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ON THE EXISTENCE AND UNIQUENESS OF POSITIVE PERIODIC SOLUTIONS OF NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we investigate a nonlinear neutral differential equation of first order with iterative terms and constant time delays. We obtain new results on the existence and uniqueness of positive periodic solutions of the nonlinear neutral differential equation such that the solution depends on the functions of that the nonlinear neutral differential equation. The proof of the new results depends on some fixed point theorems and the Green's functions. We also provide a numerical example to demonstrate the conditions of the new results can be satisfied and applied.
Keywords. Neutral differential equation; Positive periodic solutions; Time delays; Green's functions.

1. INTRODUCTION

During the recent years, many authors investigated the existence and uniqueness of positive periodic solutions (EUPPSs) of various kind of neutral differential equations (NDEs) of first order. The NDEs appear in various real-world applications, for example, in biology, more specifically in blood cell production, engineering, more specifically in control models, and so on. Recently, the EUPPSs of NDEs first order and some qualitative problems with numerous NDEs were investigated by many researchers and numerous interesting and valuable results were obtained; see the literatures therein.

In 2022, Mezghiche et al. [1] considered the following scalar the NDE of first order with iterative terms, constant and time delays: $\frac{d}{dt}[x(t)-cx(t-\tau)] = A(.)-H(...)$, where $A(.) = f(t,x(t-\tau(t)))-a(t)x(t)$, and $H(...) = H(t,x(t),x^{[2]}(t),...,x^{[n]}(t))$, and they proved two new results in relation to the EUPPSs such that the solution depends upon the functions of the NDE above.

In this paper, in place of the NDE above, we investigate the following NDE of first order with iterative terms and two multiple time delays:

$$\frac{d}{dt}\left[x(t)-\sum_{i=1}^{2}c_i(t)x(t-\tau_i)\right] = -a(t)x(t)+\sum_{i=1}^{2}f_i(t,x(t),x(t-\tau_i))-H(...), \tag{1.1}$$

where $x\in R$, $\tau_i>0$, $\tau_i\in R$, $H(.) = H(t,x(t),x^{[2]}(t),...,x^{[n]}(t))$, $x^{[n]}(t)$ is the iterative term and stands for x composed with itself n times, for example, $x^{[2]}(t) = x(x(t))$, $c_i\in C([0,T],\ (0,1))$,

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$a \in C(R,(0,\infty))$, and $f \in C([0,T]\times R^2,\ (0,\infty))$ and $H \in C([0,T]\times R^n,(0,\infty))$ are periodic functions, i.e., $c_i(t+T) = c_i(t)$, $a(t+T) = a(t)$, $f_i(t+T,x(t),x(t-\tau_i)) = f_i(t,x(t),x(t-\tau_i))$ and $H(t+T,x(t),x^{[2]}(t),...,x^{[n]}(t)) = H(t,x(t),x^{[2]}(t),...,x^{[n]}(t))$. Furthermore, we presume that the functions f_i and $H(...)$ satisfy the Lipschitz condition in their respective arguments;

$$\begin{gathered} |f_1(t,U) - f_1(t,V)| \leq k_1 |u_1 - v_1| + k_2 |u_2 - v_2|, \\ U = u_1, u_2 \text{ and } V = v_1, v_2, \\ |f_2(t,\bar{U}) - f_2(t,\bar{V})| \leq k_3 |u_3 - v_3| + k_4 |u_4 - v_4|, \\ \bar{U} = u_3, u_4 \text{ and } \bar{V} = v_3, v_4, \end{gathered} \tag{1.2}$$

and

$$|H(t,X) - H(t,Y)| \leq \sum_{i=1}^{n} l_i |x_i - y_i|, \tag{1.3}$$

where k_i and l_i are positive constants, and $X = (x_1,...,x_n)$ and $Y = (y_1,...,y_n)$.

The main motivation of this paper is raised from the results of Mezghiche et al. [1] and some related results in the literature [1]-[19]. We now outline some papers on the qualitative behaviors of various NDEs.

In [14], Serra investigated the investigation of the problem of the EPSs for the following NDE $x'(t) + ax'(t-\tau) = f(t,x(t))$ by using continuation theorems. In [15], Luo et al. studied the EUPPSs for the following NDE $\frac{d}{dt}[x(t) - cx(t-\tau(t))] = F(.)$, where $F(.) = f(t,x(t-\tau(t)) - a(t)x(t)$. Their main results were proved based on a fixed-point theorem. In [16], Tunç considered the following nonlinear NDE $\frac{d}{dt}[x(t) + px(t-\tau)] = G(.)$, where $G(.) = -h(x(t)) + q(t)\tanh x(t-\tau)$, and the nonlinear NDE with multiple delays $\frac{d}{dt}[x(t) + \sum_{i=1}^{2} c_i(t)x(t-\tau_i)] = K(.)$, where $K(.) = q(t)\tanh x(t-\sigma) - p(t)x(t)$. The author also investigated the qualitative features in relation to the solutions of these NDEs of first order by using suitable Lyapunov-Krasovskii functionals. In [17], Candan discussed the EUPPSs with ω-period of the NDE of the first order $\frac{d}{dt}[x(t) - P(t)x(t-\tau)] = L(.)$, where $L(.) = f(t,x(t-\tau)) - Q(t)x(t)$. Candan employed the KFP theorem and also construct an example for the NDE. Using the KFP theorem, Zhao and Liu [18] discussed the same problem in relation to the iterative functional differential equations (FDEs) $x'(t) = M(t) + F(t)$, where $M(t) = c_1(t)x(t) + c_2(t)x^{[2]}(t) + ... + c_n(t)x^{[n]}(t)$. In [19], Mansouri et al. considered non-autonomous iterative scalar FDEs of the type $x'(t) = H(.)$ They discussed the EUPPSs by using Schauder's FP and BFP theorems, respectively.

