J. Nonlinear Var. Anal. 7 (2023), No. 2, pp. 279-290
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.7.2023.2.08

OPTIMALITY THEOREMS FOR LINEAR FRACTIONAL OPTIMIZATION PROBLEMS INVOLVING INTEGRAL FUNCTIONS DEFINED ON $C^n[0,1]$

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Abstract. We consider a linear fractional optimization problem involving integral functions defined on $C^n[0,1]$ and obtain an optimality theorem for the problem which holds without any constraint qualification. We give an example to demonstrate how to use the optimality theorem for finding the optimal solutions.

Keywords. Constraint qualifications; Integral functions; Linear fractional optimization problem; Optimality theorems; Optimal solutions.

1. INTRODUCTION-PRELIMINARIES

Convex optimization problems involving integral functions need constraint qualifications for obtaining their optimality theorems. For example, the Slater condition becomes a constraint qualification for the problems. But it is well-known that the Slater condition is really often violated. So, in this paper, we intend to obtain the optimality theorems for the problems which hold without any constraint qualification.

Jeyakumar et al. [1] proved the Lagrange multiplier optimality theorems for convex optimization problems, which held without any constraint qualification and which were expressed by sequences. Such optimality theorems were studied for many kinds of convex optimization problems and linear fractional optimization problems; see, e.g., [2, 3, 4, 5, 6, 7, 8, 9]. In particular, Kim et al. [6] investigated optimality theorems for a linear fractional optimization problem involving integral functions defined on $L_n^2[0,1]$, which hold without any constraint qualification.

In this paper, we consider a linear fractional optimization problem (P) involving integral functions defined on $C^n[0,1]$ and obtain an optimality theorem for the problem (P) which holds without any constraint qualification and which is expressed by sequences. We give an example to demonstrate that certain set may not be closed and to illustrate how to use the optimality theorems for finding the optimal solutions.

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Received December 8, 2022; Accepted January 5, 2023.

Consider the following linear fractional optimization problem:

$$\begin{array}{lll} \text{(P)} & \text{Minimize} & \dfrac{\int_0^1 c(t)^T x(t)dt + \alpha}{\int_0^1 d(t)^T x(t)dt + \beta} \\ & \text{subject to} & x(\cdot) \in K, \\ & & a_i(t)^T x(t) = b_i(t),\ i = 1, \cdots, m, \text{ for any } t \in [0,1], \end{array}$$

where $c, d, a_i,\ i = 1, \cdots, m$, are given in $C^n[0,1]$, $b_i,\ i = 1, \cdots, m$ are given in $C[0,1]$, and K is a closed convex cone in $C^n[0,1]$. Here we denote $C^n[0,1] = \{x \mid x : [0,1] \to \mathbb{R}^n : \text{ continuous}\}$ and $C[0,1] = \{z \mid z : [0,1] \to \mathbb{R} : \text{continuous}\}$.

We next use the norm on $C^n[0,1]$ defined by $||x|| = \max_{t\in[0,1]} ||x(t)||$. We define the nonnegative dual cone of K as

$$K^* = \{v \in C^n[0,1]^* \mid v(x) \geqq 0 \ \ \forall x \in K\},$$

where $C^n[0,1]^* = \{x^* \mid x^* : C^n[0,1] \to \mathbb{R} : \text{ continuous and linear}\}$. We also use the norm on $C^n[0,1]^*$ defined by

$$||x^*|| = \sup\{|x^*(x)|/||x|| \mid x \in C^n[0,1] \text{ and } x \neq 0\}.$$

Now we give some notations and preliminary results that are needed in this paper. Let E be a normed linear space over $\mathbb{R}$ with norm $x \mapsto \|x\|$, and let E^* the dual of E. The conjugate function of a function $f : E \to \overline{\mathbb{R}}$ is the function $f^* : E^* \to \overline{\mathbb{R}}$ defined by

$$f^*(x^*) := \sup_{x\in E} \{\langle x^*, x\rangle - f(x)\} \ \ (x^* \in E^*).$$

A function $g : E \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if, for all $t \in [0,1]$, $g((1-t)x + ty) \leq (1-t)g(x) + tg(y)$ for all $x, y \in E$. Let $g : E \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. We denote the domain and the epigraph of g by $\operatorname{dom} g := \{x \in E : g(x) < +\infty\}$ and $\operatorname{epi} g := \{(x,r) \in E \times \mathbb{R} : g(x) \leq r\}$, respectively. We say a function g is lower semicontinuous if $\liminf_{y\to x} g(y) \geq g(x)$ for all $x \in E$.

Following the proof of [10, Theorem 2.123 (i) and (ii)], we can prove the following proposition stated in a normed space with a strong topology (norm topology). The proposition was proved on a normed space with weak*-topology in [11], and was stated on a Banach space with weak*-topology in [12].

