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THE SAMPLING COMPLEXITY ON NONCONVEX SPARSE PHASE RETRIEVAL PROBLEM

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Abstract. This paper discusses the k-sparse complex signal recovery from quadratic measurements via the ℓ_p-minimization model, where $0 < p \leq 1$. We establish the ℓ_p restricted isometry property over simultaneously low-rank and sparse matrices, which is a weaker restricted isometry property to guarantee the successful recovery in the ℓ_p case. The main result is to demonstrate that ℓ_p-minimization can recover complex k-sparse signals from $m \gtrsim k + pk\log(n/k)$ complex Gaussian quadratic measurements with high probability. The resulting sufficient condition is met by fewer measurements for smaller p and reaches $m \gtrsim k$ when p turns to zero. Furthermore, an iteratively-reweighted algorithm is proposed. Numerical experiments also demonstrate that ℓ_p minimization with $0 < p < 1$ performs better than ℓ_1 minimization.

Keywords. Nonconvex optimization; Restricted isometry property; Sampling complexity; Sparse phase retrieval.

1. Introduction

Phase retrieval aims at the reconstruction of some signal from the squared modulus of its linear transform. More concretely, suppose that we observe the signal $\boldsymbol{x} \in \mathbb{F}^n$ $(\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\})$ from the model

$$\boldsymbol{y} = |\boldsymbol{A}\boldsymbol{x}|^2 + \boldsymbol{\varepsilon}, \tag{1.1}$$

where $\boldsymbol{A} \in \mathbb{F}^{m\times n}$ is some known measurement matrix and $\boldsymbol{\varepsilon} \in \mathbb{R}^m$ is some noise term, and our goal is to reconstruct the unknown signal $\boldsymbol{x}$ based on $\boldsymbol{y}$ and $\boldsymbol{A}$. Such kind of task arises in many real applications, such as X-ray crystallography, astronomy, optics, and coherent diffraction imaging when the sensors and detectors can only record the intensity of light wave [1, 2, 3]. Since

$$|\boldsymbol{A}\boldsymbol{x}|^2 = |\boldsymbol{A}(c\boldsymbol{x})|^2$$

for any $|c| = 1$, the recovery of $\boldsymbol{x}$ is equivalent as the recovery of the set $\widetilde{\boldsymbol{x}}$, where

$$\widetilde{\boldsymbol{x}} := \{c\boldsymbol{x} \mid |c| = 1,\ c \in \mathbb{F}\}.$$

In noiseless case, i.e., $\boldsymbol{\varepsilon} = 0$, it demonstrates that $m \geq 2n-1$ (resp. $m \geq 4n-4$) generic measurements are sufficient to exactly recover $\boldsymbol{x} \in \mathbb{F}^n$ up to a unimodular constant when $\mathbb{F} = \mathbb{R}$ (resp. $\mathbb{F} = \mathbb{C}$) [4, 5].

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Recently, solving the phase retrieval problem received extensive attention in recent decades. A remarkable result of Candès and Li for random Gaussian measurements is that we can recover $\boldsymbol{x}$ by some semidefinite programming framework called PhaseLift [6, 7]. It applies the "lifting" technique, that is, lifting the signal $\boldsymbol{x}$ into rank-one matrix $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^*$ and then the quadratic measurements can be linearized as below:

$$y_j = |\langle \boldsymbol{a}_j, \boldsymbol{x}\rangle|^2 + \varepsilon_j = \boldsymbol{a}_j^* \boldsymbol{X} \boldsymbol{a}_j + \varepsilon_j,$$

where the measurement vectors $\boldsymbol{a}_j$ $(j = 1, \ldots, m)$ are the column elements of $\boldsymbol{A}^*$. Based on the convex relaxation model:

$$\min_{\boldsymbol{Z}\in\mathbb{F}^{n\times n}} \sum_{j=1}^{m} \left|\boldsymbol{a}_j^* \boldsymbol{Z} \boldsymbol{a}_j - y_j\right|, \qquad \text{s.t. } \boldsymbol{X} \succeq 0,$$

they proved that the solution is exact up to some global phase when $m \gtrsim n$ [6]. For some nonconvex iterative methods for phase retrieval, we refer to [8, 9, 10, 11].

In this paper, we focus on the model (1.1) in the case that $m \ll n$. Although (1.1) does not yield injective measurements, it can also become well-posed when the unknown signal $\boldsymbol{x}$ is k-sparse. It was demonstrated that $m \geq 4k-2$ for $\mathbb{F} = \mathbb{C}$ (resp. $m \geq 2k$ for $\mathbb{F} = \mathbb{R}$) generic measurements can obtain the solution uniquely modulo phase [12].

The injectivity of the measurements does not imply that efficient recovery is possible. Inspired by the success of the ℓ_1 minimization model in compressed sensing, it is natural to take the sparsity assumption and try to efficiently recover signals from fewer than n intensity measurements. The following ℓ_1 minimization model in the noiseless case is considered:

$$(\ell_1 \text{ minimization}) \qquad \min_{\boldsymbol{z}\in\mathbb{F}^n} \|\boldsymbol{z}\|_1 \qquad \text{s.t. } |\boldsymbol{A}\boldsymbol{z}|^2 = |\boldsymbol{A}\boldsymbol{x}|^2. \tag{1.2}$$

Based on the ℓ_1 minimization model, one can recover $\boldsymbol{x}$ when $m \gtrsim k\log(n/k)$ and $\boldsymbol{A} \in \mathbb{F}^{m\times n}$ is random real or complex Gaussian matrix [13, 14]. Although the constrained model in (1.2) is non-convex, many efficient algorithms were developed to solve it [9, 15]. Beyond the ℓ_1 minimization model, other nonconvex algorithms were also proposed to solve sparse phase retrieval problem, such as Sparse Truncated Amplitude Flow (SPARTA) [16], Compressive Phase Retrieval with Alternating Minimization (CoPRAM) [17], and Sparse phase retrieval via PhaseLiftOff [18].

A natural question is: whether the sampling complexity can be improved further. In this paper, we aim to find the number of measurements in the ℓ_p $(0 < p \leq 1)$ minimization model, that is,

$$(\ell_p \text{ minimization}) \qquad \min_{\boldsymbol{z}\in\mathbb{C}^n} \|\boldsymbol{z}\|_p^p \qquad \text{s.t. } |\boldsymbol{A}\boldsymbol{z}|^2 = |\boldsymbol{A}\boldsymbol{x}|^2. \tag{1.3}$$

The main result of this paper is that, for the case of random complex Gaussian measurements, the number of measurements in (1.3) can be improved to

$$m \gtrsim C_1(p)k + pC_2(p)k\log(n/k),$$

where C_1 and C_2 are determined explicitly and bounded by p. Therefore, when p turns to zero, the dependence of the sufficient number of measurements m on the signal dimension n vanishes, and the order of measurements becomes $O(k)$, which meets the injectivity order in [12].

2. NOTATIONS

Denote $\mathbb{H}^{n\times n}$ as the $n\times n$ Hermitian matrices set. For matrix $\boldsymbol{X}\in\mathbb{C}^{n_1\times n_2}$, we use $\boldsymbol{X}_{S,T}$ to denote the submatrix of $\boldsymbol{X}$ with the rows indexed in S and the columns indexed in T. We use $\boldsymbol{X}_{j,:}$ and $\boldsymbol{X}_{:,l}$ to denote the j-th row and the l-th column of $\boldsymbol{X}$, respectively. Take $\|\boldsymbol{X}\|_{0,2}$ as the number of non-zero columns in $\boldsymbol{X}$. Set

$$\|\boldsymbol{X}\|_1 := \sum_{j,l}\sqrt{\mathrm{R}(X_{j,l})^2+\mathrm{I}(X_{j,l})^2},$$

$$\|\boldsymbol{X}\|_F := \sqrt{\sum_{j,l}(\mathrm{R}(X_{j,l})^2+\mathrm{I}(X_{j,l})^2)},$$

and

$$\|\boldsymbol{X}\|_p^p := \sum_{j,l}(\mathrm{R}(\boldsymbol{X}_{j,l})^2+\mathrm{I}(\boldsymbol{X}_{j,l})^2)^{p/2}, (0<p\leq 1),$$

where $\mathrm{R}(X_{j,l})$ and $\mathrm{I}(X_{j,l})$ are the real and image parts of $X_{j,l}$, respectively. For any $\boldsymbol{X},\boldsymbol{Y}\in\mathbb{C}^{n_1\times n_2}$, set $\langle\boldsymbol{X},\boldsymbol{Y}\rangle := \mathrm{Tr}(\boldsymbol{X}^*\boldsymbol{Y})$.

We use $A\gtrsim B$ to denote $A\geq cB$, where c is some positive absolute constant. The notation $\lesssim$ can be defined similarly. Without specific notation, we use C,c, and their superscript (subscript) forms to denote universal constants and their values may vary with different contexts.

3. MAIN RESULTS

In standard compressed sensing, it was proved that when ℓ_1-minimization is replaced by ℓ_p-minimization, that is,

$$\min_{\boldsymbol{z}}\|\boldsymbol{z}\|_p^p \qquad \text{s.t. } \boldsymbol{Az}=\boldsymbol{Ax}, \tag{3.1}$$

fewer measurements are required for exact reconstruction [19]. Suppose that the measurement matrix $\boldsymbol{A}$ satisfies the restricted p-isometry property, i.e., for all $\boldsymbol{x}$ such that $\|\boldsymbol{x}\|_0\leq k$,

$$(1-\delta_k)\|\boldsymbol{x}\|_2^p\leq\|\boldsymbol{Ax}\|_p^p\leq(1+\delta_k)\|\boldsymbol{x}\|_2^p,$$

the unique minimizer of (3.1) is exactly $\boldsymbol{x}$ provided that

$$\delta_{ak}+b\delta_{(a+1)k}<b-1$$

with $a=\lceil b^{2/(2-p)}k\rceil/k$ and $\|\boldsymbol{x}\|_0\leq k$.

