J. Nonlinear Var. Anal. 7 (2023), No. 3, pp. 421-435
Available online at http://jnva.biemdas.com
https://doi.org/10.23952/jnva.7.2023.3.07

# EMBEDDING OF $Q_p$ SPACES INTO TENT SPACES AND THE EXTENDED CESÀRO OPERATOR

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**Abstract.** In this paper, the boundedness and compactness of the identity operator from $Q_x$ spaces into tent spaces $\mathscr{T}^q_{\lambda,s}$ are completely characterized in the unit ball of $\mathbb{C}^n$ when $q > 2$. As an application, the boundedness of the extended Cesàro operator $T_g$ from $Q_x$ to the space $F(p,q,s,p)$ is obtained. Moreover, the essential norm and compactness of $T_g$ are also investigated.
**Keywords.** $Q_p$ space; Carleson measure; Dirichlet type spaces; Extended Cesàro operator.

## 1. INTRODUCTION

Let $\mathbb{B}$ be the open unit ball of $\mathbb{C}^n$, and let $\mathbb{S}$ be the boundary of $\mathbb{B}$. When $n = 1$, $\mathbb{B}$ is the open unit disk in complex plane $\mathbb{C}$ and always denoted by $\mathbb{D}$. For any two points $z = (z_1, z_2, \cdots, z_n)$ and $w = (w_1, w_2, \cdots, w_n)$ in $\mathbb{C}^n$, we define $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ and

$$|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

Let $a \in \mathbb{B}\backslash\{0\}$. Set

$$S_a = \left\{ z \in \mathbb{B} : |a| \le |z| < 1, |1 - \langle \frac{z}{|z|}, \frac{a}{|a|}\rangle| < 1 - |a| \right\}.$$

The set $S_a$ is called the Carleson block and introduced in [2]. From [2], we see that the Carleson block plays an essential role when studying some holomorphic function spaces in the unit ball.

Let $H(\mathbb{B})$ denote the space of all holomorphic functions on $\mathbb{B}$. For any $f \in H(\mathbb{B})$, its complex gradient $\nabla f$ and invariant gradient $\widetilde{\nabla} f$ are defined by

$$\nabla f(z) = (\frac{\partial f}{\partial z_1}(z), \frac{\partial f}{\partial z_2}(z), \cdots, \frac{\partial f}{\partial z_n}(z)) \text{ and } \widetilde{\nabla} f(z) = \nabla(f \circ \sigma_z)(0),$$

respectively. Here, $\sigma_z$ is the Möbius transformation of $\mathbb{B}$. For all $a, z \in \mathbb{B}$,

$$1 - |\sigma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z, a\rangle|^2}, \ |1 - \langle a, \sigma_a(z)\rangle| = \frac{1-|a|^2}{|1-\langle a, z\rangle|}. \tag{1.1}$$

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Received December 23, 2022; Accepted April 28, 2023.

We refer the readers to [26] for more information about $\sigma_z$.

The Möbious invariant space $Q_p$ was introduced by Aulasikari, Xiao and Zhao in the unit disk (see [1]). Based on [1], the $Q_p$ space was extended to the unit ball in $\mathbb{C}^n$ by Ouyang, Yang and Zhao in [16] by using the invariant gradient $\widetilde{\nabla} f$ of a holomorphic function $f$, i.e.,

$$Q_p = \left\{ f \in H(\mathbb{B}) : \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\widetilde{\nabla} f|^2 g^p(z,a)dV(z) < \infty \right\},$$

where $dV$ is the normalized volume measure on $\mathbb{B}$ and $g(z,a) = \log|\sigma_a(z)|^{-1}$ is the Green's function for $\mathbb{B}$ with logarithmic singularity at $a$. In [16], the authors proved that, on the unit ball of $\mathbb{C}^n$,

$$Q_p = \begin{cases} BMOA & \text{when} \quad p = 1; \\ \mathscr{B} & \text{when} \quad p \in (1, \frac{n}{n-1}); \\ \mathbb{C} & \text{when} \quad p \leq \frac{n-1}{n} \text{ or } p \geq \frac{n}{n-1}. \end{cases}$$

In [8], Li and Ouyang demonstrated that, if $\frac{n-1}{n} < p < \frac{n}{n-1}$, a equivalent norm on $Q_p$ is given by

$$\|f\|_{Q_p}^2 = |f(0)|^2 + \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\Re f(z)|^2 (1-|z|^2)^{1-n} (1-|\sigma_a(z)|^2)^{np} dV(z) < \infty,$$

where $\Re f(z) = \sum_{i=1}^n z_i \frac{\partial f}{\partial z_i}$ denotes the radial derivative $f$.

Let $0 < p < \infty, \alpha > -1$. The Dirichlet type space $\mathscr{D}_\alpha^p$ on the unit ball is the space consisting of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{\mathscr{D}_\alpha^p}^p = |f(0)|^p + \int_{\mathbb{B}} |\Re f(z)|^p (1-|z|^2)^\alpha dV(z) < \infty.$$

Let $0 < p < \infty, 0 \leq s < \infty$, $-n-1 < q < \infty$, and $q+s > -1$. The general space $F(p,q,s)$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{F(p,q,s)}^p = |f(0)|^p + \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\Re f(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dV(z) < \infty.$$

Obviously, $\mathscr{D}_\alpha^p = F(p,\alpha,0)$ and $Q_p = F(2,1-n,np)$. The space $F(p,q,s)$ was first introduced by Zhao [25] in the unit disc and called general function space because it can obtain many function spaces, such as $BMOA$ space, $Q_p$ space, Bergman space $A_\alpha^p$, Hardy space $H^2$, Bloch space $\mathscr{B}$, and Dirichlet type spaces $\mathscr{D}_\alpha^p$, if it takes special parameters of $p,q,s$. See [23, 25] and the references therein for more results about general function spaces in the unit disk and the unit ball.

Let $g \in H(\mathbb{B})$. The extended Cesàro operator $T_g$ is given by

$$T_g f(z) = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t}, \qquad f \in H(\mathbb{B}),\ z \in \mathbb{B}.$$

$T_g$ is also called Riemann-Stieltjes operator. The operator $T_g$ was introduced in [4], and was studied in [4, 5, 6, 7, 9, 10, 11, 12, 14, 15, 17, 19, 20, 22, 27].