In this paper, we aim to obtain two new results in relation to the EUPPSs of the NDE (1.1). As it is seen from the literature review given above, the qualitative behaviors of solutions of numerous NDEs and FDEs have been investigated recently by using various methods. However, it is noticed that NDE (1.1) is different from the NDEs above. In view of these, we would like to point out that the NDE (1.1) is a new mathematical model for FDEs and it includes some of the NDEs above. Investigating the qualitative behaviors of solutions of the NDE (1.1) is a new contribution and improvement in relation to the related topic and literature. Next, in this article Banach's fixed point theorem (BFP theorem) and Krasnoselskii's fixed point theorem (KFP theorem) and the Green's functions method were used together. These methods are completely different from the second method of Lyapunov and Lyapunov-Krasovskii method. The technique and method of this article are interesting and novel approaches. Hence, our results do new admixtures to the qualitative features of NDEs and enhance certain novel qualitative outcomes in relation to NDEs that have been achieved in the relevant literature for the case of the constant time delays. The generalization and improvement of some former relative results of the literature are also the new contributions of this article for multiple the constant time delays.

2. Preliminaries

For $T,\ M,\ L>0$ and $m\geq 0$, let $P_T=\{x: x\in C(R,R)\}$, where x is a periodic function with the period T, be a Banach space endowed with $\|x\|=\sup_{t\in[0,T]}|x(t)|$, c_i is a periodic function with the period T, $0\leq c_i(t)\leq c_i$, $0<c_i<1$, $i=1,2$, $c=\max\{c_1,c_2\}$, $P_T(L,m,M)=P_T(.)$ with $P_T(.)=\{x: x\in P_T\}$ such that $m\leq x(t)\leq M$, $M>0$, $M\in R$, and $|x(t_2)-x(t_1)|\leq L|t_2-t_1|$, $L>0$, $L,t_1,t_2\in R$. Hence, it can be guaranteed that $P_T(.)$ is closed, convex, bounded, and $P_T(.)\subset P_T$. Then, $P_T(.)$ is compact (see Mezghiche et al. [1]).

For convenience, we introduce the notations below:

$$a_1=\max_{t\in[0,T]} a(t),\ f_1=\max_{t\in[0,T]}|f(t,0)|,\ H_1=\max_{t\in[0,T]}|\bar{H}(.)|,\bar{H}=H(t,0,0,...,0),$$

and

$$\eta_1=\frac{\exp(-A)}{\exp(A)-1},\eta_2=\frac{\exp(A)}{\exp(A)-1},\Lambda=\sum_{i=1}^{n} l_i\sum_{j=0}^{i-1}L^j,$$

where $A=\int_0^T a(u)du$.

In this paper, we construct the following conditions below and demonstrate that they can be satisfied.

Let $\delta_1,\ \delta_2,\ c,\ k,\ m,\ M,\ L,\ \Lambda$, and $L\in R$, which are positive and

$$f(t,u_i,v_i)\geq\delta_i>0, i=1,\ 2,\ \forall t\in R,\forall u_1,u_2,v_1,v_2\in(0,\infty), \tag{2.1}$$

$$2[cM+\eta_2T(kM+f_1)]\leq M, \tag{2.2}$$

and

$$2\eta_1T\delta_0-\eta_2T(M\Lambda+H_1)-2cT\eta_2a_1M+2cm\geq m, \tag{2.3}$$

where $\delta_0=\min\{\delta_1,\delta_2\}$ and

$$2\left\{\eta_2(2+a_1T)\left[\frac{1}{2}H_1+f_1+M\left(k+\frac{1}{2}\Lambda+ca_1\right)\right]+Lc\right\}\leq L. \tag{2.4}$$

We first give the following lemma, which is needed later.

