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NONCYCLIC φ-CONTRACTIONS IN HYPERBOLIC UNIFORMLY CONVEX METRIC SPACES

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Abstract. This paper aims to study the convergence of Picard's iteration to a best proximity pair for a class of noncyclic mappings with the help of projections in hyperbolic uniformly convex metric spaces. Some sufficient conditions are provided to guarantee the existence of a common best proximity pair for a pair of noncyclic mappings. Moreover, the existence and convergence of best proximity pairs for asymptotic pointwise noncyclic orbital contractions is studied. The main conclusions are supported with illustrative examples.

Keywords. Asymptotic pointwise noncyclic orbital contractions; Best proximity pair; Hyperbolic uniformly convex metric space; Noncyclic contraction.

1. INTRODUCTION

Banach contraction principle plays a pivotal role in numerous branches of applied mathematics and physical sciences. This important result was extended in various directions. One of the extensions was given by Kirk, Srinivasan, and Veeramani [1] for a cyclic mapping. One recalls that a mapping $T : A\cup B \to A\cup B$, where A and B are two nonempty subsets of a metric space (X,d), is said to be cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$. Whereas, a mapping $T : A\cup B \to A\cup B$ is said to be *noncyclic* if $T(A) \subseteq A$ and $T(B) \subseteq B$.

Very recently, the following extension of the Banach contraction principle was proved in [2].

Theorem 1.1. [2, Proposition 3.1] *Let A and B be two nonempty closed subsets of a complete metric space (X,d) and $T : A\cup B \to A\cup B$ be a noncyclic mapping. Suppose that there exists $\alpha \in (0,1)$ such that $d(Tx,Ty) \le \alpha d(x,y)$ for all $(x,y) \in A\times B$. Then T has a unique fixed point in $A\cap B$.*

Indeed, the contractive condition on the noncyclic mapping T ensures that $A\cap B$ is nonempty and it follows from the Banach contraction principle that T has a unique fixed point in $A\cap B$. The situation becomes different when we assume that $A\cap B = \emptyset$. Then it is interesting to study

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the existence of *best proximity pairs* for the non-self mapping T, that is, a point $(p,q) \in A \times B$ such that

$$p = Tp, \quad q = Tq \quad \text{and} \quad d(p,q) = \operatorname{dist}(A,B) := \inf\{d(x,y) : (x,y) \in A \times B\}.$$

In this case, the existence of a best proximity pair for noncyclic mapping T is equivalent to the existence of a solution to the following minimization problem: Find

$$\min_{x\in A} d(x,Tx), \ \min_{y\in B} d(y,Ty) \ \text{and} \ \min_{(x,y)\in A\times B} d(x,y). \tag{1.1}$$

An existence result for best proximity pairs was first established in [3]. Recall that a noncyclic mapping $T : A\cup B \to A\cup B$ is said to be *relatively nonexpansive* if $d(Tx,Ty) \leq d(x,y)$ for all $(x,y) \in A\times B$.

Theorem 1.2. *Let A and B be nonempty, bounded, closed, and convex subsets of a uniformly convex Banach space X and T a noncyclic relatively nonexpansive mapping defined on $A\cup B$. Then T has a best proximity pair.*

We mention that the existence result of Theorem 1.2 is based on the fact that every nonempty, bounded closed, and convex pair of subsets of a uniformly convex Banach space X has *proximal normal structure* (see [3, Proposition 2.1 and Theorem 2.2]).

Definition 1.1. Let A and B be nonempty subsets of a metric space (X,d). A self-mapping $T : A\cup B \to A\cup B$ is said to be a noncyclic contraction if T is a noncyclic mapping satisfying

$$d(Tx,Ty) \leq \alpha d(x,y) + (1-\alpha)\mathrm{dist}(\mathrm{A},\mathrm{B}),$$

for some $\alpha \in (0,1)$ and for every $(x,y) \in A\times B$.

Clearly, this class of noncyclic mappings is stronger than the class of relatively nonexpansive mappings. The following existence theorem was established in [4] for noncyclic contractions without the geometric notion of the proximal normal structure on the convex pair (A,B) of subsets of a Banach space.

Theorem 1.3. [4, Theorem 3.10] *Let (A,B) be a nonempty, weakly compact, and convex pair in a strictly convex Banach space X. Assume that $T : A\cup B \to A\cup B$ is a noncyclic contraction mapping. Then T has a best proximity pair.*

Note that in Theorem 1.3 the only problem is the existence of best proximity pairs. Gabeleh [5] proved the convergence of Picard's iteration to best proximity pairs for the noncyclic contraction mappings in the framework of uniformly convex Banach spaces. In this paper, we generalize the results of [5] in twofold. The first one is to extend the class of noncyclic contractions to noncyclic φ-contractions by considering an appropriate control function φ, and the second one is to shift from uniformly convex Banach spaces to uniformly convex hyperbolic metric spaces. We also refer to [6, 7] for more information related to the existence and convergence of best proximity points (pairs).

This article is organized as follows. In Section 2, we recall some basic concepts, which are required to prove our existence and convergence results. The Section 3 presents existence and convergence of best proximity pairs for the noncyclic φ-contraction in the setting of hyperbolic uniformly convex metric spaces. In Section 4, we discuss the existence of best proximity pairs for the asymptotic pointwise noncyclic orbital contractions. The existence and convergence results for common best proximity pairs for the classes of mappings are established in Section 5, the last section. We also mention that all the concepts and results are equipped with examples for better understanding of this paper.

2. PRELIMINARIES

In this section, we recall some definitions and notations which will be used next.

The notion of convexity in metric spaces was introduced by Takahashi [8].

Definition 2.1. [8] Let (X,d) be a metric space and $I := [0,1]$. A mapping $\mathscr{W} : X\times X\times I \to X$ is said to be a convex structure on X provided that, for each $(x,y;\lambda)\in X\times X\times I$ and $u\in X$,

$$d(u,\mathscr{W}(x,y;\lambda)) \le \lambda d(u,x) + (1-\lambda)d(u,y).$$

It follows from this definition that $\mathscr{W}(x,x,\lambda) = x$ whenever $x\in X$ and $\lambda\in I$.

A metric space (X,d) together with a convex structure $\mathscr{W}$ is called a convex metric space, which is denoted by $(X,d,\mathscr{W})$. For instance, Banach spaces and each of their convex subsets are convex metric spaces. But a Fréchet space is not necessarily a convex metric space. The examples of convex metric spaces which are not imbedded in any Banach space can be found in [8].

Definition 2.2. [9] A convex metric space $(X,d,\mathscr{W})$ is called a hyperbolic space if, for each $x,y,z,w\in X$ and $\lambda_1,\lambda_2\in I$, the following conditions hold:
(i) $d\big(\mathscr{W}(x,y;\lambda_1),\mathscr{W}(x,y;\lambda_2)\big) = |\lambda_1-\lambda_2|\cdot d(x,y)$,
(ii) $\mathscr{W}(x,y;\lambda) = \mathscr{W}(y,x;1-\lambda)$,
(iii) $d\big(\mathscr{W}(x,z;\lambda),\mathscr{W}(y,w;\lambda)\big) \le \lambda d(x,y) + (1-\lambda)d(z,w)$.

Definition 2.3. [8] A subset K of a convex metric space $(X,d,\mathscr{W})$ is said to be a convex set provided that $\mathscr{W}(x,y;\lambda)\in K$ for all $x,y\in K$ and $\lambda\in I$.