Motivated by [18], we define a more general function spaces $F(p,q,s,t)$ in the unit ball of $\mathbb{C}^n$. For $p > 0, q > -n-1, s > 0, t \geq 0$ such that $q+s > -1$, let $F(p,q,s,t)$ consist of all holomorphic functions $f$ such that

$$\|f\|^p_{F(p,q,s,t)} = |f(0)|^p + \sup_{a\in\mathbb{B}} \frac{1}{\left(\log \frac{2}{1-|a|^2}\right)^t} \int_{\mathbb{B}} |\Re f(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dV(z) < \infty.$$

It is easy to check that $F(p,q,s,t)$ is a Banach space under norm $\|\cdot\|_{F(p,q,s,t)}$ with $p \geq 1$.

Let $0 < \lambda, q < \infty$, $s \geq 0$, and $\mu$ be a positive Borel measure on $\mathbb{B}$. The tent space $\mathscr{T}^q_{\lambda,s}(\mu)$ consists of all measure functions $f$ satisfying

$$\|f\|^q_{\mathscr{T}^q_{\lambda,s}(\mu)} := \sup_{a\in\mathbb{B}} \frac{1}{(1-|a|)^\lambda (\log \frac{2}{1-|a|})^s} \int_{S_a} |f(z)|^q d\mu(z) < \infty, \tag{1.2}$$

which extends the tent space on the unit disk [13] to the unit ball.

In [17], Peng and Ouyang studied the boundedness and compactness of the operator $T_g : Q_x \to Q_y$ when $\frac{n-1}{n} < x \leq y < \frac{n}{n-1}$. Among others, they demonstrated that $T_g : Q_x \to Q_y$ is bounded if and only if

$$\sup_{a\in\mathbb{B}} \left(\log \frac{2}{1-|a|^2}\right)^2 \int_{\mathbb{B}} |\Re g(z)|^2 (1-|z|^2)^{1-n} (1-|\sigma_a(z)|^2)^{ny} dV(z) < \infty.$$

In this paper, under some mild conditions, by using Carleson block instead of non-isotropic balls, we prove that the identity operator $I_d : Q_p \to \mathscr{T}^q_{\lambda,s}(\mu)$ is bounded if and only if

$$\sup_{a\in\mathbb{B}} \frac{\mu(S_a)}{(1-|a|)^\lambda (\log \frac{2}{1-|a|})^{s-q}} < \infty.$$

As an application, we prove that $T_g : Q_x \to F(p,q,s,p)$ is bounded if and only if $g \in F(p,q,s)$. Moreover, we also estimate the essential norm of $T_g : Q_x \to F(p,q,s,p)$ and characterize the compactness of $T_g : Q_x \to F(p,q,s,p)$.

In this paper, constants are denoted by $C$, which are positive and may differ from one occurrence to the next. We say that $A \lesssim B$ if there exists a constant $C$ such that $A \leq CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

## 2. Embedding $Q_x$ into a Tent Space

In this section, we describe the boundedness of the identity operator from $Q_x (\frac{n-1}{n} < x < 1)$ space into a tent space on the unit ball of $\mathbb{C}^n$. For this purpose, we state some well-known results for the proof of main results in this paper. The following estimate is well known and useful; see, e.g., See [26, Theorem 1.12].

**Lemma 2.1.** *Suppose $t > 0$ and $c > -1$. For all $z \in \mathbb{B}$,*

$$\int_{\mathbb{B}} \frac{(1-|w|^2)^c dV(w)}{|1-\langle z,w\rangle|^t} \approx \frac{1}{(1-|z|^2)^{t-c-n-1}}$$

*with $t > c+n+1$.*

**Lemma 2.2.** [24, Theorem 3.1] *Suppose $s,t > 0$ and $c > -1$. For all $a,z \in \mathbb{B}$,*

$$\int_{\mathbb{B}} \frac{(1-|w|^2)^c dV(w)}{|1-\langle a,w\rangle|^s |1-\langle z,w\rangle|^t} \approx \frac{1}{(1-|z|^2)^{t-c-n-1} |1-\langle a,z\rangle|^s}$$

*with $t > c+n+1 > s$.*

In [3], Du and the first author of this paper characterized the embedding of Dirichlet type spaces $\mathscr{D}_\alpha^p$ into Lebesgue spaces $L^q(d\mu)$ in the unit ball of $\mathbb{C}^n$ by using Carleson blocks. Among others, they proved the following result, which is used in this paper.

**Lemma 2.3.** [3, Theorem 1.1] *Suppose $0 < p < q < \infty$, $\alpha > -1$, and $\mu$ is a positive Borel measure on $\mathbb{B}$. Suppose that $p < \alpha + n + 1$. Then the identity operator $I_d : \mathscr{D}_\alpha^p \to L^q(d\mu)$ is bounded if and only if*

$$M_1 = \sup_{a\in\mathbb{B}} \frac{\mu(S_a)}{(1-|a|)^{\frac{q}{p}(\alpha+n+1-p)}} < \infty.$$

*Moreover,*

$$\|I_d\|^q_{\mathscr{D}_\alpha^p \to L^q(d\mu)} \approx M_1.$$

Now we are in a position to state and prove our main result in this section.

**Theorem 2.1.** *Let $\frac{n-1}{n} < x \le 1$, $2 < q < \infty$, $0 < s \le q < \infty$, $\frac{nxq}{2} \le \lambda$, and $\mu$ be a positive Borel measure. Then the identity operator $I_d : Q_x \to \mathscr{T}^q_{\lambda,s}(\mu)$ is bounded if and only if*

$$M_2 = \sup_{a\in\mathbb{B}} \frac{\mu(S_a)}{(1-|a|)^\lambda (\log\frac{2}{1-|a|})^{s-q}} < \infty. \tag{2.1}$$

*Moreover,*

$$\|I_d\|^q_{Q_x\to\mathscr{T}^q_{\lambda,s}(\mu)} \approx M_2.$$

*Proof.* Suppose that $I_d : Q_x \to \mathscr{T}^q_{\lambda,s}(\mu)$ is bounded. For any $a \in \mathbb{B}$, let

$$f_a(z) = \log\frac{2}{1-\langle z,a\rangle}.$$

Then $\|f_a\|_{Q_x} \lesssim 1$ by [16]. As $a$ approaches $\mathbb{S}$,

$$\begin{aligned}\frac{\mu(S_a)}{(1-|a|)^\lambda(\log\frac{2}{1-|a|})^{s-q}} &\approx \frac{1}{(1-|a|)^\lambda(\log\frac{2}{1-|a|})^{s}}\int_{S_a}|f_a(z)|^q d\mu(z)\\ &\lesssim \|f_a\|^q_{\mathscr{T}^q_{\lambda,s}}\\ &\lesssim \|f_a\|^q_{Q_x}\|I_d\|^q_{Q_x\to\mathscr{T}^q_{\lambda,s}(\mu)}.\end{aligned}$$

So (2.1) holds and $M_2 \lesssim \|I_d\|^q_{Q_x\to\mathscr{T}^q_{\lambda,s}(\mu)}$.

