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ABSOLUTE VALUE EQUATIONS WITH DATA UNCERTAINTY IN THE l_1 AND l_∞ NORM BALLS

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Abstract. Absolute value equations (AVEs) have attracted much attention in recent studies. However, the problem data may be contaminated by noises that yield a meaningless solution, even if these coefficients are uncertain within a certain range. To address this issue, we import the idea of robust optimization and present their robust counterpart models with data uncertainty in the l_1 and l_∞ norm balls. In particular, we prove that these models are equivalent to the linear programming problems. Numerical experiments demonstrate that the true solution of these AVEs can be recovered by solving the equivalent linear programming models with open-resource packages JuMP and HiGHS in Julia language.

Keywords. Absolute value equations; Data uncertainty; Robust counterpart model; Robust optimization.

1. Introduction

The target problem of this article is the absolute value equation (AVE) with uncertain data, which is different from the traditional one. To explain the motivation, we first recall that the standard absolute value equations (AVE), which is

$$Ax + B|x| = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and $|x|$ denotes the absolute value vector of $x \in \mathbb{R}^n$ in the component sense, i.e., $|x| := (|x_1|, |x_2| \dots |x_n|)^T$. If $B = -I$, where I is the identity matrix, then problem (1.1) can be cast into the following special form

$$Ax - |x| = b. \tag{1.2}$$

During the past two decades, these AVE models have attracted considerable attention in many different fields, such as finance, control, and operations research. Since they were introduced by Rohn for the first time in [1], many literatures have studied theories and numerical algorithms of these AVEs from different perspectives [2, 3, 4, 5, 6]. One of the important results in AVEs is the existence and uniqueness of their solutions. Mangasarian and Meyer [2] proved

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the equivalence relation between problem (1.2) and the linear complementarity problem (LCP). They proved that problem (1.2) has a unique solution for any $b \in R^n$ if the singular values of A exceed 1; has exactly 2^n distinct solutions if $b < 0$ and $\|A\|_\infty < \frac{\min_i |b_i| / \max_i |b_i|}{2}$; and has no solution if $0 \neq b \geq 0$ and $\|A\|_2 < 1$. After that, based on these observations, a number of numerical methods for solving AVEs were investigated, which can be divided into the following cases: (a) If the given AVE has a unique solution, the goal of numerical methods aims to obtain the associated solution more efficiently. For example, Mangasarian [4] proposed a generalized Newton method, and demonstrated the sufficient conditions for its linear convergence. In addition, a hybrid linear equations-linear programming formulation for solving the given AVE was proposed in [5]. Rohn and Hooshyarbakhsh [7] presented an iterative method and discussed sufficient conditions for unique solvability. (b) If the given AVE has more than one solution, the key concern turns to find the minimum norm solution. For example, Ketabchi and Moosaei [8] presented an algorithm to compute the minimum norm solution of problem (1.2), in which they proved that the given problem can be reduced as an unconstrained minimization problem with a differentiable convex objective function by using the exterior penalty method and proposed a quasi-Newton method for solving the corresponding unconstrained optimization problem. Moosaei and Ketabchi [9] found the minimum norm solution of the given AVE by solving a quadratic programming problem with the quadratic and linear constraints, and a Simulated Annealing algorithm was designed to solve the associated subproblem. (c) If the given AVE has no solution, the best choice is to consider the optimal correction problem associated with infeasible AVEs. For example, Hossein [10] studied the optimum correction of AVEs through making minimal changes, and translated the corrected problem into a non-differentiable, non-convex, and unconstrained fractional quadratic programming problem, and proposed a bisection algorithm. Ketabchi [11] discussed the optimum correction of linear inequality systems and AVEs, and presented a feasible direction method for solving the given problem. Ketabchi [12] demonstrated that the optimum correction problem can be transformed into a nonconvex and fractional quadratic problem, and proposed a genetic algorithm. Hossein Moosaei [13] used the Tikhonov regularization to investigate the optimum correction of AVEs, and poroved that the corresponding global optimal solution can be found by using a subgradient method. With the further extension of AVEs, numerous scholars also discussed other variants. For the problem (1.2) associated with convex cones, one can refer to [14, 15, 16, 17] and references therein.

Notice that all the literatures aforementioned discuss these AVEs with deterministic data. Similar as the recent studies in the linear equations and optimization problems, it might be difficult to determine the exact problem data for AVEs. In the extreme case, the slight perturbation of these data may change the associated solutions greatly even if the data are uncertain within a certain range. However, there are few results about AVEs with uncertainty data, especially for problem (1.1), which motives us to fill this gap. There are two main approaches to tackle mathematical problems with uncertainty data. One is robust optimization for the worst-case situation, and the other is stochastic optimization in the average viewpoint. In this paper, we import the idea from robust optimization theory [18, 19] and study their robust counterpart models to protect these AVEs against data uncertainty in the l_1 and l_∞ norm balls, in which the associated models can be cast into LP problems and solved by open-resource solvers JuMP [20] and HiGHS [21]. Numerical experiments indicate that the true solution of these AVEs can also be recovered.

