J. Nonlinear Var. Anal. 7 (2023), No. 3, pp. 345-365
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.7.2023.3.02

EXISTENCE OF CONSTANT SIGN AND NODAL SOLUTIONS FOR A CLASS OF (p,q)-LAPLACIAN-KIRCHHOFF PROBLEMS

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Abstract. This paper is dedicated to studying a (p,q)-Laplacian-Kirchhoff type equation. We prove the existence of three bounded solutions (one positive, one negative, and one nodal with precisely two nodal domains) by applying the Nehari manifold along with a quantitative deformation lemma and truncation technique.
Keywords. (p,q)-Laplacian; Kirchhoff type equation; Nodal solution; Nehari manifold.

1. Introduction and Statement of Results

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$. In this paper, we study the existence of constant sign and nodal solutions to the following (p,q)-Laplacian-Kirchhoff type equation

$$\begin{cases} -\left(1+a\int_\Omega |\nabla u|^p dx\right)\Delta_p u - \left(1+b\int_\Omega |\nabla u|^q dx\right)\Delta_q u + V(x)(u^{p-1}+u^{q-1}) = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a,\ b>0$, $1<p<q<N<2q<q^*$, $q^*=\frac{Nq}{N-q}$ is the critical Sobolev exponent, and $\Delta_t u = \operatorname{div}(|\nabla u|^{t-2}\nabla u)$ with $t\in\{p,q\}$ denotes the usual t-Laplace operator.

Throughout the paper, we require the following conditions:

(A_1) $V\in\mathscr{C}(\Omega,\mathbb{R})$ and there exists $V_0>0$ such that $V_0=\inf_{x\in\Omega}V(x)$;

(A_2) $f\in\mathscr{C}^1(\mathbb{R},\mathbb{R})$ and there exists a constant $r\in(2q,q^*)$ such that $\lim_{|t|\to+\infty}\frac{f(t)}{|t|^{r-1}}=0$;

(A_3) $\lim\limits_{t\to\pm\infty}\frac{f(t)}{|t|^{2q-2}t}=+\infty$;

(A_4) $\lim\limits_{t\to 0}\frac{f(t)}{|t|^{p-2}t}=0$;

(A_5) $\frac{f(t)}{|t|^{2q-1}}$ is a strictly increasing function on $(-\infty,0)$ and on $(0,+\infty)$;

If $a=b=0$ and $V(x)=0$, equation (1.1) is reduced to a (p,q)-Laplacian equation

$$\begin{cases} -\Delta_p u - \Delta_q u = f(x,u), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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Received May 20, 2022; Accepted May 5, 2023.

which appears as a steady state of a general reaction-diffusion system

$$u_t = \text{div}\,[K(u)\nabla u] + h(x,u), \tag{1.3}$$

where $K(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. It appears not only in the field of physics, but also in the fields of biophysics, plasma physics, and chemical reaction design. In most instances, function u describes a concentration and the first term on the right of (1.3) corresponds to the diffusion with diffusion coefficient $K(u)$, while the second term on the right of (1.3) relates to the source and loss process. Usually, in these types of applications, $h(x,u)$ is a polynomial of u with variable coefficients; see, e.g., [1] and the references therein for further details on the applications. Mugnai and Papageorgiou in [2] demonstrated that the superlinear (p,q)-Laplacian equations (1.2) without the Ambrosetti-Rabinowitz condition has at least three nontrivial solutions. When weight V is continuous, positive, and coercive, an existence result on problem (1.2) in $\mathbb{R}^N$ was obtained in [3]. For recent results on (p,q)-Laplacian equations and problem (1.2), we refer to [4, 5, 6, 7, 8] and the references therein.

As we know, when $a = b = 0$ and $p = q$, equation (1.1) turns out to be a p-Schrödinger equation of the form

$$-\Delta_p u + V(x)u^{p-1} = f(x,u), \tag{1.4}$$

which was studied both in bounded domain [9, 10] and in $\mathbb{R}^N$ [11, 12, 13].

When $p = q = 2$, equation (1.4) is reduced to a well-known Schrödinger equation, which was wildly studied and some recent results on problem (1.4) were obtained in [14, 15]. When $a = b \neq 0,\ p = q > 1$, we obtain the following Kichhoff type equation involving p-Laplace operator

$$\begin{cases} -\left(1 + a\int_\Omega |\nabla u|^p dx\right)\Delta_p u + V(x)u^{p-1} = f(x,u), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

It is widely known that the variational problems involving nonlocal operators are much more difficult and challenging. In the last decade, much attention was focused on the non-local operators; see, e.g., [16, 17, 18, 19, 20, 21] and the references therein. For instance, combining the constraint variational method and the quantitative deformation lemma, Rasouli, Fani, and Khademloo [21] proved that problem (1.5) possesses one sign-changing solution when $V(x) = 0$ and $f(x,u) = f(u)$. The existence result of multiple solutions and sign-changing ground state solutions for the above problem was obtained in [20] and [22], respectively. In [19], by the variational methods, penalization techniques and Lyusternik-Schnirelmann theory, Jia and Li [19] proved the existence, multiplicity, and concentration of solutions for the equation (1.5) in $\mathbb{R}^N$.

In this paper, we investigate multiple solutions for problem (1.1) in a more general case that $p \neq q$. We emphasize that we are considering the sum of two nonlocal operators $\left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)\Delta_p u$ and $\left(\int_{\mathbb{R}^N} |\nabla u|^q dx\right)\Delta_p u$ with $p < q$. The appearance of the nonlocal operators has caused some mathematical difficulties, which makes the research on this type of problem particularly interesting. In the non-local problems, only a few recent works deal with (p,q)-Laplacian. Although problem (1.1) has a variational structure, the main difficulty in the application of classical variational tools is the lack of homogeneity for (p,q)-Laplace operators. Our results encompass and improve the corresponding results presented in [21, 22] to the (p,q)-Laplacian Kirchhoff type equations.

Inspired by the facts above and [23, 24], we here demonstrate the existence of three solutions to problem (1.1) by applying the Nehari manifold along with truncation and comparison techniques, and critical point theory, which presents the novelty of the research on problem (1.1). As far as we know, there are few results on the (p,q)-Laplacian Kirchhoff type equations [23, 25], but there are no results on multiple solutions to problem (1.1). So this work may be the first result in this direction.

Throughout the paper, we assume that

$$1 < p < q < N, \quad \frac{Nq}{N+q} < p. \tag{1.6}$$

It is easy to see that the second inequality in (1.6) implies $q < p^*$. We now state our main results.

Theorem 1.1. *Assume that conditions (A_1)-(A_5) and (1.6) hold. Then problem (1.1) admits at least three nontrivial solutions u_+, u_-, w, where u_+ is positive, u_- is negative, and w is nodal with two nodal domains.*

Theorem 1.2. *Assume that conditions (A_1)-(A_5) and (1.6) hold. Then problem (1.1) admits a solution u^* such that $I(u^*) = inf_{\mathcal{N}} I = c$. Moreover, $I(w) > 2c$, where w is nodal solution obtained in Theorem 1.1.*

Remark 1.1. When $p = q$ and $V(x) = 0$, by Theorem 1.1, we have the existence of constant sign and nodal solutions to the following problem

$$\begin{cases} -\left(1 + a\int_\Omega |\nabla u|^p dx\right)\Delta_p u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

In [21, 22], the authors only studied the existence of sign-changing ground state solutions to problem (1.7). From this point of view, Theorem 1.1 could be viewed as some extension and completeness of related results in [21, 22].

Theorem 1.3. *Under the assumptions of Theorem 1.1, each solution to problem (1.1) is bounded.*

Remark 1.2. In order to obtain the boundedness of solutions to problem (1.1), we use a variant of the Moser iteration argument [26].

The rest of this paper is organized as follows. In Section 2, we present some notation and technical lemmas. We prove Theorem 1.1 and Theorem 1.2 in Section 3. Section 4 is devoted to the proof of Theorem 1.3.

2. Preliminaries

To prove our main results, we need the following notation and useful results. Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$ be a bounded domain and $1 < p < q < \infty$. For $1 \leq s \leq \infty$, we denote by $|\cdot|_s$ the norm in $L^s(\Omega)$. For $1 < t < \infty$ and $t < N$, under the condition (A_1), we consider the Sobolev space $W_0^{1,t}(\Omega)$ endowed with the norm $\|u\|_t^t = \int_\Omega (|\nabla u|^t + V(x)|u|^t)dx$. By condition (A_1), we obtain the continuous embeddings $W_0^{1,t}(\Omega) \hookrightarrow L^{\bar{t}}(\Omega)$ for $t \leq \bar{t} \leq t^*$, which implies there is a constant $c_{\bar{t}} > 0$ such that

$$|u|_{\bar{t}} \leq c_{\bar{t}}\|u\|_t, \quad \forall u \in W_0^{1,t}(\Omega), \tag{2.1}$$

where t^* denotes the critical Sobolev exponent of t, i.e., $t^* = Nt/(N-t)$. These embeddings are compact for $\bar{t} < t^*$. We define $X = W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$ equipped with the norm $\|u\|_X = \|u\|_p + \|u\|_q$.