Definition 2.4. [8] A convex metric space $(X,d,\mathscr{W})$ is said to have property (C) if every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection.

For example, every bounded, closed, and convex subset of a reflexive Banach space X has property (C). Let A and B be two nonempty subsets of a convex metric space $(X,d,\mathscr{W})$. We say that a pair (A,B) in a convex metric space $(X,d,\mathscr{W})$ satisfies a property if both A and B satisfy that property. For instance, (A,B) is closed if and only if both A and B are closed. The *closed and convex hull* of a set A is denoted by $\overline{\mathrm{con}}(A)$ and defined as below

$$\overline{\mathrm{con}}(A) := \bigcap\{C : C \text{ is a closed and convex subset of } X \text{ such that } C\supseteq A\}.$$

The *proximal pair* of the pair (A,B) is denoted by (A_0,B_0) and is defined by

$$A_0 := \{x\in A : d(x,y') = \mathrm{dist}(A,B),\ \text{for some } y'\in B\},$$

$$B_0 := \{y\in B : d(x',y) = \mathrm{dist}(A,B),\ \text{for some } x'\in A\}.$$

Note that if (A,B) is a nonempty bounded, closed, and convex pair in a reflexive Banach space X, then the pair (A_0,B_0) is too and it is easy to see that $\mathrm{dist}(A_0,B_0) = \mathrm{dist}(A,B)$.

Here, we recall two geometric notions on convex metric spaces which have important roles next.

Definition 2.5. [10] A convex metric space $(X,d,\mathscr{W})$ is said to strictly convex provided that, for each $x,y,z \in X$ with $x \neq y$ if $d(z,x) \leq r$ and $d(z,y) \leq r$, then $d(z,\mathscr{W}(x,y,\frac{1}{2})) < r$.

Clearly, every strictly convex Banach space is a strictly convex metric space.

Definition 2.6. [11] A convex metric space $(X,d,\mathscr{W})$ is said to be uniformly convex if, for every $\varepsilon \in (0,2]$, there exists $\delta = \delta(\varepsilon) \in (0,1]$ such that, for all $r > 0$ and $x,y,z \in X$ with $d(z,x) \leq r, d(z,y) \leq r$ and $d(x,y) \geq r\varepsilon$, $d\big(z,\mathscr{W}(x,y,\frac{1}{2})\big) \leq (1-\delta)r$. The function $\delta : (0,2] \to (0,1]$ is called a modulus of convexity of a convex metric space $(X,d,\mathscr{W})$.

Obviously, uniformly convex Banach spaces are uniformly convex metric spaces. Also any $\mathrm{CAT}(0)$ space is uniformly convex (see [12] for more information).

Theorem 2.1. [11, Theorem 1] *Let $(X,d,\mathscr{W})$ be a complete and uniformly convex metric space. Then X has the property* (C) *in the sense of Definition 2.4.*

Let (X,d) be a metric space and A be a nonempty subset of X. The *metric projection operator* $\mathscr{P}_A : X \to 2^A$ is defined as

$$\mathscr{P}_A(x) := \Big\{y \in A : d(x,y) = \operatorname{dist}(x,A)\Big\}, \tag{2.1}$$

where 2^A denotes the set of all subsets of A.

Lemma 2.1. [13, Lemma 3.1] *Let A be a nonempty, closed, and convex subset of a strictly convex metric space $(X,d,\mathscr{W})$. If X has the property* (C)*, then metric projection $\mathscr{P}_A : X \to 2^A$ is single-valued.*

In what follows, we present a useful lemma which guarantees the nonemptyness of the proximal pairs.

Lemma 2.2. [13, Lemma 3.2] *Let (A,B) be a nonempty, closed, and convex pair in a convex metric space $(X,d,\mathscr{W})$. If A is bounded and X has the property* (C)*, then (A_0,B_0) is nonempty, bounded, closed, and convex.*

Remark 2.1. It is worth noticing that, in the main statement of Lemma 2.2, the condition of boundedness of the set A was omitted whereas it was used in the process of the proof.

The following geometric concept was introduced in [14].

Definition 2.7. Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. The pair (A,B) is said to have the P-property if and only if

$$\begin{cases} d(x_1,y_1) = \operatorname{dist}(A,B) \\ d(x_2,y_2) = \operatorname{dist}(A,B) \end{cases} \Rightarrow d(x_1,x_2) = d(y_1,y_2),$$

where $x_1,x_2 \in A_0$ and $y_1,y_2 \in B_0$.

The following lemma presents some sufficient conditions to realizing the P-property.

Lemma 2.3. [13, Lemma 3.6] *Let (A,B) be a nonempty, closed, and convex pair in a strictly convex hyperbolic metric space $(X,d,\mathscr{W})$. If A is bounded and X has the property* (C)*, then (A,B) has the* P*-property.*

Here, we recall another geometric notion, which was introduced in [15].

Definition 2.8. Let (A,B) be a nonempty pair in a metric space (X,d). Then (A,B) is said to satisfy property UC if the following holds:
If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that

$$\lim_n d(x_n,y_n) = \operatorname{dist}(A,B) = \lim_n d(z_n,y_n),$$

then $\lim_n d(x_n,z_n) = 0$.

The following two lemmas related to the property UC will be used in the sequel.

Lemma 2.4. (Compare with [16, Lemma 3.8]) *Let (A,B) be a nonempty pair in a uniformly convex metric space $(X,d,\mathscr{W})$ such that A is convex. Then (A,B) has the property* UC.

Proof. Assume that (A,B) has no property UC. Then there exist sequences $\{x_n\}$ and $\{z_n\}$ in A, a sequence $\{y_n\}$ in B, and $\varepsilon_0 > 0$ such that, for every $k \in \mathbb{N}$, there is $n_k \geq k$ so that $d(x_{n_k}, z_{n_k}) \geq \varepsilon_0$, whereas $\lim_{n\to\infty} d(x_n,y_n) = \lim_{n\to\infty} d(z_n,y_n) = \operatorname{dist}(A,B)$. Suppose $r > \operatorname{dist}(A,B)$, put $\varepsilon_1 := \frac{\varepsilon_0}{r}$, and consider $\varepsilon > 0$ such that $0 < \varepsilon < \min\{r - \operatorname{dist}(A,B), \frac{\delta(\varepsilon_1)}{1-\delta(\varepsilon_1)}\operatorname{dist}(A,B)\}$. Let $N \in \mathbb{N}$ be such that $\max\{d(x_{n_k},y_{n_k}), d(z_{n_k},y_{n_k})\} \leq \operatorname{dist}(A,B)+\varepsilon$ for all $n_k \geq N$. Uniformly convexity of X and convexity of the set A imply that

$$\begin{aligned}
\operatorname{dist}(A,B) &\leq d\big(y_{n_k}, \mathscr{W}(x_{n_k}, z_{n_k}, \frac{1}{2})\big) \\
&\leq \big(1-\delta(\varepsilon_1)\big)\big(\operatorname{dist}(A,B)+\varepsilon\big) \\
&< \big(1-\delta(\varepsilon_1)\big)\operatorname{dist}(A,B) + \delta(\varepsilon_1)\operatorname{dist}(A,B) \\
&= \operatorname{dist}(A,B), \quad \forall k \geq N,
\end{aligned}$$

which is a contradiction. □

Lemma 2.5. [15] *Let (A,B) be a nonempty pair of subsets of a metric space (X,d). Assume that (A,B) satisfies the property* UC. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B, respectively, such that either of the following holds:*

$$\lim_{m\to\infty} \sup_{n\geq m} d(x_m,y_n) = \operatorname{dist}(A,B) \text{ or } \lim_{n\to\infty} \sup_{m\geq n} d(x_m,y_n) = \operatorname{dist}(A,B).$$

Then $\{x_n\}$ is a Cauchy sequence.