Lemma 2.1. *$x\in P_T(.)\cap C^1(R,R)$ and x is a solution to the NDE (1.1) $\Leftrightarrow x\in P_T(.)$ satisfies the following integral equation:*

$$x(t)=\int_t^{t+T}G(t,s)\left\{\sum_{i=1}^{2}f_i(s,x(s),x(s-\tau_i))-H(...)-\sum_{i=1}^{2}c_i(s)a(s)x(s-\tau_i)\right\}ds+\sum_{i=1}^{2}c_i(t)x(t-\tau_i),$$

where $H(...)=H(s,x(s),x^{[2]}(s),...,x^{[n]}(s))$ and

$$G(t,s)=\frac{\exp\left(\int_t^s a(u)du\right)}{\exp\left(\int_t^{t+T}a(u)du\right)-1}.$$

Proof. Assume that $x \in P_T(.) \cap C^1(R,R)$. Next, let x be a solution of to NDE (1.1). Multiplying both sides of the NDE of this research by the term $\exp\left(\int_0^t a(u)du\right)$, we find

$$\frac{d}{dt}\left[(x(t)-\sum_{i=1}^{2}c_i(t)x(t-\tau_i))\exp(\int_0^t a(u)du)\right]$$
$$=\left[\sum_{i=1}^{2}f_i(t,x(t),x(t-\tau_i))-H(.)\right]\exp(\int_0^t a(u)du)-\sum_{i=1}^{2}c_i(t)a(t)x(t-\tau_i)\exp(\int_0^t a(u)du).$$

Hence, integrating the NDE above on the interval $[t,t+T]$ and using periodic property of the solution, we derive

$$\int_t^{t+T}\frac{d}{dt}[x(s)-\sum_{i=1}^{2}c_i(s)x(s-\tau_i)\exp(\int_0^s a(u)du)]ds$$
$$=\int_t^{t+T}[\sum_{i=1}^{2}f_i(s,x(s),x(s-\tau_i))-H(\dots)-\sum_{i=1}^{2}c_i(s)a(s)x(s-\tau_i)]\exp(\int_0^s a(u)du))ds.$$

It follows that

$$x(t)-\sum_{i=1}^{2}c_i(t)x(t-\tau_i)$$
$$=\int_t^{t+T}\frac{e^{\int_0^s a(u)du}}{e^{\int_0^{t+T} a(u)du}-e^{\int_0^t a(u)du}}\left[\sum_{i=1}^{2}f_i(s,x(s),x(s-\tau_i))-H(\dots)-\sum_{i=1}^{2}c_i(s)a(s)x(t-\tau_i)\right]ds$$
$$=\int_t^{t+T}G(t,s)\left[\sum_{i=1}^{2}f_i(s,x(s),x(s-\tau_i))-H(\dots)-\sum_{i=1}^{2}c_i(s)a(s)x(t-\tau_i)\right]ds,$$

which in turn yields that

$$x(t)=\int_t^{t+T}G(t,s)\left[\sum_{i=1}^{2}f_i(s,x(s),x(s-\tau_i))-H(\dots)\right.$$
$$\left.-\sum_{i=1}^{2}c_i(s)a(s)x(t-\tau_i)\right]ds+\sum_{i=1}^{2}c_i(t)x(t-\tau_i).$$

According to the discussion above, this relation is the desired result and the end of the proof. □

Remark 2.1. It is known from Zhao and Liu [18] that $\|x^{[v]}-y^{[v]}\| \le \sum_{k=0}^{v-1}L^k\|x-y\|$, where $v \in \mathbb{N}^+$ and $x,y \in P_T(.)$.

Remark 2.2. Observe that $0<\eta_1 \le G(t,s) \le \eta_2$, $G(t,s)$ is T periodic in $t,s \in R$, and

$$\int_{t_1}^{t_1+T}|G(t_2,s)-G(t_1,s)|\,ds \le Ta_1\eta_2|t_2-t_1|,$$

where t,s,t_1, and t_2 are in R. By virtue of Remark 2.1 and estimates (1.2) and (1.3), we find $|f_i(s,x(s),x(s-\tau_i))| \le kM+f_1, (i=1,2)$ and $|H(\dots)| \le M\Lambda+H_1$ for all $x \in P_T(.)$.

3. Qualitative Results

In this section, we utilize KFP theorem for the NDE (1.1). For this, according to Lemma 2.1, let $N : P_T(.) \to P_T$ be an operator, which is described by $(Nx)(t) = (F_1x)(t) + (F_2x)(t)$, where $F_1 : P_T(.) \to P_T$ and $F_2 : P_T(.) \to P_T$ are defined by

$$(F_1x)(t) = \int_t^{t+T} G(t,s)\{\sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i)) - H(\ldots) - \sum_{i=1}^{2} c_i(s)a(s)x(s-\tau_i)\}ds$$

and $(F_2x)(t) = \sum_{i=1}^{2} c_i(t)x(t-\tau_i)$. Hence, the existence of solutions of the NDE (1.1) is determined by means of whether the N has a fixed point or not. Because of the inequality $0 \le c_i \le c < 1$, F_2 is a contraction. We now need to achieve that operator F_1 is continuous, compact, and $F_1x + F_2y \in P_T(.)$ for all $x, y \in P_T(.)$.

We now prove that F_1 is compact and the continuous. Let $H(..) = H(s, y(s), y^{[2]}(s), \ldots, y^{[n]}(s))$.

