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GEOMETRIC INEQUALITIES FOR SOLVING VARIATIONAL INEQUALITY PROBLEMS IN CERTAIN BANACH SPACES

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Abstract. In this paper, we develop some new geometric inequalities in p-uniformly convex and uniformly smooth real Banach spaces with $p > 1$. We use the inequalities as tools to obtain the strong convergence of the sequence generated by a subsgradient method to a solution that solves fixed point and variational inequality problems. Furthermore, the convergence theorem established can be applicable in, for example, $L_p(\Omega)$, where $\Omega \subset \mathbb{R}$ is bounded set and $l_p(\mathbb{R})$ for $p \in (2, \infty)$. Finally, numerical implementations of the proposed method in the real Banach space $L_5([-1,1])$ are presented.

Keywords. Relatively nonexpansive mapping; Subgradient method; Variational inequality.

1. Introduction

A variational inequality problem (VIP) involving a single-valued monotone operator $\mathscr{A}$, defined on a subset C of a normed space E is to find a point $\varpi \in C$ that satisfies:

$$\langle y - \varpi, \mathscr{A}\varpi \rangle \geq 0, \ \forall\, y \in C. \tag{1.1}$$

As usual, let $VI(C, \mathscr{A})$ be the set of points that satisfy (1.1).

Remark 1.1. By monotonicity of $\mathscr{A}$ (which is $\langle \varpi - y, \mathscr{A}\varpi - \mathscr{A}y \rangle \geq 0, \forall\, \varpi, y \in E$) and inequality (1.1), one can easily verify that if ϖ satisfies (1.1) then, for any $y \in C$, $\langle y - \varpi, \mathscr{A}y \rangle \geq 0$.

Several problems in applied mathematics, such as financial equilibriums, optimization problems, transportation problems can be expressed as a VIP. In applications, many models arising from image recovery, signal processing, to mention but few, can be modeled as the VIP; see, e.g., [1, 2, 3, 4, 5]. These interesting connections between the VIP and other real problems

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attracted the attention of many authors. Consequently, various iterative techniques were proposed; see, e.g., [6, 7, 8, 9, 10]).

The two famous techniques for solving the VIP are the projection technique and the regularization technique. The classical projection technique involving monotone mappings is the gradient algorithm, which requires the strong monotonicity or the inverse-strong monotonicity assumption on the cost operator. To dispense with this restriction, the extragradient method (EGM), which requires the Lipschitz continuity and monotonicity of the cost operator, was introduced in [10]. However, to implement the EGM, one is required to compute two projections in each iterative step, which maybe very expensive if the set is not simple, i.e., without closed forms. Over the years, the subgradient extragradient method (SEGM) was introduced (see [6, 8]) as an improvement of the EGM. The main feature of the SEGM is the replacement of one metric projection in the EGM with a projection onto a constructable half-space which can be obtained by an explicit formula (see, e.g., [11] for other alternatives of the EGM and the regularization technique in the literature).

Results on monotone VIPs abound in vast literatures. However, most of the results concerning monotone VIPs were only established in real Hilbert spaces. Recently, there is considerable research effort to extend these results to Banach spaces. In 2018, Chidume et al. [12] proposed an iterative algorithm in 2-uniformly convex and uniformly smooth Banach spaces that solves the VIP (1.1). They compared the performance of their method with the famous algorithm of Nakajo [13], which was in the same Banach space setting as their method. Numerically, the algorithm of Chidume et al. [12] was less time consuming compared to that of Nakajo [13].

Remark 1.2. While 2-uniformly convex and uniformly smooth Banach spaces are more generic compared to Hilbert spaces (for example, they include L_p and l_p for $1 < p \leq 2$) *they exclude some classical real Banach spaces.* It is well-known that L_p and l_p with $2 < p < \infty$ are *not* 2-uniformly convex.

Our interest here is to propose hybrid subgradient methods that solve the VIP (1.1) involving monotone-type operators whose solutions are fixed points or $\mathscr{G}$-fixed points of some nonexpansive-type operators in the setting of some real Banach spaces. To achieve this, we first establish some new geometric inequalities, which are of independent interest. Furthermore, our proposed methods complement, among others, the methods of Cai et al. [14], Chidume et al. [12], and Nakajo [13] to solve the VIP (1.1) in p-uniformly convex and uniformly smooth real Banach spaces (for any $p > 1$). Finally, we present a numerical example to demonstrate that our algorithms are implementable.

2. PRELIMINARIES

Definition 2.1. Let $\mathscr{E}$ be a real normed space with dual space $\mathscr{E}^*$ and $p > 1$. The *generalized duality map* $\mathscr{G}_p : \mathscr{E} \to 2^{\mathscr{E}^*}$ is defined by

$$\mathscr{G}_p\varpi := \{\varpi^* \in \mathscr{E}^* : \langle \varpi, \varpi^* \rangle = \|\varpi\|\|\varpi^*\|,\ \|\varpi^*\| = \|\varpi\|^{p-1},\ \forall\ \varpi \in \mathscr{E}\ \}.$$

If $p = 2$, $\mathscr{G}_2$ is denoted by $\mathscr{G}$.

Definition 2.2 (Chidume [15])**.** Let $\mathscr{E}$ be a real Banach space, which is reflexive, smooth, and strictly convex. Define the following functions: $\phi_p : \mathscr{E} \times \mathscr{E} \to \mathbb{R}^+$ defined by

$$\phi_p(\varpi, y) := \|\varpi\|^p - p\langle \varpi, \mathscr{G}_p y\rangle + (p-1)\|y\|^p,\ \forall \varpi, y \in \mathscr{E},$$

and $\psi_p : \mathscr{E} \times \mathscr{E}^* \to \mathbb{R}^+$ defined by

$$\psi_p(\varpi, \varpi^*) := \|\varpi\|^p - p\langle \varpi, \varpi^* \rangle + (p-1)\|\varpi^*\|^{\frac{p}{p-1}}, \ \forall \varpi \in \mathscr{E}, \ \varpi^* \in \mathscr{E}^*.$$

Then $\psi_p(\varpi, \varpi^*) = \phi_p(\varpi, \mathscr{G}_p^{-1}\varpi^*)$, $\forall \varpi \in \mathscr{E}$, $\varpi^* \in \mathscr{E}^*$.

