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POLYNOMIAL DIFFERENTIATION COMPOSITION OPERATORS FROM WEIGHTED BERGMAN SPACES TO WEIGHTED-TYPE SPACES ON THE UNIT BALL

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Abstract. We introduce a polynomial differentiation composition operator on spaces of holomorphic functions on the open unit ball in the n-dimensional complex vector space, and characterize the boundedness and compactness of the operator from the classical weighted Bergman space to the weighted-type space and the little weighted-type space of holomorphic functions on the unit ball.

Keywords. Holomorphic functions; Polynomial differentiation composition operator; Product-type operator; Weighted Bergman space; Weighted-type space; Compact operator.

1. Introduction

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $k, l \in \mathbb{N}_0$, $k \le l$, then $j = \overline{k,l}$ is the notation used instead of $j = k, \dots, l$. We also regard that $\sum_{j=p}^{q} b_j = 0$ and $\prod_{j=p}^{p-1} b_j = 1$, when $p, q \in \mathbb{N}_0$ and $q < p$.

The open unit ball in the n-dimensional complex vector space $\mathbb{C}^n$, $n \in \mathbb{N}$, we denote by $\mathbb{B}$. In the case $n = 1$, the open unit ball is the open unit disk in the complex plane $\mathbb{C}$ and is denoted by $\mathbb{D}$. The Euclidean inner product in $\mathbb{C}^n$ is defined by

$$\langle z, w \rangle = z_1\overline{w}_1 + z_2\overline{w}_2 + \cdots + z_n\overline{w}_n,$$

where $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ are two points in $\mathbb{C}^n$. The corresponding norm is defined by $|z| = \langle z, z \rangle^{1/2}$.

The Lebesgue measure on $\mathbb{B}$ is denoted by $dV(z)$, whereas for $\alpha > -1$ is defined a normalized measure on $\mathbb{B}$ as follows $dV_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dV(z)$, where c_α is chosen such that $V_\alpha(\mathbb{B}) = \int_{\mathbb{B}} dV_\alpha(z) = 1$. By D_j we denote the partial derivative operator $D_j f(z) = \frac{\partial f}{\partial z_j}(z)$, where $j \in \{1, 2, \dots, n\}$. Let Ω be a domain in $\mathbb{C}^n$. Then by $H(\Omega)$ we denote the space of holomorphic functions on Ω, whereas by $S(\Omega)$ we denote the class of holomorphic self-maps of the domain. Some basic facts on the topic can be found, for example, in the classical books [14] and [15].

The composition operator induced by function $\varphi \in S(\Omega)$ is defined by

$$C_\varphi f = f \circ \varphi, \tag{1.1}$$

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where $f \in H(\Omega)$.

The multiplication operator M_u induced by function u is defined by

$$M_u f = uf, \quad f \in H(\Omega). \tag{1.2}$$

Here we regard/assume that $u \in H(\Omega)$. In the complex plane is defined the (iterated) differentiation operator D^m on $H(\Omega)$ in the standard way $D^m f(z) = f^{(m)}(z)$, where $m \in \mathbb{N}_0$, $f \in H(\Omega)$, $z \in \Omega$, and $D^1 = D$ is the classical differentiation operator, that is,

$$Df = f'. \tag{1.3}$$

Recall that $D^0 = I$, where I is the identity operator. Product type operators containing (1.1), (1.2), or (1.3), as well as some other linear operators were studied a lot. In addition to weighted composition operators, the products DC_φ and $C_\varphi D$ attracted much attention recently; see, for example, [9, 13] and the related references therein.

The weighted differentiation composition operators $D^m_{\varphi,u} := M_u C_\varphi D^m$ and their special cases (predominately in the case that $u \equiv 1$) have been studied extensively on subspaces of $H(\mathbb{D})$; see, for example, [10, 25, 26, 27] and the related references therein.

Let $\mathfrak{R}$ be the radial differentiation operator, that is, $\mathfrak{R}f = \sum_{j=1}^n z_j D_j f$. By using the operator, in [20], the author defined the following one $\mathfrak{R}^m_{\varphi,u} := M_u C_\varphi \mathfrak{R}^m$, and it was further studied later, e.g., in [22]. Note that it is related to the operator $D^m_{\varphi,u}$ acting on the spaces of holomorphic function in a domain in the complex plane. For other product-type operators containing differentiation operators, we refer to [11, 18] and the references therein. A sum of operators of the form $M_u C_\varphi D^j$ were investigated first in [23] and [24]. For some subsequent studies of the operator; we refer to [1, 4, 6]. The problem of studying sums of related operators on subspaces of $H(\mathbb{B})$ naturally appears. For a recently introduced operator of this type; see [21].

