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THE DEFORMATIONS PROBLEM FOR THE STIELTJES STRINGS SYSTEM WITH A NONLINEAR CONDITION

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Abstract. In the present paper, we investigate a problem, describing the deformations process for the Stieltjes strings system located along a geometric star - shaped graph under the influence of an external force. The case when the force can be concentrated at separate points, including a node of the graph, is considered. The non-linear condition arises due to the presence of a limiter on the strings displacement in the node. Using variational methods, the necessary and sufficient conditions for the extremum of an energy functional are established; existence and uniqueness theorems for the solution are proved; an explicit formula for the solution is obtained; and the dependence solution on the length of the limiter is studied.

Keywords. Energy functional; Function of bounded variation; Geometric graph; Stieltjes integral.

1. INTRODUCTION

The mathematical models are described in terms of a branching argument, i.e., the argument taking values from some geometric graph arise in the analysis of processes in complex physical systems. The examples of such systems are given by elastic meshes, rod lattices, electrical circuits, acoustic networks, waveguides, hydraulic systems, and so on. Active mathematical interest to study such problems has led to a large number of related papers. Here, we refer to [1]-[16]. However, in all these works, only the problems with linear boundary conditions were considered.

In the present paper, we study a boundary value problem on a star - shaped graph, consisting of n one-dimensional segments of length l_i, connected at one point (called a node), with a

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nonlinear condition at the node. This problem has the form

$$\begin{cases} -(p_iu_i')(x)+\int_0^x u_i\,dQ_i=F_i(x)-F_i(+0)-(p_iu_i')(+0), i=1,2,...,n, \\ \sum\limits_{i=1}^n p_i(+0)u_i'(+0)+f\in N_{[-m,m]}u(0), \\ u_1(0)=u_2(0)=...=u_n(0)=u(0), \\ (p_iu_i')(l_i-0)+\gamma_iu_i(l_i)=f_i, i=1,2,...,n. \end{cases} \tag{1.1}$$

Problem (1.1) describes the deformations process for a system of n Stieltjes strings located in an equilibrium position along this graph. Here the functions $u_i(x)$ determine the deformations of each string; $F_i(x)$ describe the distribution of the external force; and f is equal to the external force concentrated at $x=0$. The functions $p_i(x)$ characterize the elasticity of strings; $Q_i(x)$ describe the elastic response of the environment; f_i are equal to the external forces concentrated at the points l_i; and γ_i coincide with the elasticity of the springs attached to the points l_i ($i=1,2,...,n$). Here $N_{[-m,m]}u(0)$ denotes the outward normal cone at the point $u(0)$ to the segment $[-m,m]$, defined by the number set

$$N_{[-m,m]}u(0)=\{\xi\in R:\xi(c-u(0))\leq 0\quad\forall c\in[-m,m]\}.$$

The nonlinear condition

$$\sum_{i=1}^n p_i(+0)u_i'(+0)+f\in N_{[-m,m]}u(0) \tag{1.2}$$

arises due to the presence of a limiter, represented by the segment $[-m,m]$, on the displacement of strings in the node, i.e., we assume $|u(0)|\leq m$. From (1.2), it follows that if the external force is such that the inequality $|u(0)|<m$ holds, then $\sum\limits_{i=1}^n p_i(+0)u_i'(+0)=-f$. Otherwise, if $u(0)=m$, then $\sum\limits_{i=1}^n p_i(+0)u_i'(+0)+f\geq 0$, and if $u(0)=-m$, then $\sum\limits_{i=1}^n p_i(+0)u_i'(+0)+f\leq 0$.

The paper is structured as follows. The second section contains necessary definitions and results. The third section contains an exact description of the investigated problem and a proof of the necessary condition for a potential energy functional extremum. In the fourth section, the main results of the paper are established. In particular, the existence and uniqueness theorems for the solution are proved, a sufficient condition for the energy functional extremum is established, and the dependence solution on the length of the limiter is analyzed.

2. PRELIMINARIES

In this section, we recall some notions and facts which we need in the sequel.

We assume that the star-shaped graph Γ is oriented from the node and consists of segments, parameterized as $[0,l_i]$, numbered arbitrarily ($i=1,2,....,n$). The point 0 (node) corresponds to the inner vertex of the graph, the points l_i correspond to the boundary vertices of the graph, and the intervals $(0,l_i)$ are the edges of the graph. Denote by $\partial\Gamma$ the set of boundary vertices of the graph; Υ_i is the graph edge($i=1,2,....,n$); and $R(\Gamma)=\bigcup_{i=1}^n\Upsilon_i$. A scalar function $z(x)$ defined on a graph Γ is a map $z:\Gamma\to R$. The restriction of $z(x)$ to $(0,l_i]$ is denoted by $z_i(x)$.

We say that a function $z(x)$ defined on the edge Υ_i has bounded variation on this edge if

1) the function $z(x)$ has finite one-sided limits at the boundary points $x=0$ and $x=l_i$ of the edge Υ_i;

2) there is a constant c_i such that for any partition of the edge Υ_i by points $0=x_0^i<x_1^i<...<$

$x^i_{n_i} = l_i$ the sum $\sum_{j=0}^{n_i-1} |z(x^i_{j+1}) - z(x^i_j)| \le c_i$ (at the boundary points of the edge, the function z is determined by its limit values)

The jump of the function z at the boundary vertex $a \in \partial\Gamma$ is defined as $\Delta z(a) = z(a) - z(a-0)$; the jump of the function z at any interior point ξ of the graph edge is equal to $\Delta z(\xi) = z(\xi+0) - z(\xi-0)$; and the jump of the function at the node is defined as $\Delta z(0) = \sum_{i=1}^{n} (z_i(0+0) - z(0))$.

We say that a function $z(x)$ defined on Γ is absolutely continuous on Γ if it is absolutely continuous on every edge Υ_i, and if it is continuous at the node, i.e., the one-sided limits along the edges coincide with $z(0)$, finally, if it is continuous at points from $\partial\Gamma$.

The integro - differential equation

$$-(\widetilde{p}v')(x) + \int_0^x v\, d\widetilde{Q} = \widetilde{F}(x) - \widetilde{F}(0) - (\widetilde{p}v')(+0), \quad x \in \overline{[0,l]}_\sigma, \tag{2.1}$$

introduced in [17] and [18], will play an important role in this paper. The solutions of Equation (2.1) belong to the class of absolutely continuous functions on $[0,l]$, whose derivatives have bounded variation on $[0,l]$. We assume that the functions $\widetilde{p}$, $\widetilde{F}$, $\widetilde{Q}$ have bounded variation on the segment $[0,l]$ and $\inf_{[0,l]} \widetilde{p} > 0$; and functions $\widetilde{p}$, $\widetilde{F}$, $\widetilde{Q}$ are continuous at points $x = 0$ and $x = l$. We need to explain the set $\overline{[0,l]}_\sigma$, which x belongs to, so that Equation (2.1) has the correct meaning at singular points, where the derivative $v'(x)$ and the functions $\widetilde{p}$, $\widetilde{F}$, and $\widetilde{Q}$ can be discontinuous. Let us describe the construction from [17, 18] for the definition of the extension. Denote by S the set of all points at which the functions $\widetilde{p}$, $\widetilde{F}$, and $\widetilde{Q}$ have non-zero simple jumps, that is distinct left-hand and right-hand limits. Consider the Jordan representation of the bounded variation functions $\widetilde{p}$, $\widetilde{Q}$, $\widetilde{F}$ in the form $\widetilde{p} = \widetilde{p}^+ - \widetilde{p}^-$, $\widetilde{Q} = \widetilde{Q}^+ - \widetilde{Q}^-$, and $\widetilde{F} = \widetilde{F}^+ - \widetilde{F}^-$. Denote by $\sigma(x)$ the following sum of nondecreasing functions

$$\sigma(x) = x + \widetilde{p}^+(x) + \widetilde{p}^-(x) + \widetilde{Q}^+(x) + \widetilde{Q}^-(x) + \widetilde{F}^+(x) + \widetilde{F}^-(x).$$