Definition 2.3. The *generalized projection* $\Pi_C : \mathscr{E} \to C$ is defined by $\bar{y} = \Pi_C(y) \in C$ with $\phi_p(\tilde{\varpi}, \varpi) = \inf_{y \in C} \phi_p(y, \varpi)$ in a strictly convex, smooth, and reflexive real Banach space $\mathscr{E}$.

Definition 2.4. A mapping $S : C \to \mathscr{E}$ is called *relatively nonexpansive* if the set of its *asymptotic fixed points* equals the set of its fixed points and $\phi_p(\varpi, Sy) \leq \phi_p(\varpi, y)$ for any $\varpi \in F(S)$ and $y \in C$.

Lemma 2.1 (Chidume [15])**.** *Let $\mathscr{E}$ be a real Banach space that is reflexive, smooth, and strictly convex. Then, for $p > 1$, $\psi_p(\varpi, \varpi^*) + p\langle \mathscr{G}_p^{-1}\varpi^* - \varpi, y^* \rangle \leq \psi_p(\varpi, \varpi^* + y^*)$, $\forall \varpi \in \mathscr{E}$, $\varpi^*, y^* \in \mathscr{E}^*$.*

Lemma 2.2 (Chidume [15])**.** *Under the same setting in of space in Lemma 2.1, there exists a constant $c_p > 0$ such that, for all $z, \varpi, y \in \mathscr{E}$, $\phi_p\big(z, \mathscr{G}_p^{-1}(\zeta \mathscr{G}_p\varpi + (1-\zeta)\mathscr{G}_p y)\big) \leq \zeta \phi_p(z, \varpi) + (1-\zeta)\phi_p(z, y) - c_p w_p(\zeta)\|\mathscr{G}_p\varpi - \mathscr{G}_p y\|^p$, where $w_p(\zeta) = \zeta^p(1-\zeta) + \zeta(1-\zeta)^p$.*

Remark 2.1. In Lemmas 2.3, 2.4, and 2.5, the Banach space is p-uniformly convex and smooth with $p > 1$.

Lemma 2.3 (Xu [16])**.** *Under the setting of Remark 2.1, there exists a constant $d_p > 0$ such that, for every $\varpi, y \in \mathscr{E}$, the following inequality holds: $\|\varpi + y\|^p \geq \|\varpi\|^p + p\langle y, \mathscr{G}_p\varpi \rangle + d_p\|y\|^p$.*

Lemma 2.4 (Xu, [16])**.** *Under the setting of Remark 2.1, there exists a constant $c_p > 0$ such that, for every $\varpi, y \in \mathscr{E}$, the following inequality holds: $\langle \varpi - y, \mathscr{G}_p\varpi - \mathscr{G}_p y \rangle \geq c_p\|\varpi - y\|^p$.*

Lemma 2.5 (Chidume [15])**.** *Let $\{u_n\}$ and $\{y_n\}$ be sequences in the space mentioned in Remark 2.1. Then, $\phi_p(u_n, y_n) \to 0$ implies $||u_n - y_n|| \to 0$ as $n \to \infty$.*

Lemma 2.6 (Xu, [17])**.** *Let $\{a_n\}$ be a nonnegative real sequence with the condition $a_{n+1} \leq \alpha_n\beta_n + (1-\alpha_n)a_n$, $n \geq 0$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences such that (i) $\{\alpha_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; (ii) $\limsup_{n\to\infty} \beta_n \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.*

3. Main Results

In the sequel, C is assumed to be a nonempty, convex, and closed subset of some real Banach space $\mathscr{E}$.

Lemma 3.1. *Under the same space setting given in Definition 2.3, let $\varpi \in \mathscr{E}$ and $y \in C$. Then $\tilde{\varpi} = \Pi_C\varpi$ if and only if $\langle \tilde{\varpi} - y, \mathscr{G}_p\varpi - \mathscr{G}_p\tilde{\varpi} \rangle \geq 0$, $\forall y \in C$. Furthermore, for $p > 1$, let $\mathscr{E}$ be a the real Banach space stated in Remark 2.1. Then $\phi_p(y, \tilde{\varpi}) + \phi_p(\tilde{\varpi}, \varpi) \leq \phi_p(y, \varpi)$ for all $\varpi \in \mathscr{E}$, $y \in C$.*

Proof. Observe that

$$\begin{aligned}
&\phi_p(y,\tilde{\varpi})+\phi_p(\tilde{\varpi},\varpi)\\
&=\|y\|^p-p\langle y,\mathscr{G}_p\tilde{\varpi}\rangle+(p-1)\|\tilde{\varpi}\|^p+\|\tilde{\varpi}\|^p-p\langle\tilde{\varpi},\mathscr{G}_p\varpi\rangle+(p-1)\|\varpi\|^p\\
&=\|y\|^p-p\langle y,\mathscr{G}_p\varpi\rangle-p\langle y,\mathscr{G}_p\tilde{\varpi}-\mathscr{G}_p\varpi\rangle+p\|\tilde{\varpi}\|^p-p\langle\tilde{\varpi},\mathscr{G}_p\varpi\rangle+(p-1)\|\mathscr{G}_p\varpi\|^p\\
&=\phi_p(y,\varpi)-p\langle y,\mathscr{G}_p\tilde{\varpi}-\mathscr{G}_p\varpi\rangle+p\langle\tilde{\varpi},\mathscr{G}_p\tilde{\varpi}\rangle-p\langle\tilde{\varpi},\mathscr{G}_p\varpi\rangle\\
&=\phi_p(y,\varpi)+p\langle\tilde{\varpi}-y,\mathscr{G}_p\tilde{\varpi}-\mathscr{G}_p\varpi\rangle\\
&\leq\phi_p(y,\varpi).
\end{aligned}$$

The proof of the lemma is complete. □

Lemma 3.2. *Let $\varpi, y, z \in \mathscr{E}$, where E is the space mentioned in Remark 2.1. Then*

$$\phi_p(\varpi,z)+\phi_p(z,y)+p\langle z-\varpi,\mathscr{G}_py-\mathscr{G}_pz\rangle=\phi_p(\varpi,y).$$