Conversely, suppose that (2.1) holds. Let $f \in Q_x$ and $a \in \mathbb{B}$. By [16, Theorem 3.8] and the fact that $Q_x \subset \mathscr{B}$, one has

$$|f(a)| \lesssim \|f\|_{Q_x}\log\frac{2}{1-|a|}. \tag{2.2}$$

Thus

$$J_1 := \frac{1}{(1-|a|)^\lambda(\log\frac{2}{1-|a|})^s}\int_{S_a}|f(z)|^q d\mu(z) \lesssim \frac{\mu(S_a)\|f\|^q_{Q_x}}{(1-|a|)^\lambda(\log\frac{2}{1-|a|})^{s-q}} \le M_2\|f\|^q_{Q_x}.$$

Since $0 < s \le q < \infty$, one has

$$\sup_{a\in\mathbb{B}} \frac{\mu(S_a)}{(1-|a|)^\lambda} \le \sup_{a\in\mathbb{B}} \frac{\mu(S_a)}{(1-|a|)^\lambda (\log\frac{2}{1-|a|})^{s-q}} = M_2 < \infty.$$

By Lemma 2.3, we have $\|I_d\|^q_{\mathscr{D}^2_{\frac{2\lambda}{q}+1-n}\to L^q(d\mu)} \lesssim M_2$. Let $y$ be fixed and large enough. As $a$ approaches $\mathbb{S}$, we obtain

$$\begin{aligned}
J_2 :&= \frac{1}{(1-|a|)^\lambda (\log\frac{2}{1-|a|})^s} \int_{S_a} |f(z)-f(a)|^q d\mu(z) \\
&\le \frac{1}{(1-|a|)^\lambda} \int_{S_a} |f(z)-f(a)|^q d\mu(z) \\
&\lesssim (1-|a|^2)^y \int_{\mathbb{B}} \frac{|f(z)-f(a)|^q}{|1-\langle z,a\rangle|^{y+\lambda}} d\mu(z) \\
&\lesssim M_2 (1-|a|^2)^y \left( |f(0)-f(a)|^2 + \int_{\mathbb{B}} \left| \Re\left( \frac{f(z)-f(a)}{(1-\langle z,a\rangle)^{\frac{y+\lambda}{q}}} \right) \right|^2 (1-|z|^2)^{\frac{2\lambda}{q}+1-n} dV(z) \right)^{\frac{q}{2}}.
\end{aligned}$$

Using (2.2), we have

$$J_{21} := (1-|a|^2)^y |f(a)-f(0)|^q \lesssim (1-|a|^2)^y \left(\log\frac{2}{1-|a|}\right)^q \|f\|^q_{Q_x} \lesssim \|f\|^q_{Q_x}.$$

It follows from (1.1) that

$$\begin{aligned}
J_{22} &:= (1-|a|^2)^{\frac{2y}{q}} \int_{\mathbb{B}} \left| \frac{\Re f(z)}{(1-\langle z,a\rangle)^{\frac{y+\lambda}{q}}} \right|^2 (1-|z|^2)^{\frac{2\lambda}{q}+1-n} dV(z) \\
&= \int_{\mathbb{B}} |\Re f(z)|^2 (1-|z|^2)^{1-n} (1-|\sigma_a(z)|^2)^{nx} \frac{(1-|a|^2)^{\frac{2y}{q}-nx}(1-|z|^2)^{\frac{2\lambda}{q}-nx}}{|1-\langle z,a\rangle|^{\frac{2y+2\lambda}{q}-2nx}} dV(z). \\
&\lesssim \|f\|^2_{Q_x},
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
J_{23} &:= (1-|a|^2)^{\frac{2y}{q}} \int_{\mathbb{B}} |f(z)-f(a)|^2 \left| \Re\left( \frac{1}{(1-\langle z,a\rangle)^{\frac{y+\lambda}{q}}} \right) \right|^2 (1-|z|^2)^{\frac{2\lambda}{q}+1-n} dV(z) \\
&\lesssim \int_{\mathbb{B}} |f(z)-f(a)|^2 \frac{(1-|a|^2)^{\frac{2y}{q}}(1-|z|^2)^{\frac{2\lambda}{q}+1-n}}{|1-\langle z,a\rangle|^{\frac{2\lambda+2y}{q}+2}} dV(z) \\
&\le \int_{\mathbb{B}} \frac{|f(z)-f(a)|^2}{|1-\langle z,a\rangle|^2} (1-|z|^2)^{1-n} (1-|\sigma_a(z)|^2)^{nx} dV(z) \\
&:= I_a(f).
\end{aligned}$$

We claim $I_a(f) \lesssim \|f\|^2_{Q_x}$ for a moment and prove it later. Then,

$$J_2 \lesssim M_2 (J_{21} + J_{22}^{\frac{q}{2}} + J_{23}^{\frac{q}{2}}) \lesssim M_2 \|f\|^q_{Q_x}.$$

Therefore, for any $f \in Q_x$ and $a \in \mathbb{B}$, we obtain

$$\frac{1}{(1-|a|)^{\lambda}(\log\frac{2}{1-|a|})^{s}}\int_{S_a}|f(z)|^q d\mu(z) \lesssim J_1+J_2 \lesssim M_2\|f\|_{Q_x}^q,$$

which implies $\|I_d\|^q_{Q_x\to\mathscr{T}^q_{\lambda,s}(\mu)} \lesssim M_2$.