However, such kind of restricted isometry property can not be directly extended to phase retrieval problem. In the sparse phase retrieval problem, we consider a different notion of restricted isometry property, based on the fact that the quadratic measurements can be lifted in matrix space and $\boldsymbol{xx}^*$ is simultaneously low-rank and sparse. We describe the following restricted p-isometry property over low-rank and sparse matrices:

Definition 3.1. The map $\mathscr{A}:\mathbb{H}^{n\times n}\to\mathbb{R}^m$ satisfies the restricted p-isometry property (abbreviated as ℓ_p-RIP) of order (r,k) if there exist positive constants $\underline{C}$ and $\overline{C}$ such that

$$\underline{C}\|\boldsymbol{X}\|_F^p\leq\|\mathscr{A}(\boldsymbol{X})\|_p^p\leq\overline{C}\|\boldsymbol{X}\|_F^p$$

holds for all $\boldsymbol{X}\in\mathbb{H}^{n\times n}$ with $\mathrm{rank}(\boldsymbol{X})\leq r$ and $\|\boldsymbol{X}\|_{0,2}\leq k$.

When referring to the phase retrieval problem, $\mathscr{A}(\cdot)$ can be defined as

$$\mathscr{A}(\boldsymbol{X}) = (\boldsymbol{a}_1^* \boldsymbol{X} \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m^* \boldsymbol{X} \boldsymbol{a}_m).$$

Then the ℓ_p minimization in the noiseless case becomes:

$$\min_{\boldsymbol{z} \in \mathbb{C}^n} \|\boldsymbol{z}\|_p^p \qquad \text{s.t. } \mathscr{A}(\boldsymbol{z}\boldsymbol{z}^*) = \mathscr{A}(\boldsymbol{x}\boldsymbol{x}^*). \tag{3.2}$$

The following theorem illustrates the sufficient condition for exact recovery of ℓ_p minimization under the ℓ_p-RIP.

Theorem 3.1. *Assume that $\mathscr{A}(\cdot)$ satisfies the ℓ_p-RIP of order $(2, 2ak)$, that is,*

$$\underline{C} \|\boldsymbol{X}\|_F^p \leq \|\mathscr{A}(\boldsymbol{X})\|_p^p \leq \overline{C} \|\boldsymbol{X}\|_F^p$$

for any $\boldsymbol{X} \in \mathbb{H}^{n \times n}$ with rank$(\boldsymbol{X}) \leq 2$ and $\|\boldsymbol{X}\|_{0,2} \leq 2ak$. If

$$\underline{C} > \overline{C} \left(\frac{1}{a^{2-p}} + \frac{4}{a^{1-p/2}} \right)$$

for some large enough $a > 1$, then the solution $\boldsymbol{x}^\#$ to the model (3.2) satisfies

$$\boldsymbol{x}^\# (\boldsymbol{x}^\#)^* = \boldsymbol{x}\boldsymbol{x}^*.$$

Henceforth, we discuss the ℓ_p-RIP for complex Gaussian measurements. The following theorem demonstrates the sharp bounds on the ℓ_p expectation $\mathbb{E}|\boldsymbol{a}_i^* \boldsymbol{X} \boldsymbol{a}_i|^p$, that is, the upper bound and lower bound can be achieved for proper choices of $\boldsymbol{X}$. It can be taken out for research separately for interested readers.

Theorem 3.2. *Assume that $\boldsymbol{X} \in \mathbb{H}^{n \times n}$ with rank$(\boldsymbol{X}) \leq 2$ and $\|\boldsymbol{X}\|_F = 1$. Consider $\boldsymbol{a}_i$ $(i = 1, \ldots, m)$ independently drawn from complex Gaussian random vectors, i.e., $\boldsymbol{a}_i \sim \mathscr{N}(\boldsymbol{0}, \frac{1}{2}\boldsymbol{I}) + \mathscr{N}(\boldsymbol{0}, \frac{1}{2}\boldsymbol{I})\mathrm{i}$. Let $X_i = |\boldsymbol{a}_i^\top \boldsymbol{X} \boldsymbol{a}_i|^p$ with $0 < p \leq 1$ and $\mu = \mathbb{E}X_1$. Then*

$$\frac{1}{p+1} 2^{-p/2} \Gamma(p+2) \leq \mathbb{E}|\boldsymbol{a}^* \boldsymbol{X} \boldsymbol{a}|^p \leq 2^{-p/2} \Gamma(p+2), \tag{3.3}$$

and

$$\mathbb{P}\left(\left| \sum_{i=1}^m (X_i - \mu) \right| \geq t \right) \leq 2 \exp\left(-\frac{t^2}{cm\tau^2} \right), \ \text{ with } \tau \leq p + p^{1/2-p}, \tag{3.4}$$

where c is some positive absolute constant.

Remark 3.1. For any $0 < p \leq 1$, we have $2^{-p/2}\Gamma(p+2) \leq 2$ and $\frac{1}{p+1} 2^{-p/2} \Gamma(p+2) \geq 2^{-3/2}$. Then $2^{-3/2} \leq \mathbb{E}|\boldsymbol{a}^* \boldsymbol{X} \boldsymbol{a}|^p \leq 2$, for any $\boldsymbol{X} \in \mathbb{H}^{n \times n}$ with rank$(\boldsymbol{X}) \leq 2$ and $\|\boldsymbol{X}\|_F = 1$.

Furthermore, when p turn to 0, we can achieve that

$$2^{-p/2}\Gamma(p+2) \to 1 \qquad \text{and} \qquad \frac{1}{p+1} 2^{-p/2} \Gamma(p+2) \to 1,$$

which leads to more flexible choices of $\underline{C}$ and $\overline{C}$ in the ℓ_p-RIP described in Definition 3.1.

Now the concentration inequality of $\|\mathscr{A}(\boldsymbol{X})\|_p^p$ for low-rank and sparse matrix $\boldsymbol{X}$ can be demonstrated below.

Theorem 3.3. *Set* $\mathscr{X} := \{\boldsymbol{X} \in \mathbb{H}^{n\times n} \mid \|\boldsymbol{X}\|_F = 1,\ rank(\boldsymbol{X}) \le 2,\ \|\boldsymbol{X}\|_{0,2} \le L\}$ *and take* $\varepsilon, \eta > 0$. *Consider* $\boldsymbol{a}_i$ $(i = 1,\dots,m)$ *independently drawn from complex Gaussian random vectors, i.e.,* $\boldsymbol{a}_i \sim \mathscr{N}(\boldsymbol{0}, \frac{1}{2}\boldsymbol{I}) + \mathscr{N}(\boldsymbol{0}, \frac{1}{2}\boldsymbol{I})\mathrm{i}$. *Then, for any* $\boldsymbol{X} \in \mathscr{X}$,

$$m\left(2^{-3/2}(1-4\eta) - (1+\eta)\frac{4\varepsilon^p}{1-2\varepsilon^p}\right) \le \sum_{i=1}^{m} |\boldsymbol{a}_i^\top \boldsymbol{X}\boldsymbol{a}_i|^p \le 2m\left(1+\frac{\eta+2\varepsilon^p}{1-2\varepsilon^p}\right),$$

with probability at least

$$1-2\left(\frac{9\sqrt{2}en}{\varepsilon L}\right)^{5L}\exp\left(-\frac{m\eta^2}{2cp^{1-2p}}\right),$$

where c is some positive absolute constant.

Based on Theorem 3.3 and Theorem 3.1, we find that ℓ_p minimization can recover the unknown k-sparse signal from fewer measurements with a small value of p than that was needed in the aforementioned results [13, 14].

Theorem 3.4. *Consider* $\boldsymbol{a}_i$ $(i = 1,\dots,m)$ *independently drawn from complex Gaussian random vectors, i.e.,* $\boldsymbol{a}_i \sim \mathscr{N}(\boldsymbol{0}, \frac{1}{2}\boldsymbol{I}) + \mathscr{N}(\boldsymbol{0}, \frac{1}{2}\boldsymbol{I})\mathrm{i}$. *When*

$$m \gtrsim k + pk\log(n/k),$$

the solution $\boldsymbol{x}^\#$ *to model (3.2) satisfies*

$$\boldsymbol{x}^\#(\boldsymbol{x}^\#)^* = \boldsymbol{x}\boldsymbol{x}^*,$$

with probability exceeding $1 - 1/\binom{n}{s}$.

4. Numerical Experiments

Many numerical experiments were made to demonstrate the empirical success of the ℓ_1 minimization model. For example, Moravec, Romberg, and Baraniuk proposed an iterative projection algorithm to solve (1.2) [20]. The ADM algorithm was introduced in [15]. However, the proposed ADM algorithms cannot be guaranteed to converge. In this paper, we focus on the numerical behavior of

$$\min_{\boldsymbol{Z}\succeq 0} \|\boldsymbol{Z}\|_p^p \qquad \text{s.t. } \operatorname{trace}(\boldsymbol{Z}) \le \operatorname{trace}(\boldsymbol{x}\boldsymbol{x}^*),\ \boldsymbol{a}_j^*\boldsymbol{Z}\boldsymbol{a}_j = y_j,\ j = 1,\dots,m, \tag{4.1}$$

when $0 < p \le 1$. When $p = 1$, the SDP solver is applied in the small-scale case. In large-scale cases, it can use a fast matrix-factorization-based algorithm to solve the semidefinite programming [21]. Based on the factorization method, the scaled gradient method is used to give fast and robust convergence in matrix recovery problem [22].