Proof. Observe that

$$\begin{aligned}
&\phi_p(\varpi,z)+\phi_p(z,y)+p\langle z-\varpi,\mathscr{G}_py-\mathscr{G}_pz\rangle\\
&=\|\varpi\|^p-p\langle\varpi,\mathscr{G}_pz\rangle+(p-1)\|z\|^p+\|z\|^p-p\langle z,\mathscr{G}_py\rangle+(p-1)\|y\|^p+\langle z,\mathscr{G}_py\rangle-\langle\varpi,\mathscr{G}_py\rangle\\
&\quad-p\langle z,\mathscr{G}_pz\rangle+p\langle\varpi,\mathscr{G}_pz\rangle\\
&=\phi_p(\varpi,y),
\end{aligned}$$

which concludes the proof. □

Lemma 3.3. *For any ϖ and y in $\mathscr{E}$, where E is the space defined in Remark 2.1, the following inequality holds $d_p\|\varpi-y\|^p\leq\phi_p(\varpi,y)$, where d_p is the constant in Lemma 2.3.*

Proof. Replacing y by $\varpi - y$ and ϖ by y in the inequality of Lemma 2.3, we obtain, for all $\varpi, y \in \mathscr{E}$,

$$\begin{aligned}
d_p\|\varpi-y\|^p&\leq||\varpi||^p-||y||^p-p\langle\varpi-y,\mathscr{G}_py\rangle\\
&=||\varpi||^p-p\langle\varpi,\mathscr{G}_py\rangle-||y||^p+p\langle y,\mathscr{G}_py\rangle\\
&=\phi_p(\varpi,y),
\end{aligned}$$

establishing the lemma. □

Lemma 3.4. *Let ϖ_1, $\varpi_2 \in \mathscr{E}$, where E is the space mentioned in Remark 2.1. Then, there exists a constant $\kappa_p > 0$ such that $\|\Pi_C\varpi_1-\Pi_C\varpi_2\|\leq\kappa_p\|\mathscr{G}_p\varpi_1-\mathscr{G}_p\varpi_2\|^{\frac{1}{p-1}}$.*

Proof. Let $\varpi_1, \varpi_2 \in \mathscr{E}$, $\Pi_C\varpi_1 = \tilde{\varpi}_1$, and $\Pi_C\varpi_2 = \tilde{\varpi}_2$. By Lemma 3.1, we have

$$\langle\tilde{\varpi}_2-\tilde{\varpi}_1,\mathscr{G}_p\varpi_1-\mathscr{G}_p\tilde{\varpi}_1\rangle\leq 0\quad\text{and}\quad\langle\tilde{\varpi}_1-\tilde{\varpi}_2,\mathscr{G}_p\varpi_2-\mathscr{G}_p\tilde{\varpi}_2\rangle\leq 0.$$

Adding these two inequalities, we obtain $\langle\tilde{\varpi}_1-\tilde{\varpi}_2,(\mathscr{G}_p\varpi_2-\mathscr{G}_p\varpi_1)-(\mathscr{G}_p\tilde{\varpi}_2-\mathscr{G}_p\tilde{\varpi}_1)\rangle\leq 0$. Thus $\langle\tilde{\varpi}_1-\tilde{\varpi}_2,\mathscr{G}_p\varpi_2-\mathscr{G}_p\varpi_1\rangle-\langle\tilde{\varpi}_1-\tilde{\varpi}_2,\mathscr{G}_p\tilde{\varpi}_2-\mathscr{G}_p\tilde{\varpi}_1\rangle\leq 0$. This implies that $\langle\tilde{\varpi}_1-\tilde{\varpi}_2,\mathscr{G}_p\tilde{\varpi}_1-\mathscr{G}_p\tilde{\varpi}_2\rangle\leq\langle\tilde{\varpi}_1-\tilde{\varpi}_2,\mathscr{G}\varpi_1-\mathscr{G}\varpi_2\rangle$. It follows from Lemma 2.4 that $c_p\|\tilde{\varpi}_1-\tilde{\varpi}_2\|^p\leq\|\tilde{\varpi}_1-\tilde{\varpi}_2\|\cdot\|\mathscr{G}_p\varpi_1-\mathscr{G}_p\varpi_2\|$, and then $\|\tilde{\varpi}_1-\tilde{\varpi}_2\|\leq\kappa_p\|\mathscr{G}_p\varpi_1-\mathscr{G}_p\varpi_2\|^{\frac{1}{p-1}}$, where $\kappa_p=\left(\frac{1}{c_p}\right)^{\frac{1}{p-1}}$. The proof is complete. □

The theorem and corollary below are given in the setting of real Banach spaces which are both uniformly smooth and p-uniformly convex with $p \geq 2$.

Theorem 3.1. *Let $\mathscr{A}:\mathscr{E}\to\mathscr{E}^*$ be a map that is monotone on C and L-Lipschitzian on $\mathscr{E}$, and let $S:\mathscr{E}\to\mathscr{E}$ be a map that is relatively nonexpansive. Define a sequence $\{\varpi_n\}$ in $\mathscr{E}$ by*

$$\begin{cases} u,\varpi_1\in\mathscr{E};\\ y_n=\Pi_C\mathscr{G}_p^{-1}(\mathscr{G}_p\varpi_n-\zeta\mathscr{A}\varpi_n);\\ C_n=\{z\in C:\langle z-y_n,\mathscr{G}_p\varpi_n-\zeta\mathscr{A}\varpi_n-\mathscr{G}_py_n\rangle\le 0\};\\ \rho_n=\Pi_{C_n}\mathscr{G}_p^{-1}(\mathscr{G}_p\varpi_n-\zeta\mathscr{A}y_n);\\ z_n=\mathscr{G}_p^{-1}(\vartheta_n\mathscr{G}_pu+(1-\vartheta_n)\mathscr{G}_p\rho_n);\\ \varpi_{n+1}=\mathscr{G}_p^{-1}(\zeta\mathscr{G}_p\varpi_n+(1-\zeta)\mathscr{G}_pSz_n); \end{cases}\tag{3.1}$$

where $0\le\zeta<\frac{d_p}{pL}$, d_p is the constant in Lemma 3.3, and $\{\vartheta_n\}\subset[0,1]$ such that $\lim_{n\to\infty}\vartheta_n=0$ and $\sum_{n=1}^{\infty}\vartheta_n=\infty$. If $\Omega:=F(S)\cap VI(C,\mathscr{A})\neq\emptyset$, then the sequence $\{\varpi_n\}$ generated by (3.1) converges strongly to a point $\varpi\in\Omega$.