Now we prove that $I_a(f) \lesssim \|f\|^2_{Q_x}$. Let $\beta$ be large enough and

$$1<\varepsilon<\min\{nx-n+3, nx+1\}.$$

From [26, page 51], Hölder's inequality, and Lemma 2.1, for any $f \in H(\mathbb{B})$ and $a \in \mathbb{B}$, we see that

$$\begin{aligned}
&|f\circ\sigma_a(z)-f\circ\sigma_a(0)|^2\\
\lesssim&\left(\int_{\mathbb{B}}\frac{|\Re(f\circ\sigma_a)(w)|(1-|w|^2)^{\beta}dV(w)}{|1-\langle z,w\rangle|^{n+\beta}}\right)^2\\
\le&\int_{\mathbb{B}}\frac{|\Re(f\circ\sigma_a)(w)|^2(1-|w|^2)^{\beta+\varepsilon}dV(w)}{|1-\langle z,w\rangle|^{n+\beta}}\int_{\mathbb{B}}\frac{(1-|w|^2)^{\beta-\varepsilon}dV(w)}{|1-\langle z,w\rangle|^{n+\beta}}\\
\approx&\frac{1}{(1-|z|^2)^{\varepsilon-1}}\int_{\mathbb{B}}\frac{|\Re(f\circ\sigma_a)(w)|^2(1-|w|^2)^{\beta+\varepsilon}dV(w)}{|1-\langle z,w\rangle|^{n+\beta}}.
\end{aligned}$$

Therefore, by Fubini's theorem and Lemma 2.2, we obtain

$$\begin{aligned}
I_a(f)=&\int_{\mathbb{B}}|f\circ\sigma_a(z)-f\circ\sigma_a(0)|^2\frac{(1-|z|^2)^{nx-n+1}}{|1-\langle z,a\rangle|^2}dV(z)\\
\lesssim&\int_{\mathbb{B}}|\Re(f\circ\sigma_a)(w)|^2(1-|w|^2)^{\beta+\varepsilon}\left(\int_{\mathbb{B}}\frac{(1-|z|^2)^{nx-n+2-\varepsilon}}{|1-\langle z,w\rangle|^{n+\beta}|1-\langle z,a\rangle|^2}dV(z)\right)dV(w)\\
\approx&\int_{\mathbb{B}}\frac{|\Re(f\circ\sigma_a)(w)|^2(1-|w|^2)^{nx+3-n}}{|1-\langle a,w\rangle|^2}dV(w)\\
=&\int_{\mathbb{B}}|\Re(f\circ\sigma_a)(w)|^2(1-|w|^2)^{1-n}(1-|\sigma_a(w)|^2)^{nx}\frac{(1-|w|^2)^2}{|1-\langle a,w\rangle|^2}dV(w)\\
\lesssim&\|f\circ\sigma_a-f\circ\sigma_a(0)\|^2_{Q_x}\\
\lesssim&\|f\|^2_{Q_x}.
\end{aligned}$$

The proof is complete. □

## 3. Extended Cesàro Operator $T_g:\ Q_x \to F(p,q,s,p)$

In this section, we study the boundednss, compactness, and the essential norm of the extended Cesàro operator $T_g:\ Q_x \to F(p,q,s,p)$. We need the following equivalent characterization of functions in $F(p,q,s,t)$.

**Proposition 3.1.** *Let $f \in H(\mathbb{B})$, $t \ge 0$, $0<p,s<\infty$, and $-n-1<q<\infty$ such that $q+s>-1$. Then $f \in F(p,q,s,t)$ if and only if*

$$M_3(f)=\sup_{a\in\mathbb{B}}\frac{1}{(1-|a|)^s(\log\frac{2}{1-|a|^2})^t}\int_{S_a}|\Re f(z)|^p(1-|z|^2)^{q+s}dV(z)<\infty. \tag{3.1}$$

*Moreover, if $f(0)=0$, then $\|f\|^p_{F(p,q,s,t)} \approx M_3(f)$.*

*Proof.* Let $f \in F(p,q,s,t)$ and $f(0)=0$. For any $a \in \mathbb{B}$ and $z \in S_a$, $1-|a| \approx |1-\langle a,z\rangle|$. Combining with $1-|\sigma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle a,z\rangle|^2}$, we have

$$\begin{aligned}
&\frac{1}{(1-|a|)^s(\log\frac{2}{1-|a|^2})^t}\int_{S_a}|\Re f(z)|^p(1-|z|^2)^{q+s}dV(z)\\
\approx&\frac{1}{\left(\log\frac{2}{1-|a|^2}\right)^t}\int_{S_a}|\Re f(z)|^p(1-|z|^2)^q(1-|\sigma_a(z)|^2)^s dV(z)\\
\lesssim&\sup_{b\in\mathbb{B}}\frac{1}{\left(\log\frac{2}{1-|b|^2}\right)^t}\int_{\mathbb{B}}|\Re f(z)|^p(1-|z|^2)^q(1-|\sigma_b(z)|^2)^s dV(z).
\end{aligned}$$

Therefore, $M_3(f) \lesssim \|f\|^p_{F(p,q,s,t)}$.

Conversely, assume that (3.1) holds. Fix $\alpha>2$. For any $a \in \mathbb{B}\backslash\{0\}$, let

$$S_{a,\alpha} = \left\{z \in \mathbb{B} : |a| \le |z| < 1, |1-\langle \frac{z}{|z|}, \frac{a}{|a|}\rangle| < \alpha(1-|a|)\right\}.$$

It is easy to check that (see [2, Proposition 1] for example) (3.1) holds for $S_{a,\alpha}$.

Suppose that $a$ is fixed. Let $k$ be the largest integer such that $1-2^{k-1}(1-|a|)>0$,

$$a^{(j)} = (1-2^j(1-|a|))\frac{a}{|a|}, j=0,1,2,\cdots,k-1,$$

$a^{(k)}=0$, and $S_{a^{(k)},\alpha}=\mathbb{B}$. Then, for any given $j=1,2,\cdots,k$, for any $z \in S_{a^{(j)},\alpha}\backslash S_{a^{(j-1)},\alpha}$, if $z \neq 0$, either

$$|a^{(j)}| \le |z| < |a^{(j-1)}|, \ \left|1-\langle\frac{z}{|z|},\frac{a}{|a|}\rangle\right| < \alpha(1-|a^{(j)}|), \tag{3.2}$$

or

$$|a^{(j-1)}| \le |z| < 1, \ \ \alpha(1-|a^{(j-1)}|) \le \left|1-\langle\frac{z}{|z|},\frac{a}{|a|}\rangle\right| < \alpha(1-|a^{(j)}|). \tag{3.3}$$

If $z$ satisfies (3.2) and $z \neq 0$, then $|1-\langle a,z\rangle| \ge 1-|z| > 1-|a^{(j-1)}| = 2^{j-1}(1-|a|)$. Similarly, if $z$ satisfies (3.3) and $z \neq 0$, then

$$\begin{aligned}
|1-\langle a,z\rangle| &\ge \left|1-\langle\frac{a}{|a|},\frac{z}{|z|}\rangle\right| - \frac{|\langle a,z\rangle|(1-|a||z|)}{|a||z|}\\
&\ge \alpha 2^{j-1}(1-|a|)-(1-|a|)-|a|(1-|z|)\\
&\ge \alpha 2^{j-1}(1-|a|)-(1-|a|)-2^{j-1}(1-|a|)\\
&\gtrsim 2^{j-1}(1-|a|).
\end{aligned}$$