The energy functional associated with problem (1.1) is given by

$$I(u) = \frac{1}{p}\|u\|_p^p + \frac{a}{2p}|\nabla u|_p^{2p} + \frac{1}{q}\|u\|_q^q + \frac{b}{2q}|\nabla u|_q^{2q} - \int_\Omega F(u)dx,$$

for all $u \in X$. We see that I is of class $\mathscr{C}^1$ in X and, for any $u,\ v \in X$,

$$\begin{aligned}\langle I'(u), v\rangle = {}& (1+a|\nabla u|_p^p)\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx + (1+b|\nabla u|_q^q)\int_\Omega |\nabla u|^{q-2}\nabla u\cdot\nabla v dx \\ &+ \int_\Omega V(x)(|u|^{p-2}+|u|^{q-2})uv dx - \int_\Omega f(u)v dx.\end{aligned} \tag{2.2}$$

It is known that the solutions to problem (1.1) are the critical points of the functional I. Moreover, if $u \in X$ is a solution to problem (1.1) and $u^{\pm} \neq 0$, then u is the sign-changing solution to problem (1.1), where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. It is easy to see that

$$I(u) = I(u^+) + I(u^-) + \tfrac{a}{p}|\nabla u^+|_p^p|\nabla u^-|_p^p + \tfrac{b}{q}|\nabla u^+|_q^q|\nabla u^-|_q^q, \tag{2.3}$$

$$\langle I'(u), u^+\rangle = \langle I'(u^+), u^+\rangle + a|\nabla u^+|_p^p|\nabla u^-|_p^p + b|\nabla u^+|_q^q|\nabla u^-|_q^q, \tag{2.4}$$

$$\langle I'(u), u^-\rangle = \langle I'(u^-), u^-\rangle + a|\nabla u^+|_p^p|\nabla u^-|_p^p + b|\nabla u^+|_q^q|\nabla u^-|_q^q. \tag{2.5}$$

In the following, we consider two functionals $I_{\pm} : X \to \mathbb{R}$ defined by

$$I_{\pm}(u) = \frac{1}{p}\|u\|_p^p + \frac{a}{2p}|\nabla u|_p^{2p} + \frac{1}{q}\|u\|_q^q + \frac{b}{2q}|\nabla u|_q^{2q} - \int_\Omega F(u^{\pm})dx.$$

Define the Nehari manifold associated with I by

$$\mathscr{N} = \left\{u \in X\setminus\{0\} : \langle I'(u), u\rangle = 0\right\}.$$

Cerami condition [27] is a variant of (PS) condition: the functional I satisfies the Cerami condition if: any Cerami sequence, i.e., $\{u_n\} \subset X$ satisfies $|I(u_n)| \leq M_1, \quad$ for some $M_1 > 0$ and $(1+\|u_n\|_X)I'(u_n) \to 0$, has a (strongly) convergent subsequence. The abstract tool used in this article is the following version of the mountain pass theorem [28].

Theorem 2.1. *Let X be a Banach space and $h : X \to \mathbb{R}$ be a local Lipschitz function with $h(0) = 0$. Suppose that there exist an element $e \in X$ and constants $\rho, \eta > 0$ such that*
(i) $h(u) \geq \eta$ for all $u \in X$ with $\|u\|_X = \rho$;
(ii) there exists $e \in X$ with $\|e\|_X > \rho$ such that $h(e) < 0$;
(iii) h satisfies Cerami condition.
Then h has a critical value $c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} h(\gamma(t))$, where $\Gamma = \{\gamma \in \mathscr{C}([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$.

In what follows, we give the following compactness result, which is crucial for the existence of solutions to problem (1.1).

Lemma 2.1. *Assume that conditions (A_1)-(A_5) and (1.6) hold. Then $I_{\pm}$ satisfy the Cerami condition.*

Proof. We discuss the proof only for I_+, while the proof for I_- follows similarly.Let $\{u_n\}$ be a Cerami sequence such that

$$|I_+(u_n)| \le M_1, \quad \text{for some } M_1 > 0, \tag{2.6}$$

and

$$(1+\|u_n\|_X)I_+^{'}(u_n) \to 0. \tag{2.7}$$

From (2.2) and (2.7), for any $v \in X$ and $\varepsilon_n \to 0$, we obtain that

$$\begin{aligned} &\Big|(1+a|\nabla u_n|_p^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla v dx + (1+b|\nabla u_n|_q^q)\int_\Omega |\nabla u_n|^{q-2}\nabla u_n\cdot\nabla v dx \\ &\quad + \int_\Omega V(x)(|u_n|^{p-2}+|u_n|^{q-2})u_n v dx - \int_\Omega f(u_n^+)v dx\Big| \le \frac{\varepsilon_n\|v\|_X}{1+\|u_n\|_X} \end{aligned} \tag{2.8}$$

Let $v = u_n^- \in X$ in (2.8) and note that

$$(1+a|\nabla u_n|_p^p)\int_\Omega |\nabla u_n^-|^p dx + (1+b|\nabla u_n|_q^q)\int_\Omega |\nabla u_n^-|^q dx + \int_\Omega V(x)(|u_n^-|^p+|u_n^-|^q)dx \le \varepsilon_n,$$

which implies that $\|u_n^-\|_X \to 0$ as $n \to +\infty$. Hence,

$$u_n^- \to 0 \quad \text{in } X. \tag{2.9}$$

From (2.4), (2.6), and (2.9), we obtain

$$\begin{aligned} &I_+(u_n^+) - \frac{1}{2q}\langle I_+^{'}(u_n), u_n^+\rangle \\ &= \left(\frac{1}{p}-\frac{1}{2q}\right)\|u_n^+\|_p^p + \left(\frac{1}{2p}-\frac{1}{2q}\right)a|\nabla u_n^+|_p^{2p} + \frac{1}{2q}\|u_n^+\|_q^q + \int_\Omega\left[\frac{1}{2q}f(u_n^+)u_n^+ - F(u_n^+)\right]dx \\ &\le M_2, \end{aligned}$$

for some $M_2 > 0$.

Claim 1: The sequence $\{u_n^+\} \subseteq X$ is bounded.

We argue by contradiction and suppose that by passing to a subsequence if necessary

$$\|u_n^+\|_X \to +\infty, \quad \text{as } n \to +\infty. \tag{2.10}$$

Let $\phi_n = \frac{u_n^+}{\|u_n^+\|_X}$ for all $n \in \mathbb{N}$. It is easy to see that $\|\phi_n\|_X = 1$ and $\phi_n \ge 0$ for all $n \in \mathbb{N}$. Hence, we may assume that, up to a subsequence, there exists $\phi \in X$ such that

$$\begin{aligned} &\phi_n \rightharpoonup \phi, && \text{weakly in } X, \\ &\phi_n \to \phi, && \text{strongly in } L^s(\Omega),\ 1 \le s < q^*. \end{aligned} \tag{2.11}$$

First, we suppose that $\phi \ne 0$ and set $\Omega_\phi^+ = \{x \in \Omega : \phi(x) > 0\}$. Obviously, $|\Omega_\phi^+| > 0$. Then, it follows from (2.11) that $u_n^+ \to +\infty$ for a.e. $x \in \Omega_\phi^+$ as $n \to +\infty$. Together with (A_3) and Fatou's Lemma, we can infer

$$\lim_{n\to+\infty}\int_{\Omega_\phi^+}\frac{F(u_n^+)}{\|u_n^+\|_X^{2q}}dx = +\infty. \tag{2.12}$$

It follows from (A_2) and (A_3) that there exists $M_3>0$ such that, for any $t\in\mathbb{R}$, $F(t)\geq -M_3$, which together with (2.10) and (2.12) yields

$$\begin{aligned}\int_\Omega \frac{F(u_n^+)}{\|u_n^+\|_X^{2q}}dx &= \int_{\Omega_\phi^+}\frac{F(u_n^+)}{\|u_n^+\|_X^{2q}}dx+\int_{\Omega\setminus\Omega_\phi^+}\frac{F(u_n^+)}{\|u_n^+\|_X^{2q}}dx\\ &\geq \int_{\Omega_\phi^+}\frac{F(u_n^+)}{\|u_n^+\|_X^{2q}}dx-\frac{M_3}{\|u_n^+\|_X^{2q}}|\Omega|\to+\infty.\end{aligned}\tag{2.13}$$

On the other hand, it follows from (2.9) and (2.6) that

$$\int_\Omega \frac{F(u_n^+)}{\|u_n^+\|_X^{2q}}dx\leq \frac{\|\phi_n\|_p^p}{p\|u_n^+\|_X^{2q-p}}+\frac{a|\nabla\phi_n|_p^{2p}}{2p\|u_n^+\|_X^{2q-2p}}+\frac{\|\phi_n\|_q^q}{q\|u_n^+\|_X^q}+\frac{b|\nabla\phi_n|_q^{2q}}{2q}+\frac{M_1}{\|u_n^+\|_X^{2q}},$$

yielding a contradiction to (2.13).

Next, we deal with the case $\phi=0$. Set $v_n=(qk)^{\frac{1}{q}}\phi_n$, where $k\geq 1$ and $n\in\mathbb{N}$. From the definition of ϕ_n, we obtain

$$\begin{aligned} &v_n\rightharpoonup 0, &&\text{weakly in } X,\\ &v_n\to 0, &&\text{strongly in } L^s(\Omega),\ 1\leq s<q^*.\end{aligned}$$

Hence,

$$\lim_{n\to+\infty}\int_\Omega F(v_n)dx=0.\tag{2.14}$$

We let $t_n\in[0,1]$ such that

$$I(t_nu_n^+)=\max\{I(tu_n^+):0\leq t\leq 1\}.\tag{2.15}$$

In view of $\|u_n^+\|_X\to+\infty$, there exists $n_1\in\mathbb{N}$ such that, for all $n\geq n_1$,

$$0<\frac{(qk)^{\frac{1}{q}}}{\|u_n^+\|_X}\leq 1.\tag{2.16}$$

From $p<q$ and $\|\phi_n\|_p\leq\|\phi_n\|_X=1$, we observe that $\|\phi_n\|_p^p\geq\|\phi_n\|_p^q$. Using this inequality and

$$z^q+w^q\geq C_q(z+w)^q,\quad \text{for all } z,\ w\geq 0,\ q>1,\tag{2.17}$$

and (2.14)-(2.16), we derive that

$$\begin{aligned} I(t_nu_n^+)&\geq I(v_n)\\ &=\frac{1}{p}(qk)^{\frac{p}{q}}\|\phi_n\|_p^p+\frac{a}{2p}(qk)^{\frac{2p}{q}}|\nabla\phi_n|_p^{2p}+k\|\phi_n\|_q^q+\frac{b}{2q}(qk)^2|\nabla\phi_n|_q^{2q}-\int_\Omega F(v_n)dx\\ &\geq\min\left\{\frac{1}{p}q^{\frac{p}{q}},1\right\}k^{\frac{p}{q}}\left[\|\phi_n\|_p^p+\|\phi_n\|_q^q\right]-\int_\Omega F(v_n)dx\\ &\geq\min\left\{\frac{1}{p}q^{\frac{p}{q}},1\right\}k^{\frac{p}{q}}C_q+o_n(1).\end{aligned}$$