Lemma 3.1. *The operator F_1, defined by $P_T(.)$ to P_T, is continuous and compact.*

Proof. Since $P_T(.) \subset P_T$ and P_T is a compact, F_1 is compact, which can be seen from its continuity. Next, we prove that F_1 is continuous. Indeed,

$$\begin{aligned}
(F_1x)(t) - (F_1y)(t) &= \int_t^{t+T} G(t,s)\{\sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i)) - H(\ldots) \\
&\quad - \sum_{i=1}^{2} c_i(s)a(s)x(s-\tau_i)\}ds - \int_t^{t+T} G(t,s)\{\sum_{i=1}^{2} f_i(s,y(s),y(s-\tau_i)) \\
&\quad - H(..) - \sum_{i=1}^{2} c_i(s)a(s)y(s-\tau_i)\}ds \\
&\le \int_t^{t+T} |f_1(s,x(s),x(s-\tau_1)) - f_1(s,y(s),y(s-\tau_1))|\, G(t,s)ds \\
&\quad + \int_t^{t+T} |f_2(s,x(s),x(s-\tau_2)) - f_2(s,y(s),y(s-\tau_2))|\, G(t,s)ds \\
&\quad + \int_t^{t+T} G(t,s)\, |H\,(\ldots) - H(..\,)|\, ds \\
&\quad + \int_t^{t+T} a(s) \left| \sum_{i=1}^{2} c_i(s)x(s-\tau_i) - \sum_{i=1}^{2} c_i(s)y(s-\tau_i) \right| G(t,s)ds,
\end{aligned}$$

where $x, y \in P_T(.)$.

Next, it follows from (1.2), (1.3), and $0 < \eta_1 \le G(t,s) \le \eta_2$ that

$$\begin{aligned}\|F_1x - F_1y\| \le & \int_t^{t+T} k_1\,|x(s)-y(s)|\,G(t,s)ds + \int_t^{t+T} k_2\,|x(s-\tau_1)-y(s-\tau_1)|\,G(t,s)ds \\ & + \int_t^{t+T} k_3\,|x(s)-y(s)|\,G(t,s)ds + \int_t^{t+T} k_4\,|x(s-\tau_2)-y(s-\tau_2)|\,G(t,s)ds \\ & + \int_t^{t+T} \eta_2 \sum_{i=1}^{n} l_i\,\|x_i - y_i\|ds + \int_t^{t+T} \{a_1c_1\eta_2\,\|x-y\| + a_1c_2\eta_2\,\|x-y\|\}ds \\ & \le 4k\eta_2T\,\|x-y\| + \eta_2T\Lambda\,\|x-y\| + 2\eta_2ca_1T\,\|x-y\| \\ & = 2\eta_2T\left(2k + ca_1 + \frac{\Lambda}{2}\right)\|x-y\|,\end{aligned}$$

where $k = \max\{k_1,k_2,k_3,k_4\}$. This result verifies that F_1 is Lipschitz continuous. This proves that F_1 is compact. □

Lemma 3.2. *If* (2.1)-(2.4) *are fulfilled, then* $F_1x + F_2y \in P_T(.)$, *where* $x,y \in P_T(.)$.

Proof. Let $x,y \in P_T(.)$. Using the estimates above, we derive

$$\begin{aligned}(F_1x)(t) + (F_2y)(t) = & \int_t^{t+T} G(t,s)\{\sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i)) - H(...) \\ & - \sum_{i=1}^{2} c_i(s)a(s)x(s-\tau_i)\}ds + \sum_{i=1}^{2} c_i(t)y(t-\tau_i) \\ & \le \sum_{i=1}^{2} c_i(t)y(t-\tau_i) + \int_t^{t+T} G(t,s)\sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i))ds \\ & \le 2cM + 2\eta_2(kM + f_1)T \le M.\end{aligned}$$

Similarly, we obtain from (2.1) and (2.3) that

$$\begin{aligned}(F_1x)(t) + (F_2y)(t) = & \int_t^{t+T} G(t,s)\{\sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i)) - H(...) \\ & - \sum_{i=1}^{2} c_i(s)a(s)x(s-\tau_i)\}ds + \sum_{i=1}^{2} c_i(t)y(t-\tau_i) \\ & \ge \int_t^{t+T} 2\eta_1\delta_0 ds - \int_t^{t+T} \eta_2(M\Lambda + H_1)ds - \int_t^{t+T} \eta_2 2ca_1M ds + 2cm \\ & = 2\eta_1\delta_0T - \eta_2(M\Lambda + H_1)T - 2\eta_2ca_1MT + 2cm \ge m.\end{aligned}$$

Consequently,

$$m \le (F_1x)(t) + (F_2y)(t) \le M \tag{3.1}$$

for all $x,y \in P_T(.)$.