In order to solve (4.1), we apply an iteratively-reweighted method. We begin with the minimization of (1.2) when $p = 1$ as $\boldsymbol{Z}^{(1)}$. Let $\boldsymbol{Z}^{(n+1)}$ be the solution of

$$\min_{\boldsymbol{Z}\succeq 0} \sum_{l,k} \omega_{l,k}^{(n)} |Z_{l,k}| \qquad \text{s.t. } \operatorname{trace}(\boldsymbol{Z}) \le \operatorname{trace}(\boldsymbol{x}\boldsymbol{x}^*),\ \boldsymbol{a}_j^*\boldsymbol{Z}\boldsymbol{a}_j = y_j,\ j = 1,\dots,m, \tag{4.2}$$

in the noiseless case. Here the weights are given by

$$w_{l,k}^{(n)} = \left(\left|\boldsymbol{Z}_{l,k}^{(n)}\right| + \varepsilon_n\right)^{p-1}.$$

The iteration is continued until convergence, and the whole process is repeated with $\varepsilon_n = \frac{1}{10n}$ as the sequence $\{\varepsilon_n\}$ decreasing too fast does not improve the efficiency of the computations [23]. By similar statements in [23, Proposition 4.1], the sequence $\{\boldsymbol{Z}^{(n)}\}$ generated by (4.2) admits a convergent subsequence.

Denoting the algorithm output as $\boldsymbol{X}^{\#}$, we estimate the signal $\boldsymbol{x}^{\#}$ by extracting the largest *rank*-1 component of $\boldsymbol{X}^{\#}$. That is, when the eigenvalue decomposition of $\boldsymbol{X}^{\#}$ is taken as $\boldsymbol{X}^{\#} = \sum_{k=1}^{n} \lambda_k \boldsymbol{u}_k \boldsymbol{u}_k^*$ with $\lambda_1 \geq \ldots \geq \lambda_n$, set $\boldsymbol{x}^{\#} = \sqrt{\lambda_1}\boldsymbol{u}_1$. Besides, we use the relative error as

$$\text{Relative Error} := \frac{d(\boldsymbol{x}^{\#}, \boldsymbol{x})}{\|\boldsymbol{x}\|_2},$$

where $d(\boldsymbol{x}^{\#}, \boldsymbol{x}) = \min_{|c|=1} \|c\boldsymbol{x}^{\#} - \boldsymbol{x}\|_2$. We consider an algorithm to have successfully reconstructed a target signal $\boldsymbol{x}$ if the relative error is smaller than 10^{-3}.

In numerical experiments, we focus on the iterative-reweighted algorithm in (4.2) and construct two kinds of measurements: (i) the real Gaussian model: the sampling vectors $\boldsymbol{a}_j$ are real Gaussian random vectors, i.e., $\boldsymbol{a}_j \sim \mathcal{N}(0, \boldsymbol{I}_{n\times n})$; (ii) the complex Gaussian model: the sampling vectors $\boldsymbol{a}_j$ are complex Gaussian random vectors, i.e., $\boldsymbol{a}_j \sim \mathcal{N}(0, \frac{1}{2}\boldsymbol{I}_{n\times n}) + \mathcal{N}(0, \frac{1}{2}\boldsymbol{I}_{n\times n})\mathrm{i}$. The signal dimension $n = 50$. For each fixed sparsity level k, the support of $\boldsymbol{x}$ is drawn from the uniform distribution over the set of $\{1, \ldots, n\}$. The non-zero entries of the real (resp. complex) k-sparse signal $\boldsymbol{x}$ are drawn from Gaussian distribution $\mathcal{N}(0,1)$ (resp. $\mathcal{N}(0,1) + \mathcal{N}(0,1)\mathrm{i}$). For convenience, we normalize $\boldsymbol{x}$ into $\|\boldsymbol{x}\|_2 = 1$.

First of all, we investigate the convergence performance when p varies in $\{0.1, 0.5, 0.7\}$. The measurement number is $m = 30$ in the real Gaussian model and $m = 60$ in the complex Gaussian model. Figure 4.1 depicts the relative error versus the iteration number for the real Gaussian model and complex Gaussian model. The algorithm converges after a relatively small number of iterations. We find that the algorithm converges much faster when p decreases.

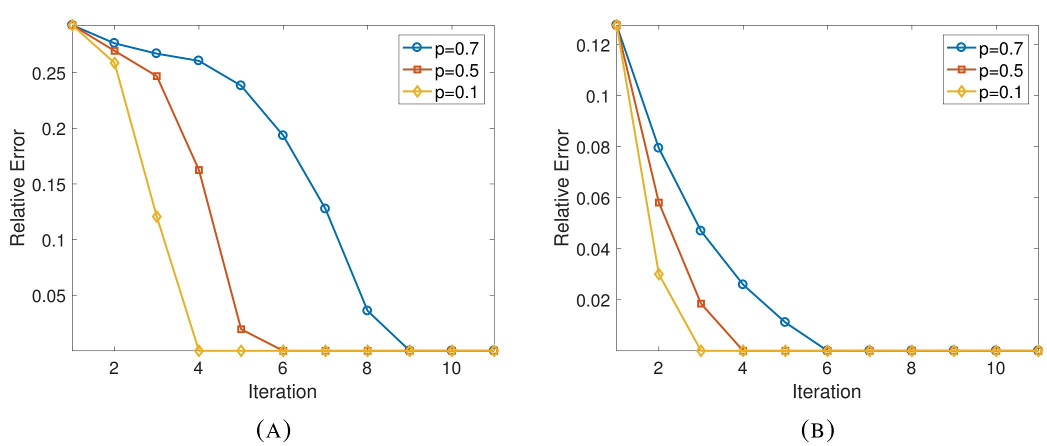
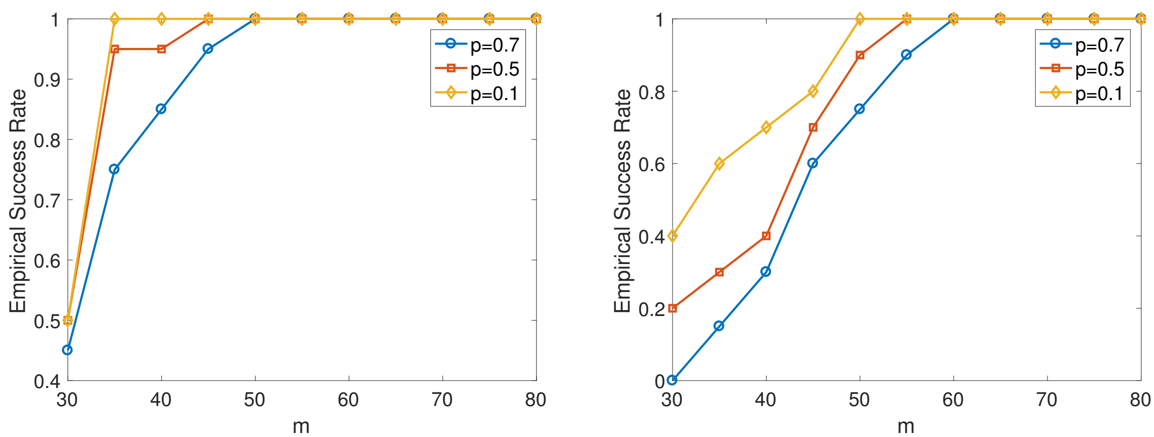


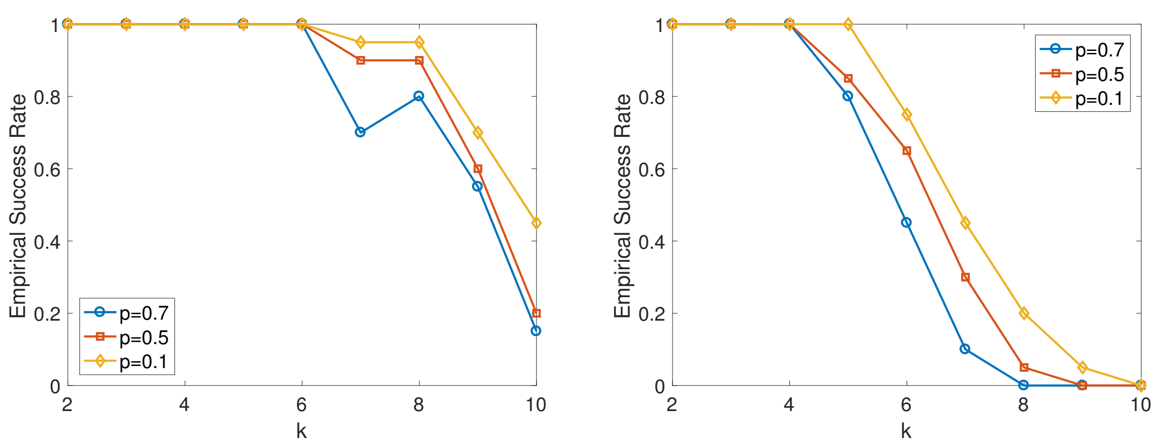
FIGURE 4.1. Comparison of the convergence result under different measurements: (a) the real Gaussian model; (b) the complex Gaussian model.

Furthermore, we test the empirical success rate against the measurement number m and the sparsity level k. We evaluate the algorithm under different choices of p by 20 trials. The plots of successful recovery probability against the sampling number m (resp. the sparsity level k) are demonstrated in Figure 4.2 (resp. Figure 4.3). The numerical results demonstrate that reducing the value of p below 1 clearly reduces the number of measurements needed for perfect recovery, and improves the success rate when the sparsity level k increases.



(A) (B)

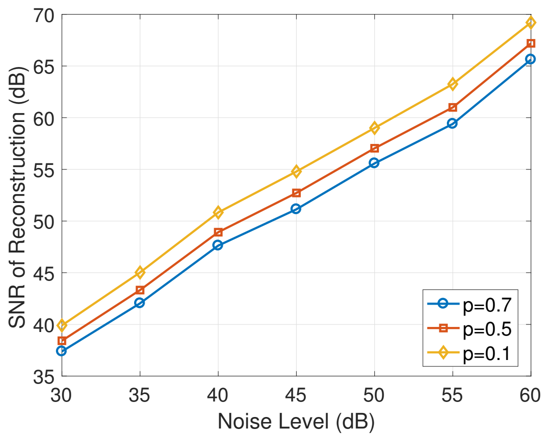
FIGURE 4.2. Comparison of the empirical success rate for fixed $k = 5$ under different measurements: (a) the real Gaussian model; (b) the complex Gaussian model.