Since $k\geq 1$ is arbitrary, we can obtain that

$$\lim_{n\to+\infty}I(t_nu_n^+)=+\infty.\tag{2.18}$$

Recalling that $I(u)\leq I_+(u)$ for all $u\in X$ with $u\geq 0$ and applying (2.6) and (2.9), we obtain

$$I(0)=0,\quad I(u_n^+)\leq M_4\tag{2.19}$$

for all $n \in \mathbb{N}$ and some $M_4 > 0$. Putting together (2.18) and (2.19), we obtain that there exists $n_2 \geq n_1$ such that $t_n \in (0,1)$ for all $n \geq n_2$. From (2.15) we have $\langle I'(t_n u_n^+), t_n u_n^+ \rangle = o_n(1)$, which is equivalent to

$$\|t_n u_n^+\|_p^p + \|t_n u_n^+\|_q^q + a|\nabla(t_n u_n^+)|_p^{2p} + b|\nabla(t_n u_n^+)|_q^{2q} - \int_\Omega f(t_n u_n^+) t_n u_n^+ dx = o_n(1). \tag{2.20}$$

It follows from assumption (A_5) and (2.20) that

$$\begin{aligned}
&I(t_n u_n^+) - \frac{1}{2q}\langle I'(t_n u_n^+), t_n u_n^+ \rangle \\
&= \left(\frac{1}{p} - \frac{1}{2q}\right)\|t_n u_n^+\|_p^p + \left(\frac{1}{2p} - \frac{1}{2q}\right) a|\nabla(t_n u_n^+)|_p^{2p} + \frac{1}{2q}\|t_n u_n^+\|_q^q \\
&\quad + \int_\Omega \left[\frac{1}{2q} f(t_n u_n^+) t_n u_n^+ - F(t_n u_n^+)\right] dx \\
&\leq \left(\frac{1}{p} - \frac{1}{2q}\right)\|u_n^+\|_p^p + \left(\frac{1}{2p} - \frac{1}{2q}\right) a|\nabla u_n^+|_p^{2p} + \frac{1}{2q}\|u_n^+\|_q^q \\
&\quad + \int_\Omega \left[\frac{1}{2q} f(u_n^+) u_n^+ - F(u_n^+)\right] dx \\
&\leq M_5,
\end{aligned}$$

which implies that $I(t_n u_n^+) \leq M_5$ for all $n \geq n_2$ and some $M_5 > 0$, which contradicts (2.18). This completes the proof of Claim 1.

By Claim 1 and (2.9), we obtain that sequence $\{u_n\}_{n\geq 1} \subseteq X$ is bounded. Therefore, up to a subsequence, we may assume that there exists $u \in X$ such that

$$\begin{aligned}
&u_n \rightharpoonup u, \qquad \text{weakly in } X, \\
&u_n \to u, \qquad \text{strongly in } L^s(\Omega),\ 1 \leq s < q^*.
\end{aligned} \tag{2.21}$$

In the light of (2.21) we have

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in } L^s(\Omega, \mathbb{R}^N) \text{ for } s = p, q. \tag{2.22}$$

Setting $v_n = u_n - u \in X$ in (2.8), taking the limit as $n \to +\infty$, and applying (2.21), we can infer that

$$|\nabla u_n|_p \to |\nabla u|_p \quad \text{and } |\nabla u_n|_q \to |\nabla u|_q. \tag{2.23}$$

Combining (2.22) and (2.23), we obtain $\nabla u_n \to \nabla u \quad$ in $L^s(\Omega, \mathbb{R}^N)$ for $s = p, q$. From this and (2.21), we have that $u_n \to u$ in X. □

Now we prove that $I_\pm$ possess a mountain pass geometry.

Lemma 2.2. *Assume that conditions* (A_1)*-*(A_5) *and (1.6) hold. Then functionals* $I_\pm$ *satisfy the following conditions:*
(i) there exist $\alpha, \rho > 0$ *such that* $I_\pm(u) \geq \alpha$ *with* $\|u\|_X = \rho$*;*
(ii) there exists $e \in X$ *with* $\|e\|_X > \rho$ *such that* $I_\pm(e) < 0$*.*

Proof. We only give the proof for the functional I_+, while the proof for the functional I_- follows similarly.

(i) From assumptions (A_2) and (A_4), for any $\varepsilon > 0$, there exists $c_\varepsilon = c(\varepsilon) > 0$ such that

$$|f(t)| \leq \varepsilon |t|^{p-1} + c_\varepsilon |t|^{r-1}, \tag{2.24}$$

$$|F(t)| \le \frac{\varepsilon}{p}|t|^p + \frac{c_\varepsilon}{r}|t|^r. \tag{2.25}$$

Choosing $\|u\|_X \le 1$ and using $1 < p < q$, we obtain $\|u\|_p < 1$. Thus $\|u\|_p^p \ge \|u\|_p^q$. It follows from (2.25), (2.1), and (2.17) that

$$\begin{aligned} I_+(u) &\ge \frac{1}{p}\|u\|_p^p + \frac{a}{2p}|\nabla u|_p^{2p} + \frac{1}{q}\|u\|_q^q + \frac{b}{2q}|\nabla u|_q^{2q} - \frac{\varepsilon}{p}|u|_p^p - \frac{c_\varepsilon}{r}|u|_r^r \\ &\ge \frac{1}{p}(1 - c_p^p\varepsilon)\|u\|_p^p + \frac{1}{q}\|u\|_q^q - \frac{c_\varepsilon}{r}c_r^r\|u\|_X^r \\ &\ge \min\left\{\frac{1}{p}\left(1 - c_p^p\varepsilon\right), \frac{1}{q}\right\} C_q\|u\|_X^q - \frac{c_\varepsilon}{r}c_r^r\|u\|_X^r \\ &\ge \tilde{c}\|u\|_X^q - \tilde{c}_\varepsilon\|u\|_X^r, \end{aligned}$$

where $\tilde{c} = \min\left\{\frac{1}{p}\left(1 - c_p^p\varepsilon\right), \frac{1}{q}\right\} C_q$, $\tilde{c}_\varepsilon = \frac{c_\varepsilon c_r^r}{r}$, and $\varepsilon \in (0, \frac{1}{c_p^p})$. Taking into account that $q < r$, there exists $\alpha > 0$ such that $I_+(u) \ge \alpha > 0 = I_+(0)$ with $\|u\|_X = \rho$.

(ii) Fix $u \in X$ with $u(x) > 0$ for a.e. $x \in \Omega$. Then applying (A_3) and Fatou's Lemma, we can obtain that

$$\begin{aligned} \frac{I_+(tu)}{\|tu\|_X^{2q}} &\le \frac{1}{p}\frac{\|u\|_p^p}{t^{2q-p}\|u\|_X^{2q}} + \frac{a}{2p}\frac{|\nabla u|_p^{2p}}{t^{2q-2p}\|u\|_X^{2q}} + \frac{1}{q}\frac{\|u\|_q^q}{t^q\|u\|_X^{2q}} + \frac{a}{2q}\frac{|\nabla u|_q^{2q}}{\|u\|_X^{2q}} \\ &\quad - \int_\Omega \frac{F(tu^+)}{(tu^+)^{2q}}\left(\frac{u^+}{\|u\|_X}\right)^{2q} dx \to -\infty, \qquad \text{as } t \to +\infty. \end{aligned}$$

□

At the end of this section, we give the proof of the existence of constant sign solutions to problem (1.1).

Proposition 2.1. *Assume that conditions (A_1)-(A_5) and (1.6) hold. Then problem (1.1) has at least two nontrivial constant sign solutions u_+, $u_- \in X$ such that $u_+(x) \ge 0$ and $u_-(x) \le 0$, for a.e. $x \in \Omega$.*

Proof. It follows from Lemmas 2.1-2.2 and Theorem 2.1 that there exist u_+, $u_- \in X$ such that $u_+ \in \mathscr{K}_{I_+}$, $u_- \in \mathscr{K}_{I_-}$, where $\mathscr{K}_{I_\pm} = \{u \in X \setminus \{0\} : I'_\pm(u) = 0\}$. Moreover, $I_+(u_+) = c_+ \ge \alpha > 0 = I_+(0)$ and $I_-(u_-) = c_- \ge \alpha > 0 = I_-(0)$, so $u_+ \ne 0$ and $u_- \ne 0$.

Finally we prove that u_+ is non-negative and u_- is non-positive. Indeed, recalling $\langle I'_+(u_+), v\rangle = 0$ and taking $v = u_+^-$ as a test function, we obtain

$$\|u_+^-\|_p^p + \|u_+^-\|_q^q + a|\nabla u_+^-|_p^{2p} + b|\nabla u_+^-|_q^{2q} = 0,$$

which implies that $u_+^- \equiv 0$, that is, $u_+ \ge 0$. The proof of $u_- \le 0$ follows similarly. □

3. Sign-changing Solution

In this section, we give the existence of a sign-changing solution to problem (1.1) by virtue of some idea due to Shuai [29]. Now, we introduce the following set:

$$\mathscr{M} = \{u \in X \setminus \{0\} : \langle I'(u), u^+\rangle = \langle I'(u), u^-\rangle = 0\}.$$

Obviously, set $\mathscr{M}$ contains all sign-changing solutions to problem (1.1).

Lemma 3.1. *Assume that conditions* (A_1)*-*(A_5) *and (1.6) hold. Then, for any* $u \in X$ *with* $u^{\pm} \neq 0$*, there exists a unique pair of* (s_u, t_u) *of positive numbers such that* $s_u u^+ + t_u u^- \in \mathscr{M}$.