So, when $j=1,2,\cdots,k$ and $z \in S_{a^{(j)},\alpha}\backslash S_{a^{(j-1)},\alpha}$, by using the fact that

$$|1-\langle a,z\rangle| \le \left|1-\langle\frac{a}{|a|},\frac{z}{|z|}\rangle\right| + \frac{|\langle a,z\rangle|(1-|a||z|)}{|a||z|} \lesssim 2^j(1-|a|),$$

we have

$$\frac{1-|a|}{|1-\langle a,z\rangle|^2} \approx \frac{1}{2^{2j}(1-|a|)}. \tag{3.4}$$

When $z \in S_{a^{(0)},\alpha}$, it is obvious that

$$\frac{1-|a|}{|1-\langle a,z\rangle|^2} \approx \frac{1}{1-|a|}. \tag{3.5}$$

Let $S_{a^{(-1)},\alpha} = \varnothing$. As $|a|$ approaches 1, by (3.4) and (3.5), we have

$$\begin{aligned}
&\frac{1}{\left(\log\frac{2}{1-|a|^2}\right)^t}\int_{\mathbb{B}} |\Re f(z)|^p(1-|z|^2)^q(1-|\sigma_a(z)|^2)^s dV(z)\\
\approx&\sum_{j=0}^{k}\frac{1}{|2^{2j}(1-|a|)|^s\left(\log\frac{2}{1-|a|^2}\right)^t}\int_{S_{a^{(j)},\alpha}\setminus S_{a^{(j-1)},\alpha}} |\Re f(z)|^p(1-|z|^2)^{q+s}dV(z)\\
\lesssim&\sum_{j=0}^{k}\frac{1}{|2^{2j}(1-|a|)|^s\left(\log\frac{2}{1-|a|^2}\right)^t}\int_{S_{a^{(j)},\alpha}} |\Re f(z)|^p(1-|z|^2)^{q+s}dV(z)\\
\lesssim&\sum_{j=0}^{k}\frac{1}{|2^{2j}(1-|a|)|^s\left(\log\frac{2}{1-|a|^2}\right)^t}\times|2^j(1-|a|)|^s\left(\log\frac{2}{|2^{j+1}(1-|a|)|}\right)^t M_3(f)\\
\lesssim&\sum_{j=0}^{\infty}\frac{1}{2^{js}}\frac{\left(\log\frac{2}{|2^{j+1}(1-|a|)|}\right)^t}{\left(\log\frac{2}{1-|a|}\right)^t}M_3(f)\\
\lesssim& M_3(f).
\end{aligned}$$

So, $\|f\|_{F(p,q,s,t)} \lesssim M_3(f)$ when $f(0)=0$. The proof is complete. □

**Theorem 3.1.** *Let $g \in H(\mathbb{B})$, $\frac{n-1}{n} < x \leq 1$, $2 < p < \infty$, $0 < s < \infty$, and $-n-1 < q < \infty$ such that $q+s > -1$ and $2s \geq nxp$. Then the operator $T_g : Q_x \to F(p,q,s,p)$ is bounded if and only if $g \in F(p,q,s)$. Moreover, if $g(0)=0$, then $\|T_g\|_{Q_x\to F(p,q,s,p)} \approx \|g\|_{F(p,q,s)}$.*

*Proof.* Suppose that $g \in F(p,q,s)$ and $g(0)=0$. By Proposition 3.1, we have

$$\|g\|^p_{F(p,q,s)} \approx \sup_{a\in\mathbb{B}}\frac{1}{(1-|a|)^s}\int_{S_a}|\Re g(z)|^p(1-|z|^2)^{q+s}dV(z) < \infty.$$

Let $d\mu_g(z) = |\Re g(z)|^p(1-|z|^2)^{q+s}dV(z)$. By Theorem 2.1, the identity operator $I_d : Q_x \to \mathscr{T}^p_{s,p}(\mu_g)$ is bounded and $\|I_d\|^p_{Q_x\to\mathscr{T}^p_{s,p}(\mu_g)} \approx \|g\|^p_{F(p,q,s)}$. Therefore, by Proposition 3.1, for any

$f \in Q_x$, we have

$$\begin{aligned}\|T_g f\|^p_{F(p,q,s,p)} &\approx \sup_{a\in\mathbb{B}} \frac{1}{(1-|a|)^s(\log\frac{2}{1-|a|})^p}\int_{S_a}|\Re(T_g f)(z)|^p(1-|z|^2)^{q+s}dV(z)\\ &= \sup_{a\in\mathbb{B}} \frac{1}{(1-|a|)^s(\log\frac{2}{1-|a|})^p}\int_{S_a}|f(z)|^p|\Re g(z)|^p(1-|z|^2)^{q+s}dV(z)\\ &= \sup_{a\in\mathbb{B}} \frac{1}{(1-|a|)^s(\log\frac{2}{1-|a|})^p}\int_{S_a}|f(z)|^p d\mu_g(z)\\ &= \|f\|^p_{\mathcal{T}^p_{s,p}(\mu_g)} \lesssim \|g\|^p_{F(p,q,s)}\|f\|^p_{Q_x}.\end{aligned}$$

That is, $\|T_g\|_{Q_x\to F(p,q,s,p)} \lesssim \|g\|_{F(p,q,s)}$.

Conversely, assume that $T_g:\ Q_x \to F(p,q,s,p)$ is bounded. For any $a \in \mathbb{B}$, set

$$f_a(z) = \log\frac{2}{1-\langle z,a\rangle}.$$

Then $\|f_a\|_{Q_x} \lesssim 1$. As $|a| \to 1$, for any $z \in S_a$, we have $|1-\langle a,z\rangle| \approx 1-|a|$ and

$$\left|\log\frac{2}{1-\langle z,a\rangle}\right| \approx \log\frac{2}{1-|a|}.$$

By Proposition 3.1, we have

$$\begin{aligned}&\frac{1}{(1-|a|)^s}\int_{S_a}|\Re g(z)|^p(1-|z|^2)^{q+s}dV(z)\\ \lesssim&\frac{1}{(1-|a|)^s(\log\frac{2}{1-|a|})^p}\int_{S_a}|\Re(T_g f_a)(z)|^p(1-|z|^2)^{q+s}dV(z)\\ \lesssim&\|T_g f_a\|^p_{F(p,q,s,p)}\\ \lesssim&\|T_g\|^p_{Q_x\to F(p,q,s,p)},\end{aligned}$$

and therefore, $\|g\|_{F(p,q,s)} \lesssim \|T_g\|_{Q_x\to F(p,q,s,p)}$ when $g(0)=0$. The proof is complete. □

Next, we give an estimation for the essential norm of $T_g : Q_x \to F(p,q,s,p)$. First, we recall some definitions. Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ be a bounded linear operator. The essential norm of $T : X \to Y$, denoted by $\|T\|_{e,X\to Y}$, is defined by

$$\|T\|_{e,X\to Y} = \inf\{\|T-K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}.$$

It is easy to see that $T : X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$.