Proposition 1.1. *Let E be a normed space. Consider a family of proper lower semicontinuous convex functions $\phi_i : E \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, where I is an arbitrary index set and suppose that $\sup_{i\in I} \phi_i$ is not identically $+\infty$. Then* $\operatorname{epi}(\sup_{i\in I} \phi_i)^* = \operatorname{clco} \bigcup_{i\in I} \operatorname{epi} \phi_i^*$.

Following the proof of [10, Theorem 2.107 and Theorem 2.123], we can prove the following proposition stated in a normed space. The proposition was stated on the Banach space (see [13, Lemma 1]).

Proposition 1.2. *Let E be a normed linear space. Let $\phi_1, \phi_2 : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and convex function. Then* $\operatorname{epi}(\phi_1 + \phi_2)^* = \operatorname{cl}(\operatorname{epi} \phi_1^* + \operatorname{epi} \phi_2^*)$.

Using [14, Theorem 1.1] and [15, Proposition 12.6], we can prove the following proposition.

Proposition 1.3. [16, 17] *Let E be a Banach space. Let $g_1 : E \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function, and let $g_2 : E \to \mathbb{R} \cup \{+\infty\}$ be a continuous convex function. Then* $\operatorname{epi}(g_1 + g_2)^* = \operatorname{epi} g_1^* + \operatorname{epi} g_2^*$.

2. Optimality Theorems

In this section, we give an optimality theorem for the problem (P) which holds without any constraint qualification and which is expressed by sequences.

Noticing that $\int_0^t \{a_i(\tau)^T x(\tau) - b_i(\tau)\}d\tau = 0$ for any $t \in [0,1]$ if and only if $a_i(t)^T x(t) - b_i(t) = 0$ for any $t \in [0,1]$, we have the following problem, which is equivalent to the problem (P) in the first section:

$$
\begin{array}{lll}
\text{(P)} & \text{Minimize} & \dfrac{\int_0^1 c(t)^T x(t)dt + \alpha}{\int_0^1 d(t)^T x(t)dt + \beta} \\
 & \text{subject to} & x(\cdot) \in K, \\
 & & \displaystyle\int_0^{\cdot} \{a_i(\tau)^T x(\tau) - b_i(\tau)\}d\tau = 0,\ i = 1, \cdots, m.
\end{array}
$$

Define

$$\triangle = \{x \in K \mid \int_0^{\cdot} \{a_i(\tau)^T x(\tau) - b_i(\tau)\}d\tau = 0,\ i = 1, \cdots, m\}.$$

So, $\triangle = \{x \in K \mid a_i(t)^T x(t) - b_i(t) = 0$ for any $t \in [0,1],\ i = 1, \cdots, m\}$. We assume that $\triangle \neq \emptyset$.

Modifying [1, Theorem 3.1], [3, Theorems 3.1 and 3.2], and [6, Theorem 2.1], we can obtain the following optimality theorem for the problem (P), which holds without any constraint qualification:

Theorem 2.1. *Let $\bar{x} \in \triangle$ and suppose that, for any $x \in \triangle$, $\int_0^1 d(t)^T x(t)dt + \beta > 0$. Then the following assertions are equivalent:*

(i) $\bar{x}$ is an optimal solution to problem (P);

(ii) $(0,0) \in \{(\int_0^1 [c(t) - q(\bar{x})d(t)]^T (\cdot)dt,\ -\alpha + q(\bar{x})\beta)\} + \{0\} \times \mathbb{R}_+$
$+\mathrm{cl}\Big(\bigcup_{\mu_i \in \mathrm{NBV}[0,1]} \{(-\sum_{i=1}^m \int_0^1 \mu_i(t) a_i(t)^T (\cdot)dt,\ -\sum_{i=1}^m \int_0^1 \mu_i(t) b_i(t)dt)\} + (-K^*) \times \mathbb{R}_+\Big)$,
where $\mathrm{NBV}[0,1] = \{\mu \mid \mu : [0,1] \to \mathbb{R}$: *a function of bounded variation, left continuous on* $[0,1)$ *and* $\mu(1) = 0\}$ *and* $q(\bar{x}) = \frac{\int_0^1 c(t)^T \bar{x}(t)dt + \alpha}{\int_0^1 d(t)^T \bar{x}(t)dt + \beta}$;

(iii) there exist $\mu_i^n \in \mathrm{NBV}[0,1],\ i = 1, \cdots, m$*,* $k_n^* \in K^*$ *such that*

$$\int_0^1 [c(t) - q(\bar{x})d(t)]^T (\cdot)dt + \lim_{n\to\infty} [-\sum_{i=1}^m \int_0^1 \mu_i^n(t) a_i(t)^T (\cdot)dt - k_n^*] = 0$$

and $\lim_{n\to\infty} k_n^*(\bar{x}) = 0$.