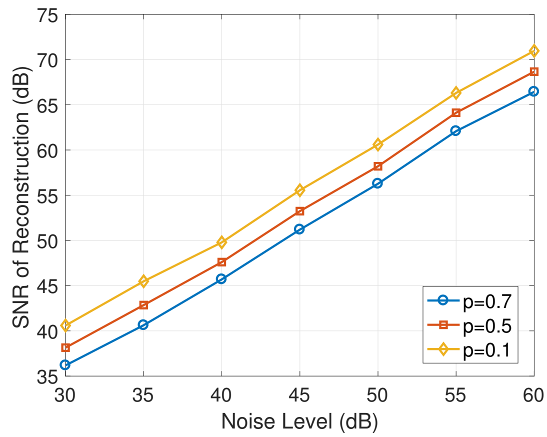
(A) (B)

FIGURE 4.3. Comparison of the empirical success rate for fixed $m = 50$ under different measurements: (a) the real Gaussian model; (b) the complex Gaussian model.

Besides, we demonstrate the performance of the algorithm under additive noise. The white Gaussian noise is followed by MATLAB function awgn$(\mathscr{A}(\boldsymbol{xx}^*), \mathrm{snr})$. Here $m = 70$, $k = 5$ and 20 trials are conducted. The SNR level varies from 30dB to 60dB. The signal-to-noise ratio of reconstruction in dB is taken as $-20log_{10}$(Relative Error). The average relative reconstruction error against SNR is shown in Figure 4.4. The desirable linear scaling between the noise level and the relative reconstruction error can be observed. Meanwhile, it provides better relative error when p decreases.



(A)



(B)

FIGURE 4.4. SNR of the signal recovery versus the noise level when $k = 5$ and $m = 70$: (a) the real Gaussian model; (b) the complex Gaussian model.

5. PROOFS

5.1. Proof of Theorem 3.1.

Proof. Denote $\boldsymbol{x}^\#$ as the solution to (3.2). Since $\exp(i\theta)\boldsymbol{x}^\#$ is also a solution to (3.2) for any $\theta \in \mathbb{R}$, we can assume that

$$\langle \boldsymbol{x}^\#, \boldsymbol{x}\rangle \in \mathbb{R} \qquad \text{and} \qquad \langle \boldsymbol{x}^\#, \boldsymbol{x}\rangle \geq 0.$$

Set $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^*$ and $\boldsymbol{H} = \boldsymbol{x}^\#(\boldsymbol{x}^\#)^* - \boldsymbol{x}\boldsymbol{x}^*$. Denote $T_0 = \mathrm{supp}(\boldsymbol{x})$. Set T_1 as the index set which contains the indices of the largest ak elements of $\boldsymbol{x}^\#_{T_0^c}$ in magnitude, and T_2 contains the indices of the next ak largest elements, and so on. For simplicity, set $T_{01} := T_0 \cup T_1$ and $\overline{\boldsymbol{H}} = \boldsymbol{H}_{T_{01},T_{01}}$. Assume that

$$\|\boldsymbol{H}_{T_{01}^c,T_{01}^c}\|_F^p \leq \sum_{i\geq 2, j\geq 2} \|\boldsymbol{x}^\#_{T_i}(\boldsymbol{x}^\#_{T_j})^*\|_F^p \leq \frac{1}{a^{2-p}}\|\overline{\boldsymbol{H}}\|_F^p, \tag{5.1}$$

and

$$\sum_{j\geq 2} \|\boldsymbol{H}_{T_i,T_j}\|_F^p = \sum_{j\geq 2} \|\boldsymbol{x}^\#_{T_i}(\boldsymbol{x}^\#_{T_j})^*\|_F^p \leq \frac{1}{a^{1-p/2}}\|\overline{\boldsymbol{H}}\|_F^p, \tag{5.2}$$

for any $i \in \{0,1\}$. Then we can directly apply the RIP bounds of $\mathscr{A}(\cdot)$, which arrives at

$$\begin{aligned}\|\mathscr{A}(\overline{\boldsymbol{H}})\|_p^p = \|\mathscr{A}(\boldsymbol{H}-\overline{\boldsymbol{H}})\|_p^p &\leq \overline{C}\left(2\sum_{j\geq 2, i=0,1} \|\boldsymbol{x}^\#_{T_i}(\boldsymbol{x}^\#_{T_j})^*\|_F^p + \sum_{i\geq 2, j\geq 2} \|\boldsymbol{x}^\#_{T_i}\boldsymbol{x}^\#_{T_j}\|_F^p\right)\\ &\leq \overline{C}\left(\frac{1}{a^{2-p}}\|\overline{\boldsymbol{H}}\|_F^p + \frac{4}{a^{1-p/2}}\|\overline{\boldsymbol{H}}\|_F^p\right) = \overline{C}\left(\frac{1}{a^{2-p}} + \frac{4}{a^{1-p/2}}\right)\|\overline{\boldsymbol{H}}\|_F^p,\end{aligned}$$

and

$$\|\mathscr{A}(\overline{\boldsymbol{H}})\|_p^p \geq \underline{C}\|\overline{\boldsymbol{H}}\|_F^p.$$

If

$$\underline{C} > \overline{C}\left(\frac{1}{a^{2-p}} + \frac{4}{a^{1-p/2}}\right),$$

we immediately obtain $\overline{\boldsymbol{H}} = \boldsymbol{0}$. It leads to

$$\|\boldsymbol{H}_{T_{01}^c,T_{01}^c}\|_F^p = 0 \qquad \text{and} \qquad \sum_{j\geq 2}\|\boldsymbol{H}_{T_j,T_i}\|_F^p = \sum_{j\geq 2}\|\boldsymbol{x}_{T_i}^{\#}(\boldsymbol{x}_{T_j}^{\#})^*\|_F^p = 0,$$

for $i \in \{0,1\}$. Therefore,

$$\|\boldsymbol{H}\|_F^p = \|\boldsymbol{x}^{\#}(\boldsymbol{x}^{\#})^* - \boldsymbol{x}\boldsymbol{x}^*\|_F^p \leq \|\boldsymbol{H}_{T_{01}^c,T_{01}^c}\|_F^p + 2\sum_{j\geq 2,\ i=0,1}\|\boldsymbol{H}_{T_j,T_i}\|_F^p + \|\overline{\boldsymbol{H}}\|_F^p = 0,$$

and the conclusion holds:

$$\boldsymbol{x}^{\#}(\boldsymbol{x}^{\#})^* = \boldsymbol{x}\boldsymbol{x}^*.$$

The only thing left is to derive (5.1) and (5.2).

On one hand, we can obtain inequality (5.1) by the following statement:

$$\begin{aligned}
\|\boldsymbol{H}_{T_{01}^c,T_{01}^c}\|_F^p &\leq \sum_{i\geq 2,j\geq 2}\|\boldsymbol{x}_{T_i}^{\#}(\boldsymbol{x}_{T_j}^{\#})^*\|_F^p \overset{(a)}{\leq} \sum_{i\geq 2,j\geq 2}\|\boldsymbol{x}_{T_i}^{\#}\|_2^p\|\boldsymbol{x}_{T_j}^{\#}\|_2^p \overset{(b)}{\leq} \frac{1}{(ak)^{2-p}}\left(\sum_{i\geq 1}\|\boldsymbol{x}_{T_i}^{\#}\|_p^p\right)^2 \\
&= \frac{1}{(ak)^{2-p}}\|\boldsymbol{x}_{T_0^c}^{\#}\|_p^{2p} = \frac{1}{(ak)^{2-p}}\|\boldsymbol{H}_{T_0^c,T_0^c}\|_p^p \\
&\overset{(c)}{\leq} \frac{1}{(ak)^{2-p}}\|\boldsymbol{H}_{T_0,T_0}\|_p^p \leq \frac{1}{a^{2-p}}\|\boldsymbol{H}_{T_0,T_0}\|_F^p \leq \frac{1}{a^{2-p}}\|\overline{\boldsymbol{H}}\|_F^p.
\end{aligned}$$

Here (a) follows from $\|\boldsymbol{x}_{T_i}^{\#}(\boldsymbol{x}_{T_j}^{\#})^*\|_F \leq \|\boldsymbol{x}_{T_i}^{\#}\|_2\|\boldsymbol{x}_{T_j}^{\#}\|_2$. Inequality (b) is based on $\|\boldsymbol{x}_{T_j}^{\#}\|_2^p \leq \frac{\|\boldsymbol{x}_{T_{j-1}}^{\#}\|_p^p}{(ak)^{1-p/2}}$ for any $j \geq 2$. (c) is according to $\|\boldsymbol{x}^{\#}\|_p \leq \|\boldsymbol{x}\|_p$ and

$$\begin{aligned}
\|\boldsymbol{H}_{T_0^c,T_0^c}\|_p^p &\leq \|\boldsymbol{H} - \boldsymbol{H}_{T_0,T_0}\|_p^p = \|\boldsymbol{x}^{\#}(\boldsymbol{x}^{\#})^* - \boldsymbol{x}_{T_0}^{\#}(\boldsymbol{x}_{T_0}^{\#})^*\|_p^p = \|\boldsymbol{x}^{\#}(\boldsymbol{x}^{\#})^*\|_p^p - \|\boldsymbol{x}_{T_0}^{\#}(\boldsymbol{x}_{T_0}^{\#})^*\|_p^p \\
&\leq \|\boldsymbol{x}\boldsymbol{x}^*\|_p^p - \|\boldsymbol{x}_{T_0}^{\#}(\boldsymbol{x}_{T_0}^{\#})^*\|_p^p \leq \|\boldsymbol{x}\boldsymbol{x}^* - \boldsymbol{x}_{T_0}^{\#}(\boldsymbol{x}_{T_0}^{\#})^*\|_p^p = \|\boldsymbol{H}_{T_0,T_0}\|_p^p.
\end{aligned}$$

