J. Nonlinear Var. Anal. 7 (2023), No. 2, pp. 235-250
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.7.2023.2.05

A HANKEL MATRIX ACTING ON FOCK SPACES

ZHENGYUAN ZHUO[1], CONGHUI SHEN[2], DONGXING LI[3], SONGXIAO LI[4,*]

[1]*School of Mathematics and Systems Science,*
Guangdong Polytechnic Normal University, Guangzhou 510665, China
[2]*College of Mathematics and Data Science, Minjiang University, Fuzhou 350108, China*
[3]*School of Financial Mathematics and Statistics, Guangdong University of Finance, Guangzhou, 510521, China*
[4]*Department of Mathematics, Shantou University, Shantou 515063, China.*

Abstract. Let ν be a positive Borel measure on the interval $[0,\infty)$. Let $\mathscr{H}_\nu = (\nu_{n,k})_{n,k\geq 0}$ be the Hankel matrix with entries $\nu_{n,k} = \int_{[0,\infty)} \frac{t^{n+k}}{n!}\, d\nu(t)$. The matrix $\mathscr{H}_\nu$ induces formally the operator $\mathscr{H}_\nu(f)(z) = \sum_{n=0}^{\infty}(\sum_{k=0}^{\infty} \nu_{n,k} a_k) z^n$ on the space of all entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$. In this paper, we investigate those positive Borel measures such that $\mathscr{H}_\nu(f)(z) = \int_{[0,\infty)} f(t) e^{tz}\, d\nu(t)$, $z \in \mathbb{C}$ for all $f \in F^p$, and among them we characterize those for which $\mathscr{H}_\nu$ is a bounded (resp., compact) operator from the Fock space F^p into the space F^q $(0 < p, q < \infty)$.
Keywords. Fock spaces; Fock Carleson measure; Hankel matrices.

1. Introduction

Let $\mathbb{C}$ be the complex plane, and let $H(\mathbb{C})$ be the space of entire functions. For $0 < p < \infty$, the Fock space F^p is defined by

$$F^p = \left\{ f \in H(\mathbb{C}) : \|f\|_p^p = \frac{p}{2\pi} \int_{\mathbb{C}} \left| f(z) e^{-\frac{1}{2}|z|^2} \right|^p dA(z) < \infty \right\},$$

where dA is the Lebesgue area measure on $\mathbb{C}$. Set

$$F^\infty = \left\{ f \in H(\mathbb{C}) : \|f\|_\infty = \operatorname*{ess\,sup}_{z\in\mathbb{C}} |f(z)| e^{-\frac{1}{2}|z|^2} < \infty \right\}.$$

In particular, F^2 is a reproducing kernel Hilbert space. The function $K_z(w) = e^{z\overline{w}}$ is the reproducing kernel for F^2 and

$$k_z(w) = \frac{K_z(w)}{\sqrt{K(z,z)}} = e^{z\overline{w} - \frac{1}{2}|z|^2}$$

is the normalized kernel.

Let f^∞ denote the space of entire functions such that

$$\lim_{z\to\infty} |f(z)| e^{-\frac{1}{2}|z|^2} = 0.$$

*Corresponding author.
E-mail addresses: zyzhuo@gpnu.edu.cn (Z. Zhuo), shenconghui2008@163.com (C. Shen), lidx@gduf.edu.cn (D. Li), jyulsx@163.com (S. Li).
Received November 26, 2022; Accepted January 8, 2023.

If $0 < p < q < \infty$, then $F^p \subset F^q \subset f^\infty \subset F^\infty$, and each inclusion is proper. Interested readers can refer to [1] for the theory of Fock spaces.

The Hilbert operator H_ν induced by the Hilbert matrix $(\frac{1}{n+k+1})_{n,k\geq 0}$ was studied on Hardy spaces [2] and Bergman spaces [3] in the unit disk. Let ν be a positive Borel measure on $[0,1)$. In [4], Galanopoulos and Peláez investigated the boundedness and compactness of the operator H_ν induced by Hankel matrix $H_\nu = (\nu_{n,k})_{n,k\geq 0}(\nu_{n,k} = \int_{[0,1)} t^{n+k}\,d\nu(t))$ on the Hardy space H^1 and the Bergman space A^2. Chatzifountas, Girela and Peláez [5] characterized the operator H_ν on Hardy spaces H^p. In [6, 7], Girela and Merchán also studied the operator H_ν acting on some analytic function spaces in the unit disk.

Recently, Ye and Zhou considered a new operator, which is called Derivative-Hilbert operator, with a close relation to the Hilbert operator H_ν, induced by Hankel matrix on analytic function spaces in [8, 9]. For more results on the operator induced by a Hankel matrix, we refer to [2, 10, 11].

Let ν be a positive Borel measure on the interval $[0,\infty)$. Let $\mathscr{H}_\nu = (\nu_{n,k})_{n,k\geq 0}$ denote the Hankel matrix with entries

$$\nu_{n,k} = \frac{1}{n!}\int_{[0,\infty)} t^{n+k}\,d\nu(t).$$

For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{C})$, we define

$$\mathscr{H}_\nu(f)(z) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\infty}\nu_{n,k}a_k\Big)z^n. \tag{1.1}$$

If the right hand side makes sense and defines a function in $H(\mathbb{C})$, the Hankel matrix $\mathscr{H}_\nu$ induces formally an operator (which will be also denoted $\mathscr{H}_\nu$) on $H(\mathbb{C})$.

One of purpose of this work is to discuss those positive Borel measures ν on $[0,\infty)$ for which the operators $\mathscr{H}_\nu$ are well defined on Fock spaces F^p $(0 < p < \infty)$. In Section 3, we prove that, for fixed $\varepsilon > \frac{1}{2}$, if $e^{\varepsilon|\cdot|^2}\nu$ is a (p,p)-Fock Carleson measure, then the power series in (1.1) is well defined on $\mathbb{C}$ for every $f \in F^p$. Furthermore, we can rewrite

$$\mathscr{H}_\nu(f)(z) = \int_{[0,\infty)} f(t)e^{tz}\,d\nu(t),\ z \in \mathbb{C}.$$

The second purpose of this work is to find out the condition of ν such that the operator $\mathscr{H}_\nu$ is bounded acting on Fock spaces by using the integral representation of $\mathscr{H}_\nu$. In Section 4, we completely characterize the measure ν for which $\mathscr{H}_\nu$ is a bounded (resp., compact) operator from the Fock space F^p into F^q $(0 < p,q < \infty)$.

Throughout this paper, for any given $p > 1$, p' denotes the conjugate exponent of p, that is, $1/p + 1/p' = 1$. We say that $A \lesssim B$ if there exists a constant C (independent of A and B) such that $A \leq CB$. The symbol $A \simeq B$ means that $A \lesssim B \lesssim A$. C denotes a finite constant that may change value from one occurrence to the next.

2. Preliminaries

In this section, we state some lemmas for the proof of our main results. The following two lemmas can be found in [12].

Lemma 2.1. *For every positive integer n,*

$$c(n!)^{\frac{p}{2}}n^{-\frac{p}{4}+\frac{1}{2}} \leq \int_0^\infty r^{np}e^{-\frac{p}{2}r^2}r\,dr \leq C(n!)^{\frac{p}{2}}n^{-\frac{p}{4}+\frac{1}{2}}.$$

Lemma 2.2. *Let $f(z)=\sum a_n z^n$ be an entire function.*

(i) *For $0<p\le 2$,*

$$\sum_{n=0}^{\infty}|a_n|^p\big(n!\big)^{\frac{p}{2}}n^{-\frac{p}{4}+\frac{1}{2}}<\infty\Rightarrow f\in F^p\Rightarrow\sum_{n=1}^{\infty}|a_n|^p\big(n!\big)^{\frac{p}{2}}n^{\frac{3p}{4}-\frac{3}{2}}<\infty.$$

(ii) *For $2\le p<\infty$,*

$$\sum_{n=0}^{\infty}|a_n|^p\big(n!\big)^{\frac{p}{2}}n^{\frac{3p}{4}-\frac{3}{2}}<\infty\Rightarrow f\in F^p\Rightarrow\sum_{n=1}^{\infty}|a_n|^p\big(n!\big)^{\frac{p}{2}}n^{-\frac{p}{4}+\frac{1}{2}}<\infty.$$

Lemma 2.3. [1, Lemma 2.32] *For any $0<p,R<\infty$, there exists a positive constant $C=C(p,R)$ such that*

$$|f(z)e^{-\frac{1}{2}|z|^2}|^p\le\frac{C}{r^2}\int_{D(z,r)}\left|f(z)e^{-\frac{1}{2}|z|^2}\right|^p dA(z)$$

for all entire functions f, all complex numbers z, and all $r\in(0,R]$. Here $D(z,r)=\{w\in\mathbb{C}:|w-z|<r\}$ denotes the Euclidean disk centered at z with radius r.