Proof. We give the proof in two steps.

Step 1. We demonstrate that $\{\varpi_n\}$ is bounded.

Let $\tau\in\Omega$. Using Lemma 3.1, Remark 1.1 and Lemma 3.2, we obtain that

$$\begin{aligned}\phi_p(\tau,\rho_n)&\le\phi_p\big(\tau,\mathscr{G}_p^{-1}(\mathscr{G}_p\varpi_n-\zeta\mathscr{A}y_n)\big)-\phi_p\big(\rho_n,\mathscr{G}_p^{-1}(\mathscr{G}_p\varpi_n-\zeta\mathscr{A}y_n)\big)\\ &=\|\tau\|^p-p\langle\tau,\mathscr{G}_p\varpi_n-\zeta\mathscr{A}y_n\rangle-\|\rho_n\|^p+p\langle\rho_n,\mathscr{G}_p\varpi_n-\zeta\mathscr{A}y_n\rangle\\ &=\phi_p(\tau,\varpi_n)-\phi_p(\rho_n,\varpi_n)+p\langle\tau-\rho_n,\zeta\mathscr{A}y_n\rangle\\ &=\phi_p(\tau,\varpi_n)-\phi_p(\rho_n,\varpi_n)+p\zeta\langle\tau-y_n,\mathscr{A}y_n\rangle+p\zeta\langle y_n-\rho_n,\mathscr{A}y_n\rangle\\ &\le\phi_p(\tau,\varpi_n)-\phi_p(\rho_n,\varpi_n)+p\zeta\langle y_n-\rho_n,\mathscr{A}y_n\rangle\\ &=\phi_p(\tau,\varpi_n)-\phi_p(\rho_n,y_n)-\phi_p(y_n,\varpi_n)+p\langle\rho_n-y_n,\mathscr{G}_p\varpi_n-\zeta\mathscr{A}y_n-\mathscr{G}_py_n\rangle.\end{aligned}\tag{3.2}$$

Using the fact that $\mathscr{A}$ is Lipschitz, $\rho_n\in C_n$, $p\ge 2$, and Lemma 3.3, we obtain that

$$\begin{aligned}&\langle\rho_n-y_n,\mathscr{G}_p\varpi_n-\zeta\mathscr{A}y_n-\mathscr{G}_py_n\rangle\\ &=\langle\rho_n-y_n,\mathscr{G}_p\varpi_n-\zeta\mathscr{A}\varpi_n-\mathscr{G}_py_n\rangle+\zeta\langle\rho_n-y_n,\mathscr{A}\varpi_n-\mathscr{A}y_n\rangle\\ &\le\zeta\langle\rho_n-y_n,\mathscr{A}\varpi_n-\mathscr{A}y_n\rangle\\ &\le\zeta\|\rho_n-y_n\|\|\mathscr{A}\varpi_n-\mathscr{A}y_n\|\\ &\le\frac{\zeta L}{2}\big(\|\rho_n-y_n\|^p+\|\varpi_n-y_n\|^p\big)\\ &\le\frac{\zeta L}{2d_p}\big(\phi_p(\rho_n,y_n)+\phi_p(y_n,\varpi_n)\big).\end{aligned}$$

Substituting this inequality in (3.2), we arrive at

$$\begin{aligned}\phi_p(\tau,\rho_n)&\le\phi_p(\tau,\varpi_n)-\phi_p(\rho_n,y_n)-\phi_p(y_n,\varpi_n)+\frac{p\zeta L}{2d_p}\big(\phi_p(\rho_n,y_n)+\phi_p(y_n,\varpi_n)\big)\\ &=\phi_p(\tau,\varpi_n)-\Big(1-\frac{p\zeta L}{2d_p}\Big)\big(\phi_p(y_n,\varpi_n)+\phi_p(\rho_n,y_n)\big) &(3.3)\\ &\le\phi_p(\tau,\varpi_n). &(3.4)\end{aligned}$$

Using Lemma 2.2 and inequality (3.4), we have

$$\begin{aligned}\phi_p(\tau,\varpi_{n+1}) &\le \zeta\phi_p(\tau,\varpi_n)+(1-\zeta)\phi_p(\tau,z_n)\\ &\le \zeta\phi_p(\tau,\varpi_n)+(1-\zeta)\big(\vartheta_n\phi_p(\tau,u)+(1-\vartheta_n)\phi_p(\tau,\rho_n)\big)\\ &\le \zeta\phi_p(\tau,\varpi_n)+(1-\zeta)\vartheta_n\phi_p(\tau,u)+(1-\zeta)(1-\vartheta_n)\phi_p(\tau,\varpi_n)\\ &= \big(1-(1-\zeta)\vartheta_n\big)\phi_p(\tau,\varpi_n)+(1-\zeta)\vartheta_n\phi_p(\tau,u)\\ &\le \max\{\phi_p(\tau,\varpi_n),\phi_p(\tau,u)\}.\end{aligned} \tag{3.5}$$

Thus, $\phi_p(\tau,\varpi_{n+1})\le\phi_p(\tau,u)$, and sequence $\{\phi_p(\tau,\varpi_n)\}$ is bounded. Hence, $\{\varpi_n\}$ is bounded. Furthermore, using Lemma 3.4, the uniform continuity of $\mathscr{G}_p$ on bounded sets, and the fact that Lipschitz maps are bounded, we have that $\{y_n\}$ and $\{\rho_n\}$ are bounded.

Step 2. We prove that $\lim_{n\to\infty}\varpi_n=\varpi$, where $\varpi=\Pi_\Omega u$.

To prove this, first we demonstrate

(*i*) $\{\|\varpi_n-y_n\|\},\ \{\|\rho_n-y_n\|\},\ \{\|\varpi_n-z_n\|\},$ and $\{\|z_n-Sz_n\|\}$ converge to zero (0).

(*ii*) $\Delta_w(\varpi_n)\subset\Omega$ (where $\Omega_w(\varpi_n)$ denotes the set of weak limits of subsequences of $\{\varpi_n\}$).