Here we define a polynomial differentiation composition operator as follows

$$P^m_{D,\varphi} f := \sum_{j=0}^m u_j C_\varphi D_{l_j} \cdots D_{l_1} f, \quad f \in H(\mathbb{B}),$$

where $m \in \mathbb{N}_0$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, and $\varphi \in S(\mathbb{B})$.

Let $p > 0$, $\alpha > -1$, and

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{B}} |f(z)|^p dV_\alpha(z) \right)^{1/p}, \tag{1.4}$$

where f is a complex-valued measurable function. The weighted Bergman space $A^p_\alpha(\mathbb{B}) = A^p_\alpha$ is defined as follows

$$A^p_\alpha(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \|f\|_{A^p_\alpha} < +\infty \right\}.$$

When $p \geq 1$, the quantity/functional in (1.4) is a norm on the space A^p_α, and with the norm it is a Banach space. When $p \in (0,1)$, it is a Frechet space with the translation invariant metric

$$d_{A^p_\alpha}(f,g) = \|f-g\|^p_{A^p_\alpha}.$$

Some results on the weighted Bergman space and the operators acting from or to the space can be found in, for example, [1, 2, 5, 7, 8, 15].

The notion of weight function, or simply weight, refers to a positive and continuous function on $\mathbb{B}$. The weighted-type space $H^\infty_\mu(\mathbb{B}) = H^\infty_\mu$, where μ is a weight, is defined as follows

$$H^\infty_\mu(\mathbb{B}) := \left\{ f \in H(\mathbb{B}) : \|f\|_{H^\infty_\mu} := \sup_{z\in\mathbb{B}} \mu(z)|f(z)| < +\infty \right\},$$

whereas the little weighted-type space is the subspace of H^∞_μ consisting of $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1} \mu(z)|f(z)| = 0,$$

and is denoted by $H^\infty_{\mu,0}(\mathbb{B}) = H^\infty_{\mu,0}$. It is a closed subspace of H^∞_μ. The space of bounded holomorphic function is obtained for $\mu(z) \equiv 1$ and denoted by $H^\infty(\mathbb{B}) = H^\infty$. The corresponding norm on the space is denoted by $\|\cdot\|_\infty$.

Let $L : X \to Y$ be a linear operator, where X and Y are two Banach spaces. If there is $M \geq 0$ such that $\|Lf\|_Y \leq M\|f\|_X$, for every $f \in X$, the operator is called bounded. We say that the operator is compact if it maps bounded sets in X into relatively compact. By B_X, we denote the unit ball in the space X. For some classical results in the topic, we refer to [3] and [16]. For some recent investigations on the boundedness and compactness of various concrete linear operators on spaces of holomorphic functions on domains in $\mathbb{C}$ or $\mathbb{C}^n$, we refer to the references included in this paper.

In this paper, we study the boundedness and compactness of the polynomial differentiation composition operator $P^m_{D,\varphi}$ from weighted Bergman spaces to weighted-type spaces on $\mathbb{B}$.

By C we denote some unspecified nonnegative constants, which can be different from one appearance to another. If we write $a \lesssim b$ (resp. $a \gtrsim b$), then it means that there is a $C > 0$ such that $a \leq Cb$ (resp. $a \geq Cb$). If the relations hold: $a \lesssim b$ and $b \lesssim a$, then we write $a \asymp b$.

2. Auxiliary Results

Here we give several lemmas which we employ in the proof of the main results in the next section. As usual, we need a characterization for the compactness. It is proved in a standard way. So, we omit the proof. One of the first results of the sort was proved in [17].

Lemma 2.1. *Let $p \geq 1$, $\alpha > -1$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, and μ be a weight function on $\mathbb{B}$. Then the bounded operator $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is compact if and only if for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in the space $A^p_\alpha(\mathbb{B})$ such that $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}$ as $k \to +\infty$, it follows that $\lim_{k\to+\infty} \|P^m_{D,\varphi} f_k\|_{H^\infty_\mu(\mathbb{B})} = 0$.*

The following result easily follows from Cauchy's estimates for derivatives of holomorphic functions on balls in $\mathbb{C}^n$, and the subharmonicity of the function $|f|^p$, when $f \in H(\mathbb{B})$ and $p > 0$ (a generalization of the result can be found, for example, in [22]).