Proof. Suppose that $\bar{x}$ is an optimal solution to problem (P). Let

$$f(x) = \int_0^1 c(t)^T x(t)dt + \alpha - q(\bar{x})[\int_0^1 d(t)^T x(t)dt + \beta]$$

and

$$h_i(x) = \int_0^{\cdot} \Big[-a_i(\tau)^T x(\tau) + b_i(\tau)\Big]d\tau,\ i = 1, \cdots, m.$$

Then $f : C^n[0,1] \to \mathbb{R}$ is continuous and linear, and $h_i : C^n[0,1] \to C[0,1]$ is continuous and affine for each $i = 1, \cdots, m$. Let

$$D = \{x \in C^n[0,1] \mid h_i(x) = 0,\ i = 1, \cdots, m\}.$$

Then $\Delta = D \cap K$. Notice that $\bar{x}$ is an optimal solution to problem (P) if and only if

$$f(x) + \delta_{\triangle}(x) \geqq f(\bar{x}) + \delta_{\triangle}(\bar{x}) \ \text{ for any } \ x \in C^n[0,1],$$

where $\delta_{\triangle}$ is the indicator function with respect to the set $\triangle$. So $(0, -f(\bar{x})) \in \text{epi}(f + \delta_{\triangle})^*$. By Proposition 1.2, one has $(0, -f(\bar{x})) \in \text{epi} f^* + \text{epi}\delta^*_{\triangle}$. Since $\triangle = D \cap K$, one has $(0, -f(\bar{x})) \in \text{epi} f^* + \text{epi}(\delta_D + \delta_K)^*$. It follows from Proposition 1.2 that

$$(0, -f(\bar{x})) \in \text{epi} f^* + \text{cl}(\text{epi}\delta^*_D + \text{epi}\delta^*_K). \tag{2.1}$$

We can easily check that

$$\text{epi}\delta^*_K = (-K^*) \times \mathbb{R}_+. \tag{2.2}$$

In view of

$$f^*(v^*) = \begin{cases} -\alpha + q(\bar{x})\beta & \text{if } \ v^* = \int_0^1 [c(t) - q(\bar{x})d(t)]^T(\cdot)dt \\ +\infty & \text{if } \ v^* \neq \int_0^1 [c(t) - q(\bar{x})d(t)]^T(\cdot)dt, \end{cases}$$

one has

$$\text{epi} f^* = \left\{ \left(\int_0^1 [c(t) - q(\bar{x})d(t)]^T(\cdot)dt, -\alpha + q(\bar{x})\beta \right) \right\} + \{0\} \times \mathbb{R}_+. \tag{2.3}$$

By Hahn-Banach theorem, for any nonzero $z_i \in C[0,1]$, there exists $\lambda_i \in C[0,1]^*$ such that $\lambda_i(z_i) > 0$. So, $\delta_D(x) = \sup_{\lambda_i \in C[0,1]^*} (\sum_{i=1}^m \lambda_i \circ h_i)(x)$ for any $x \in C^n[0,1]$. Since $\sum_{i=1}^m \lambda_i \circ h_i$ is continuous and affine, it follows from Proposition 1.1 that

$$\text{epi}\delta^*_D = \text{cl co} \bigcup_{\lambda_i \in C[0,1]^*} \text{epi}(\sum_{i=1}^m \lambda_i \circ h_i)^*.$$

By Theorem 1 (Riesz Representation Theorem) in ([18], p.113),

$$C[0,1]^* = \{x^* \mid x^*(y) = \int_0^1 y(t)d\mu(t) \ \forall y \in C[0,1], \ \mu : [0,1] \to \mathbb{R} : \text{ a function}, \ \mu \in \text{NBV}[0,1]\}.$$

So, $\lambda_i \in C[0,1]^*$ if and only if there exists $\mu_i \in \text{NBV}[0,1]$ such that, for any $y \in C[0,1]$, $\lambda_i(y) = \int_0^1 y(t)d\mu_i(t)$. So, we have

$$\begin{aligned} (\lambda_i \circ h_i)(x) &= \int_0^1 (h_i(x))(t)d\mu_i(t) \\ &= \int_0^1 \left(\int_0^t [a_i(\tau)^T x(\tau) - b_i(\tau)]d\tau \right) d\mu_i(t). \end{aligned}$$