Without loss of generality, we can assume that function $\sigma(x)$ has discontinuities only at the points of the set S. Let us introduce the metric $\rho(x,y) = |\sigma(x) - \sigma(y)|$ on the set $[0,l] \setminus S$. If $S \neq \emptyset$, then this metric space is obviously not complete. Its standard metric completion coincides (up to isomorphism) with $\overline{[0,l]}_\sigma$ and induces a topology on this space. Thus, each discontinuity point ξ of functions $\widetilde{p}$, $\widetilde{F}$, and $\widetilde{Q}$ is replaced on $\overline{[0,l]}_\sigma$ by the pair of points denoted as $\{\xi - 0, \xi + 0\}$. We will define at these points the functions $\widetilde{p}$, $\widetilde{Q}$, $\widetilde{F}$ by limit values, i.e., we suppose $\widetilde{p}(\xi \pm 0) = \lim_{x \to \xi \pm 0} \widetilde{p}(x)$, $\widetilde{F}(\xi \pm 0) = \lim_{x \to \xi \pm 0} \widetilde{F}(x)$, and $\widetilde{Q}(\xi \pm 0) = \lim_{x \to \xi \pm 0} \widetilde{Q}(x)$. We suppose $\xi - 0 > x$ for any $x < \xi$ and $\xi + 0 < x$ for any $x > \xi$. On $\overline{[0,l]}_\sigma$, the values $\widetilde{p}(\xi \pm 0)$, $\widetilde{Q}(\xi \pm 0)$ and $\widetilde{F}(\xi \pm 0)$ which were limit values on $[0,l]$ now become true values at the corresponding points of $\overline{[0,l]}_\sigma$. The continuity of the function $v(\cdot)$ enables us to preserve the usual Rimann – Stieltjes meaning for the integral term in (2.1) at $x = \xi - 0$ and $x = \xi + 0$, regarding the previous limit values as true values.

Thus, we consider Equation (2.1) in two layers: the lower level is for the values $x \in [0,l]$, when speaking about the solutions $v(x)$ themselves (under the integral), and the second level is for the values x in (2.1), where $x \in \overline{[0,l]}_\sigma$.

Notice that at each point $\xi \in S$ the equality

$$-\widetilde{p}(\xi+0)v'(\xi+0)+\widetilde{p}(\xi-0)v'(\xi-0)+v(\xi)(\widetilde{Q}(\xi+0)-\widetilde{Q}(\xi-0))=\widetilde{F}(\xi+0)-\widetilde{F}(\xi-0)$$

holds. According to [18, Theorem 1.4],

$$v'(\xi+0)=\lim_{x\to\xi+0}v'(x)=\lim_{\varepsilon\to 0+0}\frac{v(\xi+\varepsilon)-v(\xi)}{\varepsilon}=v'_+(\xi),$$

and

$$v'(\xi-0)=\lim_{x\to\xi-0}v'(x)=\lim_{\varepsilon\to 0-0}\frac{v(\xi+\varepsilon)-v(\xi)}{\varepsilon}=v'_-(\xi).$$

We will use the following results.

Lemma 2.1. *(*[17, Lemma 3.1]*) Let $A(x)$ be a function of bounded variation on $[0,l]$. Assume that for any absolutely continuous function $h(x)$ on $[0,l]$, whose derivative $h'(x)$ has bounded variation on $[0,l]$, such that $h(0)=h(l)=0$, we have $\int_0^l A\,dh=0$. Thus for all $x\in(0,l)$ the identity $A(x-0)=A(x+0)\equiv const$ holds.*

Theorem 2.1. *(*[18, Theorem 1.5]*) For any numbers v_0, w_0 and for any point $x_0\in\overline{[0,l]}_\sigma$ the problem*

$$\begin{cases} -(\widetilde{p}v')(x)+(\widetilde{p}v')(0)+\int_0^x v\,d\widetilde{Q}=\widetilde{F}(x)-\widetilde{F}(0), x\in\overline{[0,l]}_\sigma\\ v(x_0)=v_0,\\ v'(x_0)=w_0.\end{cases}$$

has a unique solution.

Let us consider the homogeneous equation

$$-\left(\widetilde{p}v'\right)(x)+\left(\widetilde{p}v'\right)(0)+\int_0^x v\,d\widetilde{Q}=0. \tag{2.2}$$

Lemma 2.2. *(*[18, Lemma 1.1]*). The space of solutions of Equation (2.2) is two-dimensional.*

Lemma 2.3. *(*[18, Proposition 2.1]*). Every non-trivial solution of Equation (2.2) can have only a finite number of zeros on $[0,l]$.*

Lemma 2.4. *(*[18, Proposition 2.2]*). Let the function $\widetilde{Q}$ be non-decreasing on $[0,l]$. Then every non-trivial solution of Equation (2.2) either does not have zeroes on $[0,l]$ or has only one zero on $[0,l]$.*

Theorem 2.2. *(*[18, Theorem 2.1]*). For any pair of solutions φ_1, φ_2 to Equation (2.2), the equality $p(x)(\varphi_1(x)\varphi_2'(x)-\varphi_2(x)\varphi_1'(x))\equiv const$ holds on $\overline{[0,l]}_\sigma$.*

Let $G\subset H$ be a closed convex set, where H is a Hilbert space and $x\in G$. The set

$$N_G(x)=\{\xi\in H:\langle\xi,c-x\rangle\le 0\quad\forall c\in G\}$$

denotes the outward normal cone to G at the point x. Notice that if x is an interior point of G, then $N_G(x)=\{0\}$. If $G=[-m,m]$, where $m>0$, then $N_G(m)=[0,+\infty)$ and $N_G(-m)=(-\infty,0]$.

3. VARIATIONAL MOTIVATION OF OUR APPROACH

Let points O, A_1, A_2,..., A_n belong to a horizontal plane π. Consider a mechanical system consisting of n strings, which in the equilibrium position are segments OA_1, OA_2, ..., OA_n. The ends of the strings are interconnected at the point O (the node). Under the influence of an external force, which is directed perpendicular to the π plane, the strings deviate from the equilibrium position. We assume that the deviation of all points of the strings is parallel to the same line, which is perpendicular to the plane π and consider small deviations from the equilibrium position. Let us introduce a coordinate system to describe the deformation process. The Ox axis for the i-th string ($i = 1,2,...,n$) contains the segment OA_i and is directed from O to A_i. The axis Oy is directed perpendicular to the plane π and passes through the point O. Thus, the point O corresponds to the origin of coordinates. The point A_i has on its axis Ox the coordinate l_i ($i = 1,2,...,n$). The graph Γ is oriented from the node and consists of edges - intervals $\Upsilon_i = (0,l_i)$, internal vertex 0 (the node) and boundary vertices l_i. Denote by $u(x)$ ($x \in \Gamma$) the function, which describes the deviation of the string system from the equilibrium position under the influence of an external force, defined by the function $F(x)$. We suppose that the strings are elastically fixed at the boundary vertices (with the help of springs γ_i). At the same time, elastic supports (springs) can also be installed at any number of points (but not more than countable) belonging to the edges. The restrictions $u_i(x)$ of the function $u(x)$ to $(0,l_i]$ determine the deformations of each string. We use a natural parameter as an argument, i.e., the distance from the corresponding point to the node. Denote by $F_i(x)$ the restriction of $F(x)$ to $(0,l_i]$. The physical meaning of $F_i(x)$ is the force applied to $(0,x]$. A concentrated force equal to f is allowed at the point $x = 0$. Notice that the jumps of the function F are equal to the forces concentrated at the corresponding points. Denote by $Q(x)$ the function describing the elastic response of the external medium. Its jumps coincide with the elasticity of the springs installed at the corresponding points. Denote by $p(x)$ the function characterizing the elastic properties of strings. We will denote by Q_i and p_i the restrictions of the functions Q and p to $(0,l_i]$. We also assume that at the node, along the Oy axis, there is a limiter on the displacement of the strings, represented by a segment $[-m,m]$. Thus, we have the condition $|u(0)| \le m$. Depending on the applied external force, the nodal point of the string system either remains inside the interval $(-m,m)$, or touches the boundaries of the limiter. Let us describe this situation in the form of a general model.