To establish (i), we start by using inequalities (3.5) and (3.3) to obtain

$$\begin{aligned}\phi_p(\tau,\varpi_{n+1}) &\le \zeta\phi_p(\tau,\varpi_n)+(1-\zeta)\big(\vartheta_n\phi_p(\tau,u)+(1-\vartheta_n)\phi_p(\tau,\rho_n)\big)\\ &\le \zeta\phi_p(\tau,\varpi_n)+(1-\zeta)\vartheta_n\phi_p(\tau,u)+(1-\zeta)(1-\vartheta_n)\\ &\quad\times\left(\phi_p(\tau,\varpi_n)-\left(1-\frac{p\zeta L}{2d_p}\right)\big(\phi_p(y_n,\varpi_n)+\phi_p(\rho_n,y_n)\big)\right)\\ &= \phi_p(\tau,\varpi_n)+(1-\zeta)\vartheta_n\phi_p(\tau,u)-\vartheta_n(1-\zeta)\phi_p(\tau,\varpi_n)\\ &\quad-(1-\zeta)\left(1-\frac{p\zeta L}{2d_p}\right)\big(\phi_p(y_n,\varpi_n)+\phi_p(\rho_n,y_n)\big)\\ &\quad+\vartheta_n(1-\zeta)\left(1-\frac{p\zeta L}{2d_p}\right)\big(\phi_p(y_n,\varpi_n)+\phi_p(\rho_n,y_n)\big).\end{aligned}$$

Let $\delta=(1-\zeta)\left(1-\frac{p\zeta L}{2d_p}\right)$. Then, we rewrite this inequality to obtain

$$\begin{aligned}\delta\big(\phi_p(y_n,\varpi_n)+\phi_p(\rho_n,y_n)\big) &\le \phi_p(\tau,\varpi_n)-\phi_p(\tau,\varpi_{n+1})+(1-\zeta)\vartheta_n\phi_p(\tau,u)\\ &\quad-\vartheta_n(1-\zeta)\phi_p(\tau,\varpi_n)+\vartheta_n\delta\big(\phi_p(y_n,\varpi_n)+\phi_p(\rho_n,y_n)\big).\end{aligned} \tag{3.6}$$

To complete the proof, we consider the following two cases.

Case 1. Assume there exists an $n_0\in\mathbb{N}$ for which $\phi_p(\tau,\varpi_{n+1})\le\phi_p(\tau,\varpi_n)$ for all $n\ge n_0$. Then, sequence $\{\phi_p(\tau,\varpi_n)\}$ is convergent.

From inequality (3.6), the fact that $\lim_{n\to\infty}\vartheta_n=0$, the convergence of $\{\phi_p(\tau,\varpi_n)\}$, and the facts that $\{\varpi_n\}$, $\{y_n\}$ and $\{\rho_n\}$ are bounded, we conclude that $\lim_{n\to\infty}\big(\phi_p(y_n,\varpi_n)+\phi_p(\rho_n,y_n)\big)=0$. By Lemma 2.5, we have $\lim_{n\to\infty}\|\varpi_n-y_n\|=\lim_{n\to\infty}\|\rho_n-y_n\|=0$, which implies that $\lim_{n\to\infty}\|\varpi_n-\rho_n\|=0$. Observe that

$$\phi_p(\varpi_n,z_n)\le\vartheta_n\phi_p(\varpi_n,u)+(1-\vartheta_n)\phi_p(\varpi_n,\rho_n)\ \Rightarrow\ \lim_{n\to\infty}\phi_p(\varpi_n,z_n)=0.$$

One has $\lim_{n\to\infty}\|\varpi_n - z_n\| = 0$. It follows from Lemma 2.2 that

$$\begin{aligned}\phi_p(\tau,\varpi_{n+1}) &\le \zeta\phi_p(\tau,\varpi_n) + (1-\zeta)\phi_p(\tau,Sz_n) - c_p w_p(\zeta)\|\mathscr{G}_p\varpi_n - \mathscr{G}_pSz_n\|^p \\ &\le \zeta\phi_p(\tau,\varpi_n) + (1-\zeta)\big(\vartheta_n\phi_p(\tau,u) + (1-\vartheta_n)\phi_p(\tau,\rho_n)\big) \\ &\quad - c_p w_p(\zeta)\|\mathscr{G}_p\varpi_n - \mathscr{G}_pSz_n\|^p \\ &\le \zeta\phi_p(\tau,\varpi_n) + (1-\zeta)\vartheta_n\phi_p(\tau,u) + (1-\zeta)(1-\vartheta_n)\phi_p(\tau,\varpi_n) \\ &\quad - c_p w_p(\zeta)\|\mathscr{G}_p\varpi_n - \mathscr{G}_pSz_n\|^p \\ &= \phi_p(\tau,\varpi_n) + (1-\zeta)\vartheta_n\big(\phi_p(\tau,u) - h(\tau,\varpi_n)\big) - c_p w_p(\zeta)\|\mathscr{G}_p\varpi_n - \mathscr{G}_pSz_n\|^p,\end{aligned}$$

which implies that

$$c_p w_p(\zeta)\|\mathscr{G}_p\varpi_n - \mathscr{G}_pSz_n\|^p \le \phi_p(\tau,\varpi_n) - \phi_p(\tau,\varpi_{n+1}) + \vartheta_n(1-\zeta)\big(\phi_p(\tau,u) - \phi_p(\tau,\varpi_n)\big).$$

Hence, $\lim_{n\to\infty}\|\mathscr{G}_p\varpi_n - \mathscr{G}_pSz_n\| = 0$. Furthermore, one deduces that $\lim_{n\to\infty}\|\varpi_n - Sz_n\| = 0$. Moreover, one also has

$$\|z_n - Sz_n\| \le \|z_n - \varpi_n\| + \|\varpi_n - Sz_n\| \;\Rightarrow\; \lim_{n\to\infty}\|z_n - Sz_n\| = 0.$$

Now, we prove (ii). Let $\tau\in\Delta_w(\varpi_n)$. Then there exists $\{\varpi_{n_k}\}\subset\{\varpi_n\}$ such that $\varpi_{n_k}\rightharpoonup\tau$. From (i), this implies that $z_{n_k}\rightharpoonup\tau$ as $k\to\infty$. Furthermore, from (i), we have $\lim_{k\to\infty}\|z_{n_k} - Sz_{n_k}\| = 0$. Thus $\tau\in F(S)$.