At the end of this section, we state the following important result.

Proposition 2.1. ([5, 17]) *Let (A,B) be a nonempty, closed, and convex pair in a strictly convex hyperbolic convex metric space $(X,d,\mathscr{W})$ with the property* (C) *such that A is bounded. Define $\mathscr{P} : A_0 \cup B_0 \to A_0 \cup B_0$ as*

$$\mathscr{P}(x) = \begin{cases} \mathscr{P}_{A_0}(x) & \text{if} \quad x \in B_0, \\ \mathscr{P}_{B_0}(x) & \text{if} \quad x \in A_0. \end{cases}$$

Then the following statements hold.

(1) *$d(x,\mathscr{P}x) = \operatorname{dist}(A,B)$ for any $x \in A_0 \cup B_0$ and $\mathscr{P}(A_0) \subseteq B_0$, $\mathscr{P}(B_0) \subseteq A_0$;*
(2) *The restriction of $\mathscr{P}$ on both A_0 and B_0 are isometry, that is, $d(\mathscr{P}x,\mathscr{P}y) = d(x,y)$ for all $(x,y) \in (A_0 \times A_0) \cup (B_0 \times B_0)$;*
(3) *The restriction of $\mathscr{P}$ on both A_0 and B_0 are affine;*
(4) *The restriction of $\mathscr{P}$ on both A_0 and B_0 are continuous.*

3. NONCYCLIC φ-CONTRACTIONS

We begin our discussions with the following definition.

Definition 3.1. Let (A,B) be a nonempty pair in a metric space (X,d). A mapping $T: A\cup B \to A\cup B$ is said to be a noncyclic φ-contraction if T is a noncyclic mapping and, for a strictly increasing mapping $\varphi:[0,\infty)\to[0,\infty)$, it satisfies

$$d(Tx,Ty) \leq d(x,y) - \varphi\big(d(x,y)\big) + \varphi\big(\mathrm{dist(A,B)}\big),$$

for every $(x,y)\in A\times B$.

It is worth noticing that if $\varphi(t) = (1-\alpha)t$ for $t\geq 0$ and $0<\alpha<1$, then T is a noncyclic contraction in the sense of Definition 1.1. The following example demonstrates that the class of noncyclic φ-contractions contains the class of noncyclic contractions as a subclass.

Example 3.1. Let $X=\mathbb{R}$ with the usual metric. For $A=B=[0,1]$, define $T: A\cup B\to A\cup B$ with

$$Tx = \begin{cases} \dfrac{x}{1+x}, & \text{if } x\in A;\\ \dfrac{x}{1-x}, & \text{if } x\in B.\end{cases}$$

Clearly, $\mathrm{dist}(A,B)=0$ and T is noncyclic mapping. Now, with $\varphi(t)=\frac{t^2}{1+t^2}$,

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{x}{1+x} - \frac{y}{1-y}\right| \leq \frac{|x-y|}{1+|x-y|}\\ &= |x-y| - \frac{|x-y|^2}{1+|x-y|} = |x-y| - \varphi(|x-y|)\\ &= d(x,y) - \varphi(d(x,y)) + \varphi\big(\mathrm{dist}(A,B)\big).\end{aligned}$$

Therefore T is a noncyclic φ-contraction with $\varphi(t)=\frac{t^2}{1+t^2}$. T is not a noncyclic contraction because, for $x=\frac{1}{2}$ and $y=\frac{2}{3}$, $\frac{d(Tx,Ty)}{d(x,y)}\geq 10$ and hence no value of $0<\alpha<1$ satisfies the condition of Definition 1.1.

Remark 3.1. If $T: A\cup B\to A\cup B$ is a noncyclic φ-contraction. Then, for any $(x,y)\in A\times B$, by the fact that the control function φ is increasing, we have

$$d(Tx,Ty) \leq d(x,y) - \varphi\big(d(x,y)\big) + \varphi\big(\mathrm{dist}(A,B)\big) \leq d(x,y),$$

that is, T is a noncyclic relatively nonexpansive mapping.

The following example demonstrates that the reverse of Remark 3.1 does not hold in general.

Example 3.2. Let $X=\{u=(u_j)_{j\geq 1}\in\mathbb{R}^{\mathbb{N}} : \sum_{j\geq 1}|u_j|^{\frac{1}{2}}<\infty\}$. Suppose that $\{e_j\}$ be a canonical basis of X. Let $d(u,v)=\sqrt{|u_1|^2+|u_2|^2+\cdots+|u_j|^2}$. Then (X,d) is a metric space. Let us define set $A=\big\{e_2+\frac{1}{2}e_1, e_2+e_j : j\in\mathbb{N}\setminus\{0,1,2\}\big\}$ and $B=\{e_1,e_2\}$. Then $dist(A,B)=\frac{1}{\sqrt{2}}$. Let us define $T: A\cup B\to A\cup B$ as follows:

$$Tu = \begin{cases} e_2+\frac{1}{2}e_1, & \text{if } u=e_2+\frac{1}{2}e_1;\\ e_2+e_3, & \text{if } u\neq e_2+\frac{1}{2}e_1 \ \&\ u\in A;\\ e_2, & \text{if } u\in B.\end{cases}$$

Then clearly T is a noncyclic mapping. Now,

- if $u = e_2 + \frac{1}{2}e_1$ and $v \in B$, then

$$d(Tu, Tv) = |e_2 + \frac{1}{2}e_1 - e_2| = \sqrt{1/2} \le d(u, v);$$

- if $u = e_2 + e_j$ and $v \in B$, then

$$d(Tu, Tv) = |e_2 + e_3 - e_2| = 1 < \sqrt{3} = d(u, v).$$

Thus T is a noncyclic relatively nonexpansive mapping. But T is not a noncyclic φ-contraction because if we take $u = e_2 + e_j \in A$, $j \in \mathbb{N}$ and $v = e_2 \in B$, then $d(Tu, Tv) - d(u, v) = 1 - 1 = 0$, and since φ is increasing, $-\varphi(d(u, v)) + \varphi(dist(A, B)) = -\varphi(1) + \varphi(\frac{1}{2}) < 0$. Thus there does not exist any φ such that $d(Tx, Ty) \le d(x, y) - \varphi\big(d(x, y)\big) + \varphi\big(\mathrm{dist}(A, B)\big)$.

The following theorem is the main result of this section.

Theorem 3.1. *Let (A, B) be a nonempty, closed, and convex pair in a complete hyperbolic uniformly convex metric space $(X, d, \mathscr{W})$ such that A is bounded. Assume that T is a noncyclic φ-contraction defined on $A \cup B$. Suppose $x_0 \in A_0$ and define*

$$\begin{cases} x_n = T^n x_0, \\ y_n = \mathscr{P} x_n, \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $\mathscr{P}$ is the projection mapping defined in (2.1). Then $\{(x_n, y_n)\} \subseteq A_0 \times B_0$ converges to a best proximity pair of the mapping T.