Let $g_i(t) = \int_0^t [a_i(\tau)^T x(\tau) - b_i(\tau)]d\tau$. Then, from [19, Theorem 6.2.3 and Theorem 6.2.10],

$$\begin{aligned} (\lambda_i \circ h_i)(x) &= \int_0^1 g_i(t)d\mu_i(t) \\ &= -\int_0^1 \mu_i(t)dg_i(t) + \mu_i(1)g_i(1) - \mu_i(0)g_i(0) \\ &= -\int_0^1 \mu_i(t)g_i'(t)dt \\ &= -\int_0^1 \mu_i(t)[a_i(t)^T x(t) - b_i(t)]dt. \end{aligned}$$

Moreover, since $\int_0^1 \mu_i(t)[a_i(t)^T(\cdot)]dt$ is continuous and linear on $C^n[0,1]$, one

$$(\lambda_i \circ h_i)^*(v^*) = \sup_{x\in C^n[0,1]} \{v^*(x) - (-\int_0^1 \mu_i(t)[a_i(t)^T x(t) - b_i(t)]dt)\}$$
$$= \begin{cases} -\int_0^1 \mu_i(t)b_i(t)dt & \text{if } v^*(\cdot) = -\int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt \\ +\infty & \text{if } v^*(\cdot) \neq -\int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt. \end{cases}$$

Thus

$$\text{epi}(\lambda_i \circ h_i)^* = \{(-\int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt,\ -\int_0^1 \mu_i(t)b_i(t)dt)\} + \{0\}\times\mathbb{R}_+.$$

Hence

$$\begin{aligned}\text{epi}\delta_D^* &= \text{clco}\Big(\bigcup_{\mu_i\in\text{NBV}[0,1]} \{(-\sum_{i=1}^m \int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt,\ -\sum_{i=1}^m \int_0^1 \mu_i(t)b_i(t)dt)\} \\ &\quad + \{0\}\times\mathbb{R}_+\Big).\end{aligned}$$

Since the set

$$\bigcup_{\mu_i\in\text{NBV}[0,1]} \{(-\sum_{i=1}^m \int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt,\ -\sum_{i=1}^m \int_0^1 \mu_i(t)b_i(t)dt)\} + \{0\}\times\mathbb{R}_+$$

is convex, one has

$$\begin{aligned}\text{epi}\delta_D^* &= \text{cl}\Big(\bigcup_{\mu_i\in\text{NBV}[0,1]} \{(-\sum_{i=1}^m \int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt,\ -\sum_{i=1}^m \int_0^1 \mu_i(t)b_i(t)dt)\} \\ &\quad + \{0\}\times\mathbb{R}_+\Big). \end{aligned} \tag{2.4}$$

From (2.1), (2.2), (2.3), and (2.4), one has

$$\begin{aligned}(0,0) &\in \{(\int_0^1 [c(t) - q(\bar{x})d(t)]^T(\cdot)dt,\ -\alpha + q(\bar{x})\beta)\} + \{0\}\times\mathbb{R}_+ \\ &+\text{cl}\Big(\bigcup_{\mu_i\in\text{NBV}[0,1]} \{(-\sum_{i=1}^m \int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt,\ -\sum_{i=1}^m \int_0^1 \mu_i(t)b_i(t)dt)\} + (-K^*)\times\mathbb{R}_+\Big).\end{aligned}$$

Thus (ii) holds.

Suppose that (ii) holds. Then, from (ii), there exist $\mu_i^n \in \text{NBV}[0,1]$, $i = 1,\cdots,m$, $k_n \in K^*$, $r \in \mathbb{R}_+$, $r_n \in \mathbb{R}_+$ such that

$$0 = \int_0^1 [c(t) - q(\bar{x})d(t)]^T(\cdot)dt + \lim_{n\to\infty}\Big[-\sum_{i=1}^m \int_0^1 \mu_i^n(t)a_i(t)^T(\cdot)dt - k_n^*\Big], \tag{2.5}$$

and

$$0 = -\alpha + q(\bar{x})\beta + r + \lim_{n\to\infty}\Big[-\sum_{i=1}^m \int_0^1 \mu_i^n(t)b_i(t)dt + r_n\Big]. \tag{2.6}$$

From (2.5) and (2.6), one has

$$\begin{aligned}0 &= -r + \lim_{n\to\infty}\Big\{-\sum_{i=1}^m \int_0^1 \mu_i^n(t)\Big(a_i(t)^T\bar{x}(t) - b_i(t)\Big)dt - k_n^*(\bar{x}) - r_n\Big\} \\ &= -r + \lim_{n\to\infty}[-k_n^*(\bar{x}) - r_n].\end{aligned}$$

Since $k_n^*(\bar{x}) \geqq 0,\ r \geqq 0$, and $r_n \geqq 0$, we have

$$r = 0,\ \lim_{n\to\infty} k_n^* \circ \bar{x} = 0, \quad \text{and} \quad \lim_{n\to\infty} r_n = 0.$$

Hence $\lim_{n\to\infty} k_n^*(\bar{x}) = 0$. It follows from (2.5) that (iii) holds.