Next, to prove that $\tau\in VI(C,\mathscr{A})$, following [18], it suffices to demonstrate that $(\tau,0)\in G(T)$ (where T is as defined above), which is equivalent to proving that $\langle v-\tau,\tau^*\rangle\ge 0$ for all $(v,\tau^*)\in G(T)$ (since T is maximally monotone). Now, let $(v,\tau^*)\in G(T)$. Then, by following [18], $(\tau^* - \mathscr{A}v)\in N_C(v)$, that is, $\langle v-y,\tau^* - \mathscr{A}v\rangle\ge 0$ for all $y\in C$. Since $y_n = \Pi_C\mathscr{G}_p^{-1}(\mathscr{G}_p\varpi_n - \zeta\mathscr{A}\varpi_n)$ and $v\in C$, by Lemma 3.1 we have that

$$\left\langle v-y_n, \frac{\mathscr{G}_py_n - \mathscr{G}_p\varpi_n}{\zeta} + \mathscr{A}\varpi_n\right\rangle \ge 0.$$

Since $y_n\in C$ and $(\tau^* - \mathscr{A}v)\in N_C(v)$, we have

$$\begin{aligned}&\langle v-y_{n_k},\tau^*\rangle \\ &\ge \langle v-y_{n_k},\mathscr{A}v\rangle \\ &\ge \langle v-y_{n_k},\mathscr{A}v\rangle - \left\langle v-y_{n_k}, \frac{\mathscr{G}_py_{n_k} - \mathscr{G}_p\varpi_{n_k}}{\zeta} + \mathscr{A}\varpi_n\right\rangle \\ &= \langle v-y_{n_k},\mathscr{A}v - \mathscr{A}y_{n_k}\rangle + \langle v-y_{n_k},\mathscr{A}y_{n_k} - \mathscr{A}\varpi_{n_k}\rangle - \left\langle v-y_{n_k}, \frac{\mathscr{G}_py_{n_k} - \mathscr{G}_p\varpi_{n_k}}{\zeta}\right\rangle \\ &\ge \langle v-y_{n_k},\mathscr{A}y_{n_k} - \mathscr{A}\varpi_{n_k}\rangle - \left\langle v-y_{n_k}, \frac{\mathscr{G}_py_{n_k} - \mathscr{G}_p\varpi_{n_k}}{\zeta}\right\rangle.\end{aligned}$$

Hence, $\langle v-\tau,\tau^*\rangle\ge 0$. Therefore, $\Delta_w(\varpi_n)\subset VI(C,\mathscr{A})$ and $\Delta_w(\varpi_n)\subset\Omega$.

Next, we prove that $\lim_{n\to\infty}\|\varpi_n - \varpi\| = 0$, where $\varpi = \Pi_\Omega u$. Since $\{\varpi_n\}$ is bounded, there exists a subsequence $\{\varpi_{n_k}\}$ of $\{\varpi_n\}$ such that $\varpi_{n_k}\rightharpoonup z$ and

$$\lim_{k\to\infty}\langle\varpi_{n_k} - \varpi,\mathscr{G}_pu - \mathscr{G}_p\varpi\rangle = \limsup_{n\to\infty}\langle\varpi_n - \varpi,\mathscr{G}_pu - \mathscr{G}_p\varpi\rangle = \limsup_{n\to\infty}\langle z_n - \varpi,\mathscr{G}_pu - \mathscr{G}_p\varpi\rangle.$$

Since $\Delta_w(\varpi_n)\subset\Omega$ and $\varpi = \Pi_\Omega u$, we have

$$\lim_{k\to\infty}\langle\varpi_{n_k} - \varpi,\mathscr{G}_pu - \mathscr{G}_p\varpi\rangle = \langle z - \varpi,\mathscr{G}_pu - \mathscr{G}_p\varpi\rangle \le 0.$$

Hence, we deduce that

$$\limsup_{n\to\infty}\langle z_n-\varpi,\mathscr{G}_pu-\mathscr{G}_p\varpi\rangle\le 0. \tag{3.7}$$

Using Lemmas 2.2 and 2.1, and inequality (3.4), we obtain

$$\begin{aligned}\phi_p(\varpi,\varpi_{n+1}) &= \phi_p\big(\varpi,\mathscr{G}_p^{-1}(\zeta\mathscr{G}_p\varpi_n+(1-\zeta)\mathscr{G}_pSz_n)\big)\\ &\le \zeta\phi_p(\varpi,\varpi_n)+(1-\zeta)\phi_p(\varpi,Sz_n)\\ &\le \zeta\phi_p(\varpi,\varpi_n)+(1-\zeta)\phi_p(\varpi,\mathscr{G}_p^{-1}(\vartheta_n\mathscr{G}_pu+(1-\vartheta_n)\mathscr{G}_p\rho_n))\\ &= \zeta\phi_p(\varpi,\varpi_n)+(1-\zeta)\psi_r(\varpi,\vartheta_n\mathscr{G}_pu+(1-\vartheta_n)\mathscr{G}_p\rho_n)\\ &\le \zeta\phi_p(\varpi,\varpi_n)+(1-\zeta)\Big(\psi_r\big(\varpi,\vartheta_n\mathscr{G}_pu+(1-\vartheta_n)\mathscr{G}_p\rho_n-\vartheta_n(\mathscr{G}_pu-\mathscr{G}_p\varpi)\big)\\ &\qquad +2\vartheta_n\langle z_n-\varpi,\mathscr{G}_pu-\mathscr{G}_p\varpi\rangle\Big)\\ &\le \zeta\phi_p(\varpi,\varpi_n)+(1-\zeta)(1-\vartheta_n)\psi_r(\varpi,J\rho_n)+2(1-\zeta)\vartheta_n\langle z_n-\varpi,Ju-J\varpi\rangle\\ &\le \zeta\phi_p(\varpi,\varpi_n)+(1-\zeta)(1-\vartheta_n)\phi_p(\varpi,\varpi_n)+2(1-\zeta)\vartheta_n\langle z_n-\varpi,Ju-J\varpi\rangle\\ &= \big(1-(1-\zeta)\vartheta_n\big)\phi_p(\varpi,\varpi_n)+2(1-\zeta)\vartheta_n\langle z_n-\varpi,Ju-J\varpi\rangle. \end{aligned} \tag{3.8}$$

By inequality (3.7) and Lemma 2.6 we have $\lim_{n\to\infty}\phi_p(\varpi,\varpi_n)=0$. Hence, $\lim_{n\to\infty}\|\varpi_n-\varpi\|=0$.