Lemma 2.2. *Let $p > 0$, $\alpha > -1$, and $N \in \mathbb{N}_0$. Then, for every multi-index $\vec{l} = (l_1, l_2, \ldots, l_j)$ such that $|\vec{l}| = N$, there is $C_{\vec{l}} > 0$ such that*

$$\left| \frac{\partial^N f(z)}{\partial z^{l_1}_{k_1} \partial z^{l_2}_{k_2} \cdots \partial z^{l_j}_{k_j}} \right| \leq \frac{C_{\vec{l}} \|f\|_{A^p_\alpha(\mathbb{B})}}{(1-|z|^2)^{\frac{n+1+\alpha}{p}+N}},$$

for every $f \in A^p_\alpha(\mathbb{B})$ and $z \in \mathbb{B}$.

The following result is easily proved by [15, Proposition 1.4.10]. We omit this simple and known proof.

Lemma 2.3. *Let $p>0$ and $\alpha>-1$. Then, for each $a\ge 0$ and $w\in\mathbb{B}$, the following function*

$$f_{w,a}(z)=\frac{(1-|w|^2)^{\frac{n+1+\alpha}{p}+a}}{(1-\langle z,w\rangle)^{\frac{2(n+1+\alpha)}{p}+a}}, \tag{2.1}$$

belongs to $A^p_\alpha(\mathbb{B})$. Moreover,

$$\sup_{w\in\mathbb{B}}\|f_{w,a}\|_{A^p_\alpha(\mathbb{B})}\lesssim 1. \tag{2.2}$$

The following lemma is a known generalization of [12, Lemma 1].

Lemma 2.4. *A closed set K in $H^\infty_{\mu,0}$ is compact if and only if it is bounded and*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(z)|f(z)|=0.$$

Now we present some of the test functions which are used in the proofs of our main results.

Lemma 2.5. *Let $p>0$, $\alpha>-1$, $m\in\mathbb{N}$, and $w\in\mathbb{B}$ be such that $w_j\ne 0$, $j=\overline{1,n}$. Then, for each $s\in\{0,1,\dots,m\}$, there are $c^{(s)}_j$, $j=\overline{0,m}$ such that the function $h^{(s)}_w(z)=\sum_{k=0}^m c^{(s)}_k f_{w,k}(z)$, where $f_{w,a}$, defined in (2.1), satisfies*

$$D_{l_s}\cdots D_{l_1}h^{(s)}_w(w)=\frac{\overline{w}_{l_1}\overline{w}_{l_2}\cdots\overline{w}_{l_s}}{(1-|w|^2)^{\frac{n+1+\alpha}{p}+s}} \tag{2.3}$$

and

$$D_{l_t}\cdots D_{l_1}h^{(s)}_w(w)=0, \tag{2.4}$$

for every $t\in\{0,1,\dots,m\}\setminus\{s\}$. In addition,

$$\sup_{w\in\mathbb{B}}\|h^{(s)}_w\|_{A^p_\alpha(\mathbb{B})}\lesssim 1. \tag{2.5}$$

Proof. Let $d_k=\frac{2(n+1+\alpha)}{p}+k$, $k\in\mathbb{N}_0$, and $h_w(z)=\sum_{k=0}^m c_k f_{w,k}(z)$. We have

$$D_{l_t}\cdots D_{l_1}h_w(z)=\sum_{k=0}^m c_k\frac{d_kd_{k+1}\cdots d_{k+t-1}\overline{w}_{l_1}\overline{w}_{l_2}\cdots\overline{w}_{l_t}(1-|w|^2)^{\frac{n+1+\alpha}{p}+k}}{(1-\langle z,w\rangle)^{d_k+t}},$$

for $t\in\mathbb{N}_0$. Hence,

$$D_{l_t}\cdots D_{l_1}h_w(w)=\frac{\overline{w}_{l_1}\overline{w}_{l_2}\cdots\overline{w}_{l_t}}{(1-|w|^2)^{\frac{n+1+\alpha}{p}+t}}\sum_{k=0}^m c_k\prod_{l=0}^{t-1}d_{k+l}$$

for $t \in \mathbb{N}_0$. Consider the system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ d_0 & d_1 & \cdots & d_m \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{k=0}^{s-1} d_k & \prod_{k=0}^{s-1} d_{k+1} & \cdots & \prod_{k=0}^{s-1} d_{k+m} \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{k=0}^{m-1} d_k & \prod_{k=0}^{m-1} d_{k+1} & \cdots & \prod_{k=0}^{m-1} d_{k+m} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_s \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \tag{2.6}$$

where the unit in the vector on the right-hand side of (2.6) is on the $(s+1)$th position. By [19, Lemma 5], we see that the determinant of the system is different from zero. From this, we have that, for each $s \in \{0,1,\dots,m\}$, there is a unique solution $c_k := c_k^{(s)}$, $k = \overline{0,m}$, to (2.6). Let $h_w^{(s)}(z) := \sum_{k=0}^{m} c_k^{(s)} f_{w,k}(z)$. Then (2.3) and (2.4) hold. From (2.2), it follows that (2.5) also holds. □