Proof. Since the hyperbolic uniformly convex space $(X, d, \mathscr{W})$ is complete, it satisfies the property (C) by Theorem 2.1. Moreover, from Lemma 2.2, the pair (A_0, B_0) is nonempty, bounded, closed, and convex. It also follows from Lemma 2.3 that pair (A, B) has the P-property. Moreover, using Lemma 2.4, one sees that pair (A, B) has the property UC. It is easy to see that T is noncyclic on $A_0 \cup B_0$. We prove that the mappings $\mathscr{P}$ and T commute on $A_0 \cup B_0$. Let $x \in A_0$. Since T is relatively nonexpansive, one has $d(Tx, T\mathscr{P}x) \le d(x, \mathscr{P}x) = \mathrm{dist}(A, B)$, and so $d(Tx, T\mathscr{P}x) = \mathrm{dist}(A, B) = d(Tx, \mathscr{P}Tx)$. Since (A, B) has the P-property, $\mathscr{P}Tx = T\mathscr{P}x$. Equivalently, we can see that $\mathscr{P}$ and T commute on B_0. Due to commutativity of $\mathscr{P}$ and T on $A_0 \cup B_0$, we have $y_n = \mathscr{P}(T^n x_0) = T(\mathscr{P}T^{n-1}x_0) = T(\mathscr{P}x_{n-1}) = T(y_{n-1})$ for all $n \in \mathbb{N}$. Now let us take $\xi_n := d(x_{n+1}, y_n)$. Since T is relatively nonexpansive, we have

$$\xi_n = d(Tx_n, Ty_{n-1}) \le d(x_n, y_{n-1}) = \xi_{n-1}, \quad \forall n \in \mathbb{N}.$$

Therefore, $\{\xi_n\}$ is a decreasing and bounded sequence. Thus $\lim_{n\to\infty} \xi_n = t_0$ for some $t_0 \ge \mathrm{dist}(A, B)$. If $\xi_{n_0} = 0$ for some $n_0 \ge 1$, then the result follows. So, we assume that $\xi_n > 0$ for each $n \ge 1$. Observe that

$$\begin{aligned} \xi_{n+1} &= d(x_{n+1}, y_n) = d(Tx_n, Ty_{n-1}) \\ &\le d(x_n, y_{n-1}) - \varphi\big(d(x_n, y_{n-1})\big) + \varphi\big(\mathrm{dist}(A, B)\big) \\ &\le \xi_n - \varphi(\xi_n) + \varphi\big(\mathrm{dist}(A, B)\big). \end{aligned}$$

Hence, $\varphi\big(\mathrm{dist}(A, B)\big) \le \varphi(\xi_n) \le \xi_n - \xi_{n+1} + \varphi\big(\mathrm{dist}(A, B)\big)$ for each $n \ge 1$. Since φ is strictly increasing and $\xi_n \ge t_0 \ge \mathrm{dist}(A, B)$ for each $n \ge 1$, $\lim_{n\to\infty} \varphi(\xi_n) = \varphi(t_0) = \varphi\big(\mathrm{dist}(A, B)\big)$. This implies that $d(x_{n+1}, y_n) \to \mathrm{dist}(A, B)$. Also $d(x_n, y_n) = d(x_n, \mathscr{P}x_n) = (A, B)$ for any $n \in \mathbb{N}$. In view of the fact that (A,B) has the property UC, we conclude that $d(x_{n+1},x_n)\to 0$. We assert that

$$\forall \varepsilon>0,\ \exists N\in\mathbb{N},\quad \text{s.t.}\quad \forall m>n\geq N:\ d(x_m,Ty_n)\leq \operatorname{dist}(A,B)+\varepsilon.$$

Using the contradiction method, there is an $\varepsilon>0$ such that, for all $k\in\mathbb{N}$, there exist $m_k>n_k\geq k$ for which $d(x_{m_k},Ty_{n_k})>\operatorname{dist}(A,B)+\varepsilon$ and $d(x_{m_k-1},Ty_{n_k})\leq \operatorname{dist}(A,B)+\varepsilon$. Then

$$\operatorname{dist}(A,B)+\varepsilon<d(x_{m_k},Ty_{n_k})\leq d(x_{m_k},x_{m_k-1})+d(x_{m_k-1},Ty_{n_k}),$$

and then $d(x_{m_k},Ty_{n_k})\to \operatorname{dist}(A,B)+\varepsilon$. Besides,

$$\begin{aligned} d(x_{m_k},Ty_{n_k}) &\leq d(x_{m_k},x_{m_k+1})+d(\underbrace{x_{m_k+1}}_{Tx_{m_k}},Ty_{n_k+1})+d(Ty_{n_k+1},Ty_{n_k})\\ &\leq d(x_{m_k},x_{m_k+1})+d(x_{m_k},y_{n_k+1})-\varphi\big(d(x_{m_k},y_{n_k+1})\big)\\ &\quad+\varphi\big(\operatorname{dist}(A,B)\big)+d(Ty_{n_k+1},Ty_{n_k}). \end{aligned}\tag{3.2}$$

Furthermore,

$$\begin{aligned} d(Ty_{n_k+1},Ty_{n_k})&=d(T\mathscr{P}x_{n_k+1},T\mathscr{P}x_{n_k})\\ &=d(\mathscr{P}Tx_{n_k+1},\mathscr{P}Tx_{n_k})=d(\mathscr{P}x_{n_k+2},\mathscr{P}x_{n_k+1}). \end{aligned}$$

Since $\mathscr{P}$ is an isometry on A_0 (Proposition 2.1; part (2)), we have

$$d(\mathscr{P}x_{n_k+1},\mathscr{P}x_{n_k+2})=d(x_{n_k+1},x_{n_k+2})\to 0.$$

Letting $k\to\infty$ in (3.2), we obtain

$$\begin{aligned} \operatorname{dist}(A,B)+\varepsilon&=\lim_{k\to\infty}d(x_{m_k},Ty_{n_k})\\ &\leq(\operatorname{dist}(A,B)+\varepsilon)-\lim_{k\to\infty}\varphi\big(d(x_{m_k},y_{n_k+1})\big)+\varphi\big(\operatorname{dist}(A,B)\big)\\ &\leq \operatorname{dist}(A,B)+\varepsilon. \end{aligned}$$

This gives us $\lim_{k\to\infty}\varphi\big(d(x_{m_k},y_{n_k+1})\big)=\varphi\big(\operatorname{dist}(A,B)\big)$, which in turn implies that

$$\varphi\big(\operatorname{dist}(A,B)+\varepsilon\big)\leq\lim_{k\to\infty}\varphi\big(d(x_{m_k},y_{n_k+1})\big)=\varphi\big(\operatorname{dist}(A,B)\big).$$

This is a contradiction by the fact that φ is strictly increasing. Therefore, (4) holds and

$$\lim_{n\to\infty}\sup_{m\geq n}d(x_m,Ty_n)=\operatorname{dist}(A,B).$$

By using Lemma 2.5, the sequence $\{x_n\}$ is Cauchy and so converges to some element $u\in A_0$. Continuity of $\mathscr{P}|_{A_0}$ (Proposition 2.1) ensures that $y_n=\mathscr{P}x_n\to\mathscr{P}u:=v$. Thereby,

$$d(Tx_n,\mathscr{P}Tu)=d(Tx_n,T\mathscr{P}u)\leq d(x_n,v)\to \operatorname{dist}(A,B).$$

Hence, $Tx_n\to Tu$. By this reality that $d(x_n,Tx_n)\to 0$, we obtain u is a fixed point of T in A_0 and so $Tv=T\mathscr{P}u=\mathscr{P}Tu=\mathscr{P}u=v$, which deduces that (u,v) is a best proximity pair of T, where $(x_n,y_n)\to(u,v)$. □

Let us illustrate Theorem 3.1 with the following examples.