The outline of this paper is as follows. In Section 2, we recall some basic concepts and background materials regarding the norm consistency of matrix norms with the l_1 and l_∞ vector norms. In Section 3 and Section 4, we present the equivalent descriptions of robust optimization counterparts for the AVEs with data uncertainty in the l_1 and l_∞ norm balls, respectively. Numerical results are provided in Section 5. Lastly, we do a summary in the final section, Section 6.

2. Preliminaries

In this section, we recall some basic concepts and background materials used in the subsequent analysis, in which the norm consistency between the l_1 and l_∞ matrix norm and the associated vector version will be considered.

We first recall the norm consistency between the l_1 matrix norm and the l_1 vector version, whose definitions are given by

$$\|A\|_1 := \max_{1\le j\le n}\sum_{i=1}^{m}|a_{ij}|, \quad \|x\|_1 := \sum_{j=1}^{n}|x_j|,$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and a_{ij} is the (i,j)-th entry of A. The arguments are quite routine, which can be found in textbooks of matrix analysis [22]. We present them just for completeness.

Lemma 2.1. *Let A be a matrix in $\mathbb{R}^{m\times n}$ and x be a vector in $\mathbb{R}^n$. Then*

$$\|Ax\|_1 \le \|A\|_1\|x\|_1.$$

Proof. From the above definitions of the l_1 vector norm and the l_1 matrix norm, we have

$$\begin{aligned}
\|Ax\|_1 &= \left|\sum_{j=1}^{n} a_{1j}x_j\right| + \left|\sum_{j=1}^{n} a_{2j}x_j\right| + \cdots + \left|\sum_{j=1}^{n} a_{mj}x_j\right| \\
&\le \sum_{j=1}^{n}|a_{1j}x_j| + \sum_{j=1}^{n}|a_{2j}x_j| + \cdots + \sum_{j=1}^{n}|a_{mj}x_j| \\
&= \sum_{j=1}^{n}|a_{1j}||x_j| + \sum_{j=1}^{n}|a_{2j}||x_j| + \cdots + \sum_{j=1}^{n}|a_{mj}||x_j| \\
&= \left(\sum_{i=1}^{m}|a_{i1}|\right)|x_1| + \left(\sum_{i=1}^{m}|a_{i2}|\right)|x_2| + \cdots + \left(\sum_{i=1}^{m}|a_{in}|\right)|x_n| \\
&\le \left(\max_{1\le j\le n}\sum_{i=1}^{m}|a_{ij}|\right)\cdot\left(\sum_{j=1}^{n}|x_j|\right) \\
&= \|A\|_1\|x\|_1.
\end{aligned}$$

Hence, the proof is complete. □

Remark 2.1. Let $D \in \mathbb{R}^{n\times n}$ denote a diagonal matrix such that $|x| = Dx$, whose each diagonal element is ± 1. For notional simplicity, we call such D the sign pattern of x. Now, we choose matrix $A \in \mathbb{R}^{m\times n}$ satisfying the following two conditions: (a) Each column of $A^T \in \mathbb{R}^{n\times m}$ only have two types of sign pattern D or $-D$, in which D is the sign pattern of $x \in \mathbb{R}^n$; (b) Each column of matrix $|A| \in \mathbb{R}^{m\times n}$ (the absolute value of A in the component sense) have the same value of l_1 norm. It is easy to see that Item (a) and Item (b) make the relations (3.10) and (3.11) become equalities. In this case, it follows from Lemma 2.1 that $\|Ax\|_1 = \|A\|_1\|x\|_1$. For example, we choose

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 1 & -2 \end{bmatrix},\ x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Then, $\|Ax\|_1 = 16$, $\|A\|_1 = 4$ and $\|x\|_1 = 4$. In this case, $\|Ax\|_1 = \|A\|_1\|x\|_1$.

Remark 2.2. It is clear that the l_1 matrix norm is consistent with the l_1 vector norm from Lemma 2.1 and Remark 2.1.

To end this section, we also review the norm consistency between the l_∞ matrix norm and the l_∞ vector version, whose definitions are given by

$$\|A\|_\infty := \max_{1\le i\le m}\sum_{j=1}^{n}|a_{ij}|, \quad \|x\|_\infty := \max_{1\le j\le n}|x_j|,$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and a_{ij} is the (i, j)-th entry of A.