Suppose that (iii) holds. Then there exist $\mu_i^n \in \mathrm{NBV}[0,1]$ and $k_n^* \in K^*,\ i = 1,\cdots,m$ such that

$$\int_0^1 [c(t) - q(\bar{x})d(t)]^T(\cdot)dt + \lim_{n\to\infty}\Big[-\sum_{i=1}^m \int_0^1 \mu_i^n(t)a_i(t)^T(\cdot)dt - k_n^*\Big] = 0$$
$$\text{and} \quad \lim_{n\to\infty} k_n^*(\bar{x}) = 0.$$

Then, for any $x \in \triangle$,

$$\begin{aligned}
0 &= \int_0^1 [c(t) - q(\bar{x})d(t)]^T(x(t) - \bar{x}(t))dt \\
&\quad + \lim_{n\to\infty}\Big[-\sum_{i=1}^m \int_0^1 \mu_i^n(t)a_i(t)^T(x(t) - \bar{x}(t))dt - k_n^*(x - \bar{x})\Big] \\
&= \int_0^1 c(t)^T x(t)dt + \alpha - q(\bar{x})\Big[\int_0^1 d(t)^T x(t)dt + \beta\Big] \\
&\quad - \Big[\int_0^1 c(t)^T \bar{x}(t)dt + \alpha\Big] + q(\bar{x})\Big[\int_0^1 d(t)^T \bar{x}(t)dt + \beta\Big] \\
&\quad + \lim_{n\to\infty}\Big[-\sum_{i=1}^m \int_0^1 \mu_i^n(t)(b_i(t) - b_i(t))dt - k_n^*(x)\Big] \\
&= \int_0^1 c(t)^T x(t)dt + \alpha - q(\bar{x})\Big[\int_0^1 d(t)^T \bar{x}(t)dt + \beta\Big] - \int_0^1 c(t)^T \bar{x}(t)dt \\
&\quad - \lim_{n\to\infty} k_n^*(x).
\end{aligned}$$

Since $\lim_{n\to\infty} k_n^*(x) \geqq 0$ for any $x \in \triangle$, one has

$$\int_0^1 c(t)^T x(t)dt + \alpha - q(\bar{x})\Big[\int_0^1 d(t)^T x(t)dt + \beta\Big] \geqq 0$$

for any $x \in \triangle$. Since $\bar{x} \in \triangle$ and $\int_0^1 d(t)^T x(t)dt + \beta > 0$ for any $x \in \triangle$, one has

$$q(\bar{x}) \leqq \frac{\int_0^1 c(t)^T x(t)dt + \alpha}{\int_0^1 d(t)^T x(t)dt + \beta}$$

for any $x \in \triangle$. In view of $q(\bar{x}) = \frac{\int_0^1 c(t)^T \bar{x}(t)dt + \alpha}{\int_0^1 d(t)^T \bar{x}(t)dt + \beta}$, one has that $\bar{x}$ is an optimal solution to problem (P). Thus (i) holds. □

Now we discuss about the closedness of the set in (ii) of Theorem 2.1. Following the proof of [2, Proposition 2.1], we can prove the following proposition.

Proposition 2.1. *Suppose that int$K \neq \emptyset$. Assume that the following constraint qualifications for problem (P) hold:*

(i) there exists $\widehat{x} \in$ int$K$ such that

$$a_i(t)^T \widehat{x}(t) - b_i(t) = 0 \ \ i = 1,\cdots,m \ \ \textit{for any } t \in [0,1];$$

(ii) for any $y=(y_1,\cdots,y_m)\in C^m[0,1]$, *there exists* $\widetilde{x}\in C^n[0,1]$ *such that*

$$\int_0^t a_i(\tau)^T\widetilde{x}(\tau)d\tau = y_i(t) \ \text{ for any } i=1,\cdots,m \ \text{ and } t\in[0,1].$$

Then the set

$$\Lambda := \bigcup_{\mu_i\in NBV[0,1]}\left\{\left(-\sum_{i=1}^m\int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt,\ -\sum_{i=1}^m\int_0^1 \mu_i(t)b_i(t)dt\right)\right\}+(-K^*)\times\mathbb{R}_+$$

is closed in $C^n[0,1]^*\times\mathbb{R}$.

From Theorem 2.1, we can obtain the following theorem.