Example 3.3. Consider the space ℓ_p, $1 < p < \infty$ with canonical basis $\{e_n\}$. Let $A = \{te_1 + e_2 : t \in [0,2]\}$ and $B = \{se_1 + e_3 : s \in \mathbb{R}\}$. Then (A,B) is closed convex pair and A is bounded. Also, $\operatorname{dist}(A,B) = 2^{1/p}$. Here $A_0 = A$ and $B_0 = \{se_1 + e_3 : s \geq 0\}$. Thus (A,B) is not proximinal.

Let us now define a mapping $T : A \cup B \to A \cup B$ as

$$Tu = \begin{cases} e_2, & \text{if } t \in \mathbb{Q} \cap [0,2] \ \& \ u \in A; \\ e_2 + \frac{t}{2}e_1, & \text{if } t \in \mathbb{R} \setminus \mathbb{Q} \cap [0,2] \ \& \ u \in A; \\ e_3, & \text{if } s < -1 \ \& \ u \in B; \\ e_3 + \frac{|s|}{2}e_1, & \text{if } s \geq -1 \ \& \ u \in B. \end{cases}$$

Then T is not continuous and noncyclic mapping on $A \cup B$. We see that T is noncyclic φ-contraction. In fact, if $u = te_1 + e_2 \in A$ and $v = se_1 + e_3 \in B$, then $\|u - v\| = ((t-s)^p + 2)^{1/p}$ and we have following cases:

- if $t \in \mathbb{Q} \cap [0,2]$ and $s < -1$, then, for some suitable choice of φ,

$$\|Tu - Tv\| = \|e_2 - e_3\| = 2^{1/p} \leq \|u - v\| - \varphi(\|u - v\|) + \varphi(\operatorname{dist}(A,B));$$

- if $t \in \mathbb{Q} \cap [0,2]$ and $s \geq -1$, then, for some suitable choice of φ,

$$\|Tu - Tv\| = \|e_2 - \frac{|s|}{2}e_1 - e_3\| = (2 + (\frac{|s|}{2})^p)^{1/p} \leq \|u - v\| - \varphi(\|u - v\|) + \varphi(\operatorname{dist}(A,B));$$

- if $t \in \mathbb{R} \setminus \mathbb{Q} \cap [0,2]$ and $s < -1$, then, for some suitable choice of φ,

$$\|Tu - Tv\| = \|e_2 + \frac{t}{2}e_1 - e_3\| = (2 + (\frac{t}{2})^p)^{1/p} \leq \|u - v\| - \varphi(\|u - v\|) + \varphi(\operatorname{dist}(A,B));$$

- if $t \in \mathbb{R} \setminus \mathbb{Q} \cap [0,2]$ and $s \geq -1$, then, for some suitable choice of φ,

$$\|Tu - Tv\| = \|e_2 + \frac{t}{2}e_1 - \frac{|s|}{2}e_1 - e_3\| = (2 + (\frac{t - |s|}{2})^p)^{1/p} \leq \|u - v\| - \varphi(\|u - v\|) + \varphi(\operatorname{dist}(A,B)).$$

Therefore, T is a noncyclic φ-contraction for some suitable choice of φ. Note that (e_2, e_3) is a best proximity pair for the mapping T. Now, we choose $u_0 = t_0e_1 + e_2 \in A_0$. Then,

♣ if $t_0 \in \mathbb{Q} \cap [0,2]$, then $u_n = T^n u_0 = e_2 \to e_2$ and $v_n = \mathscr{P}u_n = \mathscr{P}e_3 = e_3 \to e_3$;
♣ if $t_0 \in \mathbb{R} \setminus \mathbb{Q} \cap [0,2]$, then $u_n = T^n u_0 = \frac{t_0}{2^n}e_1 + e_2 \to e_2$ and $v_n = \mathscr{P}u_n = \frac{t_0}{2^n}e_1 + e_3 \to e_3$.

Thus it follows from Theorem 3.1 that (x_n, y_n) converges to best proximity pair of the mapping T on $A \cup B$.

Example 3.4. Let $A = \{(x,y) : x \in [0,2], y \in [0,x]\}$ and $B = [3,4] \times [0,2]$ be subsets of a Banach space $\mathbb{R}^2$. Clearly, $A_0 = \{(2,y) : y \in [0,2]\}$, $B_0 = \{(3,p) : p \in [0,2]\}$, and $\operatorname{dist}(A,B) = 1$. Let us define a mapping $T : A \cup B \to A \cup B$ as follows:

$$T(x,y) = \begin{cases} \left(\frac{x+2}{2}, \frac{y}{2}\right), & \text{if } (x,y) \in A; \\ \left(\frac{x+3}{2}, \frac{y}{2}\right), & \text{if } (x,y) \in B. \end{cases}$$

It is easy to demonstrate that T is noncyclic $A \cup B$ and also not continuous. The projection operator $\mathscr{P} : A_0 \cup B_0 \to A_0 \cup B_0$ is defined as $\mathscr{P}(2,y) = (3,y)$ and $\mathscr{P}(3,v) = (2,v)$ for $y, v \in$

$[0,1]$. Now for $\mathbf{x}=(x,y)\in A$ and $\mathbf{u}=(u,v)\in B$, we have

$$\begin{aligned}\|T\mathbf{x}-T\mathbf{u}\| =&\|T(x,y)-T(u,v)\| = \sqrt{\left|\frac{x+2}{2}-\frac{u+3}{2}\right|^2+\left|\frac{y}{2}-\frac{v}{2}\right|^2}\\ &\leq\sqrt{\left|\frac{x-u}{2}\right|^2+\frac{1}{2}+\left|\frac{y-v}{2}\right|^2}\leq \|(x,u)-(y,v)\|-\varphi(\|(x,u)-(y,v)\|)+\varphi\big(\mathrm{dist}(A,B)\big)\\ &=\|\mathbf{x}-\mathbf{u}\|-\varphi(\|\mathbf{x}-\mathbf{u}\|)+\varphi\big(\mathrm{dist}(A,B)\big).\end{aligned}$$

Therefore, T is noncyclic φ-contraction for $\varphi(t)=\frac{t}{2}$. It is worth noticing here that $\big((2,0),(3,0)\big)$ is a best proximity pair of T. Let $\mathbf{x}_0=(2,y)\in A_0$. Then $\mathbf{x}_n=T^n\mathbf{x}_0=(2,\frac{y}{2^n})\to(2,0)$ and $\mathbf{y}_n=\mathscr{P}\mathbf{x}_n=(3,\frac{y}{2^n})\to(3,0)$. Hence, the Picard iteration sequence defined in (3.1) converges to the best proximity pair of T.