3. Main Results

Our first result in this section deals with the boundedness of $P_{D,\varphi}^m : A_\alpha^p(\mathbb{B}) \to H_\mu^\infty(\mathbb{B})$.

Theorem 3.1. *Let $p \geq 1$, $\alpha > -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi = (\varphi_1,\dots,\varphi_n) \in S(\mathbb{B})$,*

$$\min_{j=\overline{1,n}} \inf_{z\in\mathbb{B}} |\varphi_j(z)| \geq \delta > 0, \tag{3.1}$$

and μ be a weight function on $\mathbb{B}$. Then the operator $P_{D,\varphi}^m : A_\alpha^p(\mathbb{B}) \to H_\mu^\infty(\mathbb{B})$ is bounded if and only if

$$M_j := \sup_{z\in\mathbb{B}} \frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}+j}} < +\infty, \quad j = \overline{0,m}. \tag{3.2}$$

Moreover, if the operator $P_{D,\varphi}^m : A_\alpha^p(\mathbb{B}) \to H_\mu^\infty(\mathbb{B})$ is bounded, then the following asymptotic relationship holds

$$\|P_{D,\varphi}^m\|_{A_\alpha^p(\mathbb{B})\to H_\mu^\infty(\mathbb{B})} \asymp \sum_{j=0}^{m} M_j. \tag{3.3}$$

Proof. Assume that $P_{D,\varphi}^m : A_\alpha^p(\mathbb{B}) \to H_\mu^\infty(\mathbb{B})$ is bounded. Then there is $C > 0$ such that

$$\|P_{D,\varphi}^m f\|_{H_\mu^\infty(\mathbb{B})} \leq C\|f\|_{A_\alpha^p(\mathbb{B})} \tag{3.4}$$

for every $f \in A_\alpha^p(\mathbb{B})$. By Lemma 2.5, for each $s \in \{0,1,\dots,m\}$ and $w \in \mathbb{B}$, there is a function $h_{\varphi(w)}^{(s)} \in A_\alpha^p(\mathbb{B})$ such that

$$D_{l_s}\cdots D_{l_1} h_{\varphi(w)}^{(s)}(\varphi(w)) = \frac{\overline{\varphi_{l_1}(w)\varphi_{l_2}(w)}\cdots\overline{\varphi_{l_s}(w)}}{(1-|\varphi(w)|^2)^{\frac{n+1+\alpha}{p}+s}}, \tag{3.5}$$

and

$$D_{l_t}\cdots D_{l_1} h_{\varphi(w)}^{(s)}(\varphi(w)) = 0, \tag{3.6}$$

for every $t \in \{0,1,\dots,m\} \setminus \{s\}$, and $L_s := \sup_{w\in\mathbb{B}} \|h^{(s)}_{\varphi(w)}\|_{A^p_\alpha(\mathbb{B})} < +\infty$. This together with (3.4)-(3.6), as well as (3.1), implies

$$\begin{aligned} L_s\|P^m_{D,\varphi}\|_{A^p_\alpha(\mathbb{B})\to H^\infty_\mu(\mathbb{B})} \geq & \|P^m_{D,\varphi}h^{(s)}_{\varphi(w)}\|_{H^\infty_\mu(\mathbb{B})} \\ = & \sup_{z\in\mathbb{B}} \mu(z)\left|\sum_{j=0}^m u_j(z)D_{l_j}\cdots D_{l_1}h^{(s)}_{\varphi(w)}(\varphi(z))\right| \\ \geq & \mu(w)\left|\sum_{j=0}^m u_j(w)D_{l_j}\cdots D_{l_1}h^{(s)}_{\varphi(w)}(\varphi(w))\right| \\ = & \mu(w)|u_s(w)|\frac{|\overline{\varphi_{l_1}(w)}|\cdots|\overline{\varphi_{l_s}(w)}|}{(1-|\varphi(w)|^2)^{\frac{n+1+\alpha}{p}+s}} \\ \geq & \delta^s\frac{\mu(w)|u_s(w)|}{(1-|\varphi(w)|^2)^{\frac{n+1+\alpha}{p}+s}}, \end{aligned} \tag{3.7}$$