Theorem 2.2. *Let* $\bar{x}\in\triangle$ *and assume that for any* $x\in\triangle$, $\int_0^1 d(t)^Tx(t)dt+\beta>0$. *Suppose that the set*

$$\bigcup_{\mu_i\in \mathrm{NBV}[0,1]}\left\{\left(-\sum_{i=1}^m\int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt,\ -\sum_{i=1}^m\int_0^1 \mu_i(t)b_i(t)dt\right)\right\}+(-K^*)\times\mathbb{R}_+$$

is closed in $C^n[0,1]^*\times\mathbb{R}$. *Then* $\bar{x}$ *is an optimal solution to problem (P) if and only if there exist* $\mu_i\in\mathrm{NBV}[0,1]$ *and* $k^*\in K^*$ *such that*

$$\int_0^1\left[c(t)-q(\bar{x})d(t)-\sum_{i=1}^m\mu_i(t)a_i(t)\right]^T(\cdot)dt-k^*(\cdot)=0$$

$$\text{and } \ k^*(\bar{x})=0.$$

Now we consider linear optimization problem. If $\alpha=0,\ d=0$, and $\beta=1$, then problem (P) becomes the following linear optimization problem (LP):

$$\begin{array}{lll}\text{(LP)} & \text{Minimize} & \displaystyle\int_0^1 c(t)^Tx(t)dt\\ & \text{subject to} & x(\cdot)\in K,\\ & & a_i(t)^Tx(t)=b_i(t),\ i=1,\cdots,m,\ \text{for any } t\in[0,1].\end{array}$$

If $K=C^n[0,1]$ (note that $\mathrm{int}K\neq\emptyset$), we can obtain the following theorem.

Theorem 2.3. *Let* $K=C^n[0,1]$ *and* $\bar{x}\in C^n[0,1]$. *Assume that the following constraint qualifications hold:*

(i) there exists $\widehat{x}\in C^n[0,1]$ *such that* $a_i(t)^T\widehat{x}(t)-b_i(t)=0,\ i=1,\cdots,m$ *for any* $t\in[0,1]$;

(ii) for any $y=(y_1,\cdots,y_m)\in C^m[0,1]$, *there exists* $\widetilde{x}\in C^n[0,1]$ *such that*

$$\int_0^t a_i(\tau)^T\widetilde{x}(\tau)d\tau = y_i(t),\ \text{ for any } i=1,\cdots,m \ \text{ and } t\in[0,1].$$

Then $\bar{x}$ *is an optimal solution to problem (LP) if and only if*

$$a_i(t)^T\bar{x}(t)=b_i(t),\ i=1,\cdots,m,\ \text{for any } t\in[0,1]$$

and there exist $\mu_i\in\mathrm{NBV}[0,1]$, $i=1,\cdots,m$ *such that* $c(t)-\sum_{i=1}^m\mu_i(t)a_i(t)=0$ *a.e. on* $[0,1]$.

Proof. By Proposition 2.1, the set

$$\bigcup_{\mu_i\in \mathrm{NBV}[0,1]}\left\{\left(-\sum_{i=1}^m\int_0^1 \mu_i(t)a_i(t)^T(\cdot)dt,-\sum_{i=1}^m\int_0^1 \mu_i(t)b_i(t)dt\right)\right\}+\{0\}\times\mathbb{R}_+$$

is closed. So, by Theorem 2.2, $\bar{x}$ is an optimal solution to problem (LP) if and only if

$$a_i(t)^T \bar{x}(t) = b_i(t),\ i = 1, \cdots, m,\ \text{for any } t \in [0,1]$$

and there exists $\mu_i \in \mathrm{NBV}[0,1]$ such that $\int_0^1 [c(t) - \sum_{i=1}^m \mu_i(t) a_i(t)]^T x(t) dt = 0$ for any $x \in C^n[0,1]$. By Theorem 1A.1 (Dubois-Reymond Lemma) in [20], one sees that $\bar{x}$ is an optimal solution to problem (LP) if and only if

$$a_i(t)^T \bar{x}(t) = b_i(t),\ i = 1, \cdots, m,\ \text{for any } t \in [0,1]$$

and there $\mu_i \in \mathrm{NBV}[0,1]$ such that $c(t) - \sum_{i=1}^m \mu_i(t) a_i(t) = 0$ a.e. on $[0,1]$. □

3. EXAMPLE

Now we give an example to demonstrate that the set Λ in Proposition 2.1 may not be closed and illustrate how to use the optimality condition (iii) in Theorem 2.1 to find the optimal solutions.

Example 3.1. Let $K = \{(x_1, x_2, x_3) \in C^3[0,1] \mid x_1(t) \geqq \sqrt{x_2(t)^2 + x_3(t)^2},\ \forall t \in [0,1]\}$. Then K is a closed convex cone in $C^3[0,1]$. Let $a_1(t) = (1,0,-1),\ \forall t \in [0,1]$ and $b_1(t) = 0,\ \forall t \in [0,1]$. Let $\Lambda = \bigcup_{u_1 \in NBV[0,1]} \{(-\int_0^1 u_1(t) a_1(t)^T(\cdot) dt, -\int_0^1 u_1(t) b_1(t) dt)\} + (-K^*) \times \mathbb{R}_+$, where $NBV[0,1] = \{u \mid u : [0,1] \to \mathbb{R}\ :\ \text{a function of bounded variation, left continuous on } [0,1) \text{ and } u(1) = 0\}$. Then $\Lambda \subset C^3[0,1]^* \times \mathbb{R}$, where $C^3[0,1]^*$ is the topological dual space of $C^3[0,1]$.