On the other hand, inequality (5.2) can be obtained as below. For any $i \in \{0,1\}$, we have

$$\begin{aligned}
\sum_{j\geq 2}\|\boldsymbol{H}_{T_i,T_j}\|_F^p &= \sum_{j\geq 2}\|\boldsymbol{H}_{T_j,T_i}\|_F^p = \sum_{j\geq 2}\|\boldsymbol{x}_{T_i}^{\#}(\boldsymbol{x}_{T_j}^{\#})^*\|_F^p \leq \|\boldsymbol{x}_{T_i}^{\#}\|_2^p\sum_{j\geq 2}\|\boldsymbol{x}_{T_j}\|_2^p \\
&\leq \|\boldsymbol{x}_{T_i}^{\#}\|_2^p \cdot \frac{1}{(ak)^{1-p/2}}\|\boldsymbol{x}_{T_0^c}^{\#}\|_p^p \overset{(d)}{\leq} \|\boldsymbol{x}_{T_{01}}^{\#}\|_2^p \cdot \frac{k^{1-p/2}}{(ak)^{1-p/2}}\|\boldsymbol{x}_{T_{01}}^{\#} - \boldsymbol{x}\|_2^p \\
&\overset{(e)}{\leq} \frac{1}{a^{1-p/2}}\|\overline{\boldsymbol{H}}\|_F^p.
\end{aligned}$$

Here (d) follows from

$$\|\boldsymbol{x}_{T_0^c}^{\#}\|_p^p \leq \|\boldsymbol{x}\|_p^p - \|\boldsymbol{x}_{T_0}^{\#}\|_p^p \leq \|\boldsymbol{x} - \boldsymbol{x}_{T_0}^{\#}\|_p^p \leq k^{1-p/2}\|\boldsymbol{x} - \boldsymbol{x}_{T_0}^{\#}\|_2^p \leq k^{1-p/2}\|\boldsymbol{x} - \boldsymbol{x}_{T_{01}}^{\#}\|_2^p.$$

(e) is based on Lemma 3.2 in [14], that is,

$$\|\boldsymbol{x} - \boldsymbol{x}_{T_{01}}^{\#}\|_2 \cdot \frac{1}{\sqrt{2}}\|\boldsymbol{x}_{T_{01}}^{\#}\|_2 \leq \|\boldsymbol{x}\boldsymbol{x}^* - \boldsymbol{x}_{T_{01}}^{\#}(\boldsymbol{x}_{T_{01}}^{\#})^*\|_F,$$

when $\langle \boldsymbol{x}, \boldsymbol{x}^{\#}_{T_{01}} \rangle \geq 0$. □

5.2. **Proof of Theorem 3.2.**

5.2.1. *Auxiliary Results.* First of all, we establish some technical tools.

Lemma 5.1. *Assume that λ is some positive integer and $0 < q \leq 1$. Take some $\mu > 0$. Then*

$$\int_0^{\mu^{1/q}} (x^q - \mu)^\lambda dx \leq \mu^{\lambda+1/q} q^{\lceil \tau\lambda \rceil} \lceil \tau\lambda \rceil !,$$

for any fixed τ with $0 < \tau \leq 1$.

Proof. Taking u with $\mu u = x^q$, we have

$$\begin{aligned}\int_0^{\mu^{1/q}} (x^q - \mu)^\lambda dx &= \int_0^1 \left((\mu u - \mu)^\lambda \cdot \frac{\mu^{1/q}}{q} \cdot u^{1/q-1} \right) du = \frac{\mu^{\lambda+1/q}}{q} B(\lambda+1, 1/q) \\ &= \frac{\mu^{\lambda+1/q}}{q} \cdot \frac{\Gamma(\lambda+1)\Gamma(1/q)}{\Gamma(\lambda+1+1/q)} = \mu^{\lambda+1/q} \cdot \prod_{j=1}^{\lambda} \frac{j}{j+1/q} = \mu^{\lambda+1/q} \cdot \prod_{j=1}^{\lambda} \frac{jq}{jq+1} \\ &\leq \mu^{\lambda+1/q} \prod_{j=1}^{\lceil \tau\lambda \rceil} jq = \mu^{\lambda+1/q} q^{\lceil \tau\lambda \rceil} \lceil \tau\lambda \rceil !.\end{aligned}$$

□

Lemma 5.2. *Assume that $\lambda, \mu > 0$. Then*

$$\int_\mu^{+\infty} (x-\mu)^\lambda \exp(-x^2/2) dx \leq 2^{\frac{\lambda-1}{2}} \Gamma\left(\frac{\lambda+1}{2}\right).$$

Proof. By direct calculation, we have

$$\begin{aligned}\int_\mu^{+\infty} (x-\mu)^\lambda \exp(-x^2/2) dx &= \int_0^\infty x^\lambda \exp(-(x+\mu)^2/2) dx \leq \int_0^\infty x^\lambda \exp(-x^2/2) dx \\ &= \int_0^\infty (2t)^{\frac{\lambda-1}{2}} \exp(-t) dt = 2^{\frac{\lambda-1}{2}} \Gamma\left(\frac{\lambda+1}{2}\right)\end{aligned}$$

by taking $t = x^2/2$. □

5.2.2. *Proof of (3.3) in Theorem 3.2.*

Proof. By the definition of $\boldsymbol{X}$, the eigenvalue decomposition of $\boldsymbol{X}$ becomes $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^*$, where $\boldsymbol{\Sigma} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \in \mathbb{R}^{2\times 2}$ with $\alpha^2 + \beta^2 = 1$ and $\boldsymbol{U} \in \mathbb{C}^{n\times 2}$ satisfies $\boldsymbol{U}^*\boldsymbol{U} = \boldsymbol{I}$. Under the rotation invariance property of Gaussian random vector, we have

$$\mathbb{E}|\boldsymbol{a}^\top \boldsymbol{X} \boldsymbol{a}|^p = \mathbb{E}|\alpha(z_1^2 + z_2^2) + \beta(z_3^2 + z_4^2)|^p \cdot 2^{-p}, \tag{5.3}$$

where z_1, z_2, z_3, and z_4 are independently drawn from $\mathcal{N}(0,1)$.

Now we focus on the upper and lower bounds of $\mathbb{E}|\alpha(z_1^2+z_2^2)+\beta(z_3^2+z_4^2)|^p$. Without loss of generality, we assume $\alpha \geq |\beta|$. Taking $z_1=\rho_1\cos\theta$, $z_2=\rho_1\sin\theta$, $z_3=\rho_2\cos\phi$, and $z_4=\rho_2\sin\phi$, we have

$$\begin{aligned}
&\mathbb{E}|\alpha(z_1^2+z_2^2)+\beta(z_3^2+z_4^2)|^p\\
=&\frac{1}{(2\pi)^2}\int\cdots\int_{\mathbb{R}^4}|\alpha(z_1^2+z_2^2)+\beta(z_3^2+z_4^2)|^p\exp\left(-\frac{z_1^2+z_2^2+z_3^2+z_4^2}{2}\right)dz_1dz_2dz_3dz_4\\
=&\frac{1}{(2\pi)^2}\int_0^{2\pi}d\theta\int_0^{2\pi}d\phi\int_0^{+\infty}\int_0^{+\infty}\rho_1\rho_2|\alpha\rho_1^2+\beta\rho_2^2|^p\exp\left(-\frac{\rho_1^2+\rho_2^2}{2}\right)d\rho_1d\rho_2\\
=&\int_0^{+\infty}\int_0^{+\infty}\rho_1\rho_2|\alpha\rho_1^2+\beta\rho_2^2|^p\exp\left(-\frac{\rho_1^2+\rho_2^2}{2}\right)d\rho_1d\rho_2. \qquad (5.4)
\end{aligned}$$

Denote $\rho_1=\rho\cos\varphi$ and $\rho_2=\rho\sin\varphi$. One sees that (5.4) becomes

$$\begin{aligned}
&\mathbb{E}|\alpha(z_1^2+z_2^2)+\beta(z_3^2+z_4^2)|^p\\
=&\int_0^{+\infty}\int_0^{+\infty}\rho_1\rho_2|\alpha\rho_1^2+\beta\rho_2^2|^p\exp\left(-\frac{\rho_1^2+\rho_2^2}{2}\right)d\rho_1d\rho_2.\\
=&\int_0^{+\infty}\rho^{2p+3}\exp(-\rho^2/2)d\rho\cdot\int_0^{\pi/2}\cos\varphi\sin\varphi|\alpha\cos^2\varphi+\beta\sin^2\varphi|^pd\varphi\\
=&\int_0^{+\infty}\frac{1}{2}\rho^{2p+3}\exp(-\rho^2/2)d\rho\cdot\int_0^1|\alpha+(\beta-\alpha)t|^pdt,
\end{aligned}$$

where the last equality above is based on $t=sin^2\varphi$. Thus $\mathbb{E}|\alpha(z_1^2+z_2^2)+\beta(z_3^2+z_4^2)|^p$ can be considered as

$$\mathbb{E}|\alpha(z_1^2+z_2^2)+\beta(z_3^2+z_4^2)|^p=\mathbb{E}_{\rho,t}(\rho^{2p}\cdot|\alpha+(\beta-\alpha)t|^p)=\mathbb{E}_\rho(\rho^{2p})\mathbb{E}_t(|\alpha+(\beta-\alpha)t|^p), \quad (5.5)$$

where t is taken as some random variable drawn from $\mathscr{U}[0,1]$, and the density function $p(\rho)$ of ρ satisfies

$$p(\rho)=\frac{1}{2}\rho^3\exp(-\rho^2/2), \qquad 0\leq\rho<\infty.$$

By direct calculation, we have

$$\begin{aligned}
&\mathbb{E}|\alpha(z_1^2+z_2^2)+\beta(z_3^2+z_4^2)|^p\\
&=\mathbb{E}_\rho(\rho^{2p})\mathbb{E}_t(|\alpha+(\beta-\alpha)t|^p)=2^p\Gamma(p+2)\int_0^1|\alpha+(\beta-\alpha)t|^pdt\\
&=\begin{cases}2^p\Gamma(p+2)\cdot(1/2)^{p/2} & \alpha=\beta=\frac{\sqrt{2}}{2};\\ 2^p\Gamma(p+2)\cdot\frac{1}{p+1}\cdot\frac{1}{\alpha-\beta}\cdot(\alpha^{p+1}-\beta^{p+1}) & \alpha\neq\beta \text{ and } \beta\geq 0;\\ 2^p\Gamma(p+2)\cdot\frac{1}{p+1}\cdot\frac{1}{\alpha-\beta}\cdot(\alpha^{p+1}+(-\beta)^{p+1}) & \beta<0.\end{cases}
\end{aligned}$$

and $\mathbb{E}|\alpha(z_1^2+z_2^2)+\beta(z_3^2+z_4^2)|^p$ reaches the upper and lower bounds on the case of $\beta=\frac{\sqrt{2}}{2}$ and $\beta=-\frac{\sqrt{2}}{2}$, respectively. Therefore, we can obtain

$$\frac{1}{p+1}2^{p/2}\Gamma(p+2)\leq\mathbb{E}|\alpha(z_1^2+z_2^2)+\beta(z_3^2+z_4^2)|^p\leq 2^{p/2}\Gamma(p+2),$$

which leads to

$$\frac{1}{p+1}2^{-p/2}\Gamma(p+2) \le \mathbb{E}|\boldsymbol{a}^\top \boldsymbol{X}\boldsymbol{a}|^p = \mathbb{E}|\alpha(z_1^2+z_2^2)+\beta(z_3^2+z_4^2)|^p \cdot 2^{-p} \le 2^{-p/2}\Gamma(p+2).$$