for every $w \in \mathbb{B}$. Taking the supremum in (3.7) over $\mathbb{B}$, we get $M_s < +\infty$, for each $s \in \{0,1,\dots,m\}$, and $L_s\|P^m_{D,\varphi}\|_{A^p_\alpha(\mathbb{B})\to H^\infty_\mu(\mathbb{B})} \geq \delta^s M_s$, $s = \overline{0,m}$. Hence

$$\sum_{j=0}^m M_j \lesssim \|P^m_{D,\varphi}\|_{A^p_\alpha(\mathbb{B})\to H^\infty_\mu(\mathbb{B})}. \tag{3.8}$$

Assume (3.2) holds. It follows from Lemma 2.2 that

$$\begin{aligned} \mu(z)|P^m_{D,\varphi}f(z)| = & \mu(z)\left|\sum_{j=0}^m u_j(z)D_{l_j}\cdots D_{l_1}f(\varphi(z))\right| \\ \leq & C\sum_{j=0}^m \frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}+j}}\|f\|_{A^p_\alpha(\mathbb{B})}. \end{aligned} \tag{3.9}$$

Taking in (3.9) the supremum over $\mathbb{B}$, and then over $B_{A^p_\alpha(\mathbb{B})}$, and employing (3.2), the boundedness of $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ follows. Moreover, the relation holds

$$\|P^m_{D,\varphi}\|_{A^p_\alpha(\mathbb{B})\to H^\infty_\mu(\mathbb{B})} \lesssim \sum_{j=0}^m M_j. \tag{3.10}$$

From (3.8) and (3.10), the relation in (3.3) follows. □

Now we characterize the boundedness of $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$.

Theorem 3.2. *Let $p \geq 1$, $\alpha > -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, and μ be a weight function on $\mathbb{B}$. Then the operator $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$ is bounded if and only if $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is bounded and*

$$\lim_{|z|\to 1} \mu(z)|u_j(z)| = 0, \quad j = \overline{0,m}. \tag{3.11}$$

Proof. If $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$ is bounded, then $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is also bounded. Let $f_0(z) \equiv 1$. Since $f_0 \in A^p_\alpha(\mathbb{B})$, we have $P^m_{D,\varphi}(f_0) \in H^\infty_{\mu,0}(\mathbb{B})$. Hence

$$\lim_{|z|\to 1} \mu(z)|P^m_{D,\varphi}(f_0)(z)| = \lim_{|z|\to 1} \mu(z)|u_0(z)| = 0,$$

from which it follows that $u_0 \in H^\infty_{\mu,0}(\mathbb{B})$. Let $f_1(z) = z_{l_1}$. The fact $f_1 \in A^p_\alpha(\mathbb{B})$, implies $P^m_{D,\varphi}(f_1) \in H^\infty_{\mu,0}(\mathbb{B})$, that is,

$$\lim_{|z|\to 1} \mu(z)|P^m_{D,\varphi} f_1(z)| = \lim_{|z|\to 1} \mu(z)|u_0(z)\varphi_{l_1}(z) + u_1(z)| = 0,$$

which together with $|\varphi_{l_1}(z)| < 1$ and the fact $u_0 \in H^\infty_{\mu,0}(\mathbb{B})$ implies that $\lim_{|z|\to 1} \mu(z)|u_1(z)| = 0$, that is, $u_1 \in H^\infty_{\mu,0}(\mathbb{B})$.

Suppose that we have proved (3.11) for $j = \overline{0,s}$, for some $s \in \{2,3,\dots,m-1\}$. Let $f_{s+1}(z) = z_{l_1} z_{l_2} \cdots z_{l_{s+1}}$. The fact $f_{s+1} \in A^p_\alpha(\mathbb{B})$ implies $P^m_{D,\varphi}(f_{s+1}) \in H^\infty_{\mu,0}(\mathbb{B})$. It is easy to see that $f_{s+1}(z) = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ for some $\alpha_j \in \mathbb{N}_0$, $j = \overline{1,n}$, such that $\sum_{j=1}^n \alpha_j = s+1$. Polynomial f_{s+1} is homogeneous. Thus, for each $t \in \mathbb{N}_0$ with $0 \le t \le s+1$, we have

