 0$ as $n \to \infty$. It follows that $\{(u_{\beta_n}, v_{\beta_n})\}$ is bounded in H. Going if necessary to a subsequence $\{(u_n, v_n)\}$, we may assume that there exists $(\bar{u}, \bar{v})$ in H and $(u_{\beta_n}, v_{\beta_n}) \rightharpoonup (\bar{u}, \bar{v})$ such that

$$\begin{cases} -\Delta \bar{u} + \lambda_1 \bar{u} = f(x)\left(\int_\Omega \frac{|\bar{u}(y)|^{2_\mu^*}}{|x-y|^\mu} dy\right) |\bar{u}|^{2_\mu^*-2}\bar{u}, & in\ \Omega, \\ -\Delta \bar{v} + \lambda_2 \bar{v} = g(x)\left(\int_\Omega \frac{|\bar{v}(y)|^{2_\mu^*}}{|x-y|^\mu} dy\right) |\bar{v}|^{2_\mu^*-2}\bar{v}, & in\ \Omega, \\ \bar{u}, \bar{v} \ge 0, & in\ \Omega, \\ \bar{u}, \bar{v} = 0, & on\ \partial\Omega. \end{cases} \tag{3.7}$$

Since $I_0'(u_{\beta_n}, v_{\beta_n}) \to 0$ and $\lim_{n\to\infty} I_0(u_{\beta_n}, v_{\beta_n}) = \lim_{n\to\infty} I_{\beta_n}(u_{\beta_n}, v_{\beta_n})$, we know that

$$I_0'(\bar{u}, \bar{v}) = 0 \text{ and } \lim_{n\to\infty} I_0(u_{\beta_n}, v_{\beta_n}) = \lim_{n\to\infty} c_{\beta_n} > 0.$$

By the definition of c_0 and Lemma 2.5, one has

$$\begin{aligned} 0 < \lim_{n\to\infty} c_{\beta_n} &= \lim_{n\to\infty} I_0(u_{\beta_n}, v_{\beta_n}) \le c_0 \le \min\{B_f, B_g\} \\ &< \tfrac{N-\mu+2}{4N-2\mu} \min\left\{ S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} |f|_\infty^{\frac{N-2}{N-\mu+2}}, S_{H,L}^{\frac{2N-\mu}{N-\mu+2}} |g|_\infty^{\frac{N-2}{N-\mu+2}} \right\}. \end{aligned} \tag{3.8}$$

Similar to the proof of the first part of Theorem 1.1, we have $(u_{\beta_n}, v_{\beta_n}) \to (\bar{u}, \bar{v})$ in H. By (3.8), we derive that one of the following conclusion holds:

(i) $\bar{u} \equiv 0$ and $\bar{v}$ is a positive ground state of $-\Delta v + \lambda_2 v = g(x)(\int_\Omega \frac{|v(y)|^{2_\mu^*}}{|x-y|^\mu} dy)|v|^{2_\mu^*-2}v$, $v \in H_0^1(\Omega)$;

(ii) $\bar{v} \equiv 0$ and $\bar{u}$ is a positive ground state of $-\Delta u + \lambda_1 u = f(x)(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy)|u|^{2^*_\mu - 2}u$, $u \in H_0^1(\Omega)$.

Now, suppose $\lambda_1 < \lambda_2$. It is easy to deduce that $B_f < B_g$. From the definition of c_β, one sees that (ii) occurs, that is, $\bar{v} \equiv 0$ and $\bar{u}$ is a positive ground state of $-\Delta u + \lambda_1 u = f(x)(\int_\Omega \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} dy)|u|^{2^*_\mu - 2}u$, $u \in H_0^1(\Omega)$. The case $\lambda_2 < \lambda_1$ follows similarly. The proof of this lemma is completed. □

Completion of the proof of Theorem 1.1. Combining Lemma 3.1 and Lemma 3.2, we prove the second part of Theorem 1.1.

4. Existence of Positive Higher Energy Solutions

In this section, we prove Theorem 1.2 with $\beta < 0$. At first, we regard system (1.1) as a perturbation of 3.7 and look for a special PS sequence, and then we prove Theorem 1.2.

Define $X = S_f \times S_g$, then $(u_{\lambda_1}, v_{\lambda_2}) \in X$. Without loss of generality, we assume that $B_f \leq B_g$ from now on.

Lemma 4.1. *X is compact in H, and there exist two constants $C_2 > C_1 > 0$ such that*

$$C_1 \leq \|u\|_{\lambda_1},\ \|v\|_{\lambda_2} \leq C_2,\ \forall\ (u,v) \in X.$$

Proof. By Lemma 2.2, we have that S_i $(i = f, g)$ are both compact in $H_0^1(\Omega)$. Since $B_i > 0$ $(i = f, g)$, we obtain this lemma immediately. □

Since

$$J_f(u_{\lambda_1}) = \max_{t>0} J_f(tu_{\lambda_1}) = B_f. \tag{4.1}$$

Clearly, there exist $0 < t_0 < 1 < t_1$ such that

$$J_f(tu_{\lambda_1}) \leq \frac{B_f}{4},\ \forall\, t \in (0, t_0] \cup [t_1, +\infty). \tag{4.2}$$

Similarly, there exist $0 < s_0 < 1 < s_1$ such that

$$J_g(sv_{\lambda_2}) \leq \frac{B_g}{4},\ \forall\, s \in (0, s_0] \cup [s_1, +\infty). \tag{4.3}$$