Now we show that Λ may not be closed. Let $a^*(\cdot) = \int_0^1 (0,-1,0)^T(\cdot) dt$, that is, $a^*(x_1,x_2,x_3) = \int_0^1 (-x_2(t)) dt\ \forall (x_1,x_2,x_3) \in C^3[0,1]$. Suppose to the contrary that $(a^*, 0) \in \Lambda$. Then there exists $u_1 \in NBV[0,1]$ such that $a^* \in \{-\int_0^1 u_1(t) a_1(t)^T(\cdot) dt\} + (-K^*)$. Thus,

$$\int_0^1 (0,-1,0)^T(\cdot) dt + \int_0^1 u_1(t) a_1(t)^T(\cdot) dt \in -K^*$$

and hence

$$\int_0^1 (u_1(t), -1, -u_1(t))^T(\cdot) dt \in -K^*.$$

It follows that

$$\int_0^1 \Big[-u_1(t) x_1(t) + x_2(t) + u_1(t) x_3(t)\Big] dt \geqq 0\ \ \forall (x_1,x_2,x_3) \in K,$$

which implies that, for any $(a,b,c) \in \widetilde{K} := \{(a,b,c) \in \mathbb{R}^3 \mid a \geqq \sqrt{b^2 + c^2}\}$,

$$0 \leqq a \int_0^1 (-u_1(t)) dt + b \cdot 1 + c \int_0^1 u_1(t) dt.$$

Since $\widetilde{K}$ is self-dual, one has $(\int_0^1 (-u_1(t)) dt, 1, \int_0^1 u_1(t) dt) \in \widetilde{K}$. Thus

$$\int_0^1 (-u_1(t)) dt \geqq \sqrt{1^2 + (\int_0^1 u_1(t) dt)^2},$$

which is a contradiction. Hence $(a^*, 0) \notin \Lambda$.

Now we prove that $(a^*, 0) \in cl\Lambda$, where $cl\Lambda$ is the closure of Λ. Let

$$k_n^*(\cdot) = \int_0^1 \Big(\sqrt{n^2(1-t)^2 + (1 + \frac{1}{n(1+t)})^2}, 1 + \frac{1}{n(1+t)}, -n(1-t)\Big)^T(\cdot) dt.$$

Notice that

$$\forall t \in [0,1], \Big(\sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}, 1+\frac{1}{n(1+t)}, -n(1-t)\Big) \in \widetilde{K}.$$

So, $\forall (x_1, x_2, x_3) \in K$ and $\forall t \in [0,1]$,

$$\sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2} x_1(t) + (1+\frac{1}{n(1+t)}) x_2(t) - n(1-t) x_3(t) \geqq 0.$$

Hence, $\forall (x_1, x_2, x_3) \in K$, $k_n^*(x_1, x_2, x_3) \geqq 0$ and so $k_n^* \in K^*$. Let $u_1^n(t) = -n(1-t)$ $\forall t \in [0,1]$, and let $a_n^*(\cdot) = \int_0^1 (-u_1^n(t)) a_1(t)^T(\cdot)dt - k_n^*(\cdot)$. Then $u_1^n \in NBV[0,1]$ and

$$\begin{aligned} a_n^*(\cdot) &= \int_0^1 n(1-t)(1,0,-1)^T(\cdot)dt - k_n^*(\cdot) \\ &= \int_0^1 \Big(n(1-t) - \sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}, -1-\frac{1}{n(1+t)}, 0\Big)^T(\cdot)dt. \end{aligned}$$

So, we have,

$$\begin{aligned} \|a_n^* - a^*\| &= \sup_{\|x\| \leqq 1} |(a_n^* - a^*)(x)| \\ &= \sup_{\|x\| \leqq 1} \Big|\int_0^1 (n(1-t) - \sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}, -\frac{1}{n(1+t)}, 0)^T \\ &\qquad (x_1(t), x_2(t), x_3(t))dt\Big| \\ &\leqq \sup_{\|x\| \leqq 1} \Big|\int_0^1 (n(1-t) - \sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}) x_1(t)dt \\ &\quad + \sup_{\|x\| \leqq 1} \Big|\int_0^1 \frac{1}{n(1+t)} x_2(t)\Big| dt \\ &\leqq \int_0^1 \Big|n(1-t) - \sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}\Big| dt + \int_0^1 \frac{1}{n(1+t)} dt \\ &= \int_0^1 \frac{(1+\frac{1}{n(1+t)})^2}{n(1-t) + \sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}} dt + \frac{1}{n} log 2. \end{aligned}$$