Proof. Let $u \in X$ with $u^{\pm} \neq 0$ and denote

$$K = a \int_{\Omega} |\nabla u^+|^p dx \int_{\Omega} |\nabla u^-|^p dx, \quad H = b \int_{\Omega} |\nabla u^+|^q dx \int_{\Omega} |\nabla u^-|^q dx.$$

From the definition of $\mathscr{M}$, we can obtain that $su^+ + tu^- \in \mathscr{M}$ is equivalent to

$$\langle I'(su^+ + tu^-),\ su^+ \rangle = \langle I'(su^+ + tu^-),\ tu^- \rangle = 0.$$

That is,

$$\begin{cases} s^p \|u^+\|_p^p + s^q \|u^+\|_q^q + as^{2p} |\nabla u^+|_p^{2p} + bs^{2q} |\nabla u^+|_q^{2q} + s^p t^p K + s^q t^q H - \displaystyle\int_{\Omega} f(su^+) su^+ dx = 0, \\ t^p \|u^-\|_p^p + t^q \|u^-\|_q^q + at^{2p} |\nabla u^-|_p^{2p} + bt^{2q} |\nabla u^-|_q^{2q} + t^p s^p K + s^q t^q H - \displaystyle\int_{\Omega} f(tu^-) tu^- dx = 0. \end{cases} \tag{3.1}$$

Hence, our aim is to verify that there exists only one positive solution (s,t) of system (3.1). Now, we consider the following system with a parameter $\lambda \in [0,1]$.

$$\begin{cases} s^p \|u^+\|_p^p + s^q \|u^+\|_q^q + as^{2p} |\nabla u^+|_p^{2p} + bs^{2q} |\nabla u^+|_q^{2q} + \lambda (s^p t^p K + s^q t^q H) \\ \qquad\qquad - \displaystyle\int_{\Omega} f(su^+) su^+ dx = 0, \\ t^p \|u^-\|_p^p + t^q \|u^-\|_q^q + at^{2p} |\nabla u^-|_p^{2p} + bt^{2q} |\nabla u^-|_q^{2q} + \lambda (t^p s^p K + s^q t^q H) \\ \qquad\qquad - \displaystyle\int_{\Omega} f(tu^-) tu^- dx = 0. \end{cases} \tag{3.2}$$

Define

$$\Lambda := \left\{ \lambda : \lambda \in [0,1] \text{ such that (3.2) is uniquely solvable in } \mathbb{R}^+ \right\},$$

and let

$$\begin{aligned} g_\lambda(s,t) = {} & s^p \|u^+\|_p^p + s^q \|u^+\|_q^q + as^{2p} |\nabla u^+|_p^{2p} + bs^{2q} |\nabla u^+|_q^{2q} + \lambda (s^p t^p K + s^q t^q H) \\ & - \int_{\Omega} f(su^+) su^+ dx \\ y_\lambda(s,t) = {} & t^p \|u^-\|_p^p + t^q \|u^-\|_q^q + at^{2p} |\nabla u^-|_p^{2p} + bt^{2q} |\nabla u^-|_q^{2q} + \lambda (t^p s^p K + s^q t^q H) \\ & - \int_{\Omega} f(tu^-) tu^- dx. \end{aligned}$$

We now demonstrate $0 \in \Lambda$. Since $g_0(s,t)$ is independent of t and $y_0(s,t)$ is independent of s, we only need to demonstrate that there exists a unique $t > 0$ such that $g_0(s,t) = 0$, and the case for $y_0(s,t)$ is similar. It follows from $u^+ \neq 0$, (A_2)-(A_4) that $g_0(0,t) = 0$, $g_0(s,t) > 0$ for $s > 0$ small and $g_0(s,t) < 0$ for s large. Assume that there exist $0 < s_1 < s_2$ such that $g_0(s_1,t) = g_0(s_2,t) = 0$. This indicates that

$$s_1^{p-2q} \|u^+\|_p^p + s_1^{-q} \|u^+\|_q^q + as_1^{2p-2q} |\nabla u^+|_p^{2p} + b|\nabla u^+|_q^{2q} = \int_{\Omega} \frac{f(s_1 u^+)}{s_1^{2q-1}} u^+ dx$$

and

$$s_2^{p-2q} \|u^+\|_p^p + s_2^{-q} \|u^+\|_q^q + as_2^{2p-2q} |\nabla u^+|_p^{2p} + b|\nabla u^+|_q^{2q} = \int_{\Omega} \frac{f(s_2 u^+)}{s_2^{2q-1}} u^+ dx.$$

Subtracting the above two equations, we arrive at

$$0<\left(\frac{1}{s_1^{2q-p}}-\frac{1}{s_2^{2q-p}}\right)\|u^+\|_p^p+\left(\frac{1}{s_1^q}-\frac{1}{s_2^q}\right)\|u^+\|_q^q+a\left(\frac{1}{s_1^{2q-2p}}-\frac{1}{s_2^{2q-2p}}\right)|\nabla u^+|_p^{2p}$$
$$=\int_\Omega\left[\frac{f(s_1u^+)}{(s_1u^+)^{2q-1}}-\frac{f(s_2u^+)}{(s_2u^+)^{2q-1}}\right](u^+)^{2q}dx<0,$$

which yields a contradiction due to (A_5) and $0<s_1<s_2$.

Claim Set Λ is open and closed in $[0,1]$.

First, we prove that Λ is open in $[0,1]$. If $\lambda_0\in\Lambda$ and $(\tilde{s},\tilde{t})\in\mathbb{R}^+\times\mathbb{R}^+$ is the unique solution to (3.2) with $\lambda=\lambda_0$, direct calculations present that

$$\begin{aligned}\frac{\partial g_\lambda(s,t)}{\partial s}|_{(\tilde{s},\tilde{t})}=&(p-2)\tilde{s}^{p-1}\|u^+\|_p^p+(q-2)\tilde{s}^{q-1}\|u^+\|_q^q+(2p-2)a\tilde{s}^{2p-1}|\nabla u^+|_p^{2p}\\&+(2q-2)b\tilde{s}^{2q-1}|\nabla u^+|_q^{2q}+\lambda_0[(p-1)\tilde{s}^{p-1}\tilde{t}^pK+(q-1)\tilde{s}^{q-1}\tilde{t}^qH]\\&-\int_\Omega f'(\tilde{s}u^+)\tilde{s}u^+dx,\end{aligned}$$

$$\begin{aligned}\frac{\partial y_\lambda(s,t)}{\partial t}|_{(\tilde{s},\tilde{t})}=&(p-2)\tilde{t}^{p-1}\|u^-\|_p^p+(q-2)\tilde{t}^{q-1}\|u^-\|_q^q+(2p-2)a\tilde{t}^{2p-1}|\nabla u^-|_p^{2p}\\&+(2q-2)b\tilde{t}^{2q-1}|\nabla u^-|_q^{2q}+\lambda_0[(p-1)\tilde{t}^{p-1}\tilde{s}^pK+(q-1)\tilde{t}^{q-1}\tilde{s}^qH]\\&-\int_\Omega f'(\tilde{t}u^-)\tilde{t}u^-dx,\end{aligned}$$

and

$$\frac{\partial g_\lambda(s,t)}{\partial t}|_{(\tilde{s},\tilde{t})}=\lambda_0(p\tilde{t}^{p-1}\tilde{s}^pK+q\tilde{t}^{q-1}\tilde{s}^qH),\quad \frac{\partial y_\lambda(s,t)}{\partial s}|_{(\tilde{s},\tilde{t})}=\lambda_0(p\tilde{s}^{p-1}\tilde{t}^pK+q\tilde{s}^{q-1}\tilde{t}^qH).$$

Define the matrix

$$B=\begin{pmatrix}\frac{\partial g_\lambda(\tilde{s},\tilde{t})}{\partial s} & \frac{\partial g_\lambda(\tilde{s},\tilde{t})}{\partial t}\\ \frac{\partial y_\lambda(\tilde{s},\tilde{t})}{\partial s} & \frac{\partial y_\lambda(\tilde{s},\tilde{t})}{\partial t}\end{pmatrix}.$$

It follows from (A_5) that

$$f'(s)s^{2q}-(2q-1)f(s)s^{2q-1}>0,$$

which yields

$$\frac{\partial g_\lambda(s,t)}{\partial s}|_{(\tilde{s},\tilde{t})}<-(2q-p+1)\tilde{s}^{p-1}\|u^+\|_p^p-\lambda_0[(2q-p)\tilde{s}^{p-1}\tilde{t}^pK+q\tilde{s}^{q-1}\tilde{t}^qH],$$

and

$$\frac{\partial y_\lambda(s,t)}{\partial t}|_{(\tilde{s},\tilde{t})}<-(2q-p+1)\tilde{t}^{p-1}\|u^-\|_p^p-\lambda_0[(2q-p)\tilde{t}^{p-1}\tilde{s}^pK+q\tilde{t}^{q-1}\tilde{s}^qH].$$

Therefore, from (1.6), we derive that

$$\begin{aligned}\det B>&\left\{(2q-p+1)\tilde{s}^{p-1}\|u^+\|_p^p+\lambda_0[(2q-p)\tilde{s}^{p-1}\tilde{t}^pK+q\tilde{s}^{q-1}\tilde{t}^qH]\right\}\\&\left\{(2q-p+1)\tilde{t}^{p-1}\|u^-\|_p^p+\lambda_0[(2q-p)\tilde{t}^{p-1}\tilde{s}^pK+q\tilde{t}^{q-1}\tilde{s}^qH]\right\}\\&-\lambda_0^2(p\tilde{t}^{p-1}\tilde{s}^pK+q\tilde{t}^{q-1}\tilde{s}^qH)(p\tilde{s}^{p-1}\tilde{t}^pK+q\tilde{s}^{q-1}\tilde{t}^qH)\\>&0.\end{aligned}$$

Thus it follows from the implicit function theorem that there exists an open neighborhood U_0 of λ_0 and $E_0 \subset \mathbb{R}^+ \times \mathbb{R}^+$ of $(\tilde{s},\tilde{t})$ such that system (3.2) has a unique solution in $U_0 \times E_0$.