Lemma 2.4. [1, Theorem 2.7] *Let $0<p\le\infty$ and $f\in F^p$. Then $|f(z)|\le\|f\|_p e^{\frac{1}{2}|z|^2}$ for all $z\in\mathbb{C}$.*

Lemma 2.5. [1, Lemma 2.33] *Let $0<p\le\infty$. Each k_a is a unit vector in F^p.*

Lemma 2.6. [13, Lemma 2.4] *Let $0<p\le\infty$. For $\lambda=\{\lambda_j\}_{j=1}^{\infty}\in l^p$, set $S(\lambda)(z)=\sum_{j=1}^{\infty}\lambda_j k_{a_j}(z)$, $z\in\mathbb{C}$, then S is a bounded operator from l^p to F^p.*

Lemma 2.7. [1, Theorem 2.29] *Suppose $1\le p_0\le p_1\le\infty$ and $0\le\theta\le 1$. Then $[F^{p_0},F^{p_1}]_\theta=F^p$, where $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$.*

Lemma 2.8. [1, Corollary 2.25 and Theorem 2.26] *Set*

$$\langle f,g\rangle=\frac{1}{\pi}\int_{\mathbb{C}}f(w)\overline{g(w)}e^{-|z|^2}\,dA(w).$$

If $1\le p<\infty$ and let p' be the conjugate exponent of p, then the dual space of F^p can be identified with $F^{p'}$ under the pairing $\langle f,g\rangle$. If $0<p<1$, then the dual space of F^p can be identified with F^∞ under the pairing $\langle f,g\rangle$. The dual space of f^∞ can be identified with F^1 under the pairing $\langle f,g\rangle$.

Let $0<p,q<\infty$ and let $\mu\ge 0$. Recall that μ a (p,q)-Fock Carleson measure if there exists some constant C such that, for all $f\in F^p$,

$$\left(\int_{\mathbb{C}}\left|f(z)e^{-\frac{1}{2}|z|^2}\right|^q d\mu(z)\right)^{\frac{1}{q}}\le C\|f\|_p.$$

When $p=q$, μ is exactly the Fock Carleson measure for F^p (see [1, 14]). Also, μ is called a vanishing (p,q)-Fock Carleson measure if

$$\lim_{j\to\infty}\int_{\mathbb{C}}\left|f_j(z)e^{-\frac{1}{2}|z|^2}\right|^q d\mu(z)=0$$

whenever $\{f_j\}$ is a bounded sequence in F^p that converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $j\to\infty$.

Let $\widehat{\mu}_r(z) = \frac{\mu(D(z,r))}{|D(z,r)|}$, where $|E|$ means the area of any measurable set $E \subset \mathbb{C}$ with respect to the normalized area measure dA. Given $r > 0$, a sequence $\{a_k\}$ in $\mathbb{C}$ is called an r-lattice if $\cup_{k=1}^{\infty} D(a_k, r)$ covers $\mathbb{C}$ and the disks $\{D(a_k, r/3)\}_{k=1}^{\infty}$ are pairwise disjoint. For any $\delta > 0$, there exists a positive integer m (depending only on r and δ) such that every point in $\mathbb{C}$ belongs to at most m of the sets $D(a_k, \delta)$; see [1].

The following two lemmas characterize the (p,q)-Fock Carleson measure and vanishing (p,q)-Fock Carleson measure for $0 < p,q < \infty$, which can be found in [13].

Lemma 2.9. *Let $0 < p \leq q < \infty$, and let $\mu \geq 0$.*
(1) *μ is a (p,q)-Fock Carleson measure if and only if $\widehat{\mu}_r(z)$ is bounded on $\mathbb{C}$ for some (or any) $r > 0$.*
(2) *μ is a vanishing (p,q)-Fock Carleson measure if and only if $\widehat{\mu}_r(z) \to 0$ as $z \to \infty$ for some (or any) $r > 0$.*

Lemma 2.10. *Let $0 < q < p < \infty$ and let $\mu \geq 0$. Set $s = \frac{p}{q}$ and s' to be the conjugate exponent of s. Then the following statements are equivalent:*
(1) *μ is a (p,q)-Fock Carleson measure;*
(2) *μ is a vanishing (p,q)-Fock Carleson measure;*
(3) *$\widehat{\mu}_r(z) \in L^{s'}(dA)$ for some (or any) $r > 0$.*
(4) *$\sum_{k=1}^{\infty} \widehat{\mu}_r(a_k)^{s'}$ for some (or any) $r > 0$, where $\{a_k\}$ is reserved for this lattice and $a_k \to \infty$ as $k \to \infty$.*

In the light of above two lemmas, the notion of (vanishing) (p,q)-Fock Carleson measures does not depend on the particular value of p,q, but depends only on the ratio $s = \frac{p}{q}$ in the case $0 < q < p < \infty$. Let Λ^s be the class of all (p,q)-Fock Carleson measures and Λ_0^s be the class of all vanishing (p,q)-Fock Carleson measures. When $0 < s \leq 1$ (equivalently, $p \leq q$), we simply write Λ and Λ_0 for Λ^s and Λ_0^s respectively. That is

$$\Lambda = \{\mu \geq 0 : \widehat{\mu}_r \in L^\infty \; for \; some \; r > 0\}$$

and

$$\Lambda_0 = \{\mu \geq 0 : \lim_{|z| \to \infty} \widehat{\mu}_r(z) = 0 \; for \; some \; r > 0\}.$$

Notice that $\Lambda^s \subset \Lambda$ and $\Lambda_0^s \subset \Lambda_0$ for all $s > 0$.

3. Conditions such that $\mathscr{H}_\nu$ is Well Defined

In this section, we investigate the sufficient conditions or necessary conditions for operator $\mathscr{H}_\nu$ to be well defined on Fock spaces.

Theorem 3.1. *Suppose $0 < p < \infty$. Let ν be a positive Borel measure on $[0,\infty)$. If $e^{\varepsilon|\cdot|^2}\nu \in \Lambda$ for any fixed $\varepsilon > \frac{1}{2}$, then the power series in (1.1) is a well defined entire function for every $f \in F^p$. Furthermore,*

$$\mathscr{H}_\nu(f)(z) = \int_{[0,\infty)} f(t) e^{tz} d\nu(t), \; z \in \mathbb{C}, \; f \in F^p. \tag{3.1}$$

Proof. Fix $f(z)=\sum_{n=0}^{\infty}a_nz^n\in F^p$ and z with $|z|\le r$, $0<r<\infty$. Since $e^{\varepsilon|\cdot|^2}\nu\in\Lambda$, it might as well assume that $e^{\varepsilon|\cdot|^2}\nu$ is (s,s)-Fock Carleson measure for any $0<s<\infty$. By Lemma 2.4, we deduce that

$$
\begin{aligned}
&\int_{[0,\infty)}|f(t)e^{tz}|\,d\nu(t)\\
&\le\|f\|_p\int_{[0,\infty)}|e^{tz}|e^{\frac{1}{2}|t|^2}\,d\nu(t)\\
&=\|f\|_p\int_{[0,\infty)}|e^{tz}|e^{(\frac{1}{2}-\varepsilon)|t|^2}e^{\varepsilon|t|^2}\,d\nu(t)\\
&\le\|f\|_p\int_{\mathbb{C}}|e^{tz}|e^{(\frac{1}{2}-\varepsilon)|t|^2}\,dA(t)\le\|f\|_pe^{\frac{|r|^2}{4\varepsilon-2}}.
\end{aligned}
$$