According to [12], the potential energy functional for the Stieltjes strings system has the form

$$\Phi(u) = \int_\Gamma \frac{pu'^2}{2}\,dx + \int_\Gamma \frac{u^2}{2}\,dQ - \int_\Gamma u\,dF. \tag{3.1}$$

We suppose that functions p, Q, F satisfy the conditions:
(i) functions p and F have bounded variation on each edge, and $\inf_{R(\Gamma)} p > 0$;
(ii) function Q_i does not decrease on each interval $(0,l_i]$, where $i = 1,2,...,n$; $\Delta Q_i(l_i) = \gamma_i > 0$; and $\sum\limits_{i=1}^{n} (Q_i(0+0) - Q(0)) = 0$.
(iii) $\sum\limits_{i=1}^{n} (F_i(0+0) - F(0)) = f$, $\Delta F_i(l_i) = f_i$, where $i = 1,2,...,n$.

In (3.1), we understand the first integral, characterizing the work of the string elasticity force as the sum of the Lebesgue integrals over the edges; the second integral, defining the work of the elastic force of the external environment, is equal to the corresponding sums of the Stieltjes integrals over the edges plus integrals over the boundary vertices. Finally, we understand the third integral, defining the work of the external force, as the corresponding sums of the Stieltjes integrals over the edges, plus the integrals over the node and boundary vertices. According to the Lagrange-Hamilton principle, the real form of the strings system minimizes the functional $\Phi(u)$. We consider the case when the restriction

$$|u(0)| \leq m \tag{3.2}$$

holds. The functional $\Phi(u)$ with Condition (3.2) we consider on the set E of absolutely continuous on Γ functions $u(x)$, whose derivatives $u_i'(x)$ $(i = 1,2,...,n)$ are functions of bounded variation on each edge.

Let a function $u_0(x)$ minimize the functional $\Phi(u)$ with Condition (3.2). Then $\Phi(u_0) \leq \Phi(u)$ for all $u \in E$, satisfying (3.2). Consider functions $h \in E$ such that $h(a) = 0$, where $a \in \partial\Gamma$, $h(0) = 0$. Suppose $u(x) = u_0(x) + \lambda h(x)$. Notice that $u \in E$, $|u(0)| = |u_0(0)| \leq m$. Thus $\Phi(u_0) \leq \Phi(u_0 + \lambda h)$. Fixing h, we consider the function $\varphi_h(\lambda)$ of the real variable λ defined as $\varphi_h(\lambda) = \Phi(u_0 + \lambda h)$. Then, for all $\lambda \in R$, $\varphi_h(0) \leq \varphi_h(\lambda)$, and by Fermat's theorem, $\frac{d}{d\lambda}\varphi_h(\lambda)|_{\lambda=0} = 0$. The last equality can be rewritten as

$$\int_{R(\Gamma)} pu_0'h'\,dx + \int_{R(\Gamma)} u_0 h\,dQ + \sum_{a\in\partial\Gamma} u_0(a)h(a)\Delta Q(a) - \int_{R(\Gamma)} h dF - fh(0) - \sum_{a\in\partial\Gamma} h(a)\Delta F(a) = 0. \tag{3.3}$$

With respect to $h(0) = 0$ and $h(a) = 0$, where $a \in \partial\Gamma$, we obtain

$$\int_{R(\Gamma)} pu_0'h'\,dx + \int\limits_{R(\Gamma)} u_0 h\,dQ - \int_{R(\Gamma)} h\,dF = 0. \tag{3.4}$$

Denoting by p_i, Q_i, and F_i the restrictions of p, Q, and F to $(0,l_i]$, we redefine the functions p_i, Q_i and F_i at the points 0 and l_i by limit values. Denote by $g_i(x) = \int_0^x u_{0i}\,dQ_i, (i = 1,2,...,n)$. Let us consider $\int_{R(\Gamma)} u_0 h\,dQ$. We have

$$\int_{R(\Gamma)} u_0 h\,dQ = \sum_{i=1}^{n} \int_0^{l_i} h_i u_{0i}\,dQ_i = \sum_{i=1}^{n} (h_i g_i)|_0^{l_i} - \sum_{i=1}^{n} \int_0^{l_i} g_i\,dh_i.$$

Since $h_i(l_i) = h_i(0) = 0$, we obtain

$$\int_{R(\Gamma)} u_0 h\,dQ = -\sum_{i=1}^{n} \int_0^{l_i} g_i\,dh_i.$$

Similarly,

$$\int_{R(\Gamma)} h\,dF = -\sum_{i=1}^{n} \int_0^{l_i} F_i\,dh_i.$$

Then Equality (3.4) has the form

$$\sum_{i=1}^{n} \int_0^{l_i} (p_i u_{0i}' - \int_0^x u_{0i}\,dQ_i + F_i)dh_i = 0. \tag{3.5}$$

Equality (3.5) is true for all functions $h \in E$, satisfying the conditions $h_i(0) = h_i(l_i) = 0,\ i = 1,2,...,n$. Let us consider the functions h such that $h_1(0) = h_1(l_1) = 0$, $h_i(x) \equiv 0$, for $i \geq 2$. For such functions, (3.5) has the form

$$\int_0^{l_1} (p_1 u_{01}' - \int\limits_0^x u_{01} dQ_1 + F_1) dh_1 = 0. \tag{3.6}$$

Applying Lemma 2.1 to Equality (3.6), we obtain

$$(p_1 u_{01}')(x) - \int_0^x u_{01}\, dQ_1 + F_1(x) = c_1 = const,$$

which can be rewritten as

$$(p_1 u_{01}')(x) - \int_0^x u_{01}\, dQ_1 + F_1(x) = F_1(+0) + (p_1 u_1')(+0).$$

Similarly, we obtain that, for all numbers $i = 1, 2, \ldots, n$, the equalities

$$(p_i u_{0i}')(x) - \int_0^x u_{0i}\, dQ_i + F_i(x) = c_i = const \tag{3.7}$$

hold, or

$$-\left(p_i u_{0i}'\right)(x) + \int_0^x u_{0i}\, dQ_i = F_i(x) - F_i(+0) - (p_i u_{0i}')(+0), x \in \Upsilon_{i\sigma_i}, \quad i = 1, 2, \ldots, n. \tag{3.8}$$

We emphasize that in (3.8) the functions p_i, Q_i and F_i are extended to the segment $[0, l_i]$ by limit values. Thus we have the equations on each edge. Here we denote by $\sigma_i(x)$ the function

$$\sigma_i(x) = x + p_i^+(x) + p_i^-(x) + Q_i(x) + F_i^+(x) + F_i^-(x), \tag{3.9}$$

$p_i^+(x), p_i^-(x), F_i^+(x)$, and $F_i^-(x)$ are non-decreasing functions from the Jordan representation of bounded variation functions $p_i(x) = p_i^+(x) - p_i^-(x)$, $F_i(x) = F_i^+(x) - F_i^-(x)$; and $\Upsilon_{i\sigma_i} = \overline{[0, l_i]}_{\sigma_i}$. Returning to Equality (3.3), we consider the functions $h \in E$ such that $h(0) = 0$. Let us represent (3.3) as

$$\sum_{i=1}^n \int_{(0,l_i)} (p_i u_{0i}' - \int_0^x u_{0i}\, dQ_i + F_i)\, dh_i + \sum_{i=1}^n h_i(l_i) \int_{(0,l_i)} u_{0i}\, dQ_i - \sum_{i=1}^n h_i(l_i) F_i(l_i - 0) \tag{3.10}$$

$$+ \sum_{i=1}^n u_{0i}(l_i) h_i(l_i) \gamma_i - \sum_{i=1}^n h_i(l_i) f_i = 0.$$