Let $x, y \in P_T(.)$, $t_1,\ t_2 \in [0,T]$ with $t_1 < t_2$ and $H(\ldots) = H(s, x(s), x^{[2]}(s), \ldots, x^{[n]}(s))$. Then, for all $x,\ y \in P_T(.)$, we have

$$\begin{aligned}
&|((F_1x)+(F_2y))(t_2) - ((F_1x)+(F_2y))(t_1)| \\
&\le \left| \int_{t_2}^{t_2+T} G(t_2,s) \sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i)) - \int_{t_1}^{t_1+T} G(t_1,s) \sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i))ds \right| \\
&+ \left| \int_{t_2}^{t_2+T} G(t_2,s)H(\ldots)ds - \int_{t_1}^{t_1+T} G(t_1,s)H(\ldots)ds \right| + \left| \int_{t_2}^{t_2+T} G(t_2,s) \sum_{i=1}^{2} c_i(s)a(s)x(s-\tau_i)ds \right. \\
&\left. - \int_{t_1}^{t_1+T} G(t_1,s) \sum_{i=1}^{2} c_i(s)a(s)x(s-\tau_i)ds \right| + \left| \sum_{i=1}^{2} c_i(t_2)y(t_2-\tau_i) - \sum_{i=1}^{2} c_i(t_1)y(t_1-\tau_i) \right|.
\end{aligned}$$

According to the above outcomes, we have

$$\begin{aligned}
&\left| \int_{t_2}^{t_2+T} G(t_2,s) \sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i)) - \int_{t_1}^{t_1+T} G(t_1,s) \sum_{i=1}^{2} f_i(s,x(s),x(s-\tau_i))ds \right| \\
&\le 2 \left| \int_{t_2}^{t_1} \eta_2(kM+f_1)ds \right| + 2 \left| \int_{t_1+T}^{t_2+T} \eta_2(kM+f_1)ds \right| \\
&\quad + 2 \int_{t_1}^{t_1+T} |G(t_2,s) - G(t_1,s)| (kM+f_1)ds \\
&= 2(2+a_1T)\eta_2(kM+f_1)\, |t_2 - t_1|.
\end{aligned}$$

By virtue of the above discussions, we notice that

$$\begin{aligned}
&\left| \int_{t_2}^{t_2+T} G(t_2,s)H(\ldots)ds - \int_{t_1}^{t_1+T} G(t_1,s)H(.)ds \right| \\
&= \left| \int_{t_2}^{t_1} G(t_2,s)H(\ldots) + \int_{t_1}^{t_1+T} G(t_2,s)H(\ldots) + \int_{t_1+T}^{t_2+T} G(t_2,s)H(\ldots) - \int_{t_1}^{t_1+T} G(t_1,s)H(\ldots)ds \right| \\
&\le \left| \int_{t_2}^{t_1} \eta_2(M\Lambda+H_1)ds \right| + \left| \int_{t_1+T}^{t_2+T} \eta_2(M\Lambda+H_1)ds \right| + \int_{t_1}^{t_1+T} |G(t_2,s) - G(t_1,s)| (M\Lambda+H_1)ds \\
&= (2+a_1T)\eta_2(M\Lambda+H_1)\, |t_2 - t_1|,
\end{aligned}$$

where $H(\ldots) = H(s, x(s), x^{[2]}(s), \ldots, x^{[n]}(s))$. In view of $0 < \eta_1 \leq G(t,s) \leq \eta_2$, we obtain

$$\begin{aligned}
&\left| \int_{t_2}^{t_2+T} G(t_2,s) \sum_{i=1}^{2} c_i(s)a(s)x(s-\tau_i). - \int_{t_1}^{t_1+T} G(t_1,s) \sum_{i=1}^{2} c_i(s)a(s)x(s-\tau_i)ds \right| \\
&\leq \left| \int_{t_2}^{t_1} G(t_2,s) \sum_{i=1}^{2} c_i(s)a(s)x(s-\tau_i))ds \right| + \left| \int_{t_1+T}^{t_2+T} G(t_2,s) \sum_{i=1}^{2} c_i(s)a(s)x(s-\tau_i)ds \right| \\
&\quad + \int_{t_1}^{t_1+T} |G(t_2,s) - G(t_1,s)| \sum_{i=1}^{2} c_i(s)a(s)x(s-\tau_i)ds \\
&\leq \left| \int_{t_2}^{t_1} \eta_2 a_1 2cMds \right| + \left| \int_{t_1+T}^{t_2+T} \eta_2 a_1 2cMds \right| + 2cMa_1 \int_{t_1}^{t_1+T} |G(t_2,s) - G(t_1,s)|\, ds \\
&= 2(2 + a_1 T)\eta_2 cMa_1 \left| t_2 - t_1 \right|.
\end{aligned}$$

Since $x, y \in P_T(.)$, then

$$\begin{aligned}
&\left| \sum_{i=1}^{2} c_i(t_2)y(t_2-\tau_i) - \sum_{i=1}^{2} c_i(t_1)y(t_1-\tau_i) \right| \\
&\leq c\,|y(t_2-\tau_1) - y(t_1-\tau_1)| + c\,|y(t_2-\tau_2) - y(t_1-\tau_2)| = 2Lc\,|t_2 - t_1|.
\end{aligned}$$