Lemma 2.2. *Let A be a matrix in $\mathbb{R}^{m\times n}$ and x be a vector in $\mathbb{R}^n$. Then*

$$\|Ax\|_\infty \le \|A\|_\infty\|x\|_\infty.$$

Proof. From the above definitions of the l_∞ vector norm and the l_∞ matrix norm, we have

$$\begin{aligned} \|Ax\|_\infty &= \max_{1\le i\le m}|a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n| \\ &\le \max_{1\le i\le m}\left(|a_{i1}x_1| + |a_{i2}x_2| + \cdots + |a_{in}x_n|\right) \qquad (2.1) \\ &\le \max_{1\le i\le m}\left(|a_{i1}| + |a_{i2}| + \cdots + |a_{in}|\right)\max_{1\le j\le n}|x_j| \qquad (2.2) \\ &= \left(\max_{1\le i\le m}\sum_{j=1}^{n}|a_{ij}|\right)\left(\max_{1\le j\le n}|x_j|\right) \\ &= \|A\|_\infty\|x\|_\infty. \end{aligned}$$

Hence, the proof is complete. □

Remark 2.3. Let $D \in \mathbb{R}^{n\times n}$ be the sign pattern of x as defined in Remark 2.1. For notional simplicity, we divide D into two parts, i.e.,

$$D = \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix},$$

where D_1 denotes the sign pattern of subvector of x whose component is equal to $\|x\|_\infty$ and D_2 corresponds to the other part of D. Now, we choose matrix $A \in \mathbb{R}^{m\times n}$ satisfying the following two conditions: (a) Each column of $A^T \in \mathbb{R}^{n\times m}$ have only two types of D_1-part sign pattern D_1 or $-D_1$; (b) The D_2-part columns of $A \in \mathbb{R}^{m\times n}$ are all zero vector. It is easy to see that Item (a)

and Item (b) make the relations (2.1) and (2.2) become equalities. In this case, it follows from Lemma 2.2 that $\|Ax\|_\infty = \|A\|_\infty \|x\|_\infty$. For example, we choose

$$A = \begin{bmatrix} 1 & 0 & -5 \\ -2 & 0 & 6 \\ 3 & 0 & -7 \\ -4 & 0 & 8 \end{bmatrix}, \; x = \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}.$$

Then $\|Ax\|_\infty = 12$, $\|A\|_\infty = 12$ and $\|x\|_\infty = 1$. In this case, $\|Ax\|_\infty = \|A\|_\infty \|x\|_\infty$.

Remark 2.4. It is clear that the l_∞ matrix norm is consistent with the l_∞ vector norm from Lemma 2.2 and Remark 2.3.

3. AVE with Data Uncertainty in the l_1 Norm Ball

Now, we focus on our target problem, that is, the problem (1.1) with uncertain data, where A, B and b are contaminated by a certain noise level in terms of the l_1 norm ball. Our main idea is proposing a robust counterpart model for this problem to against data uncertainty. To the contrast, two types of robust counterpart models for problem (1.2) with uncertain data are presented.

3.1. **Robust counterpart model of problem (1.1).** Assume that the data A, B and b are contaminated in a certain level, in which the uncertainty set is defined as

$$\| [\Delta A \; \Delta B \; \Delta b] \|_1 \le \rho,$$

where ρ is a given contamination level, $\Delta A \in \mathbb{R}^{n\times n}, \Delta B \in \mathbb{R}^{n\times n}$, $\Delta b \in \mathbb{R}^n$ denote noises in matrices A, B and vector b, respectively. For a given $x \in \mathbb{R}^n$, the associated worst-case residual of problem (1.1) has the following form

$$r(A,B,b,\rho) := \max_{\|[\Delta A \;\Delta B \;\Delta b]\|_1 \le \rho} \{ \|(A+\Delta A)x + (B+\Delta B)|x| - (b+\Delta b)\|_1 \}. \tag{3.1}$$

Then, robust counterpart model of problem (1.1) is given by

$$\phi(A,B,b,\rho,x) := \min_{x\in\mathbb{R}^n} \{ r(A,B,b,\rho) \}. \tag{3.2}$$

The following theorem presents an equivalent form of robust counterpart model (3.2).