We now need to prove that $U_0 \subset \Lambda$. Assume that there exists $\lambda_1 \in U_0$ such that the second solution $(\bar{s},\bar{t})$ of (3.2) exits in $(\mathbb{R}^+,\mathbb{R}^+) \setminus E_0$. Then using the implicit function existence theorem, there exists a solution curve $(\lambda,(\bar{s}(\lambda),\bar{t}(\lambda)))$ in $(\lambda_1-\varepsilon,\lambda_1+\varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^+$, which satisfies (3.2) and intersects with $(\lambda_1,(\bar{s},\bar{t}))$. Let us assume $\lambda_0 < \lambda_1$ and extend the curve as much as possible. Since it cannot be defined at λ_0 and enter into $U_0 \times E_0$, there should exist a point $\lambda_2 \in (\lambda_0,\lambda_1]$ such that $(\bar{s}(\lambda),\bar{t}(\lambda))$ exists in $(\lambda_2,\lambda_1]$ and blows up as $\lambda \to \lambda_2$. This is impossible. Indeed, it follows from (A_3) that at least one of the left sides of (3.2) is strictly negative if (s,t) is sufficiently large. This is a contradiction. Hence, $U_0 \subset \Lambda$. The proof for the case $\lambda_0 > \lambda_1$ is similar.

Next, we prove that Λ is closed in $[0,1]$. Assume that there exists a sequence $\lambda_n \to \lambda_0$ and (s_n,t_n) is the solution to (3.2) with $\lambda = \lambda_n$. From the discussion above, it can be seen that (s_n,t_n) is bounded above. Hence, we may assume that there exists a solution $(s_0,t_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ of (3.2) with $\lambda = \lambda_0$ such that $(s_n,t_n) \to (s_0,t_0)$. It follows from (3.2), (2.24), and (2.1) that

$$\|s_n u^+\|_p^p \le \varepsilon c_p^p \|s_n u^+\|_p^p + c_\varepsilon c_r^r \|s_n u^+\|_p^r, \tag{3.3}$$

which yields that $s_n \ge C_1 > 0$ uniformly in n. Hence, $s_0 \ge C_1 > 0$. Similarly, we can deduce that $t_0 \ge C_2 > 0$. Therefore, $(s_0,t_0) \in \mathbb{R}^+ \times \mathbb{R}^+$. From the implicit function theorem, (s_0,t_0) is the unique solution in $\mathbb{R}^+ \times \mathbb{R}^+$. □

Lemma 3.2. *Let $u \in X$ with $u^\pm \neq 0$ be fixed. Then the vector (s_u,t_u) obtained in Lemma 3.1 is the unique maximum point to the function $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by $\varphi(s,t) = I(su^+ + tu^-)$.*

Proof. By the proof of Lemma 3.1, (s_u,t_u) is the unique critical point of φ in $\mathbb{R}^+ \times \mathbb{R}^+$. It follows from (A_3) that $\varphi(s,t) \to -\infty$ uniformly as $|(s,t)| \to \infty$. Hence, we prove that a maximum point cannot be attained on the boundary of $\mathbb{R}^+ \times \mathbb{R}^+$. Assume by contradiction that $(0,\hat{t})$ is a maximum point of φ. Then we see that

$$\begin{aligned}
\varphi(s,\hat{t}) &= I(su^+ + \hat{t}u^-)\\
&= \frac{s^p}{p}\|u^+\|_p^p + \frac{as^{2p}}{2p}|\nabla u^+|_p^{2p} + \frac{s^q}{q}\|u^+\|_q^q + \frac{bs^{2q}}{2q}|\nabla u^+|_q^{2q}\\
&\quad - \int_\Omega F(su^+)dx + \frac{as^p\hat{t}^p}{p}|\nabla u^+|_p^p|\nabla u^-|_p^p + \frac{bs^p\hat{t}^p}{q}|\nabla u^+|_q^q|\nabla u^-|_q^q\\
&\quad + \frac{\hat{t}^p}{p}\|u^-\|_p^p + \frac{a\hat{t}^{2p}}{2p}|\nabla u^-|_p^{2p} + \frac{\hat{t}^q}{q}\|u^-\|_q^q + \frac{b\hat{t}^{2q}}{2q}|\nabla u^-|_q^{2q} - \int_\Omega F(\hat{t}u^-)dx
\end{aligned}$$

is increasing with respect to s when s is small. Therefore, $(0,\hat{t})$ is not a maximum point of φ in $\mathbb{R}^+ \times \mathbb{R}^+$. □

Lemma 3.3. *Assume that conditions (A_1)-(A_5) and (1.6) hold. Then $m = \inf_{\mathscr{M}} I > 0$ is achieved.*

Proof. Let $c = \inf_{\mathscr{N}} I$. Firstly, we present that $c > 0$. Following the proof of Lemma 2.2, we can see that there exist $\alpha,\rho > 0$ such that $I(u) \ge \alpha$ for all $u \in X$ with $\|u\|_X = \rho$. Now, let $u \in \mathscr{N}$ and choose $t_u > 0$ such that $t_u\|u\|_X = \rho$. By Lemma 2.2, we have $I(u) \ge I(t_u u) \ge \alpha > 0$. Then $c > 0$. Now, for $u \in \mathscr{M}$, recalling that $u \in \mathscr{N}$, we obtain that

$$\inf_{\mathscr{M}} I(u) \ge \inf_{\mathscr{N}} I(u) = c > 0.$$

Thus $m>0$. Let $\{\omega_n\}\subseteq\mathscr{M}$ be such that $I(\omega_n)\to m$. Similar to the proof of Lemma 2.1, we can see that the sequences $\{\omega_n^+\},\{\omega_n^-\}\subseteq X$ are bounded. Then, we may assume that $\omega_n^+\rightharpoonup\omega^+\geq 0$ and $\omega_n^-\rightharpoonup\omega^-\leq 0$. Recalling that $\omega_n\in\mathscr{M}$, we obtain

$$0=\langle I'(\omega_n),\omega_n^\pm\rangle=\|\omega_n^\pm\|_p^p+\|\omega_n^\pm\|_q^q+a|\nabla\omega_n|_p^p|\nabla\omega_n^\pm|_p^p+b|\nabla\omega_n|_q^q|\nabla\omega_n^\pm|_q^q-\int_\Omega f(\omega_n^\pm)\omega_n^\pm dx. \tag{3.4}$$

Similar to (3.3), there exists $\kappa>0$ such that $\|w_n^\pm\|_p\geq\kappa$. It follows from (2.24) and (3.4) that

$$\kappa^p\leq\|w_n^\pm\|_p^p\leq\varepsilon\int_\Omega|w_n^\pm|^p dx+c_\varepsilon\int_\Omega|w_n^\pm|^r dx.$$

Since $\{\omega_n^+\}$ and $\{\omega_n^-\}$ are bounded, there exists $C>0$ such that

$$\kappa^p\leq\|w_n^\pm\|_p^p\leq\varepsilon C+c_\varepsilon\int_\Omega|w_n^\pm|^r dx.$$

Setting $\varepsilon=\frac{\kappa^p}{2C}$, we obtain that

$$\int_\Omega|w_n^\pm|^r dx\geq\frac{\kappa^p}{2c_\varepsilon}.$$

Therefore,

$$\int_\Omega|w^\pm|^r dx=\lim_{n\to\infty}\int_\Omega|w_n^\pm|^r dx\geq\frac{\kappa^p}{2c_\varepsilon}.$$

Hence, $w^\pm\neq 0$. It follows from Lemma 3.1 that there exist $s_w,t_w>0$ such that $s_w w^+ + t_w w^-\in\mathscr{M}$. We claim that $0<s_w,t_w\leq 1$. Assume by contradiction that there exists $\delta>0$ such that $s_w\geq t_w\geq 1+\delta$. Since $s_w w^+ + t_w w^-\in\mathscr{M}$, then we observe

$$\begin{aligned}
&s_w^p\|w^+\|_p^p+s_w^q\|w^+\|_q^q+as_w^{2p}|\nabla w|_p^p|\nabla w^+|_p^p+bs_w^{2q}|\nabla w|_q^q|\nabla w^+|_q^q\\
&\geq s_w^p\|w^+\|_p^p+s_w^q\|w^+\|_q^q+as_w^{2p}|\nabla w^+|_p^{2p}+bs_w^{2q}|\nabla w^+|_q^{2q}\\
&\quad+as_w^p t_w^p|\nabla w^+|_p^p|\nabla w^-|_p^p+bs_w^q t_w^q|\nabla w^+|_q^q|\nabla w^-|_q^q\\
&=\int_\Omega f(s_w w^+)s_w w^+ dx.
\end{aligned}\tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}
\int_\Omega f(w^+)w^+ dx&=\liminf_{n\to\infty}\int_\Omega f(w_n^+)w_n^+ dx\\
&=\liminf_{n\to\infty}[\|w_n^+\|_p^p+\|w_n^+\|_q^q+a|\nabla w_n|_p^p|\nabla w_n^+|_p^p+b|\nabla w_n|_q^q|\nabla w_n^+|_q^q]\\
&\geq\|w^+\|_p^p+\|w^+\|_q^q+a|\nabla w|_p^p|\nabla w^+|_p^p+b|\nabla w|_q^q|\nabla w^+|_q^q.
\end{aligned}\tag{3.6}$$