□

5.2.3. *Proof of (3.4) in Theorem 3.2*. The following theorem plays a fundamental role in the proof of (3.4).

Theorem 5.1. [24, Theorem 1.5] *Let $X_1,\dots,X_M$ be independent copies of some random variable X. Denote $\mu = \mathbb{E}X$ and*

$$\tau = sup_{k\ge 1}(3.1)^{\frac{1}{4}}\left[\frac{2^k k!}{(2k)!}\mathbb{E}(X-\mu)^{2k}\right]^{\frac{1}{2k}}.$$

Then

$$\mathbb{P}\left(\Big|\sum_{i=1}^{M}(X_i-\mu)\Big| \ge t\right) \le 2\exp\left(-\frac{t^2}{2M\tau^2}\right).$$

Proof of (3.4) in Theorem 3.2. According to (5.3), when $1/2 \le p \le 1$, we have

$$\||\boldsymbol{a}_1^\top \boldsymbol{X}\boldsymbol{a}_1|^p\|_{\psi_1} \le 4\||z|^{2p}\|_{\psi_1} \le 4\|\max\{1,|z|^{2p}\}\|_{\psi_1} \le 4\|\max\{1,z^2\}\|_{\psi_1} \le \bar{c},$$

where $z \sim \mathcal{N}(0,1)$ and $\|\cdot\|_{\psi_1} := \sup_{k\ge 1}(\mathbb{E}|\cdot|^k)^{1/k}$. Therefore,

$$\mathbb{P}\left(\left|\sum_{i=1}^{m}(X_i-\mu)\right| \ge t\right) \le 2\exp\left(-\frac{t^2}{c_1 m}\right), \tag{5.6}$$

for any $1/2 \le p \le 1$ [25]. Then we discuss the case that $0 < p < \frac{1}{2}$. In order to apply Theorem 5.1, we should estimate

$$\tau = \sup_{k\ge 1}(3.1)^{\frac{1}{4}}\left[\frac{2^k k!}{(2k)!}\mathbb{E}(X-\mu)^{2k}\right]^{\frac{1}{2k}},$$

when $X = |\boldsymbol{a}_1^\top \boldsymbol{X}\boldsymbol{a}_1|^p$ and $\mu = \mathbb{E}X$. According to (5.5), we have

$$\mathbb{E}(X-\mu)^{2k} = \mathbb{E}_{\rho,t}(f(\rho)g(t) - \mathbb{E}_\rho f(\rho)\mathbb{E}_t g(t))^{2k},$$

where $f(\rho) = \rho^{2p}$ and $g(t) = |\alpha + (\beta-\alpha)t|^p$ with the density functions on ρ and t as

$$p(\rho) = \frac{1}{2}\rho^3\exp(-\rho^2/2),\ 0 \le \rho < \infty \quad \text{and} \quad p(t) = 1,\ 0 \le t \le 1.$$

Therefore, by $(x+y)^{2k} \le 2^{2k-1}x^{2k} + 2^{2k-1}y^{2k}$, we can obtain that

$$\begin{aligned}
\mathbb{E}(X-\mu)^{2k} =& \mathbb{E}_{\rho,t}(f(\rho)g(t) - \mathbb{E}_\rho f(\rho)\mathbb{E}_t g(t))^{2k} = \mathbb{E}_\rho\left(\mathbb{E}_t(f(\rho)g(t) - \mathbb{E}_\rho f(\rho)\mathbb{E}_t g(t))^{2k} \mid \rho\right) \\
\le& 2^{2k-1}\mathbb{E}_\rho\left(\mathbb{E}_t(f(\rho)g(t) - f(\rho)\mathbb{E}_t g(t))^{2k} \mid \rho\right) \\
&+ 2^{2k-1}\mathbb{E}_\rho\left(\mathbb{E}_t(f(\rho)\mathbb{E}_t g(t) - \mathbb{E}_\rho f(\rho)\mathbb{E}_t g(t))^{2k} \mid \rho\right) \\
\le& 2^{2k-1}\mathbb{E}_\rho f(\rho)^{2k} \cdot \mathbb{E}_t(g(t) - \mathbb{E}_t g(t))^{2k} + 2^{2k-1}\mathbb{E}_t g(t)^{2k} \cdot \mathbb{E}_\rho(f(\rho) - \mathbb{E}_\rho f(\rho))^{2k}.
\end{aligned}$$

Now we calculate the upper bounds of $\mathbb{E}_\rho f(\rho)^{2k} \cdot \mathbb{E}_t(g(t) - \mathbb{E}_t g(t))^{2k}$ and $\mathbb{E}_t g(t)^{2k} \cdot \mathbb{E}_\rho(f(\rho) - \mathbb{E}_\rho f(\rho))^{2k}$ separately.

(a) Estimation of $\mathbb{E}_\rho f(\rho)^{2k}\cdot\mathbb{E}_t(g(t)-\mathbb{E}_t g(t))^{2k}$.

On one hand, we have

$$\mathbb{E}_\rho f(\rho)^{2k}=\int_0^{+\infty}\rho^{4pk}\cdot\frac{1}{2}\rho^3\exp(-\rho^2/2)d\rho=2^{2pk}\Gamma(2pk+2)\le 2^{2k}(\lfloor 2pk\rfloor+2)!.$$

On the other hand, we denote $\nu=\mathbb{E}_t g(t)$. By direct calculation, we have

$$\nu=\begin{cases}(1/2)^{p/2}, & \alpha=\beta=\frac{\sqrt{2}}{2};\\ \frac{1}{p+1}\cdot\frac{1}{\alpha-\beta}\cdot(\alpha^{p+1}-\beta^{p+1}), & \alpha\ne\beta \text{ and } \beta\ge 0;\\ \frac{1}{p+1}\cdot\frac{1}{\alpha-\beta}\cdot(\alpha^{p+1}+(-\beta)^{p+1}), & \beta<0.\end{cases}$$

If

$$\begin{aligned}&\mathbb{E}_t(g(t)-\mathbb{E}_t g(t))^{2k}\\ \le&\begin{cases}0, & \alpha=\beta=\frac{\sqrt{2}}{2};\\ p^{2k}\cdot\left(\frac{\sqrt{5}}{2}e\right)^{2k}+\sqrt{5}\cdot p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil!+p^{2k}\cdot(\sqrt{5})^{2k}, & \alpha\ne\beta \text{ and } \beta\ge 0;\\ 2p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil!+2p^{2k}(2\sqrt{2}e)^{2k}, & \beta<0,\end{cases}\end{aligned}\tag{5.7}$$

then

$$\mathbb{E}_t(g(t)-\mathbb{E}_t g(t))^{2k}\le 2p^{2k}\cdot\left(2\sqrt{2}e\right)^{2k}+\sqrt{5}\cdot p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil!,$$

which leads to

$$\begin{aligned}&\mathbb{E}_\rho f(\rho)^{2k}\mathbb{E}_t(g(t)-\mathbb{E}_t g(t))^{2k}\\ &\le 2^{2k}(\lfloor 2pk\rfloor+2)!\cdot\left(2p^{2k}\cdot\left(2\sqrt{2}e\right)^{2k}+\sqrt{5}\cdot p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil!\right).\end{aligned}\tag{5.8}$$

Therefore, we should estimate $\mathbb{E}_t(g(t)-\mathbb{E}_t g(t))^{2k}$ in (5.7) case by case.

When $\alpha=\beta=\frac{\sqrt{2}}{2}$, it is easy to see that $\mathbb{E}_t(g(t)-\mathbb{E}_t g(t))^{2k}=0$.