Let $\varphi_n(t) = \frac{(1+\frac{1}{n(1+t)})^2}{n(1-t) + \sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}}$, $t \in [0,1]$. Then

$$|\varphi_n(t)| \leqq \frac{(1+\frac{1}{n(1+t)})^2}{\sqrt{(1+\frac{1}{n(1+t)})^2}} \leqq 2, \ \forall t \in [0,1],$$

and $\lim_{n\to\infty} \varphi_n(t) = 0$, $\forall t \in [0,1)$. By Lebesque Convergence theorem,

$$\lim_{n\to\infty} \int_0^1 \varphi_n(t)dt = \int_0^1 \lim_{n\to\infty} \varphi_n(t)dt = 0.$$

Thus

$$\lim_{n\to\infty} \|a_n^* - a^*\| = 0 \tag{3.1}$$

and so $(a_n^*, 0)$ converges to $(a^*, 0)$. Since $(a_n^*, 0) \in \Lambda$, one has $(a^*, 0) \in cl\Lambda$. Since $(a^*, 0) \notin \Lambda$, one has that Λ is not closed.

Now we consider the following linear optimization problem:

$$\text{(MP)} \qquad \text{Minimize}_{(x_1,x_2,x_3)\in C^3[0,1]} \quad \int_0^1 x_2(t)dt$$
$$\text{subject to} \quad x_1(t) - x_3(t) = 0$$
$$x_1(t) \geqq \sqrt{x_2(t)^2 + x_3(t)^2} \quad \forall t \in [0,1].$$

Let $c(t) = (0,1,0)$. Then the problem becomes:

$$\text{Minimize}_{(x_1,x_2,x_3)\in C^3[0,1]} \quad \int_0^1 c(t)^T x(t)dt$$
$$\text{subject to} \quad a_1(t)^T x(t) = b_1(t)$$
$$x \in K.$$

Let

$$\triangle = \{(x_1,x_2,x_3) \in C^3[0,1] \mid x_1(t) - x_3(t) = 0,\ x_1(t) \geqq \sqrt{x_2(t)^2 + x_3(t)^2},\ \forall t \in [0,1]\}.$$

Then

$$\triangle = \{(x_1,x_2,x_3) \in C^3[0,1] \mid x_1(t) = x_3(t),\ x_1(t) \geqq 0,\ x_2(t) = 0,\ \forall t \in [0,1]\}.$$

Clearly $\triangle$ is the set of solutions to problem (MP) and the optimal value of problem (MP) is 0. Let $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) \in \triangle$ be any fixed. It follows from (3.1) that

$$\begin{aligned}
&\int_0^1 c(t)^T(\cdot)dt + \lim_{n\to\infty}\Big[-\int_0^1 u_1^n(t)a_1(t)(\cdot)dt - k_n^*(\cdot)\Big] \\
&= \int_0^1 (0,1,0)^T(\cdot)dt + \lim_{n\to\infty}\Big[\int_0^1 n(1-t)(1,0,-1)^T(\cdot)dt \\
&\quad - \int_0^1 \Big(\sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}, 1+\frac{1}{n(1+t)}, -n(1-t)\Big)^T(\cdot)dt\Big] \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} k_n^*(\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3) &= \lim_{n\to\infty}\int_0^1 \Big(\sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2}\ \widetilde{x}_1(t) \\
&\quad + (1+\frac{1}{n(1+t)})\widetilde{x}_2(t) - n(1-t)\widetilde{x}_3(t)\Big)dt \\
&= \lim_{n\to\infty}\int_0^1 \Big(\sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2} - n(1-t)\Big)\widetilde{x}_3(t)dt.
\end{aligned}$$

Let

$$\psi_n(t) = \sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2} - n(1-t).$$

Then there exists $M > 0$ such that $|\psi_n(t)\widetilde{x}_3(t)| \leqq M$, $\forall t \in [0,1]$ and $\lim_{n\to\infty} \psi_n(t)\widetilde{x}_3(t) = 0$, $\forall t \in [0,1)$. By the Lebesque Convergence Theorem, one has

$$\lim_{n\to\infty} k_n^*(\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3) = \int_0^1 \lim_{n\to\infty}\Big(\sqrt{n^2(1-t)^2 + (1+\frac{1}{n(1+t)})^2} - n(1-t)\Big)\widetilde{x}_3(t)dt = 0.$$

From Theorem 2.1, one sees that $(\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3) \in \triangle$ is an optimal solution to problem (MP).

Acknowledgments
This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (NRF-2022R1A2C1003309).

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