So the integral in (3.1) uniformly converges on any compact subset of $\mathbb{C}$, the resulting function is analytic in $\mathbb{C}$ and, for every $z\in\mathbb{C}$,

$$
\begin{aligned}
\int_{[0,\infty)}f(t)e^{tz}\,d\nu(t)&=\sum_{n=0}^{\infty}\int_{[0,\infty)}f(t)\frac{t^n}{n!}\,d\nu(t)\,z^n\\
&=\sum_{n=0}^{\infty}\frac{1}{n!}\int_{[0,\infty)}\sum_{k=0}^{\infty}a_kt^{n+k}\,d\nu(t)\,z^n.
\end{aligned}
\tag{3.2}
$$

For any $1<s<\infty$, the definition of Fock Carleson measure, Hölder's inequality, and Lemma 2.1 imply that

$$
\begin{aligned}
|\nu_{n,k}|&\le\frac{1}{n!}\int_{[0,\infty)}|t|^{n+k}\,d\nu(t)\\
&\le\frac{1}{n!}\left(\int_{[0,\infty)}|t^ke^{-\frac{1}{2}|t|^2}|^se^{\varepsilon|t|^2}\,d\nu(t)\right)^{1/s}\left(\int_{[0,\infty)}|t^ne^{(\frac{1}{2}-\varepsilon)|t|^2}|^{s'}e^{\varepsilon|t|^2}\,d\nu(t)\right)^{1/s'}\\
&\le\frac{1}{n!}\left(\int_{\mathbb{C}}|t^ke^{-\frac{1}{2}|t|^2}|^s\,dA(t)\right)^{1/s}\left(\int_{\mathbb{C}}|t^ne^{(\frac{1}{2}-\varepsilon)|t|^2}|^{s'}\,dA(t)\right)^{1/s'}\\
&\lesssim\frac{1}{n!}(k!)^{\frac{1}{2}}k^{-\frac{1}{4}+\frac{1}{2s}}\left(\frac{n!}{(\varepsilon-\frac{1}{2})^n}\right)^{\frac{1}{2}}n^{-\frac{1}{4}+\frac{1}{2s'}}.
\end{aligned}
\tag{3.3}
$$

In particular, when $k=0,1$ and $s=2$, the above inequality can be rewritten as

$$
|\nu_{n,0}|\simeq|\nu_{n,1}|\lesssim\frac{1}{n!}\left(\frac{n!}{(\varepsilon-\frac{1}{2})^n}\right)^{\frac{1}{2}}.
\tag{3.4}
$$

Suppose that $0 < p \le 2$. (3.3) with $s = 4$, (3.4), and (i) in Lemma 2.2 indicate that, for every n,

$$
\begin{aligned}
&|\sum_{k=0}^{\infty} \nu_{n,k} a_k| \le |\nu_{n,0} a_0| + \sum_{k=1}^{\infty} |\nu_{n,k} a_k| = |\nu_{n,0} a_0| + \sum_{k=1}^{\infty} |\nu_{n+1,k-1} a_k| \\
\lesssim & |a_0| \frac{1}{n!}\left(\frac{n!}{(\varepsilon-\frac{1}{2})^n}\right)^{\frac{1}{2}} + \frac{1}{n!}\left(\frac{(n+1)!}{(\varepsilon-\frac{1}{2})^{n+1}}\right)^{\frac{1}{2}} (n+1)^{\frac{1}{8}} \sum_{k=1}^{\infty} |a_k| \big((k-1)!\big)^{\frac{1}{2}} (k-1)^{-\frac{1}{8}} \\
\lesssim & |a_0| \frac{1}{n!}\left(\frac{n!}{(\varepsilon-\frac{1}{2})^n}\right)^{\frac{1}{2}} + \frac{1}{n!}\left(\frac{(n+1)!}{(\varepsilon-\frac{1}{2})^{n+1}}\right)^{\frac{1}{2}} (n+1)^{\frac{1}{8}} \sum_{k=1}^{\infty} |a_k| (k!)^{\frac{1}{2}} k^{-\frac{5}{8}} \\
\lesssim & |a_0| \frac{1}{n!}\left(\frac{n!}{(\varepsilon-\frac{1}{2})^n}\right)^{\frac{1}{2}} + \frac{1}{n!}\left(\frac{(n+1)!}{(\varepsilon-\frac{1}{2})^{n+1}}\right)^{\frac{1}{2}} (n+1)^{\frac{1}{8}} \left(\sum_{k=1}^{\infty} |a_k|^2 k! \sum_{k=1}^{\infty} k^{-\frac{5}{4}}\right)^{\frac{1}{2}} \\
\lesssim & \frac{1}{n!}\left(\frac{(n+1)!}{(\varepsilon-\frac{1}{2})^{n+1}}\right)^{\frac{1}{2}} (n+1)^{\frac{1}{8}}.
\end{aligned}
$$

Arguing as in the preceding one, if $2 < p < \infty$, (3.3) with $s = p$, (3.4), (ii) in Lemma 2.2, and Hölder's inequality show that, for every n,

$$
\begin{aligned}
|\sum_{k=0}^{\infty} \nu_{n,k} a_k| &\lesssim (|a_0| + |a_1|) \frac{1}{n!}\left(\frac{n!}{(\varepsilon-\frac{1}{2})^n}\right)^{\frac{1}{2}} + \frac{1}{n!}\left(\frac{(n+2)!}{(\varepsilon-\frac{1}{2})^{n+2}}\right)^{\frac{1}{2}} (n+2)^{-\frac{1}{4}+\frac{1}{2p'}} \\
&\quad \cdot \sum_{k=2}^{\infty} |a_k| \Big(k!\Big)^{\frac{1}{2}} (k)^{-\frac{1}{4}+\frac{1}{2p}} (k^2-k)^{-\frac{1}{2}} \\
&\lesssim (|a_0| + |a_1|) \frac{1}{n!}\left(\frac{n!}{(\varepsilon-\frac{1}{2})^n}\right)^{\frac{1}{2}} + \frac{1}{n!}\left(\frac{(n+2)!}{(\varepsilon-\frac{1}{2})^{n+2}}\right)^{\frac{1}{2}} (n+2)^{-\frac{1}{4}+\frac{1}{2p'}} \\
&\quad \cdot \left(\sum_{k=2}^{\infty} |a_k|^p (k!)^{\frac{p}{2}} (k)^{-\frac{p}{4}+\frac{1}{2}}\right)^{1/p} \left(\sum_{k=2}^{\infty} (k^2-k)^{-\frac{p'}{2}}\right)^{1/p'} \\
&\lesssim \frac{1}{n!}\left(\frac{(n+2)!}{(\varepsilon-\frac{1}{2})^{n+2}}\right)^{\frac{1}{2}} (n+2)^{-\frac{1}{4}+\frac{1}{2p'}}.
\end{aligned}
$$

In each of the cases above, we have that the series in (1.1) is well defined for all $z \in \mathbb{C}$, and

$$\sum_{k=0}^{\infty} a_k \nu_{n,k} = \frac{1}{n!} \int_{[0,\infty)} f(t) t^n \, d\nu(t).$$

By (3.2), we obtain $\mathscr{H}_\nu(f)(z) = \int_{[0,\infty)} f(t) e^{tz} \, d\nu(t)$, $z \in \mathbb{C}$. This proves the desired result. □

Theorem 3.2. *Suppose $0 < p < \infty$ and let ν be a positive Borel measure on $[0,\infty)$ which satisfies*

$$\int_{[0,\infty)} \int_{[0,t)} e^{2ts} \, d\nu(s) \, d\nu(t) < \infty, \ if \ 0 < p \le 2, \tag{3.5}$$

or

$$\int_{[0,\infty)} \int_{[0,t)} (ts)^2 e^{2ts} \, d\nu(s) \, d\nu(t) < \infty, \ if \ 2 < p \le \infty. \tag{3.6}$$

Then the power series in (1.1) is well defined for every $f \in F^p$ and (3.1) holds.