Remark 3.2. It is worth noticing that if, in Theorem 3.1, the mapping T is noncyclic contraction, then the boundedness condition of set A can be dropped. Indeed, by a similar argument of the proof of Theorem 3.5 of [2], we can conclude the existence of a best proximity pair for the mapping T, so the proximal pair (A_0,B_0) is nonempty. Now, it is interesting to ask whether Theorem 3.1 holds where the sets A and B are unbounded?

4. Asymptotic Pointwise Noncyclic Orbital Contractions

Let (A,B) be a nonempty pair in a metric space (X,d). If $T:A\cup B\to A\cup B$ is a noncyclic mapping and $x\in A\cup B$, then the orbit setting at x is defined by $\mathscr{O}_T x:=\{x,Tx,T^2x,...,T^nx,...\}$, where $T^nx=T(T^{n-1}x)$ for $n\in\mathbb{N}$ and $T^0x=x$. For any $(x,y)\in A\times B$, we set $\mathscr{O}_T(x,y):=\mathscr{O}_T(x)\cup\mathscr{O}_T(y)$, We mention here that if $(x,y)\in A\times B$, then $\mathscr{O}_T x\subseteq A$ and $\mathscr{O}_T y\subseteq B$.

Definition 4.1. Suppose that (A,B) is a nonempty pair in a metric space (X,d). A mapping $T:A\cup B\to A\cup B$ is said to be an asymptotic pointwise noncyclic orbital contraction provided that, for each $(x,y)\in A\times B$,

$$\begin{aligned} d(T^nx,T^ny)&\leq\alpha_n(x)\mathrm{diam}\big[\mathscr{O}_T(x,y)\big]+(1-\alpha_n(x))\mathrm{dist}(A,B),\quad\forall y\in B,\\ d(T^nx,T^ny)&\leq\alpha_n(y)\mathrm{diam}\big[\mathscr{O}_T(x,y)\big]+(1-\alpha_n(y))\mathrm{dist}(A,B),\quad\forall x\in A,\end{aligned}$$

where, for each $n\in\mathbb{N}$, $\alpha_n:A\cup B\to\mathbb{R}^+$ and $\limsup_{n\to\infty}\alpha_n(x)\leq\eta$ for some $0<\eta<1$ and for all $x\in A\cup B$.

Here we study the existence and convergence of best proximity pairs for asymptotic pointwise noncyclic orbital contractions.

Theorem 4.1. *Let (A,B) be a nonempty, closed and convex pair in a complete hyperbolic uniformly convex metric space $(X,d,\mathscr{W})$ such that A is bounded. Assume that T is an asymptotic pointwise noncyclic orbital contraction defined on $A\cup B$ which is relatively nonexpansive. Suppose $x_0\in A_0$ and define*

$$\begin{cases} x_n=T^nx_0,\\ y_n=\mathscr{P}x_n,\end{cases}$$

for all $n\in\mathbb{N}$, where $\mathscr{P}$ is the projection mapping defined in (2.1). If $T|_{A_0}$ is continuous, then $\{(x_n,y_n)\}\subseteq A_0\times B_0$ converges to a best proximity pair of the mapping T.

Proof. Obviously, for $x_0 \in A_0$, sequence $\{\mathrm{diam}[\mathscr{O}_T(T^n x_0, T^n \mathscr{P} x_0)]\}$ is decreasing and bounded below by $\mathrm{dist}(A,B)$. Assume that $\mathrm{diam}[\mathscr{O}_T(T^n x_0, T^n \mathscr{P} x_0)] \to \kappa_{x_0} \geq \mathrm{dist}(A,B)$. Then, for any $l_1, l_2 \in \mathbb{N}$ with $l_1 \leq l_2$,

$$d(T^{(n+l_1)}x_0, T^{(n+l_2)}(\mathscr{P}x_0)) \leq \alpha_{n+l_1}(x_0)\mathrm{diam}[\mathscr{O}_T(x_0,\mathscr{P}x_0)] + (1-\alpha_{n+l_1}(x_0))\mathrm{dist}(A,B).$$

Taking the supremum with respect to l_1 and l_2 and then letting $n \to \infty$, we obtain

$$\kappa_{x_0} \leq \eta\,\mathrm{diam}[\mathscr{O}_T(x_0,\mathscr{P}x_0)] + (1-\eta)\mathrm{dist}(A,B).$$

On the other hand, for each $m \in \mathbb{N}$ we have

$$\begin{aligned}\kappa_{x_0} &= \lim_{n\to\infty} \mathrm{diam}\Big[\mathscr{O}_T\Big(T^n(T^m x_0), T^n\big(T^m(\mathscr{P}x_0)\big)\Big)\Big] \\ &\leq \eta\,\mathrm{diam}\Big[\mathscr{O}_T\big(T^m x_0, T^m(\mathscr{P}x_0)\big)\Big] + (1-\eta)\mathrm{dist}(A,B).\end{aligned}$$

Letting $m \to \infty$, we obtain $\kappa_{x_0} \leq \eta\kappa_{x_0} + (1-\eta)\mathrm{dist}(A,B)$, which implies that $\kappa_{x_0} = \mathrm{dist}(A,B)$. Thus $\lim_{n\to\infty}\sup_{m\geq n} d(T^n x_0, T^m \mathscr{P} x_0) = \mathrm{dist}(A,B)$. In view of the fact that (A,B) has the property UC, and Lemma 2.5, the sequence $\{x_n\}$ is Cauchy, so it converges to an element $u \in A_0$. Since T is continuous, one has $Tu = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = u$, that is, $u \in A_0$ is a fixed point of T. Moreover, from the continuity of $\mathscr{P}$ on A_0, we have $\mathscr{P}x_n \to \mathscr{P}u := v \in B_0$ and that $Tv = T\mathscr{P}u = \mathscr{P}Tu = \mathscr{P}u = v$, that is, (u,v) is a best proximity pair of T and the proof is completed. □

5. Common Best Proximity Pairs

Consider the noncyclic mappings T_1 and T_2 defined on $A\cup B$, where (A,B) is a nonempty pair in metric space (X,d). One says that a point $(u,v) \in A\times B$ is a *common best proximity pair* for the noncyclic pair of mappings $(T_1;T_2)$ provided that

$$u = T_1 u = T_2 u, \quad v = T_1 v = T_2 v, \quad d(u,v) = \mathrm{dist}(A,B).$$

It is worth noticing that $(u,v) \in A\times B$ is a common best proximity pair for the noncyclic pair $(T_1;T_2)$ whenever (u,v) is a solution of the following nonlinear optimization problem:

$$\min_{(x,y)\in A\times B}\Big\{d(x,T_1x), d(y,T_2y), d(x,y)\Big\}.$$

In this section, we survey the existence of a common best proximity pair for a pair of noncyclic mappings and then we present some sufficient conditions in order to study the convergence of such points. To this end, we introduce the following notion.

Definition 5.1. Assume that $(T_1;T_2)$ is a noncyclic pair of mappings on $A\cup B$, where (A,B) is a nonempty pair in a metric space (X,d). We say that T_1 is noncyclic φ-contraction w.r.t. T_2 and the pair (A,B) if there exists a strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that

$$d(T_1x,T_1y) \leq d(T_2x,T_2y) - \varphi\big(d(T_2x,T_2y)\big) + \varphi\big(\mathrm{dist}(A,B)\big), \quad \forall (x,y) \in A\times B.$$

Let us illustrate this concept with the following example.