Define

$$\bar{\gamma}_1(t) := tu_{\lambda_1}\ \forall\, 0 \leq t \leq t_1,\ \bar{\gamma}_2(s) := sv_{\lambda_2}\ \forall\, 0 \leq s \leq s_1.$$

and

$$\bar{\gamma}(t,s) := (\bar{\gamma}_1(t), \bar{\gamma}_2(s)).$$

Then there exists a constant $C_0 > 0$ such that

$$\max_{(t,s)\in Q} \|\bar{\gamma}(t,s)\| \leq C_0, \tag{4.4}$$

where $Q := [0, t_1] \times [0, s_1]$. For $\beta \geq 0$, we define

$$\tilde{c}_\beta := \inf_{\gamma \in \tilde{\Gamma}} \max_{(t,s)\in Q} I_\beta(\gamma(t,s)),\ m_\beta := \max_{(t,s)\in Q} I_\beta(\bar{\gamma}(t,s)),$$

where

$$\tilde{\Gamma} := \Big\{ \gamma \in C(Q,H) \quad : \max_{(t,s)\in Q} \|\gamma(t,s)\| \le 2C_2 + C_0, \\ \gamma(t,s) = \bar{\gamma}(t,s),\ \forall\ (t,s) \in Q \setminus \{(t_0,t_1)\times(s_0,s_1)\} \Big\}. \tag{4.5}$$

Since $\bar{\gamma}(t,s) \in \tilde{\Gamma}$, then $\tilde{\Gamma}$ is nonempty.

Lemma 4.2. $\lim_{\beta\to 0} \tilde{c}_\beta = \lim_{\beta\to 0} m_\beta = \tilde{c}_0 = B_f + B_g$.

Proof. Since $\beta > 0$, we have $I_\beta(\bar{\gamma}(t,s)) \le I_0(\bar{\gamma}(t,s))$. Thus

$$\begin{aligned} m_\beta &\le m_0 = \max_{(t,s)\in Q} I_0(\bar{\gamma}(t,s)) = \max_{t\in[0,t_1]} J_f(\bar{\gamma}_1(t)) + \max_{s\in[0,s_1]} J_g(\bar{\gamma}_2(s)) \\ &= J_f(\bar{\gamma}_1(1)) + J_g(\bar{\gamma}_2(1)) = J_f(u_{\lambda_1}) + J_g(v_{\lambda_2}) = B_f + B_g. \end{aligned}$$

Recalling that $\bar{\gamma} \in \tilde{\Gamma}$, we know that $\tilde{c}_\beta \le m_\beta$, so

$$\limsup_{\beta\to 0} \tilde{c}_\beta \le \liminf_{\beta\to 0} m_\beta \le \limsup_{\beta\to 0} m_\beta \le m_0,\ \tilde{c}_0 \le m_0. \tag{4.6}$$

On the other hand, for every $\gamma(t,s) = (\gamma_1(t,s),\gamma_2(t,s)) \in \tilde{\Gamma}$, define $\Upsilon(\gamma) : [t_0,t_1]\times[s_0,s_1] \to \mathbb{R}^2$ by

$$\Upsilon(\gamma)(t,s) := (J_3(\gamma_1(t,s)) - J_4(\gamma_2(t,s)), J_3(\gamma_1(t,s)) + J_4(\gamma_2(t,s)) - 2),$$

where $J_3, J_4 : H_0^1(\Omega) \to \mathbb{R}$ are defined by

$$J_3(u) := \begin{cases} \dfrac{\displaystyle\int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy}{\displaystyle\int_\Omega (|\nabla u|^2 + \lambda_1 u^2)}, & \text{if } u \ne 0, \\ 0, & \text{if } u = 0, \end{cases}$$

and

$$J_4(u) := \begin{cases} \dfrac{\displaystyle\int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy}{\displaystyle\int_\Omega (|\nabla u|^2 + \lambda_2 u^2)}, & \text{if } u \ne 0, \\ 0, & \text{if } u = 0. \end{cases}$$

For every $u \in H_0^1(\Omega)$, by the Hardy-Littlewood Sobolev inequality, it follows that

$$\int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy \le |f|_\infty \left(S_{H,L}\frac{\lambda_1(\Omega)}{\lambda_1(\Omega)+\lambda_i}\right)^{22^*_\mu} \|u\|^{22^*_\mu}\ i = 1,2.$$

Clearly, J_3 and J_4 are continuous. Moreover

$$\Upsilon(\bar{\gamma})(t,s)$$

$$= \left(\frac{t^{22^*_\mu-2}\int_\Omega\int_\Omega f(x)\frac{|u_{\lambda_1}(x)|^{2^*_\mu}|u_{\lambda_2}(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy}{\int_\Omega(|\nabla u_{\lambda_1}|^2+\lambda_1 u_{\lambda_1}^2)} - \frac{s^{22^*_\mu-2}\int_\Omega\int_\Omega g(x)\frac{|v_{\lambda_2}(x)|^{2^*_\mu}|v_{\lambda_2}(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy}{\int_\Omega(|\nabla v_{\lambda_2}|^2+\lambda_2 v_{\lambda_2}^2)}, \right.$$

$$\left. \frac{t^{22^*_\mu-2}\int_\Omega\int_\Omega f(x)\frac{|u_{\lambda_1}(x)|^{2^*_\mu}|u_{\lambda_2}(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy}{\int_\Omega(|\nabla u_{\lambda_1}|^2+\lambda_1 u_{\lambda_1}^2)} + \frac{s^{22^*_\mu-2}\int_\Omega\int_\Omega g(x)\frac{|v_{\lambda_2}(x)|^{2^*_\mu}|v_{\lambda_2}(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy}{\int_\Omega(|\nabla v_{\lambda_2}|^2+\lambda_2 v_{\lambda_2}^2)} - 2 \right)$$

Since that

$$\int_\Omega(|\nabla u_{\lambda_1}|^2+\lambda_1 u_{\lambda_1}^2) = \int_\Omega\int_\Omega f(x)\frac{|u_{\lambda_1}(x)|^{2^*_\mu}|u_{\lambda_1}(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy,$$

and

$$\int_\Omega(|\nabla v_{\lambda_2}|^2+\lambda_2 v_{\lambda_2}^2) = \int_\Omega\int_\Omega g(x)\frac{|v_{\lambda_2}(x)|^{2^*_\mu}|v_{\lambda_2}(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy,$$

we obtain that $\Upsilon(\bar{\gamma})(1,1)=(0,0)$. By a direct computation, we derive that

$$\deg(\Upsilon(\bar{\gamma}),[t_0,t_1]\times[s_0,s_1],(0,0))=1.$$

For every $(t,s)\in\partial([t_0,t_1]\times[s_0,s_1]), \Upsilon(\gamma)(t,s)=\Upsilon(\bar{\gamma})(t,s)\neq(0,0)$, on has that $\deg(\Upsilon(\gamma),[t_0,t_1]\times[s_0,s_1],(0,0))$ is well define and

$$\deg(\Upsilon(\gamma),[t_0,t_1]\times[s_0,s_1],(0,0))=\deg(\Upsilon(\bar{\gamma}),[t_0,t_1]\times[s_0,s_1],(0,0))=1.$$