Proof. Assume that $0 < p \leq 2$ and ν satisfies (3.5). Fix $f(z) = \sum_{n=0}^{\infty} a_n z^n \in F^p \subset F^2$. By Cauchy-Schwarz's inequality, we obtain that for any $n \in \mathbb{N}$

$$\sum_{k=0}^{\infty} |\nu_{n,k} a_k| \leq \Big(\sum_{k=0}^{\infty} |a_k|^2 k!\Big)^{\frac{1}{2}} \Big(\sum_{k=0}^{\infty} \frac{1}{k!} |\nu_{n,k}|^2\Big)^{\frac{1}{2}}. \tag{3.7}$$

From the definition of $\nu_{n,k}$, it is easy to see that

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{k!} |\nu_{n,k}|^2 &= (\frac{1}{n!})^2 \sum_{k=0}^{\infty} \frac{1}{k!} \Big(\int_{[0,\infty)} t^{n+k} d\nu(t)\Big)^2 \\ &= (\frac{1}{n!})^2 \sum_{k=0}^{\infty} \frac{1}{k!} \int_{[0,\infty)} \int_{[0,\infty)} s^{n+k} t^{n+k} d\nu(s) d\nu(t) \\ &\lesssim (\frac{1}{n!})^2 \sum_{k=0}^{\infty} \frac{1}{k!} \int_{[0,\infty)} \int_{[0,t)} (st)^k (st)^n d\nu(s) d\nu(t) \\ &\lesssim (\frac{1}{n!})^2 \int_{[0,\infty)} \int_{[0,t)} e^{st} (st)^n d\nu(s) d\nu(t). \end{aligned} \tag{3.8}$$

By Lemma 2.2, (3.5), and (3.7), it suffices to prove that

$$\sum_{k=0}^{\infty} a_k \nu_{n,k} = \frac{1}{n!} \int_{[0,\infty)} f(t) t^n d\nu(t).$$

Furthermore,

$$\begin{aligned} &\Big|\sum_{n=0}^{\infty} (\sum_{k=0}^{\infty} \nu_{n,k} a_k) z^n\Big| \leq \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{\infty} |a_k|^2 k!\Big)^{\frac{1}{2}} \Big((\frac{1}{n!})^2 \int_{[0,\infty)} \int_{[0,t)} e^{st} (st)^n d\nu(s) d\nu(t)\Big)^{\frac{1}{2}} |z|^n \\ &\leq \Big(\sum_{k=0}^{\infty} |a_k|^2 k!\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{\infty} \frac{1}{n!} |z|^{2n}\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{\infty} \frac{1}{n!} \int_{[0,\infty)} \int_{[0,t)} e^{st} (st)^n d\nu(s) d\nu(t)\Big)^{\frac{1}{2}} \\ &= \Big(\sum_{k=0}^{\infty} |a_k|^2 k!\Big)^{\frac{1}{2}} \Big(\int_{[0,\infty)} \int_{[0,t)} e^{2st} d\nu(s) d\nu(t)\Big)^{\frac{1}{2}} \Big(\sum_{n=0}^{\infty} \frac{1}{n!} |z|^{2n}\Big)^{\frac{1}{2}}, \end{aligned}$$

for each $z \in \mathbb{C}$. This shows that the power series in (1.1) represents an analytic function in $\mathbb{C}$ and

$$\mathscr{H}_\nu(f)(z) = \int_{[0,\infty)} f(t) e^{tz} d\nu(t),\ z \in \mathbb{C}.$$

If $2 \leq p < \infty$ and ν satisfies (3.6), the proof is similar to the preceding one with replacing (3.7) and (3.8) by

$$\begin{aligned} |\sum_{k=2}^{\infty} \nu_{n,k} a_k| &\leq \sum_{k=2}^{\infty} |a_k| (k!)^{\frac{1}{2}} k^{-\frac{1}{4}+\frac{1}{2p}} k^{\frac{1}{4}-\frac{1}{2p}} (\frac{1}{k!})^{\frac{1}{2}} |\nu_{n,k}| \\ &\leq \Big(\sum_{k=2}^{\infty} |a_k|^p (k!)^{\frac{p}{2}} k^{-\frac{p}{4}+\frac{1}{2}}\Big)^{\frac{1}{p}} \Big(\sum_{k=2}^{\infty} (\frac{1}{k!})^{\frac{p'}{2}} k^{\frac{p'}{4}-\frac{p'}{2p}} |\nu_{n,k}|^{p'}\Big)^{\frac{1}{p'}}, \end{aligned}$$

and

$$\sum_{k=2}^{\infty}\Big(\frac{1}{k!}\Big)^{\frac{p'}{2}}k^{\frac{p'}{4}-\frac{p'}{2p}}|\nu_{n,k}|^{p'}=\Big(\frac{1}{n!}\Big)^{p'}\sum_{k=2}^{\infty}\Big(\frac{1}{k!}\Big)^{\frac{p'}{2}}k^{\frac{p'}{4}-\frac{p'}{2p}}\left(\int_{[0,\infty)}t^{n+k}\,d\nu(t)\right)^{2\cdot\frac{p'}{2}}$$

$$\lesssim\Big(\frac{1}{n!}\Big)^{p'}\sum_{k=2}^{\infty}k^{\frac{p'}{4}-\frac{p'}{2p}-p'}\left(\frac{1}{(k-2)!}\int_{[0,\infty)}\int_{[0,t)}(st)^k(st)^n\,d\nu(s)d\nu(t)\right)^{\frac{p'}{2}}$$

$$\lesssim\Big(\frac{1}{n!}\Big)^{p'}\left(\sum_{k=2}^{\infty}k^{(-\frac{3}{4}-\frac{1}{2p})\frac{2p'}{2-p'}}\right)^{\frac{2-p'}{2}}\left(\sum_{k=2}^{\infty}\frac{1}{(k-2)!}\int_{[0,\infty)}\int_{[0,t)}(st)^k(st)^n\,d\nu(s)d\nu(t)\right)^{\frac{p'}{2}}$$

$$\lesssim\Big(\frac{1}{n!}\Big)^{p'}\left(\sum_{k=2}^{\infty}k^{(-\frac{3}{4}-\frac{1}{2p})\frac{2p'}{2-p'}}\right)^{\frac{2-p'}{2}}\left(\int_{[0,\infty)}\int_{[0,t)}(st)^2e^{st}(st)^n\,d\nu(s)d\nu(t)\right)^{\frac{p'}{2}},$$

respectively, where $(-\frac{3}{4}-\frac{1}{2p})\frac{2p'}{2-p'}<-\frac{3}{2}$. The case of $k=0,1$ requires different treatments.

$$\sum_{n=0}^{\infty}|\nu_{n,0}a_0z^n|=a_0\sum_{n=0}^{\infty}\Big(\frac{1}{n!}\Big)^{\frac{1}{2}}|z|^n\Big(\frac{1}{n!}\Big)^{\frac{1}{2}}\int_{[0,\infty)}|t|^n\,d\nu(t)$$

$$\lesssim a_0\left(\sum_{n=0}^{\infty}\frac{1}{n!}|z|^{2n}\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\int_{[0,\infty)}\int_{[0,t)}(st)^n\,d\nu(s)d\nu(t)\right)^{\frac{1}{2}}$$

$$\lesssim a_0\left(\sum_{n=0}^{\infty}\frac{1}{n!}|z|^{2n}\right)^{\frac{1}{2}}\left(\int_{[0,\infty)}\int_{[0,t)}(st)^2e^{2st}\,d\nu(s)d\nu(t)\right)^{\frac{1}{2}}.$$

The same arguments demonstrate that

$$\sum_{n=0}^{\infty}|\nu_{n,1}a_1z^n|\lesssim a_0\left(\sum_{n=0}^{\infty}\frac{1}{n!}|z|^{2n}\right)^{\frac{1}{2}}\left(\int_{[0,\infty)}\int_{[0,t)}(st)^2e^{2st}\,d\nu(s)d\nu(t)\right)^{\frac{1}{2}}.$$

This completes the proof of the theorem. □

The following theorem together with Theorem 3.1 reveals that the necessary condition and the sufficient condition for $\mathscr{H}_\nu$ with integral representation are closely related in some situations.

Theorem 3.3. *Suppose $0<p<\infty$. Let ν be a positive Borel measure on $[0,\infty)$. If, for any $f\in F^p$ and $z\in\mathbb{C}$, the integral in (3.1) converges absolutely, then $e^{\frac{1}{2}|\cdot|^2}d\nu$ is a $(p,1)$-Fock Carleson measure.*

Proof. Fix $0<p<\infty$. The assumption implies that the integral $\int_{[0,\infty)}f(t)e^{tz}\,d\nu(t)$ converges for $z=0$, i.e., for all $f\in F^p$,

$$\left|\int_{[0,\infty)}f(t)\,d\nu(t)\right|\le\int_{[0,\infty)}|f(t)|e^{-\frac{1}{2}|t|^2}e^{\frac{1}{2}|t|^2}\,d\nu(t)<\infty.$$

Therefore, by the closed graph theorem, the identity mapping is bounded from F^p into $L^1(e^{-\frac{1}{2}|\cdot|^2}d\nu)$, which implies the desired estimate. □

Using the method employed in [4, Proposition 1.4], we obtain the following result.