Thus, it follows from (2.4) that

$$\begin{aligned}
|((F_1x) + (F_2y))(t_2) - ((F_1x) + (F_2y))(t_1)| \leq & 2(2 + a_1T)\eta_2(kM + f_1)\,|t_2 - t_1| \\
& + (2 + a_1T)\eta_2(M\Lambda + H_1)\,|t_2 - t_1| \\
& + 2(2 + a_1T)\eta_2 cMa_1\,|t_2 - t_1| \\
& + 2Lc\,|t_2 - t_1| \\
\leq & L\,|t_2 - t_1|.
\end{aligned}$$

From (3.1), we obtain the desired conclusion immediately. □

Theorem 3.1. *Let $\tau \in P_T(.)$. If (2.1)-(2.4) are satisfied, then the NDE (1.1) includes at least one PPS, which remains in $P_T(.)$.*

Proof. From Lemmas 3.1 and 3.2, the existence of at the least one solution is guaranteed for the NDE (1.1). Here, the details of the proof is omitted here. □

4. Existence and Uniqueness of Solutions

In this section, we construct two new theorems on the existence and uniqueness of solutions.

Theorem 4.1. *Let $\tau \in P_T(.)$. If (2.1)-(2.4) and the estimate*

$$2T\eta_2(2k + \frac{\Lambda}{2} + ca_1) + c < 1 \tag{4.1}$$

are fulfilled, then the NDE (1.1) has a unique PPS $x \in P_T(L, m, M)$.

Proof. Let $x,y \in P_T(.)$. According to Lemmas 3.1 and 3.2, we infer $N(P_T(.)) \subset P_T(.)$ and have

$$\|(Nx)-(Ny)\| \leq \left(2T\eta_2(2k+\frac{\Lambda}{2}+ca_1)+c\right)\|x-y\|.$$

From (4.1) and the BFP theorem, we have N is a contraction and it has a unique fixed point. It can be fulfilled that this is the unique solution of the NDE (1.1). □

Remark 4.1.

$$G_1(t,s)=\frac{e^{\int_t^s v_1(u)du}}{e^{\int_{t+T}^{t} v_1(u)du}-1}, G_1(t,s)=\frac{e^{\int_t^s v_1(u)du}}{e^{\int_{t+T}^{t} v_1(u)du}-1},$$

and

$$\int_t^{t+T} |G_1(t,s)-G_2(t,s)|\,ds \leq \mu\,\|v_1-v_2\|, \tag{4.2}$$

where

$$\mu=\frac{T^2e^{T(\|v_2\|+\max(\|v_1\|,\|v_2\|))}}{(e^{\int_{t+T}^{t} v_1(u)du}-1)(e^{\int_0^T v_2(u)du}-1)}+\frac{T^2e^{T\max(\|v_1\|,\|v_2\|)}}{e^{\int_{t+T}^{t} v_1(u)du}-1}.$$

Theorem 4.2. *If the conditions of Theorem 4.1 are fulfilled, then the unique solution of the NDE (1.1) depends upon continuously the functions a, f and H.*

Proof. Assume that G_1 and G_2 are given as in Remark 4.1 and Let

$$x_1(t)=\int_t^{t+T} G_1(t,s)\left\{\sum_{i=1}^{2} f_i(s,x_1(s),x_2(s-\tau_i))-H_1(\ldots)-\sum_{i=1}^{2} c_i(s)v_1(s)x_1(s-\tau_i)\right\}ds$$
$$+\sum_{i=1}^{2} c_i(t)x_1(t-\tau_i),$$

and

$$x_2(t)=\int_t^{t+T} G_2(t,s)\left\{\sum_{i=1}^{2} f_i(s,x_2(s),x_2(s-\tau_i))-H_2(\ldots)-\sum_{i=1}^{2} c_i(s)v_2(s)x_2(s-\tau_i)\right\}ds$$
$$+\sum_{i=1}^{2} c_i(t)x_2(t-\tau_i),$$

with

$$H_1(\ldots)=H_1(s,x_1(s),{x_1}^{[2]}(s),\ldots,{x_1}^{[n]}(s))$$

and

$$H_2(\ldots)=H_2(s,x_2(s),{x_2}^{[2]}(s),\ldots,{x_2}^{[n]}(s)),$$

be two different solutions of the NDE (1.1). Then,

$$
\begin{aligned}
&|x_1(t)-x_2(t)| \\
&\leq \int_t^{t+T} \left| G_1(t,s)\sum_{i=1}^{2} f_i(s,x_1(s),x_1(s-\tau_i) - G_2(t,s)\sum_{i=1}^{2} f_i(s,x_2(s),x_2(s-\tau_i))ds \right| \\
&\quad + \int_t^{t+T} |G_1(t,s)H_1(\ldots)-G_2(t,s)H_2(\ldots)|\,ds \\
&\quad + \int_t^{t+T} \left| G_1(t,s)\sum_{i=1}^{2} c_i(s)v_1(s)x_1(s-\tau_i) - G_2(t,s)\sum_{i=1}^{2} c_i(s)v_2(s)x_2(s-\tau_i)ds \right| \\
&\quad + \left| \sum_{i=1}^{2} c_i(t)x_1(t-\tau_i) - \sum_{i=1}^{2} c_i(t)x_2(t-\tau_i) \right|.
\end{aligned}
$$