Example 5.1. Let $X = [0,\infty)$ be a metric space with $d(x,y) = |x-y|$ for $x,y \in X$ and $A = [0,1]$, $B = [1,\infty)$. It is clear that $\mathrm{dist}(A,B) = 0$. Let us define the mapping $T_1, T_2 : A\cup B \to A\cup B$ by

$$T_1 u = \begin{cases} u^3, & \text{if } u \in A; \\ u^2, & \text{if } u \in B; \end{cases}$$

and

$$T_2u = \begin{cases} 2u^6 - 1, & \text{if } u \in A; \\ 2u^4 - 1, & \text{if } u \in B. \end{cases}$$

Then, for $\varphi(t) = \frac{t}{2}$, it is easy to see that

$$d(T_1u, T_1v) \le d(T_2u, T_2v) - \varphi\big(d(T_2u - T_2v)\big) + \varphi\big(\mathrm{dist}(A,B)\big)$$

which implies that T_1 is noncyclic φ-contraction w.r.t. T_2 and the pair (A,B).

The following theorem is the main conclusion of this section.

Theorem 5.1. *Let (A,B) be a nonempty, closed, and convex pair in a complete hyperbolic uniformly convex metric space $(X,d,\mathscr{W})$ such that A is bounded. Suppose that $(T_1;T_2)$ is a noncyclic pair on $A\cup B$ such that*

(i) *$T_1(A) \subseteq T_2(A) \subseteq A$, $T_1(B) \subseteq T_2(B) \subseteq B$,*
(ii) *$\big(T_2(A), T_2(B)\big)$ is a closed and convex pair,*
(iii) *T_1 is noncyclic φ-contraction w.r.t. T_2 and the pair (A,B),*
(iv) *T_1 and T_2 commute.*

Then T_1 and T_2 have a common best proximity pair.

Proof. By Lemma 2.2, (A_0,B_0) is nonempty, closed, and convex. We prove that

$$\mathrm{dist}\big(T_1(A), T_1(B)\big) = \mathrm{dist}(A,B) = \mathrm{dist}\big(T_2(A), T_2(B)\big).$$

By the assumption (i), we have $\mathrm{dist}(A,B) \le \mathrm{dist}\big(T_2(A),T_2(B)\big) \le \mathrm{dist}\big(T_1(A),T_1(B)\big)$. We prove that $\mathrm{dist}(T_1(A),T_1(B)) = \mathrm{dist}(A,B)$. Suppose that $\mathrm{dist}(T_1(A),T_1(B)) > \mathrm{dist}(A,B)$. It follows that $d(T_1x,T_1y) > \mathrm{dist}(A,B)$ for any $(x,y) \in A\times B$, so $d(T_2x,T_2y) > \mathrm{dist}(A,B)$ for all $(x,y)\in A\times B$. Indeed, if $d(T_2x,T_2y) = \mathrm{dist}(A,B)$, for some $(x,y)\in A\times B$, then

$$(T_1x, T_1y) \le d(T_2x,T_2y) - \varphi\big(d(T_2x,T_2y)\big) + \varphi\big(\mathrm{dist}(A,B)\big) = \mathrm{dist}(A,B),$$

which is impossible. Thus

$$\begin{aligned} d(T_1x,T_1y) &\le d(T_2x,T_2y) - \varphi\big(d(T_2x,T_2y)\big) + \varphi\big(\mathrm{dist}(A,B)\big) \\ &< d(T_2x,T_2y) - \varphi\big(d(T_2x,T_2y)\big) + \varphi\big(d(T_2x,T_2y)\big) \\ &= d(T_2x,T_2y), \end{aligned} \tag{5.1}$$

for all $(x,y)\in A\times B$. Hence, $\mathrm{dist}(T_1(A),T_1(B)) \le \mathrm{dist}(T_2(A),T_2(B))$, which ensures that

$$\mathrm{dist}(T_1(A),T_1(B)) = \mathrm{dist}(T_2(A),T_2(B)).$$

Thus $\mathrm{dist}(T_2(A),T_2(B)) > \mathrm{dist}(A,B)$. From the fact that $\big(T_2(A),T_2(B)\big) \subseteq (A,B)$ is closed and convex and $T_2(A)\subseteq A$ is bounded, we obtain that $\big((T_2(A))_0,(T_2(B))_0\big)$ is also nonempty, closed, and convex by Lemma 2.2. Assume that $(T_2x_1,T_2y_1)\in T_2(A)\times T_2(B)$ is such that $d(T_2x_1,T_2y_1) = \mathrm{dist}\big(T_2(A),T_2(B)\big)$ for some $(x_1,y_1)\in A\times B$. It follows that

$$\mathrm{dist}\big(T_1(A),T_1(B)\big) \le d(T_1x_1,T_1y_1) < d(T_2x_1,T_2y_1) = \mathrm{dist}\big(T_2(A),T_2(B)\big),$$

which is a contradiction. Therefore, (8) satisfies. In this way, we set

$$\big(T_2(A)\big)_0 = \{y\in T_2(A) : d(x,y) = \mathrm{dist}\big(T_2(A),T_2(B)\big)(=\mathrm{dist}(A,B)),\ \text{for some} \quad x\in T_2(B)\},$$

$$\big(T_2(B)\big)_0 = \{x\in T_2(B) : d(x,y) = \mathrm{dist}\big(T_2(A),T_2(B)\big)(=\mathrm{dist}(A,B)),\ \text{for some} \quad y\in T_2(A)\}.$$

We claim that the mapping $T_1T_2^{-1}$ is singleton on $\big(T_2(A)\big)_0 \cup \big(T_2(B)\big)_0$. To this end, we consider an element $x \in \big(T_2(A)\big)_0$. Then there exists $y \in \big(T_2(B)\big)_0$ such that $d(x,y) = \operatorname{dist}(A,B)$. Hence, for any $(z,w) \in (T_1T_2^{-1}x) \times (T_1T_2^{-1}y) \subseteq T_2(A) \times T_2(B)$, $z = T_1p$ and $w = T_1q$, where $p \in T_2^{-1}x$ and $q \in T_2^{-1}y$. Notice that if $d(T_1p, T_1q) > \operatorname{dist}(A,B)$, then

$$d(T_1p,T_1q) < d(T_2p,T_2q) = d(x,y) = \operatorname{dist}(A,B),$$

which is a contradiction. Therefore, $d(z,w) = d(T_1p,T_1q) = \operatorname{dist}(A,B)$ for all $(z,w) \in (T_1T_2^{-1}x) \times (T_1T_2^{-1}y)$. Since the pair (A,B) has the P-property, (Lemma 2.3) $T_1T_2^{-1}x$ is singleton. Similarly, $T_1T_2^{-1}$ is singleton on $\big(T_2(B)\big)_0$. On the other hand, if $x \in \big(T_2(A)\big)_0$ and $y \in \big(T_2(B)\big)_0$ are such that $d(x,y) = \operatorname{dist}(A,B)$, then there exist $a \in A$ and $b \in B$ for which $x = T_2a, y = T_2b$ and so $d(T_1a,T_1b) = \operatorname{dist}(A,B)$. Hence,

$$T_1T_2^{-1}x = T_1T_2^{-1}(T_2a) = T_1a \in T_2(A), \quad T_1T_2^{-1}y = T_1T_2^{-1}(T_2b) = T_1b \in T_2(B),$$