Theorem 3.4. *Suppose $1 < p < \infty$. Let ν be a Borel measure on $[0,\infty)$. If, for any $f \in F^p$ and $z \in \mathbb{C}$, the integral in (3.1) converges absolutely, then*

$$\int_{[0,\infty)} \int_{[0,t)} e^{\theta st}\, d\nu(s)\, d\nu(t) < \infty,$$

for any θ such that $0 < \theta < 1$ and $\theta \le \frac{1}{p-1}$.

Proof. Assume that the integral in (3.1) converges absolutely for each $z \in \mathbb{C}$ for any $f \in F^p$. Taking $z = 0$, there is $C > 0$ such that

$$\Big| \int_{[0,r)} f(t)\, d\nu(t) \Big| \le \int_{[0,r)} |f(t)|\, d\nu(t) < \int_{[0,\infty)} |f(t)|\, d\nu(t) < C,$$

for all $r \in (0,\infty)$. Specially, choosing $f = 1$, we have $\int_{[0,\infty)} d\nu(t) < \infty$, which means that ν is a finite Borel measure. On the other hand, an elementary calculation demonstrates that $\|K_t\|_{p'} = e^{\frac{1}{2}|t|^2}$. Using this and Hölder's inequality, we obtain that

$$\begin{aligned}\int_{[0,r)} \int_{\mathbb{C}} |f(z) e^{t\bar{z}}| e^{-|z|^2}\, dA(z)\, d\nu(t) &\le \|f\|_p \int_{[0,r)} \|K_t\|_{p'}\, d\nu(t) \\ &= \|f\|_p \int_{[0,r)} e^{\frac{1}{2}|t|^2}\, d\nu(t) < \infty.\end{aligned}$$

For any $f \in F^p$, the reproducing property and Fubini's theorem imply that

$$\begin{aligned}\int_{[0,r)} f(t)\, d\nu(t) &= \int_{[0,r)} \int_{\mathbb{C}} f(z) e^{t\bar{z}} e^{-|z|^2}\, dA(z)\, d\nu(t) \\ &= \int_{\mathbb{C}} f(z) \overline{\int_{[0,r)} e^{tz}\, d\nu(t)} e^{-|z|^2}\, dA(z) = \langle f, g_r \rangle, \qquad (3.9)\end{aligned}$$

where $g_r(z) = \int_{[0,r)} e^{tz}\, d\nu(t)$. By Lemma 2.8 and the uniform boundedness principle, we obtain $\sup_r \|g_r\|_{p'} < C$. Let $0 < \theta < 1$ and $\theta \le \frac{1}{p-1}$. By Hölder's inequality, it is easy to see that

$$\begin{aligned}&\int_{\mathbb{C}} |g_r^{\theta}(z) e^{-\frac{1}{2}|z|^2}|^p\, dA(z) = \int_{\mathbb{C}} |g_r(z) e^{-\frac{1}{2}|z|^2}|^{p\theta} e^{(\theta-1)p\frac{1}{2}|z|^2}\, dA(z) \\ =&\left(\int_{\mathbb{C}} |g_r(z) e^{-\frac{1}{2}|z|^2}|^{p'}\, dA(z) \right)^{\frac{p\theta}{p'}} \left(\int_{\mathbb{C}} e^{(\frac{p'}{p\theta})'(\theta-1)p\frac{1}{2}|z|^2}\, dA(z) \right)^{1/(\frac{p'}{p\theta})'} \\ <&C.\end{aligned}$$

This demonstrates that $g_r^{\theta} \in F^p$. Replacing g_r^{θ} into (3.9), we obtain that

$$\begin{aligned}\int_{\mathbb{C}} |g_r(z)|^{\theta+1} e^{-|z|^2}\, dA(z) &\ge \Big| \int_{\mathbb{C}} g_r^{\theta}(z) \overline{g_r(z)} e^{-|z|^2}\, dA(z) \Big| \\ &= \int_{[0,r)} \Big(\int_{[0,r)} e^{st}\, d\nu(s) \Big)^{\theta}\, d\nu(t).\end{aligned}$$

On the other hand, the Hölder's inequality implies that

$$\int_{\mathbb{C}} |g_r(z)|^{\theta+1} e^{-|z|^2}\, dA(z) = \int_{\mathbb{C}} |g_r(z) e^{-\frac{1}{2}|z|^2}|^{\theta+1} e^{(\theta-1)\frac{1}{2}|z|^2}\, dA(z) \lesssim \|g_r\|_{p'}^{\theta+1} < C.$$

Combining this with the previous inequality and letting $r \to \infty$, we obtain that

$$\int_{[0,\infty)} \Big(\int_{[0,\infty)} e^{st}\, d\nu(s) \Big)^{\theta} d\nu(t) < C.$$

Since ν is a Borel measure on $[0,\infty)$ and $0<\theta<1$, it follows from the Hölder's inequality that

$$\begin{aligned} C > \int_{[0,\infty)} \Big(\int_{[0,\infty)} e^{st}\, d\nu(s) \Big)^{\theta} d\nu(t) &\geq \int_{[0,\infty)} \int_{[0,\infty)} e^{\theta st}\, d\nu(s)\, d\nu(t) \\ &= 2\int_{[0,\infty)} \int_{[0,t)} e^{\theta st}\, d\nu(s)\, d\nu(t). \end{aligned}$$

This proves the desired result. □

4. Boundedness and Compactness of $\mathscr{H}_\nu$

In this section, we mainly characterize those measures ν for which $\mathscr{H}_\nu$ are bounded (resp., compact) operators from F^p into F^q for some q and p.

Theorem 4.1. *Suppose $0<p\leq q<\infty$. Let ν be a positive Borel measure on $[0,\infty)$ that satisfies the condition in Theorem 3.1. Then $\mathscr{H}_\nu$ is bounded from F^p into F^q if and only if $e^{|\cdot|^2}\nu \in \Lambda$.*

Proof. Suppose that $\mathscr{H}_\nu$ is a bounded operator from F^p into F^q. Given $r>0$, Lemmas 2.3 and 2.5 demonstrate that, for any $a\in[0,\infty)$, there exists a $C>0$ such that

$$\begin{aligned} C &> \|\mathscr{H}_\nu k_a\|_q \gtrsim |\mathscr{H}_\nu k_a(a) e^{-\frac{1}{2}|a|^2}| \\ &\geq \int_{[0,\infty)} |e^{t\bar{a}-\frac{1}{2}|a|^2}|^2\, d\nu(t) = \int_{[0,\infty)} e^{-|t-a|^2} e^{|t|^2}\, d\nu(t) \\ &\gtrsim \int_{|t-a|<r} e^{|t|^2}\, d\nu(t). \end{aligned} \tag{4.1}$$

This proves that $e^{|\cdot|^2}\nu\in\Lambda$ by Lemma 2.9.

Conversely, suppose $e^{|\cdot|^2}\nu\in\Lambda$. For any $f\in F^p$, $g\in F^\infty$ and $0<\rho<1$,

$$\begin{aligned} \int_{\mathbb{C}} |\mathscr{H}_\nu(f)(\rho z) g(\rho z)| e^{-|z|^2}\, dA(z) &\leq \int_{\mathbb{C}} \int_{[0,\infty)} |f(t)| e^{\rho t z}\, d\nu(t) |g(\rho z)| e^{-|z|^2}\, dA(z) \\ &\leq \int_{\mathbb{C}} \|f\|_p \int_{[0,\infty)} |e^{\rho t z} e^{-\frac{1}{2}|t|^2}| e^{|t|^2}\, d\nu(t) |g(\rho z)| e^{-|z|^2}\, dA(z) \\ &\leq \|f\|_p \|g\|_\infty \int_{\mathbb{C}} e^{|\rho z|^2} e^{-|z|^2}\, dA(z) \\ &\lesssim \|f\|_p \|g\|_\infty. \end{aligned}$$

Therefore, Fubini's theorem and the reproducing property imply that

$$\begin{aligned} \int_{\mathbb{C}} \overline{\mathscr{H}_\nu(f)(\rho z)} g(\rho z) e^{-|z|^2}\, dA(z) &= \int_{\mathbb{C}} \int_{[0,\infty)} \overline{f(t)} e^{\rho t \bar{z}}\, d\nu(t) g(\rho z) e^{-|z|^2}\, dA(z) \\ &= \frac{1}{\rho^2} \int_{[0,\infty)} \int_{\mathbb{C}} e^{t\bar{w}} g(w) e^{-\frac{1}{\rho^2}|w|^2}\, dA(w) \overline{f(t)}\, d\nu(t) \\ &= \frac{1}{\rho^2} \int_{[0,\infty)} g(\rho^2 t) \overline{f(t)}\, d\nu(t),\ 0<\rho<1,\ f\in F^p,\ g\in F^\infty. \end{aligned} \tag{4.2}$$

For clarity, we now break the proof into three cases: $0<q<1$, $q=1$, and $1<q<\infty$.