Moreover, there exists $(t_2,s_2)\in[t_0,t_1]\times[s_0,s_1]$ such that $\Upsilon(\gamma)(t_2,s_2)=(0,0)$, that is, $J_3(\gamma_1(t_2,s_2)) = J_4(\gamma_2(t_2,s_2))=1$. Thus, $\gamma_i(t_2,s_2)\in\mathcal{N}_i$, $\gamma_i(t_2,s_2)\neq 0$ $(i=1,2)$. Coupling with Hardy-Littlewood-Sobolev inequality, one has

$$\begin{aligned}\max_{(t,s)\in Q} I_0(\gamma(t_2,t_2)) &\geq I_0(\gamma(t_1,t_2)) = J_f(\gamma_1(t_2,s_2))+J_g(\gamma_2(t_2,s_2))\\ &\geq B_f+B_g=m_0.\end{aligned}$$

This implies that $\tilde{c}_0\geq m_0$. By (4.6), we have $\tilde{c}_0=m_0$. Argue by contradiction. Assume that $\liminf_{\beta\to0}\tilde{c}_\beta<m_0$. Then there exist $\varepsilon>0, \beta_n\to0$ and $\gamma_n=(\gamma_{n,1},\gamma_{n,2})\in\tilde{\Gamma}$ such that

$$\max_{(t,s)\in Q} I_{\beta_n}(\gamma_n(t,s))\leq m_0-2\varepsilon.$$

From the definition of $\tilde{\Gamma}$, one sees that there exists n_0 large enough such that

$$\max_{(t,s)\in Q}\beta_n\left|\int_\Omega\gamma_{n,1}(t,s)\gamma_{n,2}(t,s)\right|\leq C\beta_n\leq\varepsilon,\ \forall\, n\geq n_0,$$

so

$$\max_{(t,s)\in Q} I_0(\gamma_n(t,s))\leq\max_{(t,s)\in Q} I_{\beta_n}(\gamma_n(t,s))+\varepsilon\leq m_0-\varepsilon,\ \forall\, n\geq n_0,$$

which contradicts $\tilde{c}_0\leq m_0$. Hence, $\liminf_{\beta\to0}\tilde{c}_\beta\geq m_0$. By (4.6), we obtain that

$$\lim_{\beta\to0}\tilde{c}_\beta=\lim_{n\to0}m_\beta=\tilde{c}_0=B_f+B_g.$$

The proof of this lemma is completed. □

Define

$$X^{\delta} := \{(u,v) \in H : dist((u,v),X) \leq \delta\}, \quad I_{\beta}^{c} := \big\{(u,v) \in H : I_{\beta}(u,v) \leq c\big\}.$$

Lemma 4.3. *Let $m > 0$ be a fixed number, and let $\{(u_n,v_n)\} \subset X^m$ be a sequence. Then, up to a subsequence, $(u_n,v_n) \rightharpoonup (u_0,v_0) \in X^{2m}$.*

Proof. By the definition of X^m and Lemma 4.1, one has that there exists a sequence $\{(\bar{u}_n,\bar{v}_n)\} \subset X$ such that $dist((u_n,v_n),X) = dist((u_n,v_n),(\bar{u}_n,\bar{v}_n)) \leq m$. By Lemma 4.1, there exists $(\bar{u},\bar{v}) \in X$ such that, up to a subsequence, $(\bar{u}_n,\bar{v}_n) \to (\bar{u},\bar{v})$ in H. Hence, $dist((\bar{u}_n,\bar{v}_n),(\bar{u},\bar{v})) \leq m$ for n large enough. Thus, $\{(u_n,v_n)\}$ is bounded, up to a subsequence, $(u_n,v_n) \rightharpoonup (u_0,v_0)$ in H. Since, $B_{2m}(\bar{u},\bar{v})$ is weakly closed in H, we obtain $(u_0,v_0) \in B_{2m}(\bar{u},\bar{v}) \subset X^{2m}$. □

Lemma 4.4. *Set $m_1 := \frac{1}{2}(\frac{4N-2\mu}{N+2-\mu}B_f)^{\frac{N+2-\mu}{2N-\mu}}$, and let $m \in (0,m_1)$. Assume that there exist sequences $\beta_k > 0$ with $\beta_k \to 0$, and $\{(u_k,v_k)\} \in X^m$ with $\lim\limits_{k\to\infty} I_{\beta_k}(u_k,v_k) \leq \tilde{c}_0$ and $\lim\limits_{k\to\infty} I'_{\beta_k}(u_k,v_k) = 0$. Then, up to s subsequence, $\{(u_k,v_k)\} \to (u,v)$, where $(u,v) \in X$.*

Proof. By Lemma 4.3, up to s subsequence, $(u_k,v_k) \rightharpoonup (u,v) \in X^{2m}$. By the choice of m_1, we obtain that $u \not\equiv 0$ and $v \not\equiv 0$. Since $\lim_{k\to\infty} I'_{\beta_k}(u_k,v_k) = 0$ and $\{(u_k,v_k)\}$ is bounded, for every $(\varphi,\psi) \in H$, one has

$$\begin{aligned}
&\langle I'_0(u,v),(\varphi,\psi)\rangle \\
&= \int_\Omega (\nabla u \nabla \varphi + \lambda_1 u\varphi) + \int_\Omega (\nabla v \nabla \psi + \lambda_2 v\psi) \\
&\quad - \int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*-2}u(y)\varphi(y)}{|x-y|^\mu} - \int_\Omega\int_\Omega g(x)\frac{|v(x)|^{2_\mu^*}|v(y)|^{2_\mu^*-2}v(y)\psi(y)}{|x-y|^\mu} \\
&= \lim_{k\to\infty}\left[\langle I'_{\beta_k}(u_k,v_k),(\varphi,\psi)\rangle + \beta_k\int_\Omega v_k\varphi + \beta_k\int_\Omega u_k\psi\right] = 0.
\end{aligned}$$