The following lemma is demonstrated in the case of the unit disk; see [21, Lemma 3.7 ]. However, it is still valid for the case of the unit ball of $\mathbb{C}^n$. We omit the details; see [17, Lemma 3.1].

**Lemma 3.1.** *Let $X,Y$ be two Banach spaces of analytic functions on $\mathbb{B}$. Suppose that*

*(1) the point evaluation functionals on $Y$ are continuous;*
*(2) the closed unit ball of $X$ is a compact subset of $X$ in the topology of uniform convergence on compact sets;*
*(3) $T : X \to Y$ is continuous when $X$ and $Y$ are given by the topology of uniform convergence on compact sets.*

*Then, $T$ is a compact operator if and only if, for any bounded sequence $\{f_n\}$ in $X$ such that $\{f_n\}$ converges to zero uniformly on every compact set of $\mathbb{B}$, the sequence $\{Tf_n\}$ converges to zero in the norm of $Y$.*

**Lemma 3.2.** *Let $g \in H(\mathbb{B})$, $\frac{n-1}{n} < x \le 1$, $2 < p < \infty$, $0 < s < \infty$ ,and $-n-1 < q < \infty$ such that $q+s > -1$ and $2s \ge nxp$. If $\Re g \in H^\infty$, then $T_g : Q_x \to F(p,q,s,p)$ is compact.*

*Proof.* Give $\{f_k\} \subset Q_x$ such that $\{f_k\}$ converges to zero uniformly on compact subsets of $\mathbb{B}$ and $\sup_k \|f_k\|_{Q_x} \le 1$. By Proposition 3.1, we have

$$\|T_g f_k\|^p_{F(p,q,s,p)} \approx \sup_{a\in\mathbb{B}} \frac{1}{(1-|a|)^s(\log\frac{2}{1-|a|^2})^p} \int_{S_a} |f_k(z)\Re g(z)|^p(1-|z|^2)^{q+s}dV(z).$$

Let $r \in (0,1)$. By $Q_x \subset \mathscr{B}$ and (2.2), we have

$$\begin{aligned} J_1(r,f_k,a) &:= \frac{1}{(1-|a|)^s(\log\frac{2}{1-|a|^2})^p} \int_{S_a\setminus r\mathbb{B}} |f_k(z)\Re g(z)|^p(1-|z|^2)^{q+s}dV(z) \\ &\lesssim \frac{\|\Re g\|^p_{H^\infty}\|f_k\|^p_{Q_x}}{(1-|a|)^s(\log\frac{2}{1-|a|^2})^p} \int_{S_a\setminus r\mathbb{B}} \left(\log\frac{2}{1-|z|}\right)^p (1-|z|^2)^{q+s}dV(z) \\ &\approx \frac{\|\Re g\|^p_{H^\infty}\|f_k\|^p_{Q_x}(1-|a|)^n}{(1-|a|)^s(\log\frac{2}{1-|a|^2})^p} \int_{\max\{r,|a|\}}^1 \left(\log\frac{2}{1-|z|}\right)^p (1-|z|^2)^{q+s}d|z|. \end{aligned}$$

When $|a| < r$ and $n \ge s$, we have

$$\begin{aligned} J_1(r,f_k,a) &\lesssim \|\Re g\|^p_{H^\infty}\|f_k\|^p_{Q_x}(1-|a|)^{n-s}\left(\log\frac{2}{1-r}\right)^p(1-r)^{q+s+1} \\ &\lesssim \|\Re g\|^p_{H^\infty}\|f_k\|^p_{Q_x}\left(\log\frac{2}{1-r}\right)^p(1-r)^{q+s+1}. \end{aligned}$$

When $|a| < r$ and $n < s$, we obtain

$$\begin{aligned} J_1(r,f_k,a) &\lesssim \|\Re g\|^p_{H^\infty}\|f_k\|^p_{Q_x}(1-|a|)^{n-s}\left(\log\frac{2}{1-r}\right)^p(1-r)^{q+s+1} \\ &\lesssim \|\Re g\|^p_{H^\infty}\|f_k\|^p_{Q_x}\left(\log\frac{2}{1-r}\right)^p(1-r)^{q+n+1}. \end{aligned}$$

When $|a| \ge r$, since $q+n+1 > 0$, we have

$$\begin{aligned} J_1(r,f_k,a) &\lesssim \|\Re g\|^p_{H^\infty}\|f_k\|^p_{Q_x}\left(\log\frac{2}{1-|a|}\right)^p(1-|a|)^{q+n+1} \\ &\lesssim \|\Re g\|^p_{H^\infty}\|f_k\|^p_{Q_x}\left(\log\frac{2}{1-r}\right)^p(1-r)^{q+n+1}. \end{aligned}$$

Since $q+s+1 > 0$ and $q+n+1 > 0$, for any given $\varepsilon > 0$, there exists a $r_\varepsilon \in (0,1)$ such that

$$J_1(r_\varepsilon,f_k,a) < \varepsilon\|\Re g\|^p_{H^\infty}\|f_k\|^p_{Q_x}.$$

Meanwhile, it is easy to check that

$$\begin{aligned}
J_2(r_\varepsilon, f_k, a) :&= \frac{1}{(1-|a|)^s(\log\frac{2}{1-|a|^2})^p}\int_{S_a\cap r_\varepsilon\mathbb{B}} |f_k(z)\Re g(z)|^p(1-|z|^2)^{q+s}dV(z)\\
&\le \frac{\|\Re g\|_{H^\infty}^p \sup_{|z|\le r_\varepsilon}|f_k(z)|^p}{(1-|a|)^s(\log\frac{2}{1-|a|^2})^p}\int_{S_a}(1-|z|^2)^{q+s}dV(z)\\
&\lesssim \|\Re g\|_{H^\infty}^p \sup_{|z|\le r_\varepsilon}|f_k(z)|^p(1-|a|)^{q+n+1}\\
&\lesssim \|\Re g\|_{H^\infty}^p \sup_{|z|\le r_\varepsilon}|f_k(z)|^p.
\end{aligned}$$