Combining (3.5) and (3.6), we obtain that

$$\begin{aligned}
&(1-s_w^{p-2q})\|w^+\|_p^p+(1-s_w^{-q})\|w^+\|_q^q+a(1-s_w^{2p-2q})|\nabla w|_p^p|\nabla w^+|_p^p\\
&\leq\int_\Omega\left[\frac{f(w^+)}{(w^+)^{2q-1}}-\frac{f(s_w w^+)}{s_w^{2q-1}(w^+)^{2q-1}}\right](w^+)^{2q}dx,
\end{aligned}$$

which leads to a contradiction for $s_w \geq 1+\delta$ and (A_5). The claim is true. This together with the sequentially weakly lower semicontinuity of I and (A_5) concludes that

$$\begin{aligned}
m &\leq I(s_w w^+ + t_w w^-) = I(s_w w^+ + t_w w^-) - \frac{1}{2q}\langle I'(s_w w^+ + t_w w^-), s_w w^+ + t_w w^-\rangle \\
&= \left(\frac{1}{p}-\frac{1}{2q}\right)\|s_w w^+ + t_w w^-\|_p^p + \frac{1}{2q}\|s_w w^+ + t_w w^-\|_q^q + \left(\frac{1}{2p}-\frac{1}{2q}\right) a|\nabla(s_w w^+ + t_w w^-)|_p^{2p} \\
&\quad + \frac{1}{2q}\int_\Omega \left[f(s_w w^+ + t_w w^-)(s_w w^+ + t_w w^-) - 2qF(s_w w^+ + t_w w^-)\right] dx \\
&= \left(\frac{1}{p}-\frac{1}{2q}\right)\left(\|s_w w^+\|_p^p + \|t_w w^-\|_p^p\right) + \frac{1}{2q}\left(\|s_w w^+\|_q^q + \|t_w w^-\|_q^q\right) \\
&\quad + \left(\frac{1}{2p}-\frac{1}{2q}\right) a|\nabla(s_w w^+ + t_w w^-)|_p^{2p} + \frac{1}{2q}\int_\Omega \left[f(s_w w^+)s_w w^+ - 2qF(s_w w^+)\right] dx \\
&\quad + \frac{1}{2q}\int_\Omega \left[f(t_w w^-)t_w w^- - 2qF(t_w w^-)\right] dx \\
&\leq \left(\frac{1}{p}-\frac{1}{2q}\right)\left(\|w^+\|_p^p + \|w^-\|_p^p\right) + \frac{1}{2q}\left(\|w^+\|_q^q + \|w^-\|_q^q\right) + \left(\frac{1}{2p}-\frac{1}{2q}\right) a|\nabla(w^+ + w^-)|_p^{2p} \\
&\quad + \frac{1}{2q}\int_\Omega \left[f(w^+)w^+ - 2qF(w^+)\right] dx + \frac{1}{2q}\int_\Omega \left[f(w^-)w^- - 2qF(w^-)\right] dx \\
&\leq \liminf_{n\to+\infty}\left[I(w_n) - \frac{1}{2q}\langle I'(w_n), w_n\rangle\right] = m,
\end{aligned} \tag{3.7}$$

which implies $s_w = t_w = 1$. Therefore, $I(w) = m$ and $w \in \mathscr{M}$. □

The forthcoming result regards the existence of a nodal solution of problem (3.4). The proof of this result is based on the proof [30, Theorem 1.4] and the quantitative deformation lemma [26, Lemma 2.3].

Proposition 3.1. *Assume that conditions (A_1)-(A_5) and (1.6) hold. Then I has a minimizer w in $\mathscr{M}$, which is a nodal solution to problem* (1.1).

Proof. Firstly, we aim to prove that w is a critical point of I. Arguing by contradiction, assume that $I'(w) \neq 0$. Then there exist $\delta > 0$ and $\rho > 0$ such that $\|I'(v)\|_{X^*} \geq \rho$ for all $v \in X$ with $\|v-w\|_X \leq 3\delta$. Set $D := \left[\frac{1}{2},\frac{3}{2}\right] \times \left[\frac{1}{2},\frac{3}{2}\right]$. From Lemma 3.2, we obtain

$$\chi := \max_{(s,t)\in\partial D} I(sw^+ + tw^-) < m.$$

Let $\varepsilon = \min\{\frac{m-\chi}{4}, \frac{\rho\delta}{8}\}$ and $B(w,\delta) := \{v \in X : \|v-w\| \leq \delta\}$. It follows from [31, Lemma 2.3] that there exists a deformation $\eta \in \mathscr{C}([0,1]\times X, X)$ such that

(i) $\eta(1,u) = u$ if $u \notin I^{-1}([m-2\varepsilon, m+2\varepsilon])$;
(ii) $\eta(1, I^{m+\varepsilon} \cap B(w,\delta)) \subset I^{m-\varepsilon}$;
(iii) $I(\eta(1,u)) \leq I(u)$, for all $u \in X$,
where $I^{m\pm\varepsilon} := \{u \in X : I(u) \leq m \pm \varepsilon\}$. It is easy to see that

$$\max_{(s,t)\in D} I(\eta(1, sw^+ + tw^-)) < m. \tag{3.8}$$

Define $\gamma(s,t)=\eta(1,sw^{+}+tw^{-})$,

$$\Phi_1(s,t)=\left(\langle I'(sw^{+}+tw^{-}),w^{+}\rangle,\ \langle I'(sw^{+}+tw^{-}),w^{-}\rangle\right)$$

and

$$\Phi_2(s,t)=\left(\frac{1}{s}\langle I'(\gamma^{+}(s,t)),\gamma^{+}(s,t)\rangle,\ \frac{1}{t}\langle I'(\gamma^{-}(s,t)),\gamma^{-}(s,t)\rangle\right).$$

Recalling Lemma 3.1, we obtain that $\langle I'(sw^{+}+tw^{-}),w^{\pm}\rangle>0$ if $s,t\in(0,1)$ is small enough, and $\langle I'(sw^{+}+tw^{-}),w^{\pm}\rangle<0$ if $s,t\in(1,+\infty)$ is large enough. Therefore, we can derive that $\deg(\Phi_1,D,0)=1$. It follows from (3.8) and property (i) of η that $\gamma(s,t)=sw^{+}+tw^{-}$ on ∂D. Consequently, $\Phi_1=\Phi_2$ on ∂D and $\deg(\Phi_1,D,0)=\deg(\Phi_2,D,0)=1$, which implies that $\Phi_2(s_0,t_0)=0$ for some $(s_0,t_0)\in D$. Hence, $\eta(1,s_0w^{+}+t_0w^{-})=\gamma(s_0,t_0)\in\mathscr{M}$. This contradicts (3.8) and the definition of m. Therefore, w is a critical point of I, and then w is a solution to problem (1.1).

Now, we prove that w has exactly two nodal domains. Let $w=w_1+w_2+w_3$, where

$$w_1\geq 0,\quad w_2\leq 0,\quad \Omega_1\cap\Omega_2=\emptyset,\quad w_1|_{\Omega\setminus\Omega_1}=w_2|_{\Omega\setminus\Omega_2}=w_3|_{\Omega_1\cup\Omega_2}=0,$$

$$\Omega_1:=\{x\in\Omega:w_1(x)>0\},\quad \Omega_2:=\{x\in\Omega:w_2(x)<0\},$$

and $\Omega_i(i=1,2)$ are connected open subsets of Ω. Setting $v=w_1+w_2$, we have $v^{+}=w_1$, $v^{-}=w_2,v^{\pm}\neq 0$. From Lemma 3.1, there exists a unique pair of positive numbers (s_v,t_v) such that $s_vv^{+}+t_vv^{-}\in\mathscr{M}$, that is, $s_vw_1+t_vw_2\in\mathscr{M}$. Thus

$$I(s_vw_1+t_vw_2)\geq m. \tag{3.9}$$

Applying $I'(w)=0$ we obtain that $\langle I'(v),v^{\pm}\rangle<0$. By the arguments of Lemma 3.3, we obtain that $0<s_v,t_v\leq 1$. On the other hand, we can see that

$$\begin{aligned}
0&=\frac{1}{2q}\langle I'(w),w_3\rangle\\
&=\frac{1}{2q}\left(\|w_3\|_p^p+\|w_3\|_q^q+a|\nabla w_3|_p^{2p}+b|\nabla w_3|_q^{2q}-\int_\Omega f(w_3)w_3dx\right)\\
&+\frac{1}{2q}\left(a|\nabla w_1|_p^p|\nabla w_3|_p^p+a|\nabla w_2|_p^p|\nabla w_3|_p^p+b|\nabla w_1|_q^q|\nabla w_3|_q^q+b|\nabla w_2|_q^q|\nabla w_3|_q^q\right)\\
&<I(w_3)+\frac{1}{2q}\left(a|\nabla w_1|_p^p|\nabla w_3|_p^p+a|\nabla w_2|_p^p|\nabla w_3|_p^p+b|\nabla w_1|_q^q|\nabla w_3|_q^q+b|\nabla w_2|_q^q|\nabla w_3|_q^q\right).
\end{aligned} \tag{3.10}$$

Arguing similarly as (3.7), we conclude that

$$\begin{aligned}
&I(s_vw_1+t_vw_2)\\
&=I(s_vw_1+t_vw_2)-\frac{1}{2q}\langle I'(s_vw_1+t_vw_2),s_vw_1+t_vw_2\rangle\\
&=\left(\frac{1}{p}-\frac{1}{2q}\right)\left(\|s_vw_1\|_p^p+\|t_vw_2\|_p^p\right)+\frac{1}{2q}\left(\|s_vw_1\|_q^q+\|t_vw_2\|_q^q\right)\\
&\quad+\left(\frac{1}{2p}-\frac{1}{2q}\right)a|\nabla(s_vw_1+t_vw_2)|_p^{2p}+\frac{1}{2q}\int_\Omega[f(s_vw_1)s_vw_1-2qF(s_vw_1)]\,dx\\
&\quad+\frac{1}{2q}\int_\Omega[f(t_vw_2)t_vw_2-2qF(t_vw_2)]\,dx\\
&\le\left(\frac{1}{p}-\frac{1}{2q}\right)\left(\|w_1\|_p^p+\|w_2\|_p^p\right)+\frac{1}{2q}\left(\|w_1\|_q^q+\|w_2\|_q^q\right)\\
&\quad+\left(\frac{1}{2p}-\frac{1}{2q}\right)a|\nabla(w_1+w_2)|_p^{2p}+\frac{1}{2q}\int_\Omega[f(w_1)w_1-2qF(w_1)]\,dx\\
&\quad+\frac{1}{2q}\int_\Omega[f(w_2)w_2-2qF(w_2)]\,dx\\
&=I(w_1)+I(w_2)+\frac{a}{q}|\nabla w_1|_p^p|\nabla w_2|_p^p+\frac{a}{2q}|\nabla w_1|_p^p|\nabla w_3|_p^p+\frac{a}{2q}|\nabla w_2|_p^p|\nabla w_3|_p^p\\
&\quad+\frac{b}{q}|\nabla w_1|_q^q|\nabla w_2|_q^q+\frac{b}{2q}|\nabla w_1|_q^q|\nabla w_3|_q^q+\frac{b}{2q}|\nabla w_2|_q^q|\nabla w_3|_q^q.
\end{aligned}\tag{3.11}$$