So, $I'_0(u,v) = 0$. Furthermore, since $(u_k,v_k) \in X^m$ for all k, we have

$$\langle I'_0(u_k,v_k),(\varphi,\psi)\rangle = \langle I'_{\beta_k}(u_k,v_k)(\varphi,\psi)\rangle + \beta_k\int_\Omega v_k\varphi + \beta_k\int_\Omega u_k\psi = o(1)\|(\varphi,\psi)\|.$$

On the other hand,

$$\begin{aligned}
\tilde{c}_0 \geq \lim_{k\to\infty} I_{\beta_k}(u_k,v_k) \quad &= \lim_{k\to\infty} I_0(u_k,v_k) - \lim_{k\to\infty}\beta_k\int_\Omega u_kv_k \\
&= \lim_{k\to\infty} I_0(u_k,v_k) := B.
\end{aligned} \tag{4.7}$$

So $\{(u_k,v_k)\}$ is a $(PS)_B$ sequence of I_0 with $B=\lim\limits_{k\to\infty} I_0(u_k,v_k)$. Thus

$$\begin{aligned} I_0(u,v) &= \tfrac{1}{2}\int_\Omega(|\nabla u|^2+\lambda_1 u^2)-\frac{1}{22^*_\mu}\int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy \\ &\quad+\tfrac{1}{2}\int_\Omega(|\nabla v|^2+\lambda_2 v^2)-\frac{1}{22^*_\mu}\int_\Omega\int_\Omega g(x)\frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy \\ &\quad-\tfrac{1}{22^*_\mu}\Bigg[\int_\Omega(|\nabla u|^2+\lambda_1 u^2)-\int_\Omega\int_\Omega f(x)\frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy \\ &\quad+\int_\Omega(|\nabla v|^2+\lambda_2 v^2)-\int_\Omega\int_\Omega g(x)\frac{|v(x)|^{2^*_\mu}|v(y)|^{2^*_\mu}}{|x-y|^\mu}dxdy\Bigg] \\ &=\tfrac{N-\mu+2}{4N-2\mu}\int_\Omega(|\nabla u|^2+\lambda_1 u^2+|\nabla v|^2+\lambda_2 v^2) \\ &\le \tfrac{N-\mu+2}{4N-2\mu}\liminf_{k\to\infty}\left[\int_\Omega(|\nabla u_k|^2+\lambda_1 u_k^2+|\nabla v_k|^2+\lambda_2 v_k^2)\right] \\ &=\liminf_{k\to\infty}[I_0(u_k,v_k)-\tfrac{1}{22^*_\mu}\langle I_0'(u_k,v_k),(u_k,v_k)\rangle]=B. \end{aligned}$$

From Lemma 4.2 and $\tilde{c}_0\le I_0(u,v)\le B$, it follows from (4.7) that $\tilde{c}_0=I_0(u,v)=B$, which implies that $(u_k,v_k)\to(u,v)$ in H. This further indicates $(u,v)\in X$. □

Lemma 4.5. *Let m_1 be as in Lemma 4.4. For a small $\delta\in(0,\frac{m_1}{2})$, there exists constants $0<\sigma<1$ and $\beta_1\in(0,\sqrt{(\lambda_1+\lambda_1(\Omega))(\lambda_2+\lambda_1(\Omega))})$ such that $\|I_\beta'(u,v)\|\ge\sigma$ for every $(u,v)\in I_\beta^{m_\beta}\cap(X^\delta\setminus X^{\frac{\delta}{2}})$ and any $\beta\in(0,\beta_1)$.*

Proof. On the contrary, we assume that there exist a number $\delta_0\in(0,m_1)$, a positive sequence $\{\beta_k\}$ with $\lim_{n\to+\infty}\beta_k=0$, and a sequence of function $\{(u_n,v_n)\}\subset I_{\beta_k}^{m_{\beta_k}}\cap(X^{\delta_0}\setminus X^{\frac{\delta_0}{2}})$ such that $\lim_{k\to+\infty}I_{\beta_k}'(u_k,v_k)=0$. By Lemma 4.2, one has $\{(u_k,v_k)\}\subset X^{\delta_0}$, $\delta_0<m_1$, $\lim_{k\to\infty}I_{\beta_k}(u_k,v_k)\le\tilde{c}_0$, and $\lim_{k\to\infty}I_{\beta_k}'(u_k,v_k)=0$. Hence, it follows from Lemma 4.4 that there exists $(u,v)\in X$ such that $(u_k,v_k)\to(u,v)$ in H. As a consequence, $dist((u_k,v_k),X)\to0$ as $k\to+\infty$. This contradicts with the relation $(u_k,v_k)\notin X^{\frac{\delta_0}{2}}$. □

From now on, we fix a small $\delta\in(0,\frac{m_1}{2})$ and correspond $0<\sigma<1$ and $\beta_1>0$ such that the conclusion in Lemma 4.5 hold.