Case 2. Suppose that **Case 1** fails. Then one finds $\{\varpi_{m_j}\}$, a subsequence of $\subset\{\varpi_n\}$ such that $\phi_p(\tau,\varpi_{m_j+1})>\phi_p(\tau,\varpi_{m_j})$ for all $j\in\mathbb{N}$. Furthermore, by following Mainge's celebrated lemma [19], one can find a sequence $\{n_k\}\subset\mathbb{N}$, which satisfies $\lim_{k\to\infty}n_k=\infty$ and

$$\phi_p(\tau,\varpi_{n_k})\le\phi_p(\tau,\varpi_{n_k+1}) \text{ and } \phi_p(\tau,\varpi_k)\le\phi_p(\tau,\varpi_{n_k+1}), \text{ for each } k\in\mathbb{N}. \tag{3.9}$$

Following the same idea of proof as in **Case 1** above, we one can prove that

- $\lim\limits_{k\to\infty}\|\rho_{n_k}-y_{n_k}\|=\lim\limits_{k\to\infty}\|y_{n_k}-\varpi_{n_k}\|=\lim\limits_{k\to\infty}\|\varpi_{n_k}-z_{n_k}\|=\lim\limits_{k\to\infty}\|Sz_{n_k}-z_{n_k}\|=0$;
- $\Delta_w(\varpi_{n_k})\subset\Omega$.

Finally, we prove that $\lim_{k\to\infty}\varpi_k=\varpi$. Using the boundedness of $\{\varpi_{n_k}\}$, we have that

$$\lim_{j\to\infty}\langle\varpi_{n_{kj}}-\varpi,\mathscr{G}_pu-\mathscr{G}_p\varpi\rangle=\limsup_{k\to\infty}\langle\varpi_{n_k}-\varpi,\mathscr{G}_pu-\mathscr{G}_p\varpi\rangle=\limsup_{k\to\infty}\langle z_{n_k}-\varpi,\mathscr{G}_pu-\mathscr{G}_p\varpi\rangle.$$

Since $\Delta_w(\varpi_{n_k})\subset\Omega$ and $\varpi=\Pi_\Omega u$, we have $\limsup_{k\to\infty}\langle z_{n_k}-\varpi,\mathscr{G}_pu-\mathscr{G}_p\varpi\rangle\le 0$. Furthermore, it follows from inequality (3.8) that

$$\begin{aligned}\phi_p(\varpi,\varpi_{n_k+1}) &\le \big(1-(1-\zeta)\vartheta_{n_k}\big)\phi_p(\varpi,\varpi_{n_k})+2(1-\zeta)\vartheta_{n_k}\langle z_{n_k}-\varpi,\mathscr{G}_pu-\mathscr{G}_p\varpi\rangle\\ &\le \big(1-(1-\zeta)\vartheta_{n_k}\big)\phi_p(\varpi,\varpi_{n_k})+2(1-\zeta)\vartheta_{n_k}\langle z_{n_k}-\varpi,\mathscr{G}_pu-\mathscr{G}_p\varpi\rangle.\end{aligned}$$

By rewriting this inequality and using (3.9), we arrive at

$$\phi_p(\varpi,\varpi_{n_k+1})\le 2\langle z_{n_k}-\varpi,\mathscr{G}_pu-\mathscr{G}_p\varpi\rangle.$$

Therefore, $\phi_p(\varpi,\varpi_k)\le\phi_p(\varpi,\varpi_{n_k+1})\le 2\langle z_{n_k}-\varpi,\mathscr{G}_pu-\mathscr{G}_p\varpi\rangle$, and then

$$\limsup_{k\to\infty}\phi_p(\varpi,\varpi_k)\le\limsup_{k\to\infty}2\langle z_{n_k}-\varpi,\mathscr{G}_pu-\mathscr{G}_p\varpi\rangle,$$

which implies that $\limsup_{k\to\infty}\phi_p(\varpi,\varpi_k)\le 0$. Therefore, $\lim_{k\to\infty}\varpi_k=\varpi$. The proof is complete. □

Corollary 3.1. *Under the same setting as in Theorem 3.1, if $\vartheta_n = 0$ and S is the identity map, then the following algorithm defined by*

$$\begin{cases} \varpi_1, u \in \mathscr{E}; \\ y_n = \Pi_C \mathscr{G}_p^{-1}(\mathscr{G}_p \varpi_n - \zeta \mathscr{A} \varpi_n); \\ C_n = \{z \in C : \langle z - y_n, \mathscr{G}_p \varpi_n - \zeta \mathscr{A} \varpi_n - \mathscr{G}_p y_n \rangle \leq 0\}; \\ \rho_n = \Pi_{C_n} \mathscr{G}_p^{-1}(\mathscr{G}_p \varpi_n - \zeta \mathscr{A} y_n); \\ \varpi_{n+1} = \mathscr{G}_p^{-1}(\vartheta_n \mathscr{G}_p u + (1 - \vartheta_n) \mathscr{G}_p \rho_n); \end{cases} \tag{3.10}$$

where all the parameters are as defined in Theorem 3.1, converges strongly to a point $\varpi \in VI(C, \mathscr{A})$ provided that $VI(C, \mathscr{A})$ is not empty.

Next we use the result we obtained in Theorem 3.1 to approximate fixed point of some nonexpansive-type mapping. The following notions will be used in what follows:

Definition 3.1. Let $T : \mathscr{E} \to 2^{\mathscr{E}^*}$ be nonlinear map. A point $u \in \mathscr{E}$ is called a *$\mathscr{G}$-fixed point of T* if $\mathscr{G}u \in Tu$, where $\mathscr{G} : \mathscr{E} \to \mathscr{E}^*$ is the single-valued normalized duality map on $\mathscr{E}$. We denote the set of $\mathscr{G}$-fixed point of T by $F_{\mathscr{G}}(T) := \{\varpi \in \mathscr{E} : T\varpi = \mathscr{G}\varpi\}$.

Here we remark that this notion under different concepts has been studied by numerous authors; see, e.g., [21, 22, 23]. Currently, there is a growing interest in the study of $\mathscr{G}$-fixed point and we refer to [20, 24, 25, 26, 27] for some interesting results.