Theorem 3.1. *Robust counterpart model (3.2) is equivalent to the following LP problem:*

$$\begin{array}{cl} \min\limits_{t,s\in\mathbb{R},x,\omega,y\in\mathbb{R}^n} & t+\rho s \\ \text{s.t.} & Ax + B\omega - b = y, \\ & x \le \omega, \; x \ge -\omega, \; \omega \ge 0, \\ & \|y\|_1 \le t, \; t \ge 0, \\ & 1 + 2\|x\|_1 \le s, \; s \ge 0. \end{array} \tag{3.3}$$

Proof. From the definition of the l_1 vector norm, we have

$$\begin{aligned} & \|(A+\Delta A)x + (B+\Delta B)|x| - (b+\Delta b)\|_1 \\ = \; & \left\| Ax + B|x| - b + [\Delta A \; \Delta B \; \Delta b][x^T \; |x|^T \; -1]^T \right\|_1 \\ \le \; & \|Ax + B|x| - b\|_1 + \left\| [\Delta A \; \Delta B \; \Delta b][x^T \; |x|^T \; -1]^T \right\|_1 \qquad (3.4) \\ \le \; & \|Ax + B|x| - b\|_1 + \rho\,(1 + 2\|x\|_1), \qquad (3.5) \end{aligned}$$

where relation (3.4) follows from the triangular inequality and (3.5) comes from Lemma 2.1. Let D be a sign pattern of $x \in \mathbb{R}^n$ as defined in Remark 2.1. Then, the sign pattern of vector $[x^T\ |x|^T\ -1]^T$ is given by

$$H := \begin{bmatrix} D & & \\ & I & \\ & & -1 \end{bmatrix} \in \mathbb{R}^{(2n+1)\times(2n+1)}.$$

Choosing $[\Delta A\ \Delta B\ \Delta b] = \rho u v^T \in \mathbb{R}^{n\times(2n+1)}$, where $u \in \mathbb{R}^n$ is defined as

$$u := \begin{cases} \dfrac{Ax+B|x|-b}{\|Ax+B|x|-b\|_1}, & \text{if } Ax+B|x|-b \neq 0, \\ \text{any unit } l_1 \text{ norm vector}, & \text{otherwise} \end{cases}$$

and $v := \mathrm{diag}(H)$ is the diagonal vector of H, one has $\|[\Delta A\ \Delta B\ \Delta b]\|_1 = \rho$ and relations (3.4) and (3.5) become equalities. Then, worst-case residual (3.1) has the following explicit form

$$r(A,B,b,\rho) = \|Ax+B|x|-b\|_1 + \rho\,(1+2\|x\|_1).$$

Consequently, robust counterpart model (3.2) can be reduced to

$$\phi(A,B,b,\rho,x) = \min_{x\in\mathbb{R}^n} \{\|Ax+B|x|-b\|_1 + \rho\,(1+2\|x\|_1)\}.$$

By introducing some auxiliary variables, the above model is equivalent to LP problem (3.3). □

3.2. **Robust counterpart model of problem (1.2).** In this subsection, we consider the following two types of uncertainty sets for problem (1.2), the l_1 norm uncertainty ball in vector b, and the version in matrix A and vector b.

3.2.1. *The l_1 uncertainty ball in vector b.* Assume that only the data b is contaminated in a certain level, in which the uncertainty set is defined as

$$\|\Delta b\|_1 \leq \rho,$$

where ρ is a given contamination level. For a certain $x \in \mathbb{R}^n$, the associated worst-case residual of problem (1.2) has the following form

$$r(b,\rho) := \max_{\|\Delta b\|_1 \leq \rho} \{\|Ax-|x|-(b+\Delta b)\|_1\}.$$

In order to against the data uncertainty, we also consider the corresponding robust counterpart model

$$\phi(b,\rho,x) := \min_{x\in\mathbb{R}^n} r(b,\rho). \tag{3.6}$$

Theorem 3.2. *Robust counterpart problem (3.6) is equivalent to the following LP problem:*

$$\begin{array}{ll} \min\limits_{t\in\mathbb{R},x,\omega,y\in\mathbb{R}^n} & t+\rho \\ \text{s.t.} & Ax-\omega-b=y, \\ & x\leq\omega,\ x\geq-\omega,\ \omega\geq 0, \\ & \|y\|_1\leq t,\ t\geq 0. \end{array} \tag{3.7}$$

Proof. Using the triangle inequality of the l_1 vector norm yields

$$\max_{\|\Delta b\|_1 \le \rho} \|Ax - |x| - (b+\Delta b)\|_1 \le \|Ax - |x| - b\|_1 + \rho.$$

If we choose $\Delta b = \rho u \in \mathbb{R}^n$, where

$$u := \begin{cases} \dfrac{Ax - |x| - b}{\|Ax - |x| - b\|_1}, & \text{if } Ax - |x| - b \neq 0, \\ \text{any unit } l_1 \text{ norm vector}, & \text{otherwise}, \end{cases}$$

then it is clear that $\|\Delta b\|_1 = \rho$. Consequently, $r(b,\rho) = \|Ax - |x| - b\|_1 + \rho$ and $\phi(b,\rho,x) = \min_{x\in\mathbb{R}^n} \|Ax - |x| - b\|_1 + \rho$, which is equivalent to LP problem (3.7) by introducing necessary auxiliary variables. □