Lemma 4.6. *There exist $\beta_2\in(0,\beta_1)$ and $\alpha>0$ such that, for any $\beta\in(0,\beta_2)$, $I_\beta(\bar{\gamma}(t,s))\ge\tilde{c}_\beta-\alpha$ implies that $\bar{\gamma}(t,s)\in X^{\frac{\delta}{2}}$.*

Proof. Assume by contradiction that there exist $\beta_n\to0,\alpha_n\to0$ and $(t_n,s_n)\in Q$ such that

$$I_{\beta_n}(\bar{\gamma}(t_n,s_n))\ge\tilde{c}_{\beta_n}-\alpha_n \text{ and } \bar{\gamma}(t_n,s_n)\notin X^{\frac{\delta}{2}}. \tag{4.8}$$

Passing to a subsequence, we assume that $(t_n,s_n)\to(\bar{t},\bar{s})\in Q$. By Lemma 4.2, we obtain

$$I_0(\bar{\gamma}(\bar{t},\bar{s}))\ge\lim_{n\to\infty}\tilde{c}_{\beta_n}=B_f+B_g.$$

From (4.1) and (4.6), it is easy to see that $(\bar{t},\bar{s})=(1,1)$. Thus $\lim\limits_{n\to\infty}\|\bar{\gamma}(t_n,s_n)-\bar{\gamma}(1,1)\|=0$. Note that $\bar{\gamma}(1,1)=(u_{\lambda_1},v_{\lambda_2})\in X$ contradicts (4.8). This completes the proof. □

Set

$$\alpha_0 := \min\{\tfrac{\alpha}{2}, \tfrac{B_f}{4}, \tfrac{\delta}{8}\sigma^2\}. \tag{4.9}$$

From Lemma 4.2, we have that there exists $\beta_0 \in (0, \beta_2]$ such that

$$|\tilde{c}_\beta - m_\beta| < \alpha_0,\ |\tilde{c}_\beta - (B_f + B_g)| < \alpha_0,\ \forall\ \beta \in (0, \beta_0). \tag{4.10}$$

Lemma 4.7. *For fixed $\beta \in (0, \beta_0)$, there exists $\{(u_n, v_n)\}_{n=1}^{\infty} \subset X^\delta \cap I_\beta^{m_\beta}$ such that $I_\beta'(u_n, v_n) \to 0$ in H as $n \to \infty$.*

Proof. Fix a $\beta \in (0, \beta_0)$. Assume by contradiction that there exists $0 < l(\beta) < l$ such that $\|I_\beta'(u,v)\| \geq l(\beta)$ on $X^\delta \cap I_\beta^{m_\beta}$. Then there exists a pseudo-gradient vector field V_β in H which is defined on a neighborhood Z_β of $X^\delta \cap I_\beta^{m_\beta}$ such that, for any $(u,v) \in Z_\beta$, there holds

$$\|V_\beta(u,v)\| \leq 2\min\{1, \|I_\beta'(u,v)\|\},$$

and

$$\langle I_\beta'(u,v), V_\beta(u,v) \geq \min\{1, \|I_\beta'(u,v)\|\}\|I_\beta'(u,v)\|.$$

Let η_β be a Lipschitz continuous function on H such that $0 \leq \eta_\beta \leq 1, \eta_\beta = 1$ on $X^\delta \cap I_\beta^{m_\beta}$ and $\eta_\beta = 0$ on $H \setminus Z_\beta$. Let ξ_β be a Lipschitz continuous function on $\mathbb{R}$ such that $0 \leq \xi_\beta \leq 1$ and

$$\xi_\beta(t) := \begin{cases} 1 & \text{if } |t - \tilde{c}_\beta| \leq \frac{\delta}{2}, \\ 0 & \text{if } |t - \tilde{c}_\beta| \geq \delta. \end{cases}$$

Let

$$e_\beta(u,v) := \begin{cases} -\eta_\beta(u,v)\xi_\beta(I_\beta(u,v))V_\beta(u,v), & \text{if } (u,v) \in Z_\beta, \\ 0, & \text{if } (u,v) \in H \setminus Z_\beta. \end{cases}$$

Then, for every $(u,v) \in H$, there exists a global solution $\psi_\beta : H \times [0, +\infty) \to H$ for the following Cauchy initial value problem

$$\begin{cases} \frac{d}{d\tau}\psi_\beta(u,v,\tau) = e_\beta(\psi_\beta(u,v,\tau)), \\ \psi_\beta(u,v,0) = (u,v). \end{cases}$$

It is easy to see that ψ_β has the following properties:
(i) $\psi_\beta(u,v,\tau) = (u,v)$ if $\tau = 0$ or $(u,v) \in H \setminus Z_\beta$ or $|I_\beta(u,v) - \tilde{c}_\beta| \geq \alpha$,
(ii) $\|\frac{d}{d\tau}\psi_\beta(u,v,\tau)\| \leq 2$,
(iii) $\frac{d}{d\tau}I_\beta(\psi_\beta(u,v,\tau)) = \langle I_\beta'(\psi_\beta(u,v,\tau)), e_\beta(\psi_\beta(u,v,\tau))\rangle \leq 0$.

Step 1. For every $(t,s) \in Q$, we show that there exists $\tau_{t,s} \in [0, +\infty)$ such that $\psi_\beta(\bar{\gamma}(t,s), \tau_{t,s}) \in I_\beta^{\tilde{c}_\beta - \alpha_0}$.

Assume by contradiction that there exists $(t,s) \in Q$ such that $I_\beta(\psi_\beta(\bar{\gamma}(t,s), \tau)) > \tilde{c}_\beta - \alpha_0$ for every $\tau \geq 0$. Recalling $\alpha_0 < \alpha$, we deduce from Lemma 4.6 that $\bar{\gamma}(t,s) \in X^{\frac{\delta}{2}}$. Since $I_\beta(\bar{\gamma}(t,s)) \leq m_\beta < \tilde{c}_\beta + \alpha_0$, it follows from the property (iii) that

$$\tilde{c}_\beta - \alpha_0 < I_\beta(\psi_\beta(\bar{\gamma}(t,s), \tau)) \leq m_\beta < \tilde{c}_\beta + \alpha_0,\ \forall\ \tau \geq 0.$$