Let us rewrite Equality (3.7) as

$$(p_i u_{0i}')(x) - \int_0^x u_{0i}\, dQ_i + F_i(x) = (p_i u_{0i}')(l_i - 0) - \int_{(0,l_i)} u_{0i}\, dQ_i + F_i(l_i - 0). \tag{3.11}$$

Substitute this representation to (3.10), we have

$$\sum_{i=1}^n \int_{(0,l_i)} ((p_i u_{0i}')(l_i - 0) - \int_{(0,l_i)} u_{0i}\, dQ_i + F_i(l_i - 0)) dh_i + \sum_{i=1}^n h_i(l_i) \int_{(0,l_i)} u_{0i}\, dQ_i -$$

$$- \sum_{i=1}^n h_i(l_i) F_i(l_i - 0) + \sum_{i=1}^n u_{0i}(l_i) h_i(l_i) \gamma_i - \sum_{i=1}^n h_i(l_i) f_i = 0.$$

Hence

$$\sum_{i=1}^n \left((p_i u_{0i}')(l_i - 0) + \gamma_i u_{0i}(l_i) - f_i\right) h_i(l_i) = 0,$$

and due to the arbitrariness of the values of $h_i(l_i)$, we have the equalities

$$(p_i u_{0i}')(l_i - 0) + \gamma_i u_{0i}(l_i) = f_i \tag{3.12}$$

for all $i = 1, 2, ..., n$.

Let us fix any number $c \in [-m, m]$. Consider functions $h \in E$ such that $h(0) = c - u_0(0)$. Functions of the form $u = u_0 + \lambda h$ belong to the class E. Consider the condition at the node. We have

$$u(0) = u_0(0) + \lambda h(0) = u_0(0) + \lambda(c - u_0(0)) = \lambda c + (1 - \lambda) u_0(0).$$

Since $c \in [-m, m]$, $u_0(0) \in [-m, m]$, and the segment $[-m, m]$ is a convex set, we have $u(0) \in [-m, m]$ for all $\lambda \in [0, 1]$. Hence, for $\lambda \in [0, 1]$, the inequality $\Phi(u_0) \leq \Phi(u_0 + \lambda h)$ holds. Fixing the function h indicated above, we introduce the function $\varphi_h(\lambda) = \Phi(u_0 + \lambda h)$, where $\lambda \in [0, 1]$. Then $\varphi_h(0) \leq \varphi_h(\lambda)$. Hence, for the right derivative, we have the inequality $\frac{d^+}{d\lambda} \varphi_h(\lambda)|_{\lambda=0} \geq 0$, that is,

$$\sum_{i=1}^{n} \int_{(0,l_i)} (p_i u_{0i}' - \int_0^x u_{0i}\, dQ_i + F_i)\, dh_i + \sum_{i=1}^{n} h_i(l_i) \int_{(0,l_i)} u_{0i}\, dQ_i -$$

$$- \sum_{i=1}^{n} h_i(l_i) F_i(l_i - 0) + \sum_{i=1}^{n} h_i(0) F_i(+0) + \sum_{i=1}^{n} u_{0i}(l_i) h_i(l_i) \gamma_i - \sum_{i=1}^{n} h_i(l_i) f_i - f h(0) \geq 0.$$

Due to Equality (3.11), we have

$$\sum_{i=1}^{n} \left((p_i u_{0i}')(l_i - 0) + \gamma_i u_{0i}(l_i) - f_i \right) h_i(l_i) -$$

$$- \sum_{i=1}^{n} \left((p_i u_{0i}')(l_i - 0) - \int_{(0,l_i)} u_{0i}\, dQ_i + F_i(l_i - 0) \right) h_i(0) + \sum_{i=1}^{n} h_i(0) F_i(+0) - f h(0) \geq 0.$$

It follows from Equalities (3.7) that

$$(p_i u_{0i}')(l_i - 0) - \int_{(0,l_i)} u_{0i}\, dQ_i + F_i(l_i - 0) = (p_i u_{0i}')(+0) + F_i(+0).$$

Thus, due to (3.12), we obtain $- \sum\limits_{i=1}^{n} (p_i u_{0i}')(+0) h_i(0) - f h(0) \geq 0$, Taking into account $h_i(0) = h(0) = c - u_0(0)$, we rewrite the last inequality as $-(\sum\limits_{i=1}^{n} (p_i u_{0i}')(+0) + f)(c - u_0(0)) \geq 0$, that is,

$$\sum_{i=1}^{n} (p_i u_{0i}')(+0) + f \in N_{[-m,m]}(u_0(0)). \tag{3.13}$$

Thus, the following theorem has been proved.

Theorem 3.1. *Let the function u_0 minimize the functional $\Phi(u)$ with Condition (3.2). Then $u_0(x)$ is a solution to the problem*

$$\begin{cases} -(p_i u_i')(x) + \int_0^x u_i\, dQ_i = F_i(x) - F_i(+0) - (p_i u_i')(+0), i = 1, 2, ..., n, \quad x \in \Upsilon_{i\sigma_i} \\ \sum\limits_{i=1}^{n} p_i(+0) u_i'(+0) + f \in N_{[-m,m]} u(0), \\ u_1(0) = u_2(0) = ... = u_n(0) = u(0), \\ (p_i u_i')(l_i - 0) + \gamma_i u_i(l_i) = f_i, i = 1, 2, ..., n. \end{cases} \tag{3.14}$$

It follows from the equation in (3.14) that, at each point $\xi_i \in \Upsilon_i$ where at least one of the functions p_i, Q_i and F_i is discontinuous, the equality

$$-p_i(\xi_i+0)u_i'(\xi_i+0)+p_i(\xi_i-0)u_i'(\xi_i-0)+u_i(\xi_i)\Delta Q_i(\xi_i)=\Delta F_i(\xi_i)$$

holds. The jump $\Delta Q_i(\xi_i)$ corresponds to the elasticity of the support (spring) fixed at the point ξ_i of the edge with number i; and the jump $\Delta F_i(\xi_i)$ is equal to the force concentrated at the point ξ_i. Notice that Problem (3.14) can be rewritten as

$$\begin{cases} -\dfrac{d}{d\Gamma}(pu')(x)+\dfrac{dQ}{d\Gamma}(x)u(x)=\dfrac{dF}{d\Gamma}(x), \quad x\in R_\sigma(\Gamma) \\ \dfrac{d(pu')}{d\Gamma}(0)+\dfrac{dF}{d\Gamma}(0)\in N_{[-m,m]}u(0), \end{cases} \tag{3.15}$$

where

$$\frac{d}{d\Gamma}(pu')(x)=\begin{cases} \dfrac{d}{d\sigma_i}(p_iu_i')(x), \quad x\neq 0 \\ \sum\limits_{i=1}^{n} p_i(+0)u_i'(+0), \quad x=0, \end{cases}$$

$$\frac{dQ}{d\Gamma}=\begin{cases} \dfrac{dQ_i}{d\sigma_i}(x), \quad x\neq 0 \\ 0, \quad x=0, \end{cases}$$

and

$$\frac{dF}{d\Gamma}=\begin{cases} \dfrac{dF_i}{d\sigma_i}(x), \quad x\neq 0 \\ f, \quad x=0. \end{cases}$$