$$D_{j_t} \cdots D_{j_1} f_{s+1}(z) = \gamma_t z_1^{\alpha_1 - k_1(t)} \cdots z_n^{\alpha_n - k_n(t)},$$

for some $\gamma_t \in \mathbb{N}$, where $k_i(t)$ is the number of operators D_i in the product $D_{j_t} \cdots D_{j_1}$. We have $\sum_{j=1}^n k_i(t) = t$ and

$$D_{j_{s+1}} \cdots D_{j_1} f_{s+1}(z) = \gamma_{s+1}, \tag{3.12}$$

for some $\gamma_{s+1} \in \mathbb{N}$. Thus

$$\lim_{|z|\to 1} \mu(z)|P^m_{D,\varphi} f_{s+1}(z)| = \lim_{|z|\to 1} \mu(z) \left| \sum_{j=0}^{s+1} u_j(z)\gamma_j \prod_{i=1}^n (\varphi_i(z))^{\alpha_i - k_i(j)} \right| = 0,$$

which together with $|\varphi_i(z)| < 1$, $i = \overline{1,n}$, $\alpha_i \ge k_i(j)$, for $i = \overline{1,n}$, $j = \overline{0,s+1}$, the induction hypothesis $u_j \in H^\infty_{\mu,0}(\mathbb{B})$, $j = \overline{0,s}$, and (3.12) implies $\lim_{|z|\to 1} \gamma_{s+1}\mu(z)|u_{s+1}(z)| = 0$. This and the fact $\gamma_{s+1} \ne 0$ imply $u_{s+1} \in H^\infty_{\mu,0}(\mathbb{B})$. Hence, by the induction we have that (3.11) holds for $j = \overline{0,m}$.

Now suppose that $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is bounded and (3.11) holds. Let p be a polynomial. Then

$$\begin{aligned} \mu(z)\left| \sum_{j=0}^m u_j(z) D_{l_j} \cdots D_{l_1} p(\varphi(z)) \right| &\le \sum_{j=0}^m \mu(z)|u_j(z)||D_{l_j} \cdots D_{l_1} p(\varphi(z))| \\ &\le \sum_{j=0}^m \mu(z)|u_j(z)|\|D_{l_j} \cdots D_{l_1} p\|_\infty. \end{aligned} \tag{3.13}$$

Letting $|z| \to 1$ in (3.13) and employing the estimate

$$\|D_{l_j} \cdots D_{l_1} p\|_\infty = \sup_{z\in\mathbb{B}} |D_{l_j} \cdots D_{l_1} p(z)| < +\infty, \quad j = \overline{0,m},$$

and (3.11), we have $P^m_{D,\varphi} p \in H^\infty_{\mu,0}(\mathbb{B})$ for each polynomial p. The density of the set of polynomials in $A^p_\alpha(\mathbb{B})$ implies that, for every $f \in A^p_\alpha(\mathbb{B})$, there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty} \|f - p_k\|_{A^p_\alpha(\mathbb{B})} = 0$. This together with the boundedness of $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ implies

$$\|P^m_{D,\varphi} f - P^m_{D,\varphi} p_k\|_{H^\infty_\mu(\mathbb{B})} \le \|P^m_{D,\varphi}\|_{A^p_\alpha(\mathbb{B})\to H^\infty_\mu(\mathbb{B})} \|f - p_k\|_{A^p_\alpha(\mathbb{B})} \to 0$$

as $k \to +\infty$. This fact together with fact that $H^\infty_{\mu,0}(\mathbb{B})$ is a closed subspace of $H^\infty_\mu(\mathbb{B})$ implies $P^m_{D,\varphi} f \in H^\infty_{\mu,0}(\mathbb{B})$, that is, $P^m_{D,\varphi}(A^p_\alpha(\mathbb{B})) \subseteq H^\infty_{\mu,0}(\mathbb{B})$. Hence $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$ is bounded. □

The following theorem investigates the compactness of $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$.

Theorem 3.3. *Let $p \ge 1$, $\alpha > -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, μ be a weight function on $\mathbb{B}$, and (3.1) hold. Then the operator $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is compact if and only if the operator is bounded and the following condition holds*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)\big|u_j(z)\big|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}+j}} = 0, \tag{3.14}$$

for each $j \in \{0,1,\dots,m\}$.