This means that $\xi_\beta(I_\beta(\psi_\beta(\bar{\gamma}(t,s),\tau)))\equiv 1$. If $\psi_\beta(\bar{\gamma}(t,s),\tau)\in X^\delta$ for any $\tau\geq 0$, then $\eta_\beta(\psi_\beta(\bar{\gamma}(t,s),\tau))\equiv 1$ and $\|I_\beta'(\psi_\beta(\bar{\gamma}(t,s),\tau)\|\geq l(\beta)$ for any $\tau>0$. Hence,

$$\begin{aligned}\frac{d}{d\tau}I_\beta(\psi_\beta(u,v,\tau)) &= \langle I_\beta'(\psi_\beta(u,v,\tau)),e_\beta(\psi_\beta(u,v,\tau))\rangle\\ &= \langle I_\beta'(\psi_\beta(u,v,\tau)),-\eta_\beta(u,v,\tau)\xi_\beta(I_\beta(u,v,\tau))V_\beta(u,v,\tau)\rangle\\ &= -\eta_\beta(u,v,\tau)\langle I_\beta'(\psi_\beta(u,v,\tau)),\xi_\beta(I_\beta(u,v,\tau))V_\beta(u,v,\tau)\rangle\\ &\leq -\eta_\beta(u,v,\tau)\xi_\beta(I_\beta(u,v,\tau))\langle I_\beta'(\psi_\beta(u,v,\tau)),V_\beta(u,v,\tau)\rangle\\ &\leq -\eta_\beta(u,v,\tau)\xi_\beta(I_\beta(u,v,\tau))\min\{1,\|I_\beta'(u,v,\tau)\|\}\|I_\beta'(u,v,\tau)\|\\ &\leq -\eta_\beta(u,v,\tau)\xi_\beta(I_\beta(u,v,\tau))\|I_\beta'(u,v,\tau)\|^2.\end{aligned}$$

Then

$$\begin{aligned}I_\beta(\psi_\beta(\bar{\gamma}(t,s),\tfrac{\alpha}{l(\beta)^2})) &= \int_0^{\frac{\alpha}{l(\beta)^2}}\frac{d}{d\tau}I_\beta(\psi_\beta(\bar{\gamma}(t,s),\tau))d\tau+I_\beta(\psi_\beta(\bar{\gamma}(t,s),0))\\ &\leq -\int_0^{\frac{\alpha}{l(\beta)^2}} l(\beta)^2dt+\tilde{c}_\beta+\frac{\alpha}{2}=\tilde{c}_\beta-\frac{\alpha}{2},\end{aligned}$$

a contradiction. So, there exists $\tau_{t,s}>0$ such that $\psi_\beta(\bar{\gamma}(t,s),\tau_{t,s})\notin X^\delta$. Notice that $\bar{\gamma}(t,s)\in X^{\frac{\delta}{2}}$. There exist $0<\tau_{t,s,1}<\tau_{t,s,2}\leq\tau_{t,s}$ such that $\psi_\beta(\bar{\gamma}(t,s),\tau_{t,s,1})\in\partial X^{\frac{\delta}{2}}$, $\psi_\beta(\bar{\gamma}(t,s),\tau_{t,s,2})\in\partial X^\delta$ and $\psi_\beta(\bar{\gamma}(t,s),\tau)\in\partial X^\delta\setminus X^{\frac{\delta}{2}}$ for all $\tau\in(\tau_{t,s,1},\tau_{t,s,2})$. Using Lemma 4.5, one obtains $\|I_\beta'(\psi_\beta(\bar{\gamma}(t,s),\tau))\|\geq\sigma$ for any $\tau\in(\tau_{t,s,1},\tau_{t,s,2})$. By property (ii) we obtain

$$\begin{aligned}\tfrac{\delta}{2} &\leq \|\psi_\beta(\bar{\gamma}(t,s),\tau_{t,s,2})-\psi_\beta(\bar{\gamma}(t,s),\tau_{t,s,1})\|\\ &= \Big\|\int_{\tau_{t,s,1}}^{\tau_{t,s,2}}\frac{d}{d\tau}\psi_\beta(\bar{\gamma}(t,s),\tau)d\tau\Big\|\\ &\leq 2|\tau_{t,s,2}-\tau_{t,s,1}|,\end{aligned}$$

that is, $\tau_{t,s,2}-\tau_{t,s,1}\geq\frac{\delta}{4}$. So

$$\begin{aligned}&I_\beta(\psi_\beta(\bar{\gamma}(t,s),\tau_{t,s,2}))\\ &= I_\beta(\psi_\beta(\bar{\gamma}(t,s),\tau_{t,s,1}))+\int_{\tau_{t,s,1}}^{\tau_{t,s,2}}\frac{d}{d\tau}I_\beta(\psi_\beta(\bar{\gamma}(t,s),\tau))d\tau\\ &= I_\beta(\psi_\beta(\bar{\gamma}(t,s),\tau_{t,s,1}))+\int_{\tau_{t,s,1}}^{\tau_{t,s,2}}-\eta_\beta(u,v,\tau)\xi_\beta(I_\beta(u,v,\tau))\|I_\beta'(u,v,\tau)\|^2d\tau\\ &= I_\beta(\psi_\beta(\bar{\gamma}(t,s),\tau_{t,s,1}))+\int_{\tau_{t,s,1}}^{\tau_{t,s,2}}-\|I_\beta'(u,v,\tau)\|^2d\tau\\ &\leq \tilde{c}_\beta+\alpha_0-\sigma^2(\tau_{t,s,2}-\tau_{t,s,1})\\ &\leq \tilde{c}_\beta+\alpha_0-\frac{\delta}{4}\sigma^2\\ &\leq \tilde{c}_\beta-\alpha_0,\end{aligned}$$

which is a contradiction. Define

$$T(t,s):=\inf\{\tau\geq 0: I_\beta(\psi_\beta(\bar{\gamma}(t,s,\tau))\leq\tilde{c}_\beta-\alpha_0\},$$

and let $\gamma(t,s) := \psi_\beta(\bar{\gamma}(t,s),T(t,s))$. By Step 1, we know that $\gamma(t,s)$ is well-define for every $(t,s)\in Q$ and $I_\beta(\gamma(t,s))\le \tilde{c}_\beta-\alpha_0$ for any $(t,s)\in Q$.