$$d(T_1T_2^{-1}x, T_1T_2^{-1}y) = d(T_1a,T_1b) = \operatorname{dist}(A,B),$$

that is, $T_1T_2^{-1}x \in \big(T_2(A)\big)_0$. Thus $T_1T_2^{-1}\Big(\big(T_2(A)\big)_0\Big) \subseteq \big(T_2(A)\big)_0$. By a similar discussion, $T_1T_2^{-1}\Big(\big(T_2(B)\big)_0\Big) \subseteq \big(T_2(B)\big)_0$. This implies that the mapping

$$T_1T_2^{-1} : \big(T_2(A)\big)_0 \cup \big(T_2(B)\big)_0 \to \big(T_2(A)\big)_0 \cup \big(T_2(B)\big)_0$$

is noncyclic. In addition, for $(x,y) \in \big(T_2(A)\big)_0 \times \big(T_2(B)\big)_0$ with $x = T_2a$ and $y = T_2b$ for some $(a,b) \in A \times B$, we have

$$\begin{aligned} d(T_1T_2^{-1}x, T_1T_2^{-1}y) &= d(T_1a,T_1b) \\ &\le d(T_2a,T_2b) - \varphi\big(d(T_2a,T_2b)\big) + \varphi\big(\operatorname{dist}(A,B)\big) \\ &= d(x,y) - \varphi\big(d(x,y)\big) + \varphi\big(\operatorname{dist}(A,B)\big), \end{aligned}$$

which implies that the mapping $T_1T_2^{-1}$ is a noncyclic φ-contraction on $\big(T_2(A)\big)_0 \cup \big(T_2(B)\big)_0$, where $\Big(\big(T_2(A)\big)_0, \big(T_2(B)\big)_0\Big)$ is a closed and convex pair in a complete hyperbolic uniformly convex metric space $(X,d,\mathscr{W})$, and that $T_2(A)$ is bounded. It now follows from Theorem 3.1, that $T_1T_2^{-1}$ has a best proximity pair, call $(p,q) \in \big(T_2(A)\big)_0 \times \big(T(B)\big)_0$. Thus

$$T_1T_2^{-1}p = p, \quad T_1T_2^{-1}q = q \quad \text{and} \quad d(p,q) = \operatorname{dist}(A,B).$$

By this reality that T_1 and T_2 are commute, $T_2p = T_2(T_1T_2^{-1}p) = T_1(T_2T_2^{-1})p = T_1p$. Similarly, $T_2q = T_1q$. Thus

$$\begin{aligned} d(T_1p,T_1q) &\le d(T_2p,Tq) - \varphi\big(d(Tp,Tq)\big) + \varphi\big(\operatorname{dist}(A,B)\big) \\ &= d(T_2(T_1T_2^{-1}p), T_2(T_1T_2^{-1}q)) - \varphi\big(d(T_2(T_1T_2^{-1}p), T_2(T_1T_2^{-1}q))\big) + \varphi\big(\operatorname{dist}(A,B)\big) \\ &= d(T_1p,T_1q) - \varphi\big(d(T_1p,T_1q)\big) + \varphi\big(\operatorname{dist}(A,B)\big), \end{aligned}$$

so $\varphi\big(d(T_1p,T_1q)\big) = \varphi\big(\operatorname{dist}(A,B)\big)$. This deduces that $d(T_1p,T_1q) = \operatorname{dist}(A,B)$ and hence $d(Tp, Tq) = \operatorname{dist}(A,B)$. Since T_1 is noncyclic φ-contraction w.r.t. T_2 and T_1 and T_2 commute, we obtain

$$\begin{aligned} d(T_1(T_1p),T_1q) &\leq d(T_2(T_1p),T_2q) - \varphi\big(d(T_2(T_1p),T_2q)\big) + \varphi\big(\text{dist}(A,B)\big) \\ &= d(T_1(T_2p),T_1q) - \varphi\big(d(T_1(T_2p),T_1q)\big) + \varphi\big(\text{dist}(A,B)\big) \\ &= d(T_1(T_1p),T_1q) - \varphi\big(d(T_1(T_1p),T_1q)\big) + \varphi\big(\text{dist}(A,B)\big), \end{aligned}$$

and then $d(T_1(T_1p),T_1q) = \text{dist}(A,B) = d(T_1p,T_1q)$. Again, since (A,B) has the P-property, $T_1(T_1p) = T_1p$, that is, T_1p is a fixed point of the mapping T_1. Furthermore,

$$d(T_2(T_1p),T_2q) = d(T_1(T_2p),T_1q) = d(T_1(T_1p),T_1q) = \text{dist}(A,B) = d(T_2p,T_2q).$$

Thus $T_2(T_1p) = T_2p = T_1p$ which ensures that $T_1p \in A$ is a fixed point of the mapping T_2. This concludes that T_1p is a common fixed point for the mappings T_1 and T_2. By a similar argument, T_1q is a common fixed point for T_1 and T_2, that is, $(T_1p,T_1q) \in A \times B$ is a common best proximity pair for the noncyclic pair of mappings (T_1,T_2). □

In what follows, we present a convergence result of a common best proximity pair of a pair of noncyclic mappings.

Theorem 5.2. *Under the assumptions of Theorem 4.1, and the condition that T_1 is continuous on $\big(T_2(A)\big)_0 \cup \big(T_2(B)\big)_0$, for any $x_0 \in \big(T_2(A)\big)_0$, define*

$$\begin{cases} x_n = (T_1T_2^{-1})^n x_0, \\ y_n = \mathscr{P}x_n. \end{cases}$$

Then $\{(T_1x_n,T_1y_n)\} \subseteq \big(T_2(A)\big)_0 \times \big(T_2(B)\big)_0$ converges to a common best proximity pair of the mappings T_1 and T_2.

Proof. The result follows from Theorems 3.1 and 5.1 immediately. □

The following example demonstrates the usability of Theorem 5.1.

Example 5.2. Consider $X = \mathbb{R}^2$ with the Euclidian norm. Let $A = \{(0,x) : 0 \leq x \leq 1\}$ and $B = \{(1,y) : 0 \leq y \leq 1\}$. Then $\text{dist}(A,B) = 0$. Also the pair (A,B) is nonempty, closed, and convex pair and A is bounded. Define the mappings $T_1,T_2 : A \cup B \to A \cup B$ with

$$T_1(0,x) = (0, \sin\frac{x}{2}), \quad T_1(1,y) = (1, \sin\frac{y}{2}),$$
$$T_2(0,x) = (0,x), \quad T_2(1,y) = (1,y).$$

Then clearly T_1 and T_2 are noncyclic mappings. Moreover, T_1 and T_2 are commute and $T_1(A) \subseteq T_2(A) = A$ and $T_1(B) \subseteq T_2(B) = B$. Moreover,

$$\|T_1(0,x) - T_1(1,y)\| = \sqrt{1 + |\sin\frac{x}{2} - \sin\frac{y}{2}|^2} \leq \frac{1}{2}\sqrt{1 + |x-y|^2}$$

which means that T_1 is noncyclic φ-contraction w.r.t. T_2 and the pair (A,B) for $\varphi(t) = \frac{t}{2}$. Therefore, all of the conditions of Theorem 5.1 hold and hence T_1 and T_2 have a common best proximity pair, which is a point $\big((0,0),(1,0)\big)$.

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