The designation $\frac{d}{d\sigma_i}$ means differentiation with respect to σ_i-measure, generated on each interval $(0,l_i]$ by the corresponding increasing function $\sigma_i(x)$ defined by (3.9). If ξ_i is a discontinuity point of $\sigma_i(x)$, then

$$\frac{d}{d\sigma_i}(p_iu_i')(\xi_i)=\frac{(p_iu_i')(\xi_i+0)-(p_iu_i')(\xi_i-0)}{\sigma_i(\xi_i+0)-\sigma_i(\xi_i-0)}=\frac{\Delta(p_iu_i')(\xi_i)}{\Delta\sigma_i(\xi_i)};$$

$$\frac{dQ_i}{d\sigma_i}(\xi_i)=\frac{\Delta Q_i(\xi_i)}{\Delta\sigma_i(\xi_i)}; \quad \frac{dF_i}{d\sigma_i}(\xi_i)=\frac{\Delta F_i(\xi_i)}{\Delta\sigma_i(\xi_i)}.$$

If $\xi_i=l_i$, then

$$\frac{d}{d\sigma_i}(p_iu_i')(l_i)=\frac{-(p_iu_i')(l_i-0)}{\sigma_i(l_i)-\sigma_i(l_i-0)};$$

$$\frac{dQ_i}{d\sigma_i}(l_i)=\frac{Q_i(l_i)-Q_i(l_i-0)}{\sigma_i(l_i)-\sigma_i(l_i-0)}=\frac{\gamma_i}{\Delta\sigma_i(l_i)};$$

and

$$\frac{dF_i}{d\sigma_i}(l_i)=\frac{F_i(l_i)-F_i(l_i-0)}{\sigma_i(l_i)-\sigma_i(l_i-0)}=\frac{f_i}{\Delta\sigma_i(l_i)}.$$

Thus, the condition $(pu')(a-0)+u(a)\Delta Q(a)=\Delta F(a)$, where $a\in\partial\Gamma$, follows immediately from the equation. The solution u of Problem (3.15) belongs to E, so the continuity condition at the node is immediately included in the class of admissible solutions. The set $R_\sigma(\Gamma)$ is a formal union of the sets $\Upsilon_{i\sigma_i}$ $(i=1,2,...,n)$ with all possible discontinuity points p, Q, F, except for the node.

4. Main Results

In all results, it is assumed that conditions (i), (ii) and (iii) hold. Consider Problem (3.14). A solution to Problem (3.14) is a function $u \in E$, satisfying Equations (3.8) on the corresponding edges (for all $x \in \overline{[0,l_i]}_{\sigma_i}$), and satisfying Conditions (3.12), (3.13).

Lemma 4.1. *Let us fix an arbitrary number i. Any non-trivial solution of the equation*

$$-\left(p_i u_i'\right)(x) + \int_0^x u_i \, dQ_i = -(p_i u_i')(+0), \tag{4.1}$$

satisfying the condition

$$(p_i u_i')(l_i - 0) + \gamma_i u_i(l_i) = 0, \quad x \in \overline{[0,l_i]}_{\sigma_i} \tag{4.2}$$

does not have any zeros on the segment $[0,l_i]$.

Proof. Let $u_i(x)$ be a non-trivial solution to Equation (4.1). If $u_i(l_i) = 0$, then $u_i'(l_i - 0) = 0$ and $u_i(x) \equiv 0$. Assume $u_i(l_i) > 0$, and denote by ξ_i the nearest zero to l_i of the function $u_i(x)$. Then $u_i(x) > 0$, where $x \in (\xi_i, l_i]$. Thus $u_i'(\xi_i + 0) > 0$, and Identity (4.1) provides a strict positivity of the function $u_i'(x)$ everywhere to the right of ξ_i. Thus $p_i(l_i - 0)u_i'(l_i - 0) > 0$, but it contradicts (4.2). The case $u_i(l_i) < 0$ can be considered similarly. The lemma is proved. □

Theorem 4.1. *If a solution to Problem* (3.14) *exists, then it is unique.*

Proof. Let $v(x)$ and $w(x)$ be solutions to Problem (3.14). Consider the function $u(x) = w(x) - v(x)$, which satisfies the system

$$\begin{cases} -\left(p_i u_i'\right)(x) + \displaystyle\int_0^x u_i \, dQ_i = -(p_i u_i')(+0), \quad i = 1,2,\ldots,n, \\ (p_i u_i')(l_i - 0) + \gamma_i u_i(l_i) = 0, \quad i = 1,2,\ldots,n. \end{cases}$$

Suppose that for some number i the function $u_i(x)$ is non-zero. Then, according to Lemma 4.1, the function $u_i(x)$ preserves the sign on $[0,l_i]$. Assume that $u_i(x) > 0$ for all $x \in [0,l_i]$. Since $u_1(0) = \ldots = u_i(0) = \ldots = u_n(0) = u(0)$, we have $u_j(0) > 0$ for all numbers $j = 1,2,\ldots,n$. It follows from Lemma 4.1 that $u_j(x) > 0$, where $x \in [0,l_j]$ $(j = 1,2,\ldots,n)$. Since $\gamma_i > 0$, we have $u_j'(l_j - 0) < 0$. At the same time, $-\left(p_j u_j'\right)(x) = \int_x^{l_j} u_j \, dQ_j - (p_j u_j')(l_j - 0)$. Hence $(p_j u_j')(x) < 0$. Thus the inequality $(p_j u_j')(+0) < 0$ holds for all numbers $j = 1,2,\ldots,n$. On the other hand, since

$$\sum_{i=1}^n (p_i w_i')(+0) + f \in N_{[-m,m]}(w(0)), \; \sum_{i=1}^n (p_i v_i')(+0) + f \in N_{[-m,m]}(v(0)),$$

the inequalities

$$(\sum_{i=1}^n (p_i w_i')(+0) + f)(c - w(0)) \leq 0, \; (\sum_{i=1}^n (p_i v_i')(+0) + f)(c - v(0)) \leq 0$$

hold for all $c \in [-m,m]$. Taking $c = v(0)$ in the first inequality and $c = w(0)$ in the second inequality, we have

$$(\sum_{i=1}^n (p_i w_i')(+0) + f)(v(0) - w(0)) \leq 0,$$

and

$$-\Big(\sum_{i=1}^{n}(p_i v_i')(+0)+f\Big)(v(0)-w(0))\leq 0.$$

Adding the last two inequalities, we obtain

$$\sum_{i=1}^{n}((p_i w_i')(+0)-(p_i v_i')(+0))(v(0)-w(0))\leq 0.$$

Thus $\sum_{i=1}^{n}(p_i u_i')(+0)u(0)\geq 0$, but it contradicts the inequalities $(p_i u_i')(+0)<0$ $(i=1,2,...,n)$, $u(0)>0$. Similarly, the case $u_i(x)<0$ on $[0,l_i]$ is not possible. So $u(x)\equiv 0$. The theorem is proved. □

Theorem 4.2. *Let the functions $\varphi_1^i(x)$ and $\varphi_2^i(x)$ be solutions of the equation*

$$-(p_i u_i')(x)+(p_i u_i')(+0)+\int_0^x u_i\,dQ_i=0 \tag{4.3}$$

and satisfy the conditions $\varphi_1^i(0)=1$, $(p_i\varphi_1^{i'})(l_i-0)+\gamma_i\varphi_1^i(l_i)=0$; and $\varphi_2^i(0)=0$, $(p_i\varphi_2^{i'})(l_i-0)+\gamma_i\varphi_2^i(l_i)=1$, where $i=1,2,...,n$. Then if $\left|\dfrac{f+\sum_{j=1}^{n}\int_0^{l_j}\varphi_1^j(s)\,dF_j(s)}{\sum_{j=1}^{n}p_j(+0)\varphi_1^{j'}(+0)}\right|<m$, then the solution to Problem (3.14) has the form

$$u_i(x)=-\frac{f\varphi_1^i(x)}{\sum_{j=1}^{n}p_j(+0)\varphi_1^{j'}(+0)}+\frac{\varphi_1^i(x)}{p_i(+0)\varphi_2^{i'}(+0)}\int_0^x\varphi_2^i(s)\,dF_i(s)$$
$$+\frac{\varphi_2^i(x)}{p_i(+0)\varphi_2^{i'}(+0)}\int_x^{l_i}\varphi_1^i(s)\,dF_i(s)-\frac{\varphi_1^i(x)}{\sum_{j=1}^{n}p_j(+0)\varphi_1^{j'}(+0)}\left(\sum_{j=1}^{n}\int_0^{l_j}\varphi_1^j(s)\,dF_j(s)\right). \tag{4.4}$$