Step 2. We show that $\gamma(t,s)=\psi_\beta(\bar{\gamma}(t,s),\ T(t,s))\in\tilde{\Gamma}$.

For any $(t,x)\in Q\setminus(t_0,t_1)\times(s_0,s_1)$, by 4.1, (4.2), (4.3), (4.9), and (4.10), we conclude from the fact $B_f\le B_g$ that

$$\begin{aligned} I_\beta(\bar{\gamma}(t,s))\le I_0(\bar{\gamma}(t,s)) &= J_f(\bar{\gamma}_1(t))+J_g(\bar{\gamma}_2(s))\\ &\le \begin{cases} \frac{B_f}{4}+B_g & \text{if } (t,s)\in[0,t_0]\times[0,s_1]\\ B_f+\frac{B_g}{4} & \text{if } (t,s)\in[t_0,t_1]\times[0,s_0]\end{cases}\\ &\le \frac{B_f}{4}+B_g\le B_f+B_g-3\delta_0<\tilde{c}_\beta-\alpha_0,\end{aligned}$$

which implies that $T(t,s)=0$ and then $\gamma(t,s)=\bar{\gamma}(t,s)$. From the definition of $\tilde{\Gamma}$ in (4.5), it suffices to prove that $\|\gamma(t,s)\|\le 2C_2+C_0$ for every $(t,s)\in Q$ and $T(t,s)$ is continuous with respect to (t,s).

Case 1. If $I_\beta(\bar{\gamma}(t,s))\le\tilde{c}_\beta-\alpha_0$, then $T(t,s)=0$ and hence $\gamma(t,s)=\bar{\gamma}(t,s)$. From (4.4), we know that $\|\gamma(t,s)\|\le C_0\le 2C_2+C_0$.

Case 2. If $I_\beta(\bar{\gamma}(t,s))>\tilde{c}_\beta-\alpha_0$, then $\bar{\gamma}(t,s)\in X^{\frac{\delta}{2}}$ and

$$\tilde{c}_\beta-\alpha_0<I_\beta(\psi_\beta(\bar{\gamma}(t,s),\tau))\le m_\beta<\tilde{c}_\beta+\alpha_0,\ \forall\ \tau\in[0,T(t,s)).$$

This implies $\xi_\beta(I_\beta(\psi_\beta(\bar{\gamma}(t,s),\tau)))\equiv 1$ for all $\tau\in[0,T(t,s))$. If $(\psi_\beta(\bar{\gamma}(t,s),T(t,s)))\notin X^\delta$, then there exist $0<\tau_{t,s,1}<\tau_{t,s,2}<T(t,s)$ as in Step 1. So we deduce that $I_\beta((\psi_\beta(\bar{\gamma}(t,s),\tau_{t,s,2}))\le\tilde{c}_\beta-\alpha_0$ as above, which contradicts with the definition of $T(t,s)$. Moreover, $\gamma(t,s)=\psi_\beta(\bar{\gamma}(t,s),$ $T(t,s))\in X^\delta$. Then there exists $(u,v)\in X$ such that $\|\gamma(t,s)-(u,v)\|\le\delta\le\frac{C_0}{2}$. By Lemma 4.1, one has

$$\|\gamma(t,s)\|\le\|(u,v)\|+\frac{C_0}{2}\le 2C_2+C_0.$$

Next, we prove the continuity of $T(t,s)$. For all $(\bar{t},\bar{s})\in Q$, assume that $I_\beta(\gamma(\bar{t},\bar{s}))<\tilde{c}_\beta-\alpha_0$. Then $T(\bar{t},\bar{s})=0$ from the definition of $T(t,s)$. So $I_\beta(\bar{\gamma}(\bar{t},\bar{s}))<\tilde{c}_\beta-\alpha_0$. By the continuity of $\bar{\gamma}$, there exists $\varepsilon>0$ such that, for any $(t,s)\in(\bar{t}-\varepsilon,\bar{t}+\varepsilon)\times(\bar{s}-\varepsilon,\bar{s}+\varepsilon)\cap Q$, $I_\beta(\bar{\gamma}(t,s))<\tilde{c}_\beta-\alpha_0$, that is, $T(s,t)=0$, and T is continuous at $(\bar{t},\bar{s})$. Now we assume that $I_\beta(\gamma(\bar{t},\bar{s}))=\tilde{c}_\beta-\alpha_0$. Then from the previous proof we see that $\gamma(\bar{t},\bar{s})=\psi_\beta(\bar{\gamma}(\bar{t},\bar{s}),T(\bar{t},\bar{s}))\in X^\delta$, and so

$$\|I_\beta^{'}(\psi_\beta(\bar{\gamma}(\bar{t},\bar{s}),T(\bar{t},\bar{s})))\|\ge l(\beta)>0.$$

Then, for any $w>0$, $I_\beta(\psi_\beta(\bar{\gamma}(\bar{t},\bar{s})),T(\bar{t},\bar{s})+w))<\tilde{c}_\beta-\alpha_0$. By the continuity of ψ_β, there exists $\varepsilon>0$ such that, for any $(t,s)\in(\bar{t}-\varepsilon,\bar{t}+\varepsilon)\times(\bar{s}-\varepsilon,\bar{s}+\varepsilon)\cap Q$, $I_\beta(\psi_\beta(\bar{\gamma}(\bar{t},\bar{s}),T(\bar{t},\bar{s})+$ $w))<\tilde{c}_\beta-\alpha_0$, so $T(t,s)\le T(\bar{t},\bar{s})+w$. It follows that $0\le\limsup_{(t,s)\to(\bar{t},\bar{s})}T(t,s)\le T(\bar{t},\bar{s})$. If $T(\bar{t},\bar{s})=0$, one has $\lim_{(t,s)\to(\bar{t},\bar{s})}T(t,s)=T(\bar{t},\bar{s})$. If $T(\bar{t},\bar{s})>0$, then, for any $w\in(0,T(\bar{t},\bar{s}))$, $I_\beta(\psi_\beta(\bar{\gamma}(\bar{t},\bar{s}),T(\bar{t},\bar{s})+w))>\tilde{c}_\beta-\alpha_0$. By the continuity of ψ_β again, it follows that $\liminf\limits_{(t,s)\to(\bar{t},\bar{s})}T(t,s)=$ $T(\bar{t},\bar{s})$. So T is continuous at $(\bar{t},\bar{s})$. The proof of Step 2 is completed.