It follows from (3.9)-(3.11) that

$$\begin{aligned}
m\le I(s_vw_1+t_vw_2)&<I(w_1)+I(w_2)+I(w_3)\\
&\quad+\frac{a}{q}|\nabla w_1|_p^p|\nabla w_2|_p^p+\frac{a}{q}|\nabla w_1|_p^p|\nabla w_3|_p^p+\frac{a}{q}|\nabla w_2|_p^p|\nabla w_3|_p^p\\
&\quad+\frac{b}{q}|\nabla w_1|_q^q|\nabla w_2|_q^q+\frac{b}{q}|\nabla w_1|_q^q|\nabla w_3|_q^q+\frac{b}{q}|\nabla w_2|_q^q|\nabla w_3|_q^q\\
&=I(w)=m,
\end{aligned}$$

which is a contradiction. This implies $u_3=0$. Therefore, w has exactly two nodal domains. □

Proof of Theorem 1.1. Theorem 1.1 follows immediately from a combination of Propositions 2.1 and 3.1. □

Lemma 3.4. *Assume that conditions (A_1)-(A_5) and (1.6) hold. Then, for any $u\in X\setminus\{0\}$, there exists a unique $t_0=t_0(u)>0$ such that $t_0u\in\mathcal{N}$.*

Proof. Let $u\in X\setminus\{0\}$ and define $h(t):=I(tu)$. Therefore, from the discussion in Lemma 2.2, we obtain that there exists $t_0>0$ such that $h'(t_0)=0$ and $t_0u\in\mathcal{N}$. Now, our goal is to prove that t_0 is the unique critical point of h. Arguing by contradiction, we assume that there are two positive constants $t_1<t_2$ such that $t_1u\in\mathcal{N}$, $t_2u\in\mathcal{N}$. Thus we obtain

$$t_1^{p-2q}\|u\|_p^p+t_1^{-q}\|u\|_q^q+at_1^{2p-2q}|\nabla u|_p^{2p}+b|\nabla u|_q^{2q}=\int_\Omega\frac{f(t_1u)(t_1u)}{t_1^{2q}}dx$$

and

$$t_2^{p-2q}\|u\|_p^p+t_2^{-q}\|u\|_q^q+at_2^{2p-2q}|\nabla u|_p^{2p}+b|\nabla u|_q^{2q}=\int_\Omega \frac{f(t_2u)(t_2u)}{t_2^{2q}}dx.$$

Subtracting the above two equations, we have

$$\begin{aligned}&(t_1^{p-2q}-t_2^{p-2q})\|u\|_p^p+(t_1^{-q}-t_2^{-q})\|u\|_q^q+a(t_1^{2p-2q}-t_2^{2p-2q})|\nabla u|_p^{2p}\\&=\int_\Omega\left[\frac{f(t_1u)(t_1u)}{t_1^{2q}}-\frac{f(t_2u)(t_1u)}{t_2^{2q}}\right]dx.\end{aligned}$$

Noting that $t_1<t_2,\ p<q$ and using (A_5), we obtain

$$\begin{aligned}0&<(t_1^{p-2q}-t_2^{p-2q})\|u\|_p^p+(t_1^{-q}-t_2^{-q})\|u\|_q^q+a(t_1^{2p-2q}-t_2^{2p-2q})|\nabla u|_p^{2p}\\&=\int_\Omega\left[\frac{f(t_1u)(t_1u)}{t_1^{2q}}-\frac{f(t_2u)(t_1u)}{t_2^{2q}}\right]dx<0.\end{aligned}$$

This contradiction indicates that $t_0>0$ is unique for any $u\in X\setminus\{0\}$. □

Proof of Theorem 1.2. Arguing similarly as Lemma 3.3, we can conclude that there exists $u^*\in\mathcal{N}$ such that $I(u^*)=c>0$. From [32, Corollary 2.9], the critical points of the functional I on $\mathcal{N}$ are the critical points of the functional I in X. That means $I'(u^*)=0$. Therefore, u^* is a ground state solution of (1.1). It follows from Theorem 1.1 that $w=w^++w^-$ is a nodal solution with two nodal domains. From Lemma 3.4, there exists unique t_{w^+},t_{w^-} such that $t_{w^+}w^+\in\mathcal{N},t_{w^-}w^-\in\mathcal{N}$. By (2.3) and Lemma 3.2, we obtain

$$\begin{aligned}2c&\le I(t_{w^+}w^+)+I(t_{w^-}w^-)\\&<I(t_{w^+}w^++t_{w^-}w^-)<I(w^++w^-)=m.\end{aligned}$$

The proof of Theorem 1.2 is completed. □

4. Regularity of Solutions of Problem (1.1)

In this section, we aim to obtain the regularity of nonnegative solutions to problem (1.1). More precisely, exploiting the Moser iteration argument [26], we are able to prove the boundedness of weak solutions to problem (1.1).

Proof of Theorem 1.3. Without loss of generality, we can assume that $u\ge 0$. For any $k,\ L>0$, we choose $\varphi=uu_L^{kp}$ as test function in (2.2), where $u_L:=\min\{u,L\}$. Thus

$$\begin{aligned}&(1+a|\nabla u|_p^p)\int_\Omega|\nabla u|^pu_L^{kp}dx+kp(1+a|\nabla u|_p^p)\int_\Omega|\nabla u|^{p-2}\nabla u\nabla u_Lu_L^{kp-1}udx\\&+(1+b|\nabla u|_q^q)\int_\Omega|\nabla u|^qu_L^{kp}dx+kp(1+b|\nabla u|_q^q)\int_\Omega|\nabla u|^{q-2}\nabla u\nabla u_Lu_L^{kp-1}udx\\&+\int_\Omega V(x)(|u|^{p-2}+|u|^{q-2})u\cdot u_L^{kp}dx=\int_\Omega f(u)uu_L^{kp}dx.\end{aligned}\tag{4.1}$$

From (A_2), we can see that

$$|f(u)|\le M_{13}|\nabla u|^{\frac{r-1}{r}}+M_{14}|u|^{r-1}+M_{15},$$

where $M_i > 0 (i = 13, 14, 15)$. Together with Young's inequality, we obtain that

$$\begin{aligned}
&\int_\Omega f(u)u \cdot u_L dx \\
&\leq M_{13} \int_\Omega \varepsilon^{\frac{p^*-1}{p^*}} |\nabla u|^{p\frac{p^*-1}{p^*}} u_L^{kp\frac{p^*-1}{p^*}} \varepsilon^{-\frac{p^*-1}{p^*}} u_L^{kp(1-\frac{p^*-1}{p^*})} u dx \\
&\quad + (M_{14} + M_{15}) \int_\Omega u^{p^*} u_L^{kp} dx + M_{15}|\Omega| \\
&\leq \varepsilon M_{13} \int_\Omega |\nabla u|^p u_L^{kp} dx + \left(M_{13}\varepsilon^{-(p^*-1)} + M_{14} + M_{15}\right) \int_\Omega u^{p^*} u_L^{kp} dx + M_{15}|\Omega|.
\end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), we obtain

$$\begin{aligned}
&\int_\Omega |\nabla u|^p u_L^{kp} dx + kp \int_{\{x\in\Omega: u(x)\leq L\}} |\nabla u|^p u_L^{kp} dx \\
&\leq \varepsilon M_{13} \int_\Omega |\nabla u|^p u_L^{kp} dx + \left(M_{13}\varepsilon^{-(p^*-1)} + M_{14} + M_{15}\right) \int_\Omega u^{p^*} u_L^{kp} dx + M_{15}|\Omega|.
\end{aligned} \tag{4.3}$$

Choosing $\varepsilon = \frac{1}{2M_{13}}$ in (4.3), we can see that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla u|^p u_L^{kp} dx + kp \int_{\{x\in\Omega: u(x)\leq L\}} |\nabla u|^p u_L^{kp} dx \\
&\leq \left(M_{13}\varepsilon^{-(p^*-1)} + M_{14} + M_{15}\right) \int_\Omega u^{p^*} u_L^{kp} dx + M_{15}|\Omega|.
\end{aligned} \tag{4.4}$$

Note that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\nabla u|^p u_L^{kp} dx + kp \int_{\{x\in\Omega: u(x)\leq L\}} |\nabla u|^p u_L^{kp} dx \\
&= \frac{1}{2}\int_{\{x\in\Omega: u(x)> L\}} |\nabla u|^p u_L^{kp} dx + (kp + \frac{1}{2}) \int_{\{x\in\Omega: u(x)\leq L\}} |\nabla u|^p u_L^{kp} dx \\
&\geq \frac{kp+1}{2(k+1)^p} \int_{\{x\in\Omega: u(x)> L\}} |\nabla u|^p u_L^{kp} dx + \frac{kp+1}{2} \int_{\{x\in\Omega: u(x)\leq L\}} |\nabla u|^p u_L^{kp} dx \\
&\geq \frac{kp+1}{2(k+1)^p} \int_\Omega |\nabla (u u_L^k)|^p dx,
\end{aligned} \tag{4.5}$$

recalling Bernoulli's inequality $(k+1)^p \geq kp+1$. Putting together (4.4) and (4.5), we can infer that

$$\frac{kp+1}{2(k+1)^p} \int_\Omega |\nabla (u u_L^k)|^p dx \leq \left(M_{13}\varepsilon^{-(p^*-1)} + M_{14} + M_{15}\right) \int_\Omega u^{p^*} u_L^{kp} dx + M_{15}|\Omega|. \tag{4.6}$$

Adding non-negative terms $\frac{kp+1}{2(k+1)^p} \int_\Omega V(x)|u u_L^k|^p dx$ to both ends of the equation (4.6), we obtain that

$$\frac{kp+1}{2(k+1)^p} \|u u_L^k\|_p^p \leq \frac{(kp+1)M_{16}}{2(k+1)^p} \int_\Omega |u u_L^k|^p dx + (kp+1)M_{17} \int_\Omega u^{p\cdots\cdots *} u_L^{kp} dx + M_{18}(kp+1)|\Omega| \tag{4.7}$$

Claim 2: $u \in L^s(\Omega)$ for any finite s.