Observe from (2.2) and (4.2) that

$$
\begin{aligned}
&\int_t^{t+T} \left| G_1(t,s)\sum_{i=1}^{2} f_i(s,x_1(s),x_1(s-\tau_i)) - G_2(t,s)\sum_{i=1}^{2} f_i(s,x_2(s),x_2(s-\tau_i))ds \right| \\
&\leq \int_t^{t+T} |G_1(t,s)f_1(s,x_1(s),x_1(s-\tau_1)) - G_2(t,s)f_1(s,x_2(s),x_2(s-\tau_1))|\,ds \\
&\quad + \int_t^{t+T} |G_1(t,s)f_2(s,x_1(s),x_1(s-\tau_2)) - G_2(t,s)f_2(s,x_2(s),x_2(s-\tau_2))|\,ds \\
&\leq \int_t^{t+T} |G_1(t,s)(kM+f_1) - G_2(t,s)(kM+f_1)|\,ds \\
&\quad + \int_t^{t+T} |G_1(t,s)(kM+f_1) - G_2(t,s)(kM+f_1)|\,ds \\
&= 2(kM+f_1)\int_t^{t+T} |G_1(t,s)-G_2(t,s)|\,ds \\
&\leq 2(kM+f_1)\mu\,\|v_1-v_2\|.
\end{aligned} \tag{4.3}
$$

It can be seen from [1] that

$$
\begin{aligned}
&\int_t^{t+T} |G_1(t,s)H_1(\ldots)-G_2(t,s)H_2(\ldots)|\,ds \\
&\leq T\eta_2\|H_1-H_2\| + \mu(\Lambda M+H_1)\|v_1-v_2\| + T\eta_2\Lambda\|x-y\|.
\end{aligned} \tag{4.4}
$$

It follows from (4.2) that

$$
\begin{aligned}
&\int_t^{t+T}\left|G_1(t,s)\sum_{i=1}^{2}c_i(s)v_1(s)x_1(s-\tau_i)-G_2(t,s)\sum_{i=1}^{2}c_i(s)v_2(s)x_2(s-\tau_i)ds\right|\\
&\le \int_t^{t+T} c\|v_1\|\eta_2\|x_1-x_2\|ds+\int_t^{t+T} c\|v_1\|M|G_1(t,s)-G_2(t,s)|ds+\int_t^{t+T} cM\eta_2\|v_1-v_2\|ds\\
&\quad+\int_t^{t+T} c\|v_1\|\eta_2\|x_1-x_2\|ds+\int_t^{t+T} c\|v_1\|M|G_1(t,s)-G_2(t,s)|ds+\int_t^{t+T} cM\eta_2\|v_1-v_2\|ds\\
&\le 2\eta_2 cT\|v_1\|\|x_1-x_2\|+2cM(\|v_1\|\mu+T\eta_2)\|v_1-v_2\|.
\end{aligned}
\tag{4.5}
$$

Using $c_i(t)\le c_i,\ i=1,2, c=\max\{c_1,c_2\}$, we derive

$$
\begin{aligned}
&\left|\sum_{i=1}^{2}c_i(t)x_1(t-\tau_i)-\sum_{i=1}^{2}c_i(t)x_2(t-\tau_i)\right|\\
&\le\left|c\sum_{i=1}^{2}x_1(t-\tau_i)-\sum_{i=1}^{2}x_2(t-\tau_i)\right|\\
&\le c|x_1(t-\tau_1)-x_2(t-\tau_1)|+c|x_1(t-\tau_2)-x_2(t-\tau_2)|\\
&\le 2c\|x_1-x_2\|.
\end{aligned}
\tag{4.6}
$$

Using (4.4)-(4.6), we obtain

$$
\begin{aligned}
\|x_1-x_2\| \le& 2(kM+f_1)\mu\|v_1-v_2\|+T\eta_2\|H_1-H_2\|+\mu(\Lambda M+H_1)\|v_1-v_2\|+T\eta_2\Lambda\|x-y\|\\
&+2c(\eta_2T\|v_1\|+1)\|x_1-x_2\|+2(c\|v_1\|M\mu+T\eta_2cM)\|v_1-v_2\|\\
=&[2(kM+f_1)\mu+\mu(\Lambda M+H_1)+2c\|v_1\|M\mu+2T\eta_2cM]\|v_1-v_2\|\\
&+[2\eta_2cT\|v_1\|+2c]\|x_1-x_2\|+T\eta_2(\|H_1-H_2\|+\Lambda\|x-y\|),
\end{aligned}
$$

$$
\begin{aligned}
&\|x_1-x_2\|[1-2c(\eta_2T\|v_1\|+1)]\\
&\le[2(kM+f_1)\mu+\mu(\Lambda M+H_1)+2c\|v_1\|M\mu+2T\eta_2cM]\|v_1-v_2\|\\
&\quad+T\eta_2(\|H_1-H_2\|+\Lambda\|x-y\|),
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_1-x_2\|\le&\frac{1}{1-2c(\eta_2\|v_1\|+1)}\{[2(kM+f_1)\mu\\
&+\mu(\Lambda M+H_1)+2c\|v_1\|M\mu+2T\eta_2cM]\|v_1-v_2\|\\
&+T\eta_2(\|H_1-H_2)\|+\Lambda\|x-y\|)\},
\end{aligned}
$$

which completes the proof. □

We now give an example to demonstrate our main results.