3.2.2. *The l_1 norm uncertainty ball in matrix A and vector b.* Assume that the matrix A and vector b are contaminated in a certain level, in which the uncertainty set is defined as

$$\|[\Delta A\ \Delta b]\|_1 \le \rho,$$

where ρ is a given contamination level. For a certain $x \in \mathbb{R}^n$, the worst-case residual of problem (1.2) has the following form

$$r(A,b,\rho) := \max_{\|[\Delta A\ \ \Delta b]\|_1 \le \rho} \{\|(A+\Delta A)x - |x| - (b+\Delta b)\|_1\}.$$

Consequently, the corresponding robust counterpart model is given by

$$\phi(A,b,\rho,x) := \min_{x\in\mathbb{R}^n} r(A,b,\rho). \tag{3.8}$$

Similar to Theorem 3.1, the following result can be deduced under the l_1 norm uncertainty set in matrix A and vector b.

Theorem 3.3. *The robust counterpart problem (3.8) is equivalent to the following LP problem:*

$$\begin{array}{ll} \min\limits_{t,s\in\mathbb{R},x,\omega,y\in\mathbb{R}^n} & t + \rho s \\ \text{s.t.} & Ax - \omega - b = y, \\ & x \le \omega,\ x \ge -\omega,\ \omega \ge 0, \\ & \|y\|_1 \le t,\ t \ge 0 \\ & 1 + \|x\|_1 \le s,\ s \ge 0. \end{array} \tag{3.9}$$

Proof. From the definition of the l_1 vector norm, we have

$$\begin{aligned} & \|(A+\Delta A)x + |x| - (b+\Delta b)\|_1 \\ = \ & \left\|Ax + |x| - b + [\Delta A\ \Delta b][x^T\ \ -1]^T\right\|_1 \\ \le \ & \|Ax + |x| - b\|_1 + \left\|[\Delta A\ \Delta b][x^T\ \ -1]^T\right\|_1 & (3.10) \\ \le \ & \|Ax + |x| - b\|_1 + \rho\,(1 + \|x\|_1), & (3.11) \end{aligned}$$

where the relation (3.10) follows from the triangular inequality and (3.11) comes from Lemma 2.1. Let D be a sign pattern of $x \in \mathbb{R}^n$ as defined in Remark 2.1. Then, the sign pattern of vector $[x^T\ \ -1]^T$ is given by

$$H := \begin{bmatrix} D & \\ & -1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}.$$

Choosing $[\Delta A\ \Delta b] = \rho u v^T \in \mathbb{R}^{n\times(n+1)}$, where $u \in \mathbb{R}^n$ is defined as

$$u := \begin{cases} \dfrac{Ax+|x|-b}{\|Ax+|x|-b\|_1}, & \text{if } Ax+|x|-b \neq 0, \\ \text{any unit } l_1 \text{ norm vector}, & \text{otherwise} \end{cases}$$

and $v := \mathrm{diag}(H)$ is the diagonal vector of H, one sees that $\|[\Delta A\ \Delta b]\|_1 = \rho$ and the relations (3.10) and (3.11) become equalities. Then,

$$r(A,b,\rho) = \|Ax+|x|-b\|_1 + \rho\,(1+\|x\|_1)\,.$$

Consequently, the robust counterpart model (3.8) can be reduced to

$$\phi(A,B,b,\rho,x) = \min_{x\in\mathbb{R}^n} \{\|Ax+|x|-b\|_1 + \rho\,(1+\|x\|_1)\}\,.$$

By introducing some auxiliary variables, the above model is equivalent to LP problem (3.9). □

4. AVE WITH DATA UNCERTAINTY IN THE l_∞ NORM BALL

In this section, we replace the l_1 norm uncertainty ball with the l_∞ norm uncertainty ball. We discuss three types of data contamination and demonstrate their corresponding robust counterpart models under these cases.

We first assume that the data A, B, and b are contaminated in a certain level, in which the uncertainty set is defined as

$$\|\,[\Delta A\ \Delta B\ \Delta b]\,\|_\infty \leq \sigma,$$

where σ is a given contamination level. In this case, the robust counterpart model of problem (1.1) is given by

$$\psi(A,B,b,\sigma,x) := \min_{x\in\mathbb{R}^n} r(A,B,b,\sigma), \tag{4.1}$$

where $r(A,B,b,\sigma)$ is the worst-case residual, i.e.,

$$r(A,B,b,\sigma) := \max_{\|[\Delta A\ \Delta B\ \Delta b]\|_\infty \leq \sigma} \{\|(A+\Delta A)x + (B+\Delta B)|x| - (b+\Delta b)\|_\infty\}. \tag{4.2}$$

The following theorem presents an equivalent form of robust counterpart model (4.1).