Proof. If $P^m_{D,\varphi} : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is bounded and (3.14) holds for each $j \in \{0,1,\dots,m\}$, then we have that, for every $\varepsilon > 0$, there is $\delta \in (0,1)$ such that

$$\frac{\mu(z)\big|u_j(z)\big|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}+j}} < \varepsilon, \quad j = \overline{0,m}, \tag{3.15}$$

for every $z \in \mathbb{B}$ such that $|\varphi(z)| > \delta$. Assume that $(f_k)_{k\in\mathbb{N}}$ is such that $\sup_{k\in\mathbb{N}} \|f_k\|_{A^p_\alpha(\mathbb{B})} \le M$ and $f_k \to 0$ uniformly on compacts of $\mathbb{B}$. Let $K_\delta = \{z \in \mathbb{B} : |\varphi(z)| > \delta\}$. Then by Lemma 2.2 and (3.15), we have

$$\begin{aligned}
\|P^m_{D,\varphi} f_k\|_{H^\infty_\mu(\mathbb{B})} =& \sup_{z\in\mathbb{B}} \mu(z)\Big|\sum_{j=0}^m u_j(z) D_{l_j}\cdots D_{l_1} f_k(\varphi(z))\Big| \\
\le& \sup_{z\in K_\delta} \mu(z)\Big|\sum_{j=0}^m u_j(z) D_{l_j}\cdots D_{l_1} f_k(\varphi(z))\Big| \\
&+ \sup_{z\in\mathbb{B}\setminus K_\delta} \mu(z)\Big|\sum_{j=0}^m u_j(z) D_{l_j}\cdots D_{l_1} f_k(\varphi(z))\Big| \\
\le& C\sum_{j=0}^m \sup_{z\in K_\delta} \frac{\mu(z)\big|u_j(z)\big|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}+j}} \|f_k\|_{A^p_\alpha(\mathbb{B})} \\
&+ C\sum_{j=0}^m \sup_{z\in\mathbb{B}\setminus K_\delta} \mu(z)|u_j(z)||D_{l_j}\cdots D_{l_1} f_k(\varphi(z))| \\
\le& (m+1)MC\varepsilon + C\sum_{j=0}^m \sup_{z\in\mathbb{B}\setminus K_\delta} \mu(z)|u_j(z)| \sup_{|\varphi(z)|\le\delta} \big|D_{l_j}\cdots D_{l_1} f_k(\varphi(z))\big| \\
\le& (m+1)MC\varepsilon + C\sum_{j=0}^m \|u_j\|_{H^\infty_\mu} \sup_{|w|\le\delta} \big|D_{l_j}\cdots D_{l_1} f_k(w)\big|.
\end{aligned} \tag{3.16}$$

Cauchy's estimate together with the fact that $f_k \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to +\infty$, implies

$$D_{l_j}\cdots D_{l_1} f_k \to 0, \tag{3.17}$$

uniformly on compacts of $\mathbb{B}$ as $k \to +\infty$, for $j = \overline{0,m}$. Let $f_s(z) = \prod_{j=1}^s z_{l_j}$, $s = \overline{0,m}$. Then as in Theorem 3.2, we have $u_j \in H^\infty_\mu$, $j = \overline{0,m}$, that is, $\|u_j\|_{H^\infty_\mu(\mathbb{B})} < +\infty$, $j = \overline{0,m}$. Using (3.17), the fact that $|w| \le \delta$ is compact, letting $k \to +\infty$ in (3.16), and using the fact that $\varepsilon > 0$ is arbitrary, we have $\lim_{k\to+\infty} \|P^m_{D,\varphi} f_k\|_{H^\infty_\mu(\mathbb{B})} = 0$, which implies from Lemma 2.1 that the compactness of $P^m_{D,\varphi} M_u : A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ follows.

Suppose now that $P^m_{D,\varphi}: A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is compact. Then it is bounded. If $\|\varphi\|_\infty < 1$, then (3.14) automatically holds for each $j \in \{0,1,\dots,m\}$. Suppose now $\|\varphi\|_\infty = 1$. Let $(z_k)_{k\in\mathbb{N}} \subset \mathbb{B}$ such that $|\varphi(z_k)| \to 1$ as $k \to +\infty$, and $h^{(s)}_k := h^{(s)}_{\varphi(z_k)}$, $s = \overline{0,m}$, where $h^{(s)}_w$, $s = \overline{0,m}$, are as in Lemma 2.5. Then

$$\sup_{k\in\mathbb{N}} \|h^{(s)}_k\|_{A^p_\alpha(\mathbb{B})} < +\infty, \quad s = \overline{0,m}. \tag{3.18}$$