Case $1 < q < \infty$. Combining (4.2) with Lemma 2.8, we conclude that $\mathscr{H}_\nu$ is a bounded operator from F^p into F^q if and only if there exists a positive constant C such that

$$\left| \int_{[0,\infty)} \overline{f(t)} g(t)\, d\nu(t) \right| \leq C\|f\|_p \|g\|_{q'},\ f \in F^p,\ g \in F^{q'}.$$

Note that $p \leq 1 + \frac{p}{q'}$ and $q' \leq 1 + \frac{q'}{p}$ by the fact that $p \leq q$. By Lemma 2.9, $e^{|\cdot|^2}\nu$ is a $(p, 1+\frac{p}{q'})$-Fock Carleson measure or $(q', 1+\frac{q'}{p})$-Fock Carleson measure. Thus, by Hölder's inequality,

$$\begin{aligned}
&\int_{[0,\infty)} |f(t)g(t)|\, d\nu(t) \\
\leq & \left(\int_{[0,\infty)} |f(t)e^{-\frac{1}{2}|t|^2}|^{\frac{q'+p}{q'}} e^{|t|^2}\, d\nu(t) \right)^{\frac{q'}{q'+p}} \left(\int_{[0,\infty)} |g(t)e^{-\frac{1}{2}|t|^2}|^{\frac{q'+p}{p}} e^{|t|^2}\, d\nu(t) \right)^{\frac{p}{q'+p}} \\
\lesssim & \|f\|_p \|g\|_{q'}.
\end{aligned}$$

This implies that $\mathscr{H}_\nu$ is bounded.

Case $q = 1$. From Lemma 2.8 we see that $(f^\infty)^* = F^1$ under the pairing $\langle f, g\rangle$. It follows that $\mathscr{H}_\nu$ is a bounded operator from F^p into F^1 if and only if there exists a positive constant C such that

$$\left| \int_{[0,\infty)} \overline{f(t)} g(t)\, d\nu(t) \right| \leq C\|f\|_p \|g\|_\infty,\ f \in F^p,\ g \in f^\infty.$$

By Lemma 2.9, $e^{|\cdot|^2}\nu$ is a $(p,1)$-Fock Carleson measure. Thus,

$$\int_{[0,\infty)} |f(t)g(t)|\, d\nu(t) \leq \|g\|_\infty \int_{[0,\infty)} |f(t)e^{-\frac{1}{2}|t|^2}| e^{|t|^2}\, d\nu(t) \leq C\|f\|_p \|g\|_\infty.$$

We see that $\mathscr{H}_\nu$ is bounded.

Case $0 < q < 1$. For given some $r > 0$, there exists a sequence $\{a_j\}_{k=1}^\infty$ in $[0,\infty)$ such that $a_j = 2jr$. According to our condition and Lemma 2.9, we have

$$\int_{[a_j - r, a_j + r]} e^{|t|^2}\, d\nu(t) \leq \|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}$$

for all a_j. Combining this inequality with Lemma 2.3, we deduce

$$\begin{aligned}
|\mathscr{H}_\nu(f)(z)|^q &\leq \left(\int_{[0,\infty)} |f(t)e^{tz}e^{-|t|^2}| e^{|t|^2}\, d\nu(t) \right)^q \\
&\leq \sum_{j=1}^\infty \left(\sup_{t \in [a_j - r, a_j + r]} |f(t)e^{tz}e^{-|t|^2}| \int_{[a_j - r, a_j + r]} e^{|t|^2}\, d\nu(t) \right)^q \\
&\lesssim \|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}^q \sum_{j=1}^\infty \sup_{t \in [a_j - r, a_j + r]} |f(t)e^{tz}e^{-|t|^2}|^q \\
&\lesssim \|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}^q \sum_{j=1}^\infty \int_{D(a_j, 2r)} |f(t)e^{tz}e^{-|t|^2}|^q\, dA(t) \\
&\lesssim \|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}^q \int_{\mathbb{C}} |f(t)e^{tz}e^{-|t|^2}|^q\, dA(t).
\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{\mathbb{C}}|\mathscr{H}_\nu(f)(z)e^{-\frac{1}{2}|z|^2}|^q\,dA(z)\\ \lesssim&\|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}^q\int_{\mathbb{C}}\int_{\mathbb{C}}|f(t)e^{tz}e^{-|t|^2}e^{-\frac{1}{2}|z|^2}|^q\,dA(t)\,dA(z)\\ =&\|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}^q\int_{\mathbb{C}}|f(t)e^{-\frac{1}{2}|t|^2}|^q\int_{\mathbb{C}}|e^{tz}e^{-\frac{1}{2}|t|^2}e^{-\frac{1}{2}|z|^2}|^q\,dA(z)\,dA(t)\\ =&\|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}^q\int_{\mathbb{C}}|f(t)e^{-\frac{1}{2}|t|^2}|^q\,dA(t).\end{aligned}$$

That is,

$$\|\mathscr{H}_\nu(f)\|_q\lesssim\|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}\|f\|_q\lesssim\|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}\|f\|_p,$$

where the last step follows from Lemma 2.4. The proof of the theorem is complete. □

The proof of the following lemma is similar to that of [15, Proposition 3.11]. We omit the details.

Lemma 4.1. *Suppose that $0<p,q<\infty$ and $\mathscr{H}_\nu$ is bounded from F^p into F^q. Then $\mathscr{H}_\nu$ is a compact operator if and only if, for any bounded sequence $\{f_n\}$ in F^p which converges uniformly to 0 on every compact subset of $\mathbb{C}$, $\mathscr{H}_\nu(f_n)\to 0$ as $n\to 0$ in F^q.*

Theorem 4.2. *Suppose $0<p\le q<\infty$. Let ν be a positive Borel measure on $[0,\infty)$ that satisfies the condition in Theorem 3.1. Then $\mathscr{H}_\nu$ is a compact operator from F^p into F^q if and only if $e^{|\cdot|^2}\nu\in\Lambda_0$.*

Proof. Assume that $\mathscr{H}_\nu$ is a compact operator from F^p into F^q. Using Lemmas 4.1 and 2.5, we obtain that $\{\mathscr{H}_\nu(k_a)\}$ converges to 0 in F^q when $\alpha\to\infty$. Hence, by (4.1) we deduce that $\int_{|t-a|<r}e^{|t|^2}\,d\nu(t)\to 0$. This proves $e^{|\cdot|^2}\nu\in\Lambda_0$.

Conversely, suppose that $e^{|\cdot|^2}\nu\in\Lambda_0$. If $1<q<\infty$, similarly to the proof of Theorem 4.1, by Lemma 2.9 we see that $e^{|\cdot|^2}\nu$ is a vanishing $(p,1+\frac{p}{q'})$-Fock Carleson measure or vanishing $(q',1+\frac{q'}{p})$-Fock Carleson measure. Let $\{f_j\}$ be a bounded sequence in F^p that converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $j\to\infty$. Then by Hölder's inequality, we have

$$\int_{[0,\infty)}|f_j(t)g(t)|\,d\nu(t)\le\left(\int_{[0,\infty)}|f_j(t)e^{-\frac{1}{2}|t|^2}|^{\frac{q'+p}{q'}}e^{|t|^2}\,d\nu(t)\right)^{\frac{q'}{q'+p}}\|g\|_{q'}\to 0,$$

as $j\to\infty$ for $g\in F^{q'}$. It follows from (4.2) that $\lim_{j\to\infty}\int_{\mathbb{C}}\overline{\mathscr{H}_\nu(f_j)(z)}g(z)e^{-|z|^2}\,dA(z)\to 0$ for all $g\in F^{q'}$. Therefore $\mathscr{H}_\nu:F^p\to F^q$ is compact.