When $\alpha\ne\beta$ and $\beta\ge 0$, by taking $x=\alpha+(\beta-\alpha)t$, it obtains that

$$\begin{aligned}&\mathbb{E}_t(g(t)-\mathbb{E}_t g(t))^{2k}\\ =&\int_0^1(|\alpha+(\beta-\alpha)t|^p-\nu)^{2k}dt=\int_0^1((\alpha+(\beta-\alpha)t)^p-\nu)^{2k}dt\\ =&\frac{1}{\alpha-\beta}\int_\beta^\alpha(x^p-\nu)^{2k}dx=\frac{1}{\alpha-\beta}\int_\beta^{\nu^{1/p}}(x^p-\nu)^{2k}dx+\frac{1}{\alpha-\beta}\int_{\nu^{1/p}}^\alpha(x^p-\nu)^{2k}dx.\end{aligned}\tag{5.9}$$

Since $x^p-\nu\le(x-\nu^{1/p})\cdot p\cdot\nu^{1-1/p}$ when $x\ge\nu^{1/p}$, then

$$\begin{aligned}\frac{1}{\alpha-\beta}\int_{\nu^{1/p}}^\alpha(x^p-\nu)^{2k}dx\le&\frac{1}{\alpha-\beta}\int_{\nu^{1/p}}^\alpha(x-\nu^{1/p})^{2k}\cdot p^{2k}\cdot\nu^{2(1-1/p)k}dx\\ =&\frac{1}{\alpha-\beta}p^{2k}\cdot\nu^{2(1-1/p)k}\cdot\frac{1}{2k+1}(\alpha-\nu^{1/p})^{2k+1}.\end{aligned}\tag{5.10}$$

Similarly, we have

$$\frac{1}{\alpha-\beta}\int_\beta^{\nu^{1/p}}(x^p-\nu)^{2k}dx\le\frac{1}{\alpha-\beta}\cdot p^{2k}\cdot\beta^{2(p-1)k}\frac{1}{2k+1}(\nu^{1/p}-\beta)^{2k+1},\tag{5.11}$$

and

$$\begin{aligned}\frac{1}{\alpha-\beta}\int_{\beta}^{\nu^{1/p}}(x^p-\nu)^{2k}dx &\le \frac{1}{\alpha-\beta}\int_{0}^{\nu^{1/p}}(x^p-\nu)^{2k}dx \\ &\le \frac{1}{\alpha-\beta}\cdot\nu^{2k+1/p}p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil!\end{aligned} \tag{5.12}$$

by taking $q=p$, $\lambda=2k$, and $\tau=(1-2p)/2$ in Lemma 5.1. Plugging (5.10) and (5.12) into (5.9) when $0\le\beta\le\sqrt{5}/5$, and plugging (5.10) and (5.11) into (5.9) when $\sqrt{5}/5\le\beta<\sqrt{2}/2$, we have

$$\mathbb{E}_t(g(t)-\mathbb{E}_t g(t))^{2k}\le\begin{cases}p^{2k}\cdot\left(\frac{\sqrt{5}}{2}e\right)^{2k}+\sqrt{5}\cdot p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil!, & 0\le\beta\le\sqrt{5}/5;\\ p^{2k}\cdot\left(\frac{\sqrt{5}}{2}e\right)^{2k}+p^{2k}\cdot(\sqrt{5})^{2k}, & \sqrt{5}/5\le\beta<\sqrt{2}/2.\end{cases}$$

Therefore, when $\alpha\ne\beta$ and $\beta\ge 0$, we have

$$\mathbb{E}_t(g(t)-\mathbb{E}_t g(t))^{2k}\le p^{2k}\cdot\left(\frac{\sqrt{5}}{2}e\right)^{2k}+\sqrt{5}\cdot p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil!+p^{2k}\cdot(\sqrt{5})^{2k}.$$

When $\beta<0$, we have

$$\begin{aligned}&\mathbb{E}_t(g(t)-\mathbb{E}_t g(t))^{2k}\\ =&\int_0^1(|\alpha+(\beta-\alpha)t|^p-\nu)^{2k}dt\\ =&\int_0^{\frac{\alpha}{\alpha-\beta}}((\alpha+(\beta-\alpha)t)^p-\nu)^{2k}dt+\int_{\frac{\alpha}{\alpha-\beta}}^1((-\alpha+(\alpha-\beta)t)^p-\nu)^{2k}dt\\ =&\frac{1}{\alpha-\beta}\int_0^{\alpha}(x^p-\nu)^{2k}dx+\frac{1}{\alpha-\beta}\int_0^{-\beta}(x^p-\nu)^{2k}dx.\end{aligned}$$

Since

$$\begin{aligned}&\frac{1}{\alpha-\beta}\int_0^{\alpha}(x^p-\nu)^{2k}dx\\ \le&\int_0^{\alpha}(x^p-\nu)^{2k}dx=\int_0^{\nu^{1/p}}(x^p-\nu)^{2k}dx+\int_{\nu^{1/p}}^{\alpha}(x^p-\nu)^{2k}dx\\ \le&\nu^{2k+1/p}p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil!+p^{2k}\cdot\nu^{2(1-1/p)k}\cdot\frac{1}{2k+1}(\alpha-\nu^{1/p})^{2k+1}\\ \le&p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil!+p^{2k}\cdot(\sqrt{2}e)^{2k},\end{aligned}$$

with

$$1\ge\nu^{1/p}\ge\left(\frac{1}{p+1}\cdot 2^{-p/2}\right)^{1/p}=2^{-1/2}\cdot(p+1)^{-1/p}\ge\frac{1}{\sqrt{2}e},$$

and

$$\begin{aligned}
&\frac{1}{\alpha-\beta}\int_0^{-\beta}(x^p-\nu)^{2k}dx \\
\leq &\int_0^{-\beta}(x^p-\nu)^{2k}dx = \int_0^{\nu^{1/p}}(x^p-\nu)^{2k}dx + \int_{\nu^{1/p}}^{-\beta}(x^p-\nu)^{2k}dx \\
\leq &\nu^{2k+1/p}p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil! + p^{2k}\cdot\nu^{2(1-1/p)k}\cdot\frac{1}{2k+1}(-\beta-\nu^{1/p})^{2k+1} \\
\leq &p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil! + (2\sqrt{2}e)^{2k},
\end{aligned}$$

we have

$$\begin{aligned}
\mathbb{E}_t(g(t)-\mathbb{E}_t g(t))^{2k} &= \frac{1}{\alpha-\beta}\int_0^{\alpha}(x^p-\nu)^{2k}dx + \frac{1}{\alpha-\beta}\int_0^{-\beta}(x^p-\nu)^{2k}dx \\
&\leq 2p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil! + 2p^{2k}(2\sqrt{2}e)^{2k}.
\end{aligned}$$

(b) Estimation of $\mathbb{E}_t g(t)^{2k}\mathbb{E}_\rho(f(\rho)-\mathbb{E}_\rho f(\rho))^{2k}$.
On one hand, we have

$$\mathbb{E}_t g(t)^{2k} = \begin{cases} (1/2)^{pk}, & \alpha=\beta=\frac{\sqrt{2}}{2}; \\ \frac{1}{2pk+1}\cdot\frac{1}{\alpha-\beta}\cdot(\alpha^{2pk+1}-\beta^{2pk+1}), & \alpha\neq\beta \text{ and } \beta\geq 0; \\ \frac{1}{2pk+1}\cdot\frac{1}{\alpha-\beta}\cdot(\alpha^{2pk+1}+(-\beta)^{2pk+1}), & \beta<0. \end{cases}$$

Thus, for any α and β with $\alpha^2+\beta^2=1$, it arrives at $\mathbb{E}_t g(t)^{2k}\leq 1$.

On the other hand, one denotes $\omega=\mathbb{E}_\rho(\rho^{2p})=2^p\Gamma(p+2)$. Direct calculation yields that

$$4\geq\omega^{1/p}\geq 1,$$

and

$$\begin{aligned}
&\mathbb{E}_\rho(\rho^{2p}-\mathbb{E}_\rho(\rho^{2p}))^{2k} \\
=&\int_0^{+\infty}(x^{2p}-\omega)^{2k}\cdot\frac{x^3}{2}\cdot\exp(-x^2/2)dx \\
\leq&\underbrace{\int_0^{\omega^{1/2p}}(x^{2p}-\omega)^{2k}\cdot\frac{x^3}{2}dx}_{a}+\underbrace{\int_{\omega^{1/2p}}^{\infty}(x^{2p}-\omega)^{2k}\cdot\frac{x^3}{2}\cdot\exp(-x^2/2)dx}_{b}.
\end{aligned}$$

Thus we need to estimate the upper bounds of (a) and (b), respectively. Taking $x^2=t$ and applying Lemma 5.1, we have

$$\begin{aligned}
(a) &= \int_0^{\omega^{1/2p}}(x^{2p}-\omega)^{2k}\cdot\frac{x^3}{2}dx = \frac{1}{4}\int_0^{\omega^{1/p}}(t^p-\omega)^{2k}tdt \\
&= \omega^{2k+1/p}p^k k! \leq 4^{1+2k}p^k k!.
\end{aligned}$$

Besides, it obtains that $x^{2p}-\omega \le \left(x-\omega^{1/(2p)}\right)\cdot(2p)\cdot\omega^{1-1/(2p)}$ when $x\ge\omega^{1/(2p)}$. It follows that

$$
\begin{aligned}
(b) \le & \int_{\omega^{1/2p}}^{\infty}\left(\left(x-\omega^{1/(2p)}\right)\cdot(2p)\cdot\omega^{1-1/(2p)}\right)^{2k}\cdot\frac{x^3}{2}\cdot\exp(-x^2/2)dx\\
=&(2p)^{2k}\cdot\omega^{2k-k/p}\cdot\int_0^\infty x^{2k}\left(\frac{x+\omega^{1/(2p)}}{2}\right)^3\exp(-(x+\omega^{1/(2p)})^2/2)dx\\
\le&(2p)^{2k}\cdot\omega^{2k-k/p}\cdot\int_0^\infty x^{2k}(x^3+\omega^{3/(2p)})\exp(-x^2/2)dx\\
=&(2p)^{2k}\cdot\omega^{2k-k/p}\cdot\int_0^\infty x^{2k+3}\exp(-x^2/2)dx+(2p)^{2k}\cdot\omega^{2k-k/p+3/(2p)}\cdot\int_0^\infty x^{2k}\exp(-x^2/2)dx\\
=&(2p)^{2k}\cdot\omega^{2k-k/p}\cdot2^{k+1}\cdot\Gamma(k+2)+(2p)^{2k}\cdot\omega^{2k-k/p+3/(2p)}\cdot2^{k-1/2}\cdot\Gamma(k+1/2)\\
\le&(k+1)!\cdot4^{2k}\cdot2^{2k+1}\cdot(2p)^{2k}+(2p)^{2k}\cdot4^{2k+3}\cdot2^k\cdot\sqrt{\pi}\cdot\frac{(2k)!}{2^{2k}k!}.
\end{aligned}
$$