Recently, the notion of *relatively $\mathscr{G}$-nonexpansive maps* was introduced and discussed by Chidume *et al.* [28]. We next give the definition using the generalized duality $\mathscr{G}_p$ mapping to fit the setting in this paper.

Definition 3.2. Let $T : \mathscr{E} \to \mathscr{E}^*$ be a map. A point $\varpi^* \in \mathscr{E}$ is called an *asymptotic $\mathscr{G}_p$-fixed point of T* if there exists a sequence $\{\varpi_n\} \subset \mathscr{E}$ such that $\varpi_n \rightharpoonup \varpi^*$ and $\|\mathscr{G}_p \varpi_n - T\varpi_n\| \to 0$ as $n \to \infty$. Let $\widehat{F}_{\mathscr{G}_p}(T)$ be the set of asymptotic $\mathscr{G}_p$-fixed points of T.

Definition 3.3. A map $T : \mathscr{E} \to \mathscr{E}^*$ is said to be *relatively $\mathscr{G}_p$-nonexpansive* if

(i) $\widehat{F}_{\mathscr{G}_p}(T) = F_{\mathscr{G}_p}(T) \neq \emptyset$,
(ii) $\phi_p(u, \mathscr{G}_p^{-1} T\varpi) \leq \phi_p(u, \varpi)$, $\forall\, \varpi \in \mathscr{E}$, $u \in F_{\mathscr{G}_p}(T)$.

Remark 3.1. For a nontrivial example of relatively $\mathscr{G}_p$-nonexpansive mapping, we refer to Chidume *et al.* [28]. One can easily verify from the definition above that if an operator T is relatively $\mathscr{G}_p$-nonexpansive then the operator $\mathscr{G}_p^{-1}T$ is relatively nonexpansive in the usual sense and vice versa. Furthermore, $\varpi^* \in F_{\mathscr{G}_p}(T) \Leftrightarrow \varpi^* \in F(\mathscr{G}_p^{-1}T)$.

Remark 3.2. Under same hypothesis as in Theorem 3.1, by replacing S with a relatively $\mathscr{G}_p$-nonexpansive mapping, one can conclude a $\mathscr{G}_p$-fixed point theorem. Indeed, $\mathscr{G}_p^{-1}S$ is relatively nonexpansive due to Remark 3.1.

4. Numerical Illustration

Now we give a numerical implementation of our method in the setting of a real Banach space.

We consider real Banach spaces $L_p(\Omega)$ with $p > 2$, where $\Omega \subset \mathbb{R}$ is bounded. Let $p = 5$. Then, in Theorem 3.1, set $E = L_5([-1,1])$, so $E^* = L_{\frac{5}{4}}([-1,1])$. By Alber and Ryazantseva

[29], the duality mapping $\mathscr{G}_5$ and its inverse $\mathscr{G}_{\frac{5}{4}}$ can be computed as follows:

$$\mathscr{G}_5\varpi(t) = \|\varpi\|^{-3}|\varpi(t)|^3\varpi(t) \text{ and } \mathscr{G}_{\frac{5}{4}}\varpi(t) = \|\varpi\|^{\frac{3}{4}}|\varpi(t)|^{-\frac{3}{4}}\varpi(t),\ t \in [-1,1],$$

where $\|\varpi\| = \left(\int_{-1}^{1}|\varpi(t)|^p\right)^{\frac{1}{p}}$ with $p > 1$. Let $\mathscr{A} : L_5([-1,1]) \to L_{\frac{5}{4}}([-1,1])$ and let $S : L_5([-1,1]) \to L_5([-1,1])$ be defined by

$$\mathscr{A}\varpi(t) = 2\mathscr{G}_5\varpi(t) = 2\|\varpi\|^{-3}|\varpi(t)|^3\varpi(t) \quad \text{and} \quad S\varpi(t) = \frac{1}{2}\varpi(t).$$

By Alber and Ryazantseva [29], one sees that $\mathscr{A}$ is monotone and it is not difficult to verify that S is relatively nonexpansive and the solution set $\Omega := F(S) \cap VI(C,\mathscr{A}) = \{0\}$, where "0$''$ is the zero function in $L_5([-1,1])$. Set $C = L_5([-1,1])$. Then $C_n = L_5([-1,1])$. It follows that $\Pi_C\varpi = \varpi$ for all $\varpi \in C$. Set $\vartheta_n = \frac{1}{n+1}$, $\zeta = 0.4$, $u = 0$ and varied $\varpi_1 \in L_5([-1,1])$. Using a tolerance of 10^{-8} and maximum number of iterations $n = 100$, the results obtained from the numerical simulations are presented in the table below:

TABLE 1. Numerical results for the varied initial point ϖ_1

	$\|\varpi_{n+1} - 0\|$ for the varied initial point ϖ_1					
	Algorithm (3.1)			Algorithm (3.10)		
n	$\varpi_1(t) = 2t$	$\varpi_1(t) = \sin(t) + \cos(t)$	$\varpi_1(t) = \frac{1}{1+E\varpi p(t)}$	$\varpi_1(t) = 2t$	$\varpi_1(t) = \sin(t) + \cos(t)$	$\varpi_1(t) = \frac{1}{t+E\varpi p(t)}$
1	0.971	0.931762	0.7738	0.9076	0.8709	0.7233
10	0.2546	0.2443	0.2029	2.0612	5.4641	12.1081
20	0.0655	0.0629	0.0522	0.1659	0.1592	0.1322
30	0.0173	0.0166	0.0138	0.0875	0.0839	0.0697
40	0.0046	0.0044	0.0037	0.0476	0.0457	0.038
50	0.0012	0.0012	0.001	0.0264	0.0254	0.0211
60	0.0003	0.0003	0.0002	0.0148	0.0142	0.0118
70	9.3E-05	8.9E-05	7.4E-05	0.0084	0.0081	0.0067
80	2.5E-05	2.4E-05	2E-05	0.0047	0.0046	0.0038
90	7E-06	7E-06	6E-06	0.0027	0.0026	0.0021
100	2E-06	2E-06	2E-06	0.0016	0.0015	0.0012

5. CONCLUSION

This paper provided new inequalities in setting of real Banach spaces that are p-uniformly convex with $p > 1$. The new inequalities are used as tools in establishing some a strong convergence theorem. Finally, numerical implementations of the algorithms are presented.

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