The proof for $q=1$ is similar to that of $1<q<\infty$, we omit the details here.

Now we prove the case $q<1$. Let $\{f_n\}$ be a bounded sequence in F^p that converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $n\to\infty$. Give some $r>0$ and set $a_j=2jr$. Since $e^{|\cdot|^2}\nu\in\Lambda_0$ for any $\varepsilon>0$, there exists $J>0$ such that, for all $j>J$,

$$\int_{[a_j-r,a_j+r]}e^{|t|^2}\,d\nu(t)<\varepsilon.$$

Combining this inequality with Lemma 2.3, we obtain

$$
\begin{aligned}
|\mathscr{H}_\nu(f_n)(z)|^q \leq & \sum_{j=1}^{J} \left(\sup_{t\in[a_j-r,a_j+r]} |f_n(t)e^{tz}e^{-|t|^2}| \int_{[a_j-r,a_j+r]} e^{|t|^2}\, d\nu(t) \right)^q \\
& + \sum_{j=J+1}^{\infty} \left(\sup_{t\in[a_j-r,a_j+r]} |f_n(t)e^{tz}e^{-|t|^2}| \int_{[a_j-r,a_j+r]} e^{|t|^2}\, d\nu(t) \right)^q \\
\lesssim & \|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}^q \int_{D(0,2(J+1)r)} |f_n(t)e^{tz}e^{-|t|^2}|^q\, dA(t) + \varepsilon^q \int_{\mathbb{C}} |f_n(t)e^{tz}e^{-|t|^2}|^q\, dA(t).
\end{aligned}
$$

Thus, Fubini's theorem implies that

$$
\begin{aligned}
& \int_{\mathbb{C}} |\mathscr{H}_\nu(f_n)(z)e^{-\frac{1}{2}|z|^2}|^q\, dA(z) \\
\lesssim & \|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}^q \int_{\mathbb{C}} \int_{D(0,2(J+1)r)} |f_n(t)e^{tz}e^{-|t|^2}e^{-\frac{1}{2}|z|^2}|^q\, dA(t)\, dA(z) \\
& + \varepsilon^q \int_{\mathbb{C}} \int_{\mathbb{C}} |f_n(t)e^{tz}e^{-|t|^2}e^{-\frac{1}{2}|z|^2}|^q\, dA(t)\, dA(z) \\
= & \|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}^q \int_{D(0,2(J+1)r)} |f_n(t)e^{-\frac{1}{2}|t|^2}|^q\, dA(t) + \varepsilon^q \int_{\mathbb{C}} |f_n(t)e^{-\frac{1}{2}|t|^2}|^q\, dA(t) \\
\leq & \varepsilon^q \|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}^q + \varepsilon^q \int_{\mathbb{C}} |f_n(t)e^{-\frac{1}{2}|t|^2}|^q\, dA(t) \\
= & \varepsilon^q \|\widehat{(e^{|\cdot|^2}\nu)}_r\|_{L^\infty}^q + \varepsilon^q \|f_n\|_p^q.
\end{aligned}
$$

Therefore, by the arbitrariness of ε, we see that $\mathscr{H}_\nu : F^p \to F^q$ is compact. □

Theorem 4.3. *Suppose $0 < q < p < \infty$. Let ν be a positive Borel measure on $[0,\infty)$ that satisfies the condition in Theorem 3.1. Then the following statements are equivalent:*

(i) *$\mathscr{H}_\nu$ is a bounded operator from F^p into F^q;*

(ii) *$\mathscr{H}_\nu$ is a compact operator from F^p into F^q;*

(iii) $e^{|\cdot|^2}\nu \in \Lambda^{\frac{pq'}{p+q'}}$.

Proof. $(ii) \Rightarrow (i)$. The implication is trivial.

$(i) \Rightarrow (iii)$. We first prove the case $q \geq 1$. By the assumption that $\mathscr{H}_\nu : F^p \to F^q$ is bounded, we see that the operator $\mathscr{H}_\nu : F^{q'} \to F^{p'}$ is bounded. By Lemma 2.7 we see that $\mathscr{H}_\nu : F^{2m} \to F^{(2m)'}$ is bounded, where $\frac{1}{m} = \frac{1}{p} + \frac{1}{q'}$. By duality argument,

$$
\left| \int_{[0,\infty)} \overline{f(t)} g(t)\, d\nu(t) \right| \leq C\|f\|_{2m}\|g\|_{2m},\ f \in F^{2m},\ g \in F^{2m}.
$$

Specifically, letting $g = f$, we have

$$
\int_{[0,\infty)} |f(t)|^2\, d\nu(t) \leq C\|f\|_{2m}^2,\ f \in F^{2m},
$$

which demonstrates that $e^{|\cdot|^2}\nu \in \Lambda^m$.

Now we prove the case $q < 1$. Given any $\{\lambda_j\}_{j=1}^\infty \in l^p$ and r-lattice $\{a_j\}_{j=1}^\infty$, Lemma 2.6 demonstrates that $f(z) = \sum_{j=1}^\infty \lambda_j k_{a_j}(z) \in F^p$ with $\|f\|_p \lesssim \|\{\lambda_j\}_j\|_{l^p}$. By Khinchine's inequality and the boundedness of $\mathscr{H}_\nu$, we have

$$\begin{aligned}
&\int_{\mathbb{C}} \Big(\sum_{j=1}^{\infty} |\lambda_j \mathscr{H}_\nu(k_{a_j})(z)|^2\Big)^{\frac{q}{2}} e^{-\frac{q}{2}|z|^2}\, dA(z)\\
\lesssim &\int_0^1 \int_{\mathbb{C}} \Big|\sum_{j=1}^{\infty} \psi_j(t)\lambda_j \mathscr{H}_\nu(k_{a_j})(z)\Big|^q e^{-\frac{q}{2}|z|^2}\, dA(z)\, dt\\
\lesssim &\int_0^1 \|\mathscr{H}_\nu\|^q_{F^p\to F^q} \|\sum_{j=1}^{\infty} \psi_j(t)\lambda_j k_{a_j}\|_p^q\, dt\\
\lesssim &\|\mathscr{H}_\nu\|^q_{F^p\to F^q} \|\{\psi_j(t)\lambda_j\}_j\|^q_{l^p} \lesssim \|\mathscr{H}_\nu\|^q_{F^p\to F^q} \|\{\lambda_j\}_j\|^q_{l^p},
\end{aligned}$$

where $\psi_j(t)$ is the j-th Rademacher function on $[0,1]$. Meanwhile, by Lemma 2.3 we obtain

$$\begin{aligned}
&\int_{\mathbb{C}} \Big(\sum_{j=1}^{\infty} |\lambda_j \mathscr{H}_\nu(k_{a_j})(z)|^2\Big)^{\frac{q}{2}} e^{-\frac{q}{2}|z|^2}\, dA(z)\\
\gtrsim &\sum_{k=1}^{\infty} \int_{D(a_k,r)} \Big(\sum_{j=1}^{\infty} |\lambda_j \mathscr{H}_\nu(k_{a_j})(z)|^2\Big)^{\frac{q}{2}} e^{-\frac{q}{2}|z|^2}\, dA(z)\\
\gtrsim &\sum_{k=1}^{\infty} \int_{D(a_k,r)} |\lambda_k \mathscr{H}_\nu(k_{a_k})(z)|^q e^{-\frac{q}{2}|z|^2}\, dA(z)\\
\gtrsim &\sum_{k=1}^{\infty} |\lambda_k|^q |\mathscr{H}_\nu(k_{a_k})(a_k)|^q e^{-\frac{q}{2}|a_k|^2} \gtrsim \sum_{k=1}^{\infty} |\lambda_k|^q \widehat{(e^{|\cdot|^2}\nu)}_r(a_k)^q.
\end{aligned}$$

Setting $\beta_k = |\lambda_k|^q$, then $\{\beta_k\}_{k=1}^{\infty} \in l^{\frac{p}{q}}$. Therefore,

$$\sum_{k=1}^{\infty} \beta_k \widehat{(e^{|\cdot|^2}\nu)}_r(a_k)^q \lesssim \|\mathscr{H}_\nu\|^q_{F^p\to F^q} \|\{\beta_j\}_j\|_{l^{\frac{p}{q}}}.$$

The duality argument shows that $\{\widehat{(e^{|\cdot|^2}\nu)}_r(a_k)\}_{k=1}^{\infty} \in l^{\frac{pq}{p-q}}$. Hence, $e^{|\cdot|^2}\nu \in \Lambda^{\frac{pq'}{p+q'}}$ by Lemma 2.10.