So, by Proposition 3.1, we have

$$\lim_{k\to\infty}\|T_g f_k\|_{F(p,q,s,p)} \approx \lim_{k\to\infty}\sup_{a\in\mathbb{B}}(J_1(r_\varepsilon,f_k,a)+J_2(r_\varepsilon,f_k,a)) \le \varepsilon\|\Re g\|_{H^\infty}^p\|f_k\|_{Q_x}^p.$$

By the arbitrariness of $\varepsilon$, we arrive at $\lim_{k\to\infty}\|T_g f_k\|_{F(p,q,s,p)} = 0$. That is to say, $T_g : Q_x \to F(p,q,s,p)$ is compact. The proof is complete. □

**Theorem 3.2.** *Let $g\in H(\mathbb{B})$, $\frac{n-1}{n} < x \le 1$, $2<p<\infty$, $0<s<\infty$, and $-n-1<q<\infty$ such that $q+s>-1$ and $2s\ge nxp$. If $T_g : Q_x \to F(p,q,s,p)$ is bounded, then*

$$\|T_g\|_{e,Q_x\to F(p,q,s,p)}^p \approx \limsup_{|a|\to 1}\frac{1}{(1-|a|)^s}\int_{S_a}|\Re g(z)|^p(1-|z|^2)^{q+s}dV(z).$$

*Proof.* For any $a\in\mathbb{B}$, let

$$f_a(z) = \left(\log\frac{2}{1-|a|^2}\right)^{-1}\left(\log\frac{2}{1-\langle z,a\rangle}\right)^2,\ z\in\mathbb{B}.$$

By the proof of [17, Theorem 3.1], $\sup_{a\in\mathbb{B}}\|f_a\|_{Q_x}\lesssim 1$ and $\{f_a\}$ converges to 0 uniformly on compact subsets of $\mathbb{B}$ as $|a|\to 1$. So, for any compact operator $K : Q_x \to F(p,q,s,p)$, by Lemma 3.1 and Proposition 3.1, we have

$$\begin{aligned}
\|T_g-K\|^p &\gtrsim \limsup_{|a|\to 1}\|(T_g-K)f_a\|_{F(p,q,s,p)}^p\\
&\gtrsim \limsup_{|a|\to 1}\|T_g f_a\|_{F(p,q,s,p)}^p\\
&\gtrsim \limsup_{|a|\to 1}\frac{1}{(1-|a|)^s(\log\frac{2}{1-|a|^2})^p}\int_{S_a}|\Re(T_g f_a)(z)|^p(1-|z|^2)^{q+s}dV(z)\\
&\approx \limsup_{|a|\to 1}\frac{1}{(1-|a|)^s}\int_{S_a}|\Re g(z)|^p(1-|z|^2)^{q+s}dV(z).
\end{aligned}$$

Therefore,

$$\|T_g\|_{e,Q_x\to F(p,q,s,p)}^p \gtrsim \limsup_{|a|\to 1}\frac{1}{(1-|a|)^s}\int_{S_a}|\Re g(z)|^p(1-|z|^2)^{q+s}dV(z).$$

On the other hand, by Lemma 3.2, for any $r \in (0,1)$ and $g \in F(p,q,s)$, letting $g_r(z) = g(rz)$, we have that $T_{g_r} : Q_x \to F(p,q,s,p)$ is compact. By Theorem 3.1 and Proposition 3.1,

$$\begin{aligned}
\|T_g\|^p_{e,Q_x\to F(p,q,s,p)} &\le \limsup_{r\to 1} \|T_g - T_{g_r}\|^p \\
&\approx \limsup_{r\to 1} \|g - g_r\|^p_{F(p,q,s)} \\
&\approx \limsup_{r\to 1} \sup_{a\in\mathbb{B}} \frac{1}{(1-|a|)^s} \int_{S_a} |\Re g(z) - \Re g_r(z)|^p (1-|z|^2)^{q+s} dV(z) \\
:&= \limsup_{r\to 1} \sup_{a\in\mathbb{B}} G(g,a,r).
\end{aligned}$$

Let $\delta \in (0,1)$ be given and large enough.

**Case 1:** $|a| \le \delta$. It is easy to check that

$$G(g,a,r) \le C(\delta)\|\Re g - \Re g_r\|^p_{A^p_{q+s}} \to 0, \text{ as } r \to 1.$$

**Case 2:** $\delta < |a| \le r$. For any $z \in \mathbb{B}$, let $\varepsilon \in (0, \frac{1}{2})$ be pre-fixed,

$$D_\varepsilon(z) = \{\zeta \in \mathbb{B} : |\zeta - z| < \varepsilon(1-|z|)\}$$

and $\chi_E$ be the characteristic function of the set $E$. Then, by the subharmonicity of $|\Re g_r|$ and Fubini's theorem, we have

$$\begin{aligned}
&\int_{S_a} |\Re g_r(z)|^p (1-|z|^2)^{q+s} dV(z) \\
&\lesssim \int_{\mathbb{B}} |\Re g(\zeta)|^p (1-|\zeta|^2)^{-n-1} \left( \int_{\mathbb{B}} (1-|z|^2)^{q+s} \chi_{D_\varepsilon(rz)}(\zeta)\chi_{S_a}(z) dV(z) \right) dV(\zeta).
\end{aligned}$$

Suppose $\chi_{D_\varepsilon(rz)}(\zeta)\chi_{S_a}(z) = 1$. Then, we have $z \in S_a$ and

$$1 - |\zeta| < (1+\varepsilon)(1-r|z|) < \frac{3}{2}(1-|a|^2) < 3(1-|a|).$$

Moreover, we can choose a $\alpha > 1$ such that $\chi_{D_\varepsilon(rz)}(\zeta)\chi_{S_a}(z) = 1$. Then

$$\left| 1 - \langle \frac{\zeta}{|\zeta|}, \frac{a}{|a|} \rangle \right| < 3\alpha(1-|a|)$$

when $|a| > \delta$ and $\delta \in (0,1)$ is large enough. Let $a' = (1 - 3(1-|a|))\frac{a}{|a|}$. Then we have

$$\begin{aligned}
&\int_{S_a} |\Re g_r(z)|^p (1-|z|^2)^{q+s} dV(z) \\
&\lesssim \int_{S_{a',\alpha}} |\Re g(\zeta)|^p (1-|\zeta|^2)^{-n-1} \left( \int_{\mathbb{B}} (1-|z|^2)^{q+s} \chi_{D_\varepsilon(rz)}(\zeta)\chi_{S_a}(z) dV(z) \right) dV(\zeta).
\end{aligned}$$