Let us define $\nu := u^{p^*-p}$. Applying Hölder's inequality and (2.1), for any $R>0$, we conclude that

$$\begin{aligned}\int_\Omega u^{p^*}u_L^{kp}dx &= \int_{\{x\in\Omega:\ \nu(x)\le R\}}\nu u^p u_L^{kp}dx + \int_{\{x\in\Omega:\ \nu(x)>R\}}\nu u^p u_L^{kp}dx\\ &\le R\int_{\{x\in\Omega:\ \nu(x)\le R\}}u^p u_L^{kp}dx + \left(\int_{\{x\in\Omega:\ \nu(x)>R\}}\nu^{\frac{p^*}{p^*-p}}dx\right)^{\frac{p^*-p}{p^*}}\left(\int_\Omega u^{p^*}u_L^{kp^*}dx\right)^{\frac{p}{p^*}}\\ &\le R|uu_L^k|_p^p + \left(\int_{\{x\in\Omega:\ \nu(x)>R\}}\nu^{\frac{p^*}{p^*-p}}dx\right)^{\frac{p^*-p}{p^*}} c_{p^*}^p\|uu_L^k\|_p^p.\end{aligned} \tag{4.8}$$

Note that

$$\Psi(R) := \left(\int_{\{x\in\Omega:\nu(x)>R\}}\nu^{\frac{p^*}{p^*-p}}dx\right)^{\frac{p^*-p}{p^*}} \to 0 \text{ as } R\to\infty.$$

Combining with (4.7)-(4.8), we obtain

$$\begin{aligned}&\frac{kp+1}{2(k+1)^p}\|uu_L^k\|_p^p\\ &\le \left[\frac{(kp+1)M_{16}}{2(k+1)^p}+(kp+1)M_{17}R\right]|uu_L^k|_p^p + (kp+1)M_{17}\Psi(R)c_{p^*}^p\|uu_L^k\|_p^p + M_{18}(kp+1)|\Omega|.\end{aligned} \tag{4.9}$$

Take $R=R(k,u)>0$ such that

$$(kp+1)M_{17}\Psi(R)c_{p^*}^p = \frac{kp+1}{4(k+1)^p}.$$

Then, from (4.9), we infer that

$$\frac{kp+1}{4(k+1)^p}\|uu_L^k\|_p^p \le \left[\frac{(kp+1)M_{16}}{2(k+1)^p}+(kp+1)M_{17}R\right]|uu_L^k|_p^p + M_{18}(kp+1)|\Omega|,$$

which can be rewritten as

$$\|uu_L^k\|_p^p \le M_{19}(k,u)[|uu_L^k|_p^p+1]$$

with $M_{19}(k,u)>0$. Together with the Sobolev embedding theorem, we can obtain

$$|uu_L^k|_{p*} \le c_{p^*}\|uu_L^k\|_p \le M_{20}(k,u)[|uu_L^k|_p^p+1]^{\frac{1}{p}}. \tag{4.10}$$

Let us choose k_1 satisfying $(k_1+1)p=p^*$. Therefore, (4.10) becomes

$$|uu_L^k|_{p^*} \le M_{20}(k_1,u)[|uu_L^{k_1}|_p^p+1]^{\frac{1}{p}} \le M_{21}(k_1,u)[|u^{k_1+1}|_p^p+1]^{\frac{1}{p}} = M_{21}(k_1,u)\left[|u|_{p^*}^{p^*}+1\right]^{\frac{1}{p}} < \infty, \tag{4.11}$$

since $u_L\le u$ for a.e. $x\in\Omega$. Passing to the limit as $L\to\infty$ in (4.11), we obtain

$$|u|_{(k_1+1)p^*} = |u^{k_1+1}|_{p^*}^{\frac{1}{k_1+1}} \le M_{22}(k_1,u)\left[|u|_{p^*}^{p^*}+1\right]^{\frac{1}{(k_1+1)p}} < \infty. \tag{4.12}$$

Hence, $u\in L^{(k_1+1)p^*}(\Omega)$. We set

$$(k_{m+1}+1)p = (k_m+1)p^* \quad \text{and} \quad k_1+1 = \frac{p^*}{p}.$$

Repeating the steps from (4.10)-(4.12), we conclude that for any finite positive constant k

$$|u|_{(k+1)p^*} \le M_{23}(k,u) \tag{4.13}$$

where $M_{23}(k,u) > 0$ depends both on k and u. Therefore, $u \in L^s(\Omega)$ for any $s \in (1,\infty)$. □

Now, we prove that $u \in L^\infty(\Omega)$. Fix number $\theta \in (p, p^*)$. Applying Hölder's inequality, Claim 2 and (4.13), we see that

$$\begin{aligned} |uu_L^k|_p^p &\le |\Omega|^{\frac{\theta-p}{\theta}} \left(\int_\Omega (uu_L^k)^\theta dx\right)^{\frac{p}{\theta}} \le M_{24}|uu_L^k|_\theta^p, \\ \int_\Omega u^{p^*} u_L^{kp} dx &= \int_\Omega u^{p^*-p}(uu_L^k)^p dx \\ &\le \left(\int_\Omega u^{\frac{p^*-p}{\theta-p}\theta} dx\right)^{\frac{\theta-p}{\theta}} \left(\int_\Omega (uu_L^k)^\theta dx\right)^{\frac{p}{\theta}} \le M_{25}|uu_L^k|_\theta^p, \end{aligned} \tag{4.14}$$

where M_{25} is finite in the light of Claim 2. Combining (4.7) and (4.14), we obtain that

$$\frac{kp+1}{2(k+1)^p}\|uu_L^k\|_p^p \le \frac{kp+1}{2(k+1)^p} M_{26}|uu_L^k|_\theta^p + (kp+1)M_{27}|uu_L^k|_\theta^p + M_{28}k,$$

which implies

$$\|uu_L^k\|_p^p \le (k+1)^p M_{29}[|uu_L^k|_\theta^p + 1].$$

Together with (2.1) and Claim 2, we obtain that

$$|uu_L^k|_{p^*} \le c_{p^*}\|uu_L^k\|_p \le (k+1)M_{30}[|uu_L^k|_\theta^p + 1]^{\frac{1}{p}} \le (k+1)M_{31}[|u^{k+1}|_\theta^p + 1]^{\frac{1}{p}} < \infty. \tag{4.15}$$

Using Fatou's Lemma in (4.15), we obtain

$$|u|_{(k+1)p^*} = |u^{k+1}|_{p^*}^{\frac{1}{k+1}} \le (k+1)^{\frac{1}{k+1}} M_{31}^{\frac{1}{k+1}} [|u^{k+1}|_\theta^p + 1]^{\frac{1}{(k+1)p}}. \tag{4.16}$$

Note that

$$(k+1)^{\frac{1}{\sqrt{k+1}}} \ge 1, \quad \lim_{k\to\infty} (k+1)^{\frac{1}{\sqrt{k+1}}} = 1.$$

Thus, we can obtain a constant $M_{32} > 1$ such that

$$(k+1)^{\frac{1}{k+1}} \le M_{32}^{\frac{1}{\sqrt{k+1}}}.$$

Together with (4.16), we deduce that

$$|u|_{(k+1)p^*} \le M_{32}^{\frac{1}{\sqrt{k+1}}} M_{31}^{\frac{1}{k+1}} [|u^{k+1}|_\theta^p + 1]^{\frac{1}{(k+1)p}}. \tag{4.17}$$

Therefore, it suffices to prove the uniform boundedness concerning k. Indeed, assume that there exists a sequence $k_n \to \infty$ such that $|u^{k_n+1}|_\theta^p \le 1$, that is, $|u|_{(k_n+1)\theta} \le 1$, which implies that $|u|_\infty \le 1$. We consider the opposite case that there exists $k_0 > 0$ such that

$$|u^{k+1}|_\theta^p > 1, \quad \text{for any } k \ge k_0. \tag{4.18}$$

Putting together (4.17) and (4.18), we conclude that, for any $k \ge k_0$,

$$|u|_{(k+1)p^*} \le M_{32}^{\frac{1}{\sqrt{k+1}}} M_{31}^{\frac{1}{k+1}} [2|u^{k+1}|_\theta^p]^{\frac{1}{(k+1)p}} \le M_{33}^{\frac{1}{\sqrt{k+1}}} M_{34}^{\frac{1}{k+1}} |u|_{(k+1)\theta}. \tag{4.19}$$

We set

$$(k_{n+1}+1)\theta = (k_n+1)p^*.$$

Particularly,

$$k_n+1=(k_0+1)\left(\frac{p^*}{\theta}\right)^n \quad \text{and } \lim_{n\to\infty} k_n=+\infty. \tag{4.20}$$

Then (4.19) becomes

$$|u|_{(k_n+1)p^*}\le M_{33}^{\frac{1}{\sqrt{k_n+1}}}M_{34}^{\frac{1}{k_n+1}}|u|_{(k_n+1)\theta}=M_{33}^{\frac{1}{\sqrt{k_n+1}}}M_{34}^{\frac{1}{k_n+1}}|u|_{(k_{n-1}+1)p^*},$$

from which we see that

$$|u|_{(k_n+1)p^*}\le M_{33}^{\sum_{i=1}^n\frac{1}{\sqrt{k_n+1}}}M_{34}^{\sum_{i=1}^n\frac{1}{k_n+1}}|u|_{(k_0+1)p^*}.$$

It follows from (4.20) and Claim 2 that there exists $M_{35}>0$ such that

$$|u|_{(k_n+1)p^*}\le M_{35}|u|_{(k_0+1)p^*}<\infty.$$

Taking the limit as $n\to\infty$, we have $|u|_\infty\le C$.

Acknowledgments

J. Yang was supported by Natural Science Foundation of Hunan Province of China (No. 2023JJ 30482 and No. 2022JJ30463), Research Foundation of Hunan Provincial Education Department (No. 22A0540), and Huaihua University Double First-Class Initiative Applied Characteristic Discipline of Control Science and Engineering. H. Chen was supported by the National Natural Science Foundation of China (No. 12071486).

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