Theorem 4.1. *The robust counterpart model (4.1) is equivalent to the following LP problem:*

$$\begin{array}{cl} \displaystyle\min_{t,s\in\mathbb{R},x,\omega,y\in\mathbb{R}^n} & t+\sigma s \\ \text{s.t.} & Ax+B\omega-b=y \\ & x\leq\omega,\ x\geq-\omega,\ \omega\geq 0, \\ & \|y\|_\infty \leq t,\ t\geq 0, \\ & \max\{1,\|x\|_\infty\} \leq s,\ s\geq 0. \end{array} \tag{4.3}$$

Proof. From the definition of the l_∞ vector norm, we have

$$\begin{aligned} & \|(A+\Delta A)x + (B+\Delta B)|x| - (b+\Delta b)\|_\infty \\ = & \left\|Ax+B|x|-b+[\Delta A\ \Delta B\ \Delta b][x^T\ |x|^T\ -1]^T\right\|_\infty \\ \leq & \|Ax+B|x|-b\|_\infty + \left\|[\Delta A\ \Delta B\ \Delta b][x^T\ |x|^T\ -1]^T\right\|_\infty \qquad (4.4) \\ \leq & \|Ax+B|x|-b\|_\infty + \sigma\max\{1,\|x\|_\infty\}\,. \qquad (4.5) \end{aligned}$$

Then, we choose $[\Delta A\ \Delta B\ \Delta b] = \sigma u v^T \in \mathbb{R}^{n\times(2n+1)}$, where $u \in \mathbb{R}^n$ is defined as

$$u := \begin{cases} \dfrac{Ax + B|x| - b}{\|Ax + B|x| - b\|_\infty}, & \text{if } Ax + B|x| - b \neq 0, \\ \text{any unit } l_\infty \text{ norm vector}, & \text{otherwise.} \end{cases}$$

At the same time, the choice of $v \in \mathbb{R}^{2n+1}$ needs to consider the following three cases.

(a) If $\|x\|_\infty < 1$, then

$$v_i := \begin{cases} 1 & \text{if } i = 2n+1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) If $\|x\|_\infty = 1$, then

$$v_i := \begin{cases} \dfrac{1}{|\tilde{I}|+1} & \text{if } i \in \tilde{I}, \\ 0 & \text{otherwise.} \end{cases}$$

where $\tilde{I}$ denotes the index set of D_1-part sign patten of vector $[x^T\ |x|^T\ 1] \in \mathbb{R}^{2n+1}$ and $|\tilde{I}|$ is the carnality of $\tilde{I}$. It is easy to see that $\tilde{I} = I \cup (I+n)$, where $I := \{i \in \{1,2,\cdots,n\} : |x_i| = 1\}$ and $I+n := \{j \in \{n+1, n+2, \cdots, 2n\} : j = i+n, i \in I\}$.

(c) If $\|x\|_\infty > 1$, then

$$v_i := \begin{cases} \dfrac{1}{|\tilde{I}|} & \text{if } i \in \tilde{I}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{I}$ is defined as case (b) above.

It follows from the above discussion that $\|[\Delta A\ \Delta B\ \Delta b]\|_\infty = \sigma$ and the relations (4.4) and (4.5) become equalities. Then, the worst-case residual (4.2) has the following explicit form

$$r(A,B,b,\sigma) = \|Ax + B|x| - b\|_\infty + \sigma \max\{1, \|x\|_\infty\}.$$

Consequently, robust counterpart model (4.1) can be reduced to

$$\psi(A,B,b,\rho,x) = \min_{x\in\mathbb{R}^n} \{\|Ax + B|x| - b\|_\infty + \sigma \max\{1, \|x\|_\infty\}\}.$$

Hence, the above model is equivalent to LP problem (4.3). □

To end this section, without demonstrate the proof, we present robust counterpart models of problem (1.2) under the following two types of l_∞ norm uncertainty balls:

(i) The l_∞ uncertainty ball in vector b: $\|\Delta b\|_\infty \leq \sigma$;

(ii) The l_∞ uncertainty ball in matrix A and vector b: $\|[\Delta A\ \Delta b]\|_\infty \leq \sigma$.