For any $\zeta \in S_{a',\alpha}$, we see that $\chi_{D_\varepsilon(rz)}(\zeta) = 1$ implies

$$|\zeta| - r|z| < \varepsilon(1-r|z|), \ -|\zeta| + r|z| < \varepsilon(1-r|z|),$$

i.e.,

$$\frac{|\zeta| - \varepsilon}{r(1-\varepsilon)} < |z| < \frac{\varepsilon + |\zeta|}{r(1+\varepsilon)}.$$

Here, it is should be point out that since $z\in\mathbb{B}$, $\frac{|\zeta|-\varepsilon}{r(1-\varepsilon)}<1$. Thus

$$\big|r|z|-|\zeta|\big|\lesssim 1-|\zeta|,\ \ 1-r|z|\approx 1-|\zeta|$$

and

$$1-\frac{|\zeta|-\varepsilon}{r(1-\varepsilon)}=\frac{r-r\varepsilon-|\zeta|+\varepsilon}{r(1-\varepsilon)}=\frac{1-|\zeta|}{r(1-\varepsilon)}+\frac{(r-1)(1-\varepsilon)}{r(1-\varepsilon)}\lesssim 1-|\zeta|.$$

So, when $\zeta\in S_{a',\alpha}$, letting $\eta_z=\frac{z}{|z|}$, we have from $\chi_{D_\varepsilon(rz)}(\zeta)=1$ that $|\zeta-rz|<\varepsilon(1-r|z|)$ and

$$\begin{aligned}|\eta_z-\eta_{\frac{\zeta}{r|z|}}|&\le|\eta_{\frac{\zeta}{r|z|}}-\frac{\zeta}{r|z|}|+\frac{|\zeta-rz|}{r|z|}\le|\eta_{\frac{\zeta}{r|z|}}-\frac{\zeta}{r|z|}|+\varepsilon(\frac{1}{r|z|}-1)\\&=\frac{\big|r|z|-|\zeta|\big|}{r|z|}+\varepsilon\frac{1-r|z|}{r|z|}\lesssim(1-|\zeta|).\end{aligned}$$

So,

$$\begin{aligned}&\int_{\mathbb{B}}(1-|z|^2)^{q+s}\chi_{D_\varepsilon(rz)}(\zeta)\chi_{S_a}(z)dV(z)\\&\lesssim\int_{\frac{|\zeta|-\varepsilon}{r(1-\varepsilon)}}^{\min\left\{1,\frac{\varepsilon+|\zeta|}{r(1+\varepsilon)}\right\}}(1-|z|)^{q+s}d|z|\int_{|\eta_z-\eta_{\frac{\zeta}{r|z|}}|\lesssim(1-|\zeta|)}d\sigma(\eta_z)\\&\lesssim(1-|\zeta|)^n\left(1-\frac{|\zeta|-\varepsilon}{r(1-\varepsilon)}\right)^{q+s+1}\\&\lesssim(1-|\zeta|)^{q+s+n+1}.\end{aligned}$$

By a direct calculation, we have

$$\int_{S_a}|\Re g_r(z)|^p(1-|z|^2)^{q+s}dV(z)\lesssim\int_{S_{a',\alpha}}|\Re g(\zeta)|^p(1-|\zeta|^2)^{q+s}dV(\zeta).$$

Therefore, letting

$$H(g,a)=\frac{1}{(1-|a|)^s}\int_{S_a}|\Re g(z)|^p(1-|z|^2)^{q+s}dV(z),$$

By [2, Proposition 1], we have

$$\begin{aligned}G(g,a,r)&\lesssim H(g,a)+\frac{1}{(1-|a|)^s}\int_{S_a}|\Re g_r(z)|^p(1-|z|^2)^{q+s}dV(z)\\&\lesssim H(g,a)+H(g,a')\lesssim\sup_{|a|>\min\{\delta,3\delta-2\}}H(g,a)=\sup_{|a|>3\delta-2}H(g,a).\end{aligned}$$

**Case 3:** $|a|>\max\{r,\delta\}$. Obviously,

$$|\Re g_r(z)|\le\sup\{|\Re g(\zeta)|:|\zeta|=|a|\}.$$

Let $z^*=\frac{z}{|z|}(2|z|-1)$ and $\varepsilon$ be small enough such that $D_\varepsilon(z)\subset S_{z^*}$ for any $z\in\mathbb{B}\backslash\frac{3}{4}\mathbb{B}$. When $|\zeta|=|a|$, we have

$$\begin{aligned}|\Re g(\zeta)|^p&\lesssim\frac{1}{(1-|\zeta|)^{q+s+n+1}}\int_{D_\varepsilon(\zeta)}|\Re g(\xi)|^p(1-|\xi|^2)^{q+s}dV(\xi)\\&\lesssim\frac{H(g,\zeta^*)}{(1-|\zeta^*|)^{q+n+1}}\lesssim\frac{1}{(1-|a|)^{q+n+1}}\sup_{|\zeta^*|>2\delta-1}H(g,\zeta^*).\end{aligned}$$

So,

$$\frac{1}{(1-|a|)^s}\int_{S_a}|\Re g_r(z)|^p(1-|z|^2)^{q+s}dV(z)\lesssim \sup_{|\zeta^*|>2\delta-1} H(g,\zeta^*).$$

Therefore, letting $r\to 1$, we have

$$\begin{aligned}\|T_g\|^p_{e,Q_x\to F(p,q,s,p)}&\lesssim \sup_{|a|>\min\{3\delta-2,2\delta-1\}} H(g,a)\\ &=\sup_{|a|>3\delta-2}\frac{1}{(1-|a|)^s}\int_{S_a}|\Re g(z)|^p(1-|z|^2)^{q+s}dV(z).\end{aligned}$$

As $\delta\to 1$, we get the desired result. The proof is complete. □

By Theorem 3.2, we immediately obtain the following corollary.

**Corollary 3.1.** *Let $g\in H(\mathbb{B})$, $\frac{n-1}{n}<x\le 1$, $2<p<\infty$, $0<s<\infty$, and $-n-1<q<\infty$ such that $q+s>-1$ and $2s\ge nxp$. If $T_g: Q_x\to F(p,q,s,p)$ is bounded, then $T_g: Q_x\to F(p,q,s,p)$ is compact if and only if*

$$\lim_{|a|\to 1}\sup\frac{1}{(1-|a|)^s}\int_{S_a}|\Re g(z)|^p(1-|z|^2)^{q+s}dV(z)=0.$$

**Acknowledgements**

The first author was supported by GuangDong Basic and Applied Basic Research Foundation (No. 2022A1515010317 and No. 2023A1515010614). The authors thank Dr. Juntao Du for numerous comments which led to the improvement of the original manuscript of this paper.

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