If $m+\dfrac{\sum_{j=1}^{n}\int_0^{l_j}\varphi_1^j(s)\,dF_j(s)}{\sum_{j=1}^{n}p_j(+0)\varphi_1^{j'}(+0)}+\dfrac{f}{\sum_{j=1}^{n}p_j(+0)\varphi_1^{j'}(+0)}\leq 0$, then the solution to Problem (3.14) has the form

$$u_i(x)=m\varphi_1^i(x)+\frac{\varphi_1^i(x)}{p_i(+0)\varphi_2^{i'}(+0)}\int_0^x\varphi_2^i(s)\,dF_i(s)+\frac{\varphi_2^i(x)}{p_i(+0)\varphi_2^{i'}(+0)}\int_x^{l_i}\varphi_1^i(s)\,dF_i(s). \tag{4.5}$$

If $m-\dfrac{\sum_{j=1}^{n}\int_0^{l_j}\varphi_1^j(s)\,dF_j(s)}{\sum_{j=1}^{n}p_j(+0)\varphi_1^{j'}(+0)}-\dfrac{f}{\sum_{j=1}^{n}p_j(+0)\varphi_1^{j'}(+0)}\leq 0$, then the solution to Problem (3.14) has the form

$$u_i(x)=-m\varphi_1^i(x)+\frac{\varphi_1^i(x)}{p_i(+0)\varphi_2^{i'}(+0)}\int_0^x\varphi_2^i(s)\,dF_i(s)+\frac{\varphi_2^i(x)}{p_i(+0)\varphi_2^{i'}(+0)}\int_x^{l_i}\varphi_1^i(s)\,dF_i(s),$$

where $i = 1,2,...,n$.

Proof. Fixing any number $i = 1,2,...,n$, we notice that the problem

$$\begin{cases} -(p_i\varphi_1^{i'})(x) + \int_0^x \varphi_1^i\, dQ_i = -(p_i\varphi_1^{i'})(+0), \\ \varphi_1^i(0) = 1, \\ (p_i\varphi_1^{i'})(l_i - 0) + \gamma_i\varphi_1^i(l_i) = 0 \end{cases}$$

has a unique solution. Indeed, applying Theorem 2.1 and Lemma 2.2, we obtain $\varphi_1^i(x) = c_1^i u_1^i(x) + c_2^i u_2^i(x)$, where functions $u_1^i(x)$ and $u_2^i(x)$ are solutions of the homogeneous equation (4.3) such that $u_1^i(0) = 0$, $u_1^{i'}(+0) = 1$ and $u_2^i(0) = 1$, $u_2^{i'}(+0) = 0$. Since the function $Q_i(x)$ does not decrease on $(0,l_i]$ and $u_1^i(0) = 0$, then, according to Lemma 2.4, $u_1^i(x)$ does not have zeros on $(0,l_i]$. From the condition $u_1^{i'}(+0) = 1$, it follows that $u_1^i(x) > 0$ for all $x \in (0,l_i]$, and in particular, $u_1^i(l_i) > 0$, $(p_iu_1^{i'})(l_i - 0) > 0$. Substituting the representation for $\varphi_1^i(x)$ into the boundary conditions, we obtain $c_2^i = 1$ and $c_1^i = \frac{-u_2^i(l_i)\gamma_i - (p_iu_2^{i'})(l_i-0)}{u_1^i(l_i)\gamma_i + (p_iu_1^{i'})(l_i-0)}$. Similarly, there is a solution to the problem

$$\begin{cases} -(p_i\varphi_2^{i'})(x) + \int_0^x \varphi_2^i\, dQ_i = -(p_i\varphi_2^{i'})(+0), \\ \varphi_2^i(0) = 0, \\ (p_i\varphi_2^{i'})(l_i - 0) + \varphi_2^i(l_i)\gamma_i = 1. \end{cases}$$

Let us demonstrate that $\varphi_1^{i'}(+0) < 0$. Since $\varphi_1^i(0) = 1$, we obtain $\varphi_1^i(x) > 0$ for all $x \in [0,l_i]$. Hence $(p_i\varphi_1^{i'})(l_i - 0) < 0$ and $(p_i\varphi_1^{i'})(+0) = -\int_{(0,l_i)} \varphi_1^i\, dQ_i + (p_i\varphi_1^{i'})(l_i - 0) < 0$. Since $\varphi_2^i(0) = 0$, $\varphi_2^i(x)$ preserves sign on $(0,l_i]$. Assume that $\varphi_2^i(x) < 0$, where $x \in (0,l_i]$. We obtain $\varphi_2^{i'}(+0) < 0$. Hence $(p_i\varphi_2^{i'})(x) < 0$, and in particular $(p_i\varphi_2^{i'})(l_i - 0) < 0$, but it contradicts the condition $(p_i\varphi_2^{i'})(l_i - 0) + \varphi_2^i(l_i)\Delta Q_i(l_i) = 1$. Thus $\varphi_2^i(x) > 0$ and $\varphi_2^{i'}(+0) > 0$. Assume that

$$\left| \frac{f + \sum_{j=1}^{n} \int_0^{l_j} \varphi_1^j(s)\, dF_j(s)}{\sum_{j=1}^{n} p_j(+0)\varphi_1^{j'}(+0)} \right| < m.$$

Let us prove that the functions $u_i(x)$ defined by Equality (4.4) constitute a solution to Problem (3.14). Notice that

$$u_i(+0) = u_i(0) = u(0) = -\frac{f}{\sum_{j=1}^{n} p_j(+0)\varphi_1^{j'}(+0)} - \frac{\sum_{j=1}^{n} \int_0^{l_j} \varphi_1^j(s)\, dF_j(s)}{\sum_{j=1}^{n} p_j(+0)\varphi_1^{j'}(+0)}$$

for all $i = 1,2,...,n$. Let us fix any number $i = 1,2,...,n$. Since the functions $\varphi_1^i(x)$ and $\varphi_2^i(x)$ are absolutely continuous, and, for any $\alpha \le \beta$, the equality

$$u_i(\beta) - u_i(\alpha) = \frac{1}{p_i(+0)\varphi_2^{i'}(+0)}(\varphi_1^i(\beta) - \varphi_1^i(\alpha)) \int_0^\beta \varphi_2^i(s)\, dF_i(s) +$$

$$+\frac{1}{p_i(+0)\varphi_2^{i'}(+0)}(\varphi_2^i(\beta)-\varphi_2^i(\alpha))\int_\beta^{l_i}\varphi_1^i(s)\,dF_i(s)+$$

$$+\frac{1}{p_i(+0)\varphi_2^{i'}(+0)}\int_\alpha^\beta((\varphi_1^i(\alpha)-\varphi_1^i(s))\varphi_2^i(s)+(\varphi_2^i(s)-\varphi_2^i(\alpha))\varphi_1^i(s))\,dF_i(s)-$$

$$-\frac{(\varphi_1^i(\beta)-\varphi_1^i(\alpha))\sum\limits_{j=1}^n\int_0^{l_j}\varphi_1^j(s)\,dF_j(s)}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)}-\frac{f(\varphi_1^i(\beta)-\varphi_1^i(\alpha))}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)}$$

holds, the function $u_i(x)$ is absolutely continuous on $[0,l_i]$. Let us show that the derivative $u_i'(x)$ of the function $u_i(x)$ satisfies the equality

$$u_i'(x)=\frac{\varphi_1^{i'}(x)}{p_i(+0)\varphi_2^{i'}(+0)}\int_0^x\varphi_2^i(s)\,dF_i(s)+\frac{\varphi_2^{i'}(x)}{p_i(+0)\varphi_2^{i'}(+0)}\int_x^{l_i}\varphi_1^i(s)\,dF_i(s)- \tag{4.6}$$

$$-\frac{\varphi_1^{i'}(x)}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)}\left(\sum_{j=1}^n\int_0^{l_j}\varphi_1^j(s)\,dF_j(s)\right)-\frac{f\varphi_1^{i'}(x)}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)}.$$