Theorem 4.2. *The robust counterpart model of problem (1.2) under the l_∞ uncertainty ball in vector b is equivalent to*

$$\begin{array}{ll} \min\limits_{t\in\mathbb{R}, x,\omega,y\in\mathbb{R}^n} & t + \sigma \\ \text{s.t.} & Ax - \omega - b = y \\ & x \leq \omega,\ x \geq -\omega,\ \omega \geq 0, \\ & \|y\|_\infty \leq t,\ t \geq 0. \end{array} \tag{4.6}$$

While the robust counterpart model of problem (1.2) under the l_∞ uncertainty ball in matrix A and vector b is equivalent to

$$\begin{array}{ll} \min\limits_{t,s\in\mathbb{R},x,\omega,y\in\mathbb{R}^n} & t+\sigma s \\ \text{s.t.} & Ax-\omega-b=y, \\ & x\leq\omega,\ x\geq-\omega,\ \omega\geq 0, \\ & \|y\|_\infty\leq t,\ t\geq 0, \\ & \max\{1,\|x\|_\infty\}\leq s,\ s\geq 0. \end{array} \tag{4.7}$$

5. Numerical Experiments

In this section, we report some implementation issues and numerical experiments conducted for testing efficiency of equivalent robust counterpart models in Section 3 and Section 4. All experiments are run in Julia with packages JuMP and HiGHS on a 64-bit PC with an Intel (R) Core(TM) i7-6500U of 2.50 GHz CPU and 8.00GB of RAM equipped with CentOS Linux operating system.

The preparation of test data follows the below steps: (1) Choose random matrices A, B with entries generated independently using the uniform distribution on $[-10,10]$; (2) Choose a random vector x from the uniform distribution on $[-10,10]$; (3) Compute $b=Ax+B|x|$ or $b=Ax-|x|$.

In Table 1-Table 3, we consider feasible AVE systems under the l_1 norm uncertainty ball. The first column demonstrates the dimension of matrix A and the second to fifth columns record the residual of AVEs caused by the contamination level $\rho=0.001,0.01,0.1,1$, respectively. Similar results under the l_∞ norm uncertainty ball can be found in Table 4-Table 6. From these results, we obtain the following observations:

(a) All these models have better results under $\rho=0.001$, while the robust solutions also provide a protection against the given two types of uncertainty balls especially when ρ is large.

(b) Compared with models (3.3), (3.7), and (3.9), models (4.3), (4.6), and (4.7) have better residuals, which demonstrate that standard AVE (1.1) and its special form (1.2) have better performances against data uncertainty in the l_∞ norm ball than the one in the l_1 norm ball.

6. Conclusions

In this paper, we studied two types of AVEs with data uncertainty in the l_1 and l_∞ norm balls. In order to minimize the worst-case residual, we construct the associated robust counterpart problems and prove that these models are equivalent to LP problems. Numerical experiments verify that robust solutions can be provided in a better performance. Recently, a method of alternating projections for these AVEs are proposed by Alcantara, Chen, and Tam in [23], in which the fixed points set of the corresponding alternating projections map is characterized under nondegeneracy conditions on data A, B and local linear convergence is proved. Unlike most of the existing approaches in the literature, their algorithm is capable of handling problems with $m\neq n$. By carefully checking the techniques used in the above proofs, our conclusions can be easily adapted for the case $m\neq n$. How to combine these robust counterpart models with alternating projection methods? We leave further discussion on this topic as our future work.

TABLE 1. Residual of model (3.3)

n	$\rho = 0.001$	$\rho = 0.01$	$\rho = 0.1$	$\rho = 1$
10	3.6245e-14	1.0664e-13	5.4108e-13	6.2759e-13
20	5.7412e-13	1.5275e-12	1.6766e-12	2.9488e-12
30	1.4897e-12	1.8861e-12	2.6972e-12	3.7305e-12
40	2.6534e-12	2.8545e-12	3.6980e-12	5.4033e-12
50	1.8149e-11	2.5115e-11	3.2652e-11	4.8780e-11
60	1.9908e-11	2.0601e-11	5.7967e-11	6.6753e-11
70	2.7936e-11	3.4370e-11	6.4196e-11	7.8612e-11
80	6.4359e-11	7.3593e-11	8.4211e-11	9.8119e-11
90	6.7940e-11	8.1884e-11	2.2108e-10	3.0328e-10
100	1.1549e-10	3.4082e-10	5.5172e-10	6.7342e-10