Example 4.1. In a particular case of the NDE (1.1), we are concerned the neutral equation

$$\begin{aligned}
&\frac{d}{dt}\left[x(t)-\left(\frac{1}{1000}\sin\left(\frac{2\pi}{45}t\right)+\frac{1}{1000}\sin^2\left(\frac{2\pi}{45}t\right)\right)x(t-\tau)\right]\\
&=-\left(\frac{1}{90}+\frac{1}{90}\sin^2\left(\frac{2\pi}{45}t\right)\right)x(t)\\
&\quad+\left[\frac{1}{3\pi^4}+\frac{1}{80\pi^4}\sin^2\left(\frac{2\pi}{45}t\right)+\frac{1}{18\pi^4}\sin^2\left(\frac{2\pi}{45}t\right)x(t)+\frac{1}{10\pi^4}\sin^2\left(\frac{2\pi}{45}t\right)x(t-\tau)\right]\\
&\quad-\left[\frac{1}{20\pi^8}\sin^2\left(\frac{2\pi}{45}t\right)+\frac{1}{25\pi^8}\sin^2\left(\frac{2\pi}{45}t\right)x(t)+\frac{1}{35\pi^8}\sin^2\left(\frac{2\pi}{45}t\right)x^{[2]}(t)\right].
\end{aligned}\tag{4.7}$$

Let $m=0.04, M=1.7, L=2\pi, c=0.002$, and $P_T(L,m,M)=P_{45}(2\pi,0.04,1.7)$. Then, according to the conditions of Theorem 3.1 and Theorem 4.1, we have from (4.7) that

$$l_1=\frac{1}{25\pi^8},\ l_2=\frac{1}{35\pi^8},\ H_1=\frac{1}{20\pi^8},\ \Lambda=0,526339\times10^{-4},$$

$$\delta_0=\frac{1}{3\pi^4}, f_1=\frac{83}{240\pi^4},\ k=\frac{1}{10\pi^4},\ a_1=\frac{1}{45},\ \eta_1\cong0,42289,\ \eta_2=1.89526,$$

$$\begin{aligned}
2[cM+\eta_2T(kM+f_1)]&=2[0.002\times1.7+1.89526\times45(\frac{1}{10\pi^4}\times1.7+\frac{83}{240\pi^4})]\\
&=0.91007755413\le1,7=M,
\end{aligned}$$

$$\begin{aligned}
&2\eta_1T\delta_0-\eta_2T(M\Lambda+H_1)-2cT\eta_2a_1M+2cm\\
&=2\times0.42289\times45\times\frac{1}{3\pi^4}-1.89526\times45\left(1.7\times0.526339\times10^{-4}+\frac{1}{20\pi^8}\right)\\
&\quad-2\times0.002\times45\times1,89526\times\frac{1}{45}\times1.7+2\times0.002\times0.04\\
&=0.10943299735\ge0.04=m,
\end{aligned}$$

$$\begin{aligned}
&2\left\{\eta_2(2+a_1T)\left[\frac{1}{2}H_1+f_1+M\left(k+\frac{1}{2}\Lambda+ca_1\right)\right]+Lc\right\}\\
&=2\left\{1.89526\left(2+\frac{1}{45}\times45\right)\left[\frac{1}{2}\times\frac{1}{20\pi^8}+\frac{83}{240\pi^4}\right.\right.\\
&\quad\left.\left.+1.7\left(\frac{1}{10\pi^4}+\frac{1}{2}0.526339\times10^{-4}+0.002\times\frac{1}{45}\right)+2\times\pi\times0.002\right]\right\}\\
&=0.08674914062\le2\pi=L,
\end{aligned}$$

and

$$\begin{aligned}
&2T\eta_2(2k+\frac{\Lambda}{2}+ca_1)+c\\
&=2\times1.89526\times45\times\left(\frac{2}{10\pi^4}+0.002\frac{1}{2}+0.526339\times\frac{1}{2}\right)+0.002\\
&=0.1891803597<1.
\end{aligned}$$

This shows that the conditions of Theorems 3.1 and 4.1 are satisfied. Hence, the NDE (4.7) has a unique solution in $P_T(L,m,M)=P_{45}(2\pi,0.04,1.7)$. The solution depends upon continuously to the functions a, f and H.

5. Conclusion

In this research, a nonlinear NDE of first order, which includes iterative terms and constant time delays, is taken into consideration. Two new theorems and some lemmas, which are the new contributions of this research, are given in relation to the EUPPSs of the considered NDE. The proof is done via the BFP theorem, the KFP theorem and the Green's functions. An example is given to show and verify an application of the results. The results of this research have new contributions to the qualitative theory of NDEs, and they improve and extend some results that are available in the literature for the constant time delay.

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