Denote by $\Delta_\varepsilon u_i=u_i(x+\varepsilon)-u_i(x+0)$, where $\varepsilon>0$. Let us prove the assertion for the right derivative (for the left derivative the proof is similar). We have

$$\frac{\Delta_\varepsilon u_i}{\varepsilon}=\frac{1}{p_i(+0)\varphi_2^{i'}(+0)}\frac{\Delta_\varepsilon\varphi_1^i}{\varepsilon}\int_0^{x+\varepsilon}\varphi_2^i\,dF_i+\frac{1}{p_i(+0)\varphi_2^{i'}(+0)}\frac{\Delta_\varepsilon\varphi_2^i}{\varepsilon}\int_{x+\varepsilon}^{l_i}\varphi_1^i\,dF_i+$$

$$+\frac{1}{p_i(+0)\varphi_2^{i'}(+0)}\int_{x+0}^{x+\varepsilon}\frac{\varphi_1^i(x)\varphi_2^i(s)-\varphi_2^i(x)\varphi_1^i(s)}{\varepsilon}\,dF_i(s)-$$

$$-\frac{1}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)}\frac{\Delta_\varepsilon\varphi_1^i}{\varepsilon}\left(\sum_{j=1}^n\int_0^{l_j}\varphi_1^j(s)\,dF_j(s)\right)-\frac{f}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)}\frac{\Delta_\varepsilon\varphi_1^i}{\varepsilon}.$$

Let us show that

$$\lim_{\varepsilon\to0+}\left(\int_{x+0}^{x+\varepsilon}\frac{\varphi_1^i(x)\varphi_2^i(s)-\varphi_2^i(x)\varphi_1^i(s)}{\varepsilon}\,dF_i(s)\right)=0. \tag{4.7}$$

We have

$$\begin{aligned}&\left|\frac{1}{\varepsilon}\int_{x+0}^{x+\varepsilon}\left(\varphi_1^i(x)\varphi_2^i(s)-\varphi_2^i(x)\varphi_1^i(s)\right)\,dF_i(s)\right|\\&\le\frac{1}{\varepsilon}\left(\max_{x\le s\le x+\varepsilon}|\varphi_1^i(x)\varphi_2^i(s)-\varphi_2^i(x)\varphi_1^i(s)|\right)V_{x+0}^{x+\varepsilon}(F_i),\end{aligned}$$

where $V_{x+0}^{x+\varepsilon}(F_i)$ denotes the variation of the function F_i on $[x+0,x+\varepsilon]$. Notice that

$$\begin{aligned}|\varphi_1^i(x)\varphi_2^i(s)-\varphi_2^i(x)\varphi_1^i(s)|&\le\|\varphi_1^i\|\cdot|\varphi_2^i(s)-\varphi_2^i(x)|+\|\varphi_2^i\|\cdot|\varphi_1^i(x)-\varphi_1^i(s)|\\&\le\|\varphi_1^i\|\cdot\left|\int_x^s|\varphi_2^{i'}(\tau)|\,d\tau\right|+\|\varphi_2^i\|\cdot\left|\int_x^s|\varphi_1^{i'}(\tau)|\,d\tau\right|,\end{aligned}$$

where $\|\varphi_j^i\| = \max_{[0,l_i]} |\varphi_j^i(x)|$, $j = 1,2$. Since the functions φ_1^i, φ_2^i are absolutely continuous on $[0,l_i]$, and their derivatives have bounded variations, then $|\varphi_2^{i'}(\tau)| \le c_{0i}$ and $|\varphi_1^{i'}(\tau)| \le c_{0i}$. Thus

$$|\varphi_1^i(x)\varphi_2^i(s) - \varphi_2^i(x)\varphi_1^i(s)| \le (\|\varphi_1^i\| + \|\varphi_2^i\|)c_{0i}\varepsilon.$$

Hence

$$\frac{1}{\varepsilon} \max_{x \le s \le x+\varepsilon} |\varphi_1^i(x)\varphi_2^i(s) - \varphi_2^i(x)\varphi_1^i(s)| \le (\|\varphi_1^i\| + \|\varphi_2^i\|)c_{0i}.$$

Since $V_{x+0}^{x+\varepsilon}(F_i) \to 0$ when $\varepsilon \to +0$, we obtain Equality (4.7). Thus, Equality (4.6) is proved. From (4.6), it follows that u_i' has bounded variation on $(0,l_i)$. Thus, $u(x)$ belongs to the class E. Since $|u(0)| < m$, we show that $\sum\limits_{i=1}^n p_i(+0)u_i'(+0) = -f$. We have

$$\sum_{i=1}^n p_i(+0)u_i'(+0) + f = \sum_{i=1}^n \int_0^{l_i} \varphi_1^i(s)\,dF_i(s) - \frac{\sum\limits_{i=1}^n p_i(+0)\varphi_1^{i'}(+0)}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)} \left(\sum_{j=1}^n \int_0^{l_j} \varphi_1^j(s)\,dF_j(s) \right)$$
$$- \frac{f\sum\limits_{i=1}^n p_i(+0)\varphi_1^{i'}(+0)}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)} + f = 0.$$

Fixing an arbitrary $i = 1,2,\dots,n$, we next prove that $u_i(x)$ is a solution to the equation in (3.14). Notice that

$$\int_0^x u_i\,dQ_i = \frac{1}{p_i(+0)\varphi_2^{i'}(+0)} \left((p_i\varphi_1^{i'})(x) \int_0^x \varphi_2^i(\tau)\,dF_i(\tau) + (p_i\varphi_2^{i'})(x) \int_x^{l_i} \varphi_1^i(\tau)\,dF_i(\tau) \right)$$
$$- \int_0^{l_i} \varphi_1^i(\tau)\,dF_i(\tau) + F_i(x) - F_i(+0)$$
$$- \frac{\sum\limits_{j=1}^n \int_0^{l_j} \varphi_1^j(s)\,dF_j(s)}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)} \int_0^x \varphi_1^i(s)\,dQ_i(s) - \frac{f}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)} \int_0^x \varphi_1^i(s)\,dQ_i(s).$$

Here we have used the Fubini theorem and the properties of the functions φ_1^i, φ_2^i, as well as Theorem 2.2, replacing $p_i(\tau)(\varphi_1^i(\tau)\varphi_2^{i'}(\tau) - \varphi_2^i(\tau)\varphi_1^{i'}(\tau)) \equiv const = p_i(+0)\varphi_2^{i'}(+0)$. Hence

$$-p_i(x)u_i'(x) + \int_0^x u_i\,dQ_i = F_i(x) - F_i(+0) - \int_0^{l_i} \varphi_1^i(\tau)\,dF_i(\tau)$$
$$+ \frac{p_i(+0)\varphi_1^{i'}(+0)}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)} \left(\sum_{j=1}^n \int_0^{l_j} \varphi_1^j(s)\,dF_j(s) \right) + \frac{f p_i(+0)\varphi_1^{i'}(+0)}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)}$$
$$= F_i(x) - F_i(+0) - p_i(+0)u_i'(+0).$$