TABLE 2. Residual of model (3.7)

n	$\rho = 0.001$	$\rho = 0.01$	$\rho = 0.1$	$\rho = 1$
10	1.6629e-15	2.5580e-15	5.8422e-15	7.9488e-15
20	1.6117e-14	2.2434e-14	2.6550e-14	4.0214e-14
30	4.1117e-14	5.8602e-14	7.1124e-14	8.6224e-14
40	3.4328e-13	3.9028e-13	4.9626e-13	5.8950e-13
50	3.8228e-13	4.7965e-13	5.4476e-13	6.7778e-13
60	4.7884e-13	4.8633e-13	6.1701e-13	7.8647e-13
70	6.7515e-13	7.3153e-13	7.8259e-13	8.2609e-13
80	9.4155e-13	1.5047e-12	2.1566e-12	4.5299e-12
90	1.2716e-12	2.0595e-12	3.8830e-12	5.2116e-12
100	2.4141e-12	3.5826e-12	4.5804e-12	6.0030e-12

TABLE 3. Residual of model (3.9)

n	$\rho = 0.001$	$\rho = 0.01$	$\rho = 0.1$	$\rho = 1$
10	3.7410e-15	7.5318e-15	2.4656e-14	3.8634e-14
20	2.0579e-14	2.4563e-14	2.8493e-14	4.5491e-14
30	6.8655e-14	7.9159e-14	8.5876e-14	9.0410e-14
40	3.5706e-13	4.0507e-13	5.6972e-13	6.4107e-13
50	4.5040e-13	5.0408e-13	6.9162e-13	7.2935e-13
60	5.3780e-13	5.4321e-13	7.7480e-13	8.3949e-13
70	1.3462e-12	1.4472e-12	1.6115e-12	2.1526e-12
80	2.8366e-12	3.5871e-12	4.4512e-12	5.4616e-12
90	3.7293e-12	4.3581e-12	4.9202e-12	5.7918e-12
100	4.6416e-12	5.4755e-12	5.5614e-12	6.7961e-12

TABLE 4. Residual of model (4.3)

n	$\rho = 0.001$	$\rho = 0.01$	$\rho = 0.1$	$\rho = 1$
10	3.0202e-14	5.9686e-14	6.4304e-14	1.0427e-13
20	1.6875e-13	2.7627e-13	3.1974e-13	4.2844e-13
30	8.5834e-13	9.3922e-13	1.4343e-12	2.1115e-12
40	1.1225e-12	2.4062e-12	3.0196e-12	3.7707e-12
50	4.0180e-12	5.6670e-12	6.0397e-12	8.0632e-12
60	7.6328e-12	8.1400e-12	9.2265e-12	1.0045e-11
70	8.7681e-12	9.8226e-11	1.9182e-11	2.4490e-11
80	1.1347e-11	2.0915e-11	3.4350e-11	4.8628e-11
90	4.6181e-11	5.6181e-11	6.6485e-11	7.2154e-11
100	8.4399e-11	9.5695e-11	1.7657e-10	5.0376e-10

TABLE 5. Residual of model (4.6)

n	$\rho = 0.001$	$\rho = 0.01$	$\rho = 0.1$	$\rho = 1$
10	1.3234e-15	2.4833e-15	3.2560e-15	6.5794e-15
20	5.1514e-15	7.7809e-15	8.8963e-15	9.7550e-15
30	1.4193e-14	3.9968e-14	5.0183e-14	6.8711e-14
40	4.0407e-14	5.6113e-14	7.2922e-14	1.7320e-13
50	1.1852e-13	2.9706e-13	3.5136e-13	4.0719e-13
60	2.3997e-13	4.5725e-13	5.5189e-13	6.3663e-13
70	4.7742e-13	5.0313e-13	6.8589e-13	7.0787e-13
80	8.9747e-13	1.1475e-12	2.0826e-12	3.3226e-12
90	9.5274e-13	1.2465e-12	3.4454e-12	4.6380e-12
100	1.3115e-12	2.4559e-12	4.3615e-12	5.6725e-12

TABLE 6. Residual of model (4.7)

n	$\rho = 0.001$	$\rho = 0.01$	$\rho = 0.1$	$\rho = 1$
10	2.6557e-15	7.0166e-15	1.2612e-14	2.1902e-14
20	1.1084e-14	1.2763e-14	2.4439e-14	3.7522e-14
30	4.7310e-14	5.6319e-14	6.4823e-14	8.5270e-14
40	1.0349e-13	2.4655e-13	3.5540e-13	5.0725e-13
50	2.5319e-13	3.5287e-13	4.3611e-13	5.3450e-13
60	4.0211e-13	4.8674e-13	5.5164e-13	6.2759e-13
70	1.0762e-12	1.1540e-12	1.4690e-12	1.7974e-12
80	2.0722e-12	2.8195e-12	3.5560e-12	4.0429e-12
90	2.8966e-12	3.4051e-12	4.6663e-12	5.6619e-12
100	3.5798e-12	3.9890e-12	5.1663e-12	5.8232e-12

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