Now, by Claims 1 and 2, we have proved that there exists a path $\gamma(t,s)\in\Gamma$ such that

$$\max_{(t,s)\in Q}I_\beta(\gamma(t,s))\le\tilde{c}_\beta-\alpha_0,$$

which contradicts with the definition of $\tilde{c}_\beta$. This completes the proof. □

Proof of the Theorem 1.2. Fix $m_1 := \frac{1}{2}(\frac{4N-2\mu}{N+2-\mu}B_f)^{\frac{N+2-\mu}{2N-\mu}}$. By Lemma 4.7, there exists some β_0 such that, for any fixed $\beta \in (0,\beta_0)$, there exists $\{(u_n,v_n)\} \subset I_\beta^{m_\beta} \cap X^\delta$ with $I'_\beta(u_n,v_n) \to 0$ as $n \to \infty$. Since X is compact, it is easy to see $\{(u_n,v_n)\}$ is bounded in H. By Lemma 4.3, there exists $(u_\beta,v_\beta) \subset X^m$, up to a subsequence,

$$\begin{cases} (u_n,v_n) \rightharpoonup (u_\beta,v_\beta), & in \ \ H\ , \\ (u_n,v_n) \longrightarrow (u_\beta,v_\beta), & in \ \ L^p(\Omega)\times L^p(\Omega), \quad 2 \le p < 2^*\ , \\ (u_n(x),v_n(x)) \longrightarrow (u_\beta(x),v_\beta(x)), & a.e. \ \ in \ \ \Omega. \end{cases}$$

Thus, $I'_\beta(u_\beta,v_\beta) = 0$, by the choice of $m, u_\beta \neq 0, v_\beta \neq 0$. Hence, (u_β,v_β) is the solution to (1.1).

Recalling that $\{(u_n,v_n)\} \subset I_\beta^{m_\beta} \cap X^\delta$, we have $(u_\beta,v_\beta) \in X^\delta$, which means that $u \neq 0, v \neq 0$ for small enough $\delta > 0$. Combing this with Theorem 1.1, we know $u_\beta > 0, v_\beta > 0$. Moreover, (u_β,v_β) is a positive solution to (1.1). Moreover, $I_\beta(u_\beta,v_\beta) \le m_\beta$.

In the following, we study the asymptotic behavior of (u_β,v_β) as $\beta \to 0$. For every sequence $\{\beta_n\} \subset (0,\beta_0)$ with $\beta_n \searrow 0$ as $n \to \infty$. Let $\{(u_{\beta_n},v_{\beta_n})\} \subset H$ be a sequence positive solutions obtained above. Since

$$I_\beta(u_{\beta_n},v_{\beta_n}) = J_1(u_{\beta_n}) + J_2(v_{\beta_n}) - \beta_n \int_\Omega u_{\beta_n} v_{\beta_n}, \tag{4.11}$$

and for every $\varphi,\psi \in C_0^\infty(\Omega)$

$$0 = \langle I'_{\beta_n}(u_{\beta_n},v_{\beta_n}),(\varphi,\psi)\rangle = \langle J'_1(u_{\beta_n}),\varphi\rangle + \langle J'_2(v_{\beta_n}),\psi\rangle - \beta_n \int_\Omega (v_{\beta_n}\varphi + u_{\beta_n}\psi). \tag{4.12}$$

Moreover

$$\lim_{n\to\infty} \beta_n \int_\Omega u_{\beta_n} v_{\beta_n} \le \lim_{n\to\infty} \beta_n \left(\int_\Omega |u_{\beta_n}|^2\right)^{\frac{1}{2}} \left(\int_\Omega |v_{\beta_n}|^2\right)^{\frac{1}{2}} = 0. \tag{4.13}$$

Similarly,

$$\beta_n \int_\Omega (v_{\beta_n}\varphi + u_{\beta_n}\psi) \to 0 \text{ as } n \to \infty. \tag{4.14}$$

In view of $(u_{\beta_n},v_{\beta_n}) \in X^\delta$ and $I_{\beta_n}(u_{\beta_n},v_{\beta_n}) \le m_{\beta_n}$, one has that $\{(u_{\beta_n},v_{\beta_n})\}$ is uniformly bounded for $\beta_n \in (0,\beta_0)$ in H. From (4.11)-(4.14), one obtains

$$I_0(u_{\beta_n},v_{\beta_n}) = J_f(u_{\beta_n}) + J_g(v_{\beta_n}) \le d_{\beta_n},\ I'_0(u_{\beta_n},v_{\beta_n}) = J'_f(u_{\beta_n}) + J'_g(v_{\beta_n}) \to 0 \text{ as } \beta_n \to 0.$$

Let $\beta_n \in (0,\beta_0)$, $n \in \mathbb{N}$, be any sequence with $\beta_n \to 0$ as $n \to \infty$. Repeating the proof of Lemma 4.4, up to a subsequence, one sees that there exists a $(\tilde{u},\tilde{v})$ $(\tilde{u} \in S_f, \tilde{v} \in S_g)$ such that $(u_{\beta_n},v_{\beta_n}) \to (\tilde{u},\tilde{v})$ in H as $n \to \infty$, which implies that $\tilde{u}$ is a positive ground state solution to $-\Delta u + \lambda_1 u = f(x)(\frac{1}{|\mu|^\mu} * u^{2^*_\mu})u^{2^*_\mu - 2}u,\ u \in H_0^1(\Omega)$ and $\tilde{v}$ is a positive ground state to $-\Delta v + \lambda_2 v = g(x)(\frac{1}{|\mu|^\mu} * v^{2^*_\mu})v^{2^*_\mu - 2}v,\ v \in H_0^1(\Omega)$. This completes the proof.

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