We have that $h^{(s)}_k \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to +\infty$ for each $s \in \{0,1,\dots,m\}$. From (3.18) and Lemma 2.1, we have

$$\lim_{k\to+\infty} \|P^m_{D,\varphi} h^{(s)}_k\|_{H^\infty_\mu(\mathbb{B})} = 0, \quad s = \overline{0,m}. \tag{3.19}$$

Further, it follows from (3.7) that

$$\frac{\mu(z_k)|u_s(z_k)|}{(1-|\varphi(z_k)|^2)^{\frac{n+1+\alpha}{p}+s}} \le C\|P^m_{D,\varphi} h^{(s)}_k\|_{H^\infty_\mu(\mathbb{B})}, \quad s = \overline{0,m}. \tag{3.20}$$

Letting $k \to +\infty$ in (3.20) and using (3.19), we obtain (3.14) for $s = \overline{0,m}$. □

Theorem 3.4. *Let $p \ge 1$, $\alpha > -1$, $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{0,m}$, $\varphi \in S(\mathbb{B})$, μ be a weight function on $\mathbb{B}$, and condition (3.1) hold. Then the operator $P^m_{D,\varphi}: A^p_\alpha(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$ is compact if and only if the operator is bounded and*

$$\lim_{|z|\to 1} \frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}+j}} = 0, \quad j = \overline{0,m}. \tag{3.21}$$

Proof. Assume that (3.21) holds. We have (3.2), which together with Theorem 3.1, yields the boundedness of $P^m_{D,\varphi}: A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ immediately. Letting $|z| \to 1$ in (3.9) and using (3.21), we have $\lim_{|z|\to 1} \mu(z)|P^m_{D,\varphi} f(z)| = 0$ for $f \in A^p_\alpha(\mathbb{B})$, that is, $P^m_{D,\varphi} f \in H^\infty_{\mu,0}(\mathbb{B})$. Hence, $P^m_{D,\varphi}(A^p_\alpha(\mathbb{B})) \subset H^\infty_{\mu,0}(\mathbb{B})$, from which the boundedness of $P^m_{D,\varphi}: A^p_\alpha(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$ follows. Taking the supremum in (3.9) over $\mathbb{B}$ and $B_{A^p_\alpha(\mathbb{B})}$, and employing (3.2), we obtain

$$\sup_{f\in B_{A^p_\alpha(\mathbb{B})}} \sup_{z\in\mathbb{B}} \mu(z)|P^m_{D,\varphi} f(z)| \le C\sum_{j=0}^m M_j < +\infty,$$

where M_j, $j = \overline{0,m}$, is the quantity in (3.2). Thus, $\{P^m_{D,\varphi} f : f \in B_{A^p_\alpha(\mathbb{B})}\}$ is a bounded set in $H^\infty_{\mu,0}$. Taking the supremum in (3.9) over $B_{A^p_\alpha(\mathbb{B})}$ and letting $|z| \to 1$ in such obtained inequality, we have

$$\lim_{|z|\to 1} \sup_{f\in B_{A^p_\alpha(\mathbb{B})}} \mu(z)|P^m_{D,\varphi} f(z)| = 0.$$

From this and Lemma 2.4, it follows that $P^m_{D,\varphi}: A^p_\alpha(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$ is compact. Suppose that $P^m_{D,\varphi}: A^p_\alpha(\mathbb{B}) \to H^\infty_{\mu,0}(\mathbb{B})$ is compact. Then $P^m_{D,\varphi}: A^p_\alpha(\mathbb{B}) \to H^\infty_\mu(\mathbb{B})$ is also compact. From this and Theorem 3.3, we obtain (3.15). From Theorem 3.2, we obtain (3.11), implying that there is $\eta \in (0,1)$ such that

$$\mu(z)|u_j(z)| < \varepsilon(1-\delta^2)^{\frac{n+1+\alpha}{p}+j}, \quad j = \overline{0,m}, \tag{3.22}$$

when $\eta < |z| < 1$ for ε chosen such that (3.15) holds. From (3.22), we have

$$\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^{\frac{n+1+\alpha}{p}+j}} \le \frac{\mu(z)|u_j(z)|}{(1-\delta^2)^{\frac{n+1+\alpha}{p}+j}} < \varepsilon, \quad j=\overline{0,m}, \tag{3.23}$$

when $|\varphi(z)| \le \delta$ and $\eta < |z| < 1$. From (3.15) and (3.23), relation (3.21) easily follows for $j = \overline{0,m}$. □

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