$(iii) \Rightarrow (ii)$. Suppose $e^{|\cdot|^2}\nu \in \Lambda^{\frac{pq'}{p+q'}}$. First we consider the case $q \geq 1$. Note that $p > 1+\frac{p}{q'}$ and $q' > 1+\frac{q'}{p}$ since $p > q$. By Lemma 2.10, $e^{|\cdot|^2}\nu$ is a vanishing $(p, 1+\frac{p}{q'})$-Fock Carleson measure or vanishing $(q', 1+\frac{q'}{p})$-Fock Carleson measure. Then for any bounded sequence $\{f_n\}$ in F^p that converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$, we have

$$\int_{[0,\infty)} |f_n(t)g(t)|\, d\nu(t) \leq \left(\int_{[0,\infty)} |f_n(t)e^{-\frac{1}{2}|t|^2}|^{\frac{q'+p}{q'}} e^{|t|^2}\, d\nu(t)\right)^{\frac{q'}{q'+p}} \|g\|_{q'} \to 0,$$

as $n \to \infty$ for all $g \in F^{q'}$(or $g \in f^{\infty}$ when $q = 1$). Then (4.2) implies that

$$\lim_{n\to\infty} \int_{\mathbb{C}} \overline{\mathscr{H}_\nu(f_n)(z)} g(z) e^{-|z|^2}\, dA(z) = 0,$$

for all $g \in F^{q'}$(or $g \in f^{\infty}$). Therefore, $\mathscr{H}_\nu : F^p \to F^q$ is compact.

Finally, we consider the case $q<1$. Give some $r>0$. Set $a_j=2jr$. Then for any bounded sequence $\{f_n\}$ in F^p that converges to 0 uniformly on compact subsets of $\mathbb{C}$ as $n\to\infty$, we have

$$\begin{aligned}
|\mathscr{H}_\nu(f_n)(z)|^q &\leq \sum_{j=1}^{\infty}\left(\int_{[a_j-r,a_j+r]} e^{|t|^2}\,d\nu(t)\sup_{t\in[a_j-r,a_j+r]}|f_n(t)e^{tz}e^{-|t|^2}|\right)^q\\
&\lesssim \sum_{j=1}^{\infty}\left(\int_{[a_j-r,a_j+r]} e^{|t|^2}\,d\nu(t)\right)^q\int_{D(a_j,2r)}|f_n(w)e^{wz}e^{-|w|^2}|^q\,dA(w)\\
&\lesssim \sum_{j=1}^{\infty}\int_{D(a_j,2r)}|\hat{\mu}_{3r}(w)f_n(w)e^{wz}e^{-|w|^2}|^q\,dA(w),
\end{aligned}$$

where $\hat{\mu}_{3r}(w)=\int_{[w-3r,w+3r]}e^{|t|^2}\,d\nu(t)$. By Lemma 2.10, for any positive ε, there exists a $R>0$ such that $\int_{\mathbb{C}\setminus D(0,R)}|\hat{\mu}_{3r}(w)|^{\frac{pq}{p-q}}\,dA(w)<\varepsilon$. Thus,

$$\begin{aligned}
&\int_{\mathbb{C}}|\mathscr{H}_\nu(f_n)(z)e^{-\frac{1}{2}|z|^2}|^q\,dA(z)\\
\lesssim&\int_{\mathbb{C}}\int_{\mathbb{C}}|\hat{\mu}_{3r}(w)f_n(w)e^{wz}e^{-|w|^2}e^{-\frac{1}{2}|z|^2}|^q\,dA(w)\,dA(z)\\
=&\int_{\mathbb{C}}|\hat{\mu}_{3r}(w)f_n(w)e^{-\frac{1}{2}|w|^2}|^q\int_{\mathbb{C}}|e^{wz}e^{-\frac{1}{2}|w|^2}e^{-\frac{1}{2}|z|^2}|^q\,dA(z)\,dA(w)\\
=&\left(\int_{D(0,R)}+\int_{\mathbb{C}\setminus D(0,R)}\right)|\hat{\mu}_{3r}(w)f_n(w)e^{-\frac{1}{2}|w|^2}|^q\,dA(w)\\
\leq&\left(\int_{D(0,R)}|\hat{\mu}_{3r}(w)|^{\frac{pq}{p-q}}\,dA(w)\right)^{\frac{p-q}{p}}\left(\int_{D(0,R)}|f_n(w)e^{-\frac{1}{2}|w|^2}|^p\,dA(w)\right)^{\frac{q}{p}}\\
&+\left(\int_{\mathbb{C}\setminus D(0,R)}|\hat{\mu}_{3r}(w)|^{\frac{pq}{p-q}}\,dA(w)\right)^{\frac{p-q}{p}}\left(\int_{\mathbb{C}\setminus D(0,R)}|f_n(w)e^{-\frac{1}{2}|w|^2}|^p\,dA(w)\right)^{\frac{q}{p}}\\
\leq&\varepsilon^q\left(\int_{\mathbb{C}}|\hat{\mu}_{3r}(w)|^{\frac{pq}{p-q}}\,dA(w)\right)^{\frac{p-q}{p}}+\varepsilon^{\frac{p-q}{p}}\left(\int_{\mathbb{C}}|f_n(w)e^{-\frac{1}{2}|w|^2}|^p\,dA(w)\right)^{\frac{q}{p}},
\end{aligned}$$

which implies that $\mathscr{H}_\nu : F^p\to F^q$ is compact. □

Acknowledgments

This research was supported by Guangdong Basic and Applied Basic Research Foundation (No. 2020A1515110493 and No. 2022A151501031), Natural Science Foundation of Fujian Province (No. 2022J05237), Natural Science Research Project of Guangdong Education Department (No. 2020KQNCX042) and Guangdong Polytechnic Normal University Science Foundation (No. 2021SDKYA154).

REFERENCES

[1] K. Zhu, Analysis on Fock Spaces, Springer, New York, 2012.

[2] E. Diamantopoulos and A. Siskakis, Composition operators and the Hilbert matrix, Studia Math. 140 (2000), 191-198.

[3] E. Diamantopoulos, Hilbert matrix on Bergman spaces, Illinois J. Math. 48 (2004), 1067-1078.

[4] P. Galanopoulos, J. Peláez, A Hankel matrix acting on Hardy and Bergman spaces, Studia Math. 200 (2010), 201-220.

[5] C. Chatzifountas, D. Girela, J. Peláez, A generalized Hilbert matrix acting on Hardy spaces, J. Math. Anal. Appl. 413 (2014), 154-168.

[6] D. Girela, N. Merchán, Hankel matrix acting on spaces of analytic functions, Integral Equations Operator Theory 89 (2017), 581-594.

[7] D. Girela, N. Merchán, A generalized Hilbert operator acting on conformally invariant spaces, Banach J. Math. Anal. 12 (2018), 374-398.

[8] S. Ye, Z. Zhou, A derivative-Hilbert operator acting on the Bloch space, Complex Anal. Oper. Theory 15 (2021), 88.

[9] S. Ye, Z. Zhou, A derivative-Hilbert operator acting on Bergman spaces, J. Math. Anal. Appl. 506 (2022), 125553.

[10] G. Bao, H. Wulan, Hankel matrices acting on Dirichlet spaces, J. Math. Anal. Appl. 409 (2014), 228-235.

[11] H. Widom, Hankel matrices, Trans. Am. Math. Soc. 121 (1966), 1-35.

[12] J. Tung, On Taylor coefficients and multipliers in Fock spaces, Contemp. Math. 454 (2008), 135-147.

[13] Z. Hu, X. Lv, Toeplitz operators from one Fock space to another, Integral Equations Operator Theory 70 (2011), 541-559.

[14] J. Isralowitz, K. Zhu, Toeplitz operators on the Fock space, Integral Equations Operator Theory 66 (2010), 593-611.

[15] C.C. Cowen, B.D. MacCluer, Composition Operators on Spaces of Analytic Functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1995.