Let us consider the conditions at the boundary vertices. Using Theorem 2.2 and the conditions on the functions φ_1^i, φ_2^i, we have the equality $p_i(+0)\varphi_2^{i'}(+0) = \varphi_1^i(l_i)$. Thus we obtain

$$p_i(l_i-0)u_i'(l_i-0)+\gamma_i u_i(l_i) = \frac{\varphi_1^i(l_i)f_i}{p_i(+0)\varphi_2^{i'}(+0)} = f_i.$$

Let

$$m + \frac{\sum\limits_{j=1}^{n}\int_0^{l_j}\varphi_1^j(s)\,dF_j(s)}{\sum\limits_{j=1}^{n} p_j(+0)\varphi_1^{j'}(+0)} + \frac{f}{\sum\limits_{j=1}^{n} p_j(+0)\varphi_1^{j'}(+0)} \leq 0.$$

Similarly to the first case, one can prove that the functions defined by Equality (4.5) are a solution to Problem (3.14). Here the representation for the derivative has the form

$$u_i'(x) = m\varphi_1^{i'}(x) + \frac{\varphi_1^{i'}(x)}{p_i(+0)\varphi_2^{i'}(+0)}\int_0^x \varphi_2^i(s)\,dF_i(s) + \frac{\varphi_2^{i'}(x)}{p_i(+0)\varphi_2^{i'}(+0)}\int_x^{l_i}\varphi_1^i(s)\,dF_i(s).$$

Notice that $u_i(0) = u(0) = m$, i=1,2,...,n. Let us show that $\sum\limits_{i=1}^{n} p_i(+0)u_i'(+0)+f \in N_{[-m,m]}u(0)$. Since $u(0) = m$, we need to prove $\sum\limits_{i=1}^{n} p_i(+0)u_i'(+0)+f \geq 0$. Since $\sum\limits_{i=1}^{n} p_i(+0)\varphi_1^{i'}(+0) < 0$, we have

$$\begin{aligned}\sum_{i=1}^{n} p_i(+0)u_i'(+0)+f &= m\sum_{i=1}^{n} p_i(+0)\varphi_1^{i'}(+0) + \sum_{i=1}^{n}\int_0^{l_i}\varphi_1^i(s)\,dF_i(s)+f\\ &= \sum_{i=1}^{n} p_i(+0)\varphi_1^{i'}(+0)\left(m + \frac{f}{\sum\limits_{j=1}^{n} p_j(+0)\varphi_1^{j'}(+0)} + \frac{\sum\limits_{j=1}^{n}\int_0^{l_j}\varphi_1^j(s)\,dF_j(s)}{\sum\limits_{j=1}^{n} p_j(+0)\varphi_1^{j'}(+0)}\right)\\ &\geq 0.\end{aligned}$$

The case

$$m - \frac{\sum\limits_{j=1}^{n}\int_0^{l_j}\varphi_1^j(s)\,dF_j(s)}{\sum\limits_{j=1}^{n} p_j(+0)\varphi_1^{j'}(+0)} - \frac{f}{\sum\limits_{j=1}^{n} p_j(+0)\varphi_1^{j'}(+0)} \leq 0,$$

can be considered similarly. The theorem is proved. □

Theorem 4.3. *Let $u_0(x)$ be the solution to Problem (3.14). Then u_0 minimizes $\Phi(u)$ with respect to the condition $|u(0)| \leq m$.*

Proof. Let us prove that, for any function $u \in E$ satisfying the condition $|u(0)| \leq m$, $\Phi(u) - \Phi(u_0) \geq 0$. We represent the function $u(x)$ as $u(x) = u_0(x) + h(x)$, where $h(x) = u(x) - u_0(x)$.

Notice that $h \in E$. We have

$$\begin{aligned}\Phi(u_0+h)-\Phi(u_0) &= \int\limits_\Gamma (pu_0')h'dx + \int\limits_\Gamma \frac{ph'^2}{2}dx - \int\limits_\Gamma hdF + \int\limits_\Gamma \frac{h^2}{2}dQ + \int\limits_\Gamma hu_0dQ \\ &= -(\sum_{i=1}^n (p_iu_{0i}')(+0)+f)h(0) + \int\limits_\Gamma \frac{h^2}{2}dQ + \int\limits_\Gamma \frac{ph'^2}{2}dx \geq 0.\end{aligned}$$

Since $h(0) = u(0)-u_0(0)$ and $u(0) \in [-m,m]$, we have $-(\sum\limits_{i=1}^n (p_iu_{0i}')(+0)+f)h(0) \geq 0$. The theorem is proved. $\square$

Theorem 4.4. *If $m \to 0$, then the solution $u_m(x)$ to Problem (3.14) tends to the solution to the problem*

$$\begin{cases} -(p_iu_i')(x) + \int\limits_0^x u_idQ_i = F_i(x) - F_i(+0) - (p_iu_i')(+0), i = 1,2,...,n, \\ u_i(0) = 0, \\ (p_iu_i')(l_i-0) + \gamma_iu_i(l_i) = f_i, i = 1,2,...,n \end{cases} \tag{4.8}$$

uniformly on Γ.

Proof. Let us use the formulas from Theorem 4.2 to represent the solution $u_m(x)$ of Problem (3.14). Since $m \to 0$, we have $\left| \dfrac{-f - \sum\limits_{j=1}^n \int\limits_0^{l_j} \varphi_1^j(s)dF_j(s)}{\sum\limits_{j=1}^n p_j(+0)\varphi_1^{j'}(+0)} \right| \geq m$. Since the function $\varphi_1^i(x)$ is bounded on $[0,l_i]$ we have

$$\begin{aligned}&\left| u_{im}(x) - \frac{\varphi_1^i(x)}{p_i(+0)\varphi_2^{i'}(+0)} \int\limits_0^x \varphi_2^i(s)dF_i(s) + \frac{\varphi_2^i(x)}{p_i(+0)\varphi_2^{i'}(+0)} \int\limits_x^{l_i} \varphi_1^i(s)dF_i(s) \right| \\ &= \left| m\varphi_1^i(x) \right| \leq c_i|m| \to 0\end{aligned}$$

for all i=1,2,...,n. Thus,

$$u_{im}(x) \rightrightarrows u_i(x) = \frac{\varphi_1^i(x)}{p_i(+0)\varphi_2^{i'}(+0)} \int\limits_0^x \varphi_2^i(s)dF_i(s) + \frac{\varphi_2^i(x)}{p_i(+0)\varphi_2^{i'}(+0)} \int\limits_x^{l_i} \varphi_1^i(s)dF_i(s).$$

Similarly to Theorem 4.2, we can obtain that the functions $u_i(x)$ is the solution to Problem (4.8). The theorem is proved. $\square$

Theorem 4.5. *If $m \to +\infty$, then the solution $u_m(x)$ to Problem (3.14) tends to the solution to the problem*

$$\begin{cases} -(p_iu_i')(x) + \int_0^x u_idQ_i = F_i(x) - F_i(+0) - (p_iu_i')(+0), i = 1,2,...,n, \\ \sum\limits_{i=1}^n p_i(+0)u_i'(+0) = -f, \\ u_1(0) = u_2(0) = ... = u_n(0) = u(0), \\ (p_iu_i')(l_i-0) + \gamma_iu_i(l_i) = f_i, i = 1,2,...,n \end{cases} \tag{4.9}$$

uniformly on Γ.

Proof. Let us use the formulas from Theorem 4.2 to represent the solution $u_m(x)$ of Problem (3.14). Since $m \to +\infty$, we have $\left| \dfrac{-f - \sum\limits_{j=1}^{n} \int\limits_{0}^{l_j} \varphi_1^j(s) dF_j(s)}{\sum\limits_{j=1}^{n} p_j(+0)\varphi_1^{j'}(+0)} \right| < m$ for all numbers $m \geq m_0$. Denote by $u^*(x)$ the solution to Problem (4.9). Then for all $m \geq m_0$ we have $|u_{im}(x) - u_i^*(x)| = 0$, $i = 1, 2, ..., n$. The theorem is proved. $\square$

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