Thus

$$
\mathbb{E}_\rho(\rho^{2p}-\mathbb{E}_\rho(\rho^{2p}))^{2k}\le(k+1)!\cdot4^{2k}\cdot2^{2k+2}\cdot(2p)^{2k}+(2p)^{2k}\cdot4^{2k+3}\cdot2^k\cdot\sqrt{\pi}\cdot\frac{(2k)!}{2^{2k}k!},
$$

which leads to

$$
\begin{aligned}
&\mathbb{E}_t g(t)^{2k}\mathbb{E}_\rho(f(\rho)-\mathbb{E}_\rho f(\rho))^{2k}\\
&\le(k+1)!\cdot4^{2k}\cdot2^{2k+2}\cdot(2p)^{2k}+(2p)^{2k}\cdot4^{2k+3}\cdot2^k\cdot\sqrt{\pi}\cdot\frac{(2k)!}{2^{2k}k!}.
\end{aligned}
\tag{5.13}
$$

(c) Estimation of τ.
Based on (5.8) and (5.13), we have

$$
\begin{aligned}
&\frac{2^kk!}{(2k)!}\mathbb{E}(X-\mu)^{2k}\\
\le&\frac{2^kk!}{(2k)!}\cdot2^{2k-1}\cdot2^{2k}(\lfloor 2pk\rfloor+2)!\cdot\left(2p^{2k}\cdot\left(2\sqrt{2}e\right)^{2k}+\sqrt{5}\cdot p^{\lceil(1-2p)k\rceil}\lceil(1-2p)k\rceil!\right)\\
&+\frac{2^kk!}{(2k)!}\cdot2^{2k-1}\cdot\left((k+1)!\cdot4^{2k}\cdot2^{2k+2}\cdot(2p)^{2k}+(2p)^{2k}\cdot4^{2k+3}\cdot2^k\cdot\sqrt{\pi}\cdot\frac{(2k)!}{2^{2k}k!}\right)\\
\le&(k+1)\cdot(k+2)\cdot2^{4k}\cdot p^{2k}\cdot\left(2\sqrt{2}e\right)^{2k}+(k+1)\cdot(k+2)\cdot\sqrt{5}\cdot p^{\lceil(1-2p)k\rceil}\cdot2^{4k-1}\cdot\sqrt{2}\pi k\exp(2)\\
&+(k+1)\cdot2^{8k+1}\cdot(2p)^{2k}+2^{2k-1}\cdot(2p)^{2k}\cdot4^{2k+3}\cdot\sqrt{\pi}.
\end{aligned}
$$

The second inequality is based on Stirling's approximation inequality [26]:

$$
\frac{2^kk!}{(2k)!}\cdot(n_1!)\cdot(n_2!)\le\sqrt{2}\exp(2)\cdot\sqrt{4\pi^2n_1n_2}\le\sqrt{2}\pi k\exp(2),
$$

provided that $n_1+n_2=k$. Therefore,

$$
\tau=\sup_{k\ge1}(3.1)^{\frac14}\left[\frac{2^kk!}{(2k)!}\mathbb{E}(X-\mu)^{2k}\right]^{\frac{1}{2k}}\lesssim p+p^{1/2-p},
$$

and

$$\mathbb{P}\left(\left|\sum_{i=1}^{m}(X_i-\mu)\right|\geq t\right)\leq 2\exp\left(-\frac{t^2}{c_2 m\tau^2}\right), \tag{5.14}$$

when $0<p\leq 1/2$. Based on (5.6) and (5.14), we can conclude that

$$\mathbb{P}\left(\left|\sum_{i=1}^{m}(X_i-\mu)\right|\geq t\right)\leq 2\exp\left(-\frac{t^2}{cm\tau^2}\right), \quad \text{with } \tau\leq p+p^{1/2-p}.$$

Here $c=\max\{c_1,2c_2\}$ as $p+p^{1/2-p}\leq 2$ when $1/2<p\leq 1$. □

5.3. **Proof of Theorem 3.3.**

Proof. $\mathscr{X}$ is equivalent to $\widetilde{\mathscr{X}}$ with

$$\widetilde{\mathscr{X}}=\{\boldsymbol{X}\in\mathbb{H}^{n\times n}\mid \boldsymbol{X}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^*,\ \boldsymbol{\Sigma}\in\Lambda,\ \boldsymbol{U}\in\mathscr{U}\},$$

where

$$\Lambda=\{\boldsymbol{\Sigma}\in\mathbb{R}^{2\times 2}\mid \boldsymbol{\Sigma}=\operatorname{diag}(\lambda_1,\lambda_2),\ \lambda_1^2+\lambda_2^2=1\}$$

and

$$\mathscr{U}=\{\boldsymbol{U}\in\mathbb{C}^{n\times 2}\mid \boldsymbol{U}^*\boldsymbol{U}=\boldsymbol{I},\ \|\boldsymbol{U}\|_{0,2}=L\}=\cup_{|T|=L}\mathscr{U}_T$$

with

$$\mathscr{U}_T=\{\boldsymbol{U}\in\mathbb{C}^{n\times 2}\mid \boldsymbol{U}^*\boldsymbol{U}=\boldsymbol{I},\ \boldsymbol{U}=\boldsymbol{U}_{T,:}\}.$$

Then we can construct the ε-net of $\mathscr{X}$ in the the following construction. We use $\mathscr{Q}_T$ to denote an $\varepsilon/3$-net of $\mathscr{U}_T$, that is, for any $\boldsymbol{U}\in\mathscr{U}_T$, there exists some $\boldsymbol{Q}\in\mathscr{Q}_T\subset\mathscr{U}_T$ such that $\boldsymbol{Q}=\boldsymbol{Q}_{T,:}$ and $\|\boldsymbol{Q}-\boldsymbol{U}\|_F\leq\varepsilon/3$. We have $|\mathscr{Q}_T|\leq(9\sqrt{2}/\varepsilon)^{4L}$, where $|\mathscr{Q}_T|$ is number of elements in $\mathscr{Q}_T$ [27, Lemma 2.2]. Denoting $\mathscr{Q}_\varepsilon=\cup_{|T|=L}\mathscr{Q}_T$, we obtain that

$$|Q_\varepsilon|\leq\left(\frac{en}{L}\right)^L\left(\frac{9\sqrt{2}}{\varepsilon}\right)^{4L}\leq\left(\frac{9\sqrt{2}en}{\varepsilon L}\right)^{4L}.$$

Similarly, let Λ_ε be as an $\varepsilon/3$-net of Λ with $|\Lambda_\varepsilon|\leq(9/\varepsilon)^2$ and set

$$\mathscr{N}_\varepsilon=\{\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^*\mid \boldsymbol{U}\in\mathscr{Q}_\varepsilon \text{ and } \boldsymbol{\Sigma}\in\Lambda_\varepsilon\}.$$

Therefore, for any $\boldsymbol{X}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^*\in\mathscr{X}$, there exists some $\boldsymbol{X}_0=\boldsymbol{U}_0\boldsymbol{\Sigma}_0\boldsymbol{U}_0^*\in\mathscr{N}_\varepsilon$ such that

$$\|\boldsymbol{X}-\boldsymbol{X}_0\|_F=\|\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^*-\boldsymbol{U}_0\boldsymbol{\Sigma}_0\boldsymbol{U}_0^*\|_F\leq\varepsilon.$$

Based on (3.3) and (3.4), we have that, for any $\boldsymbol{X}\in\mathscr{Q}_\varepsilon$,

$$2^{-3/2}m(1-4\eta)\|\boldsymbol{X}\|_F^p\leq\|\mathscr{A}(\boldsymbol{X})\|_p^p\leq 2m(1+\eta)\|\boldsymbol{X}\|_F^p$$

with probability at least

$$1-2\left(\frac{9\sqrt{2}en}{\varepsilon L}\right)^{5L}\exp\left(-\frac{m\eta^2}{2cp^{1-2p}}\right),$$

where c is some positive absolute constant. Set

$$\overline{C}=\max_{\boldsymbol{X}\in\mathscr{X}}\frac{1}{m}\|\mathscr{A}(\boldsymbol{X})\|_p^p.$$

For any specific $\boldsymbol{X} \in \mathscr{X}$, there exists $\boldsymbol{X}_0 \in \mathscr{N}_\varepsilon$ satisfying $\|\boldsymbol{X}-\boldsymbol{X}_0\|_F \le \varepsilon$. As $\operatorname{rank}(\boldsymbol{X}-\boldsymbol{X}_0) \le 4$, it obtains that

$$\boldsymbol{X}-\boldsymbol{X}_0 = \boldsymbol{X}_1+\boldsymbol{X}_2$$

with $\frac{\boldsymbol{X}_1}{\|\boldsymbol{X}_1\|_F}, \frac{\boldsymbol{X}_2}{\|\boldsymbol{X}_2\|_F} \in \mathscr{X}$, and $\langle \boldsymbol{X}_1, \boldsymbol{X}_2\rangle = 0$. On one hand,

$$\begin{aligned}
\|\mathscr{A}(\boldsymbol{X})\|_p^p \le& \|\mathscr{A}(\boldsymbol{X}_0)\|_p^p + \|\mathscr{A}(\boldsymbol{X}-\boldsymbol{X}_0)\|_p^p \\
\le& \|\mathscr{A}(\boldsymbol{X}_0)\|_p^p + \|\mathscr{A}(\boldsymbol{X}_1)\|_p^p + \|\mathscr{A}(\boldsymbol{X}_1)\|_p^p \\
\le& 2m(1+\eta)+\overline{C}m(\|\boldsymbol{X}_1\|_F^p+\|\boldsymbol{X}_2\|_F^p) \le 2m(1+\eta)+2^{1-p}\overline{C}m(\|\boldsymbol{X}_1\|_F+\|\boldsymbol{X}_2\|_F)^p \\
\le& 2m(1+\eta)+2^{1-p/2}\overline{C}m\|\boldsymbol{X}_1+\boldsymbol{X}_2\|_F^p \le 2m(1+\eta)+2\varepsilon^p\overline{C}m.
\end{aligned}$$

It implies that

$$\overline{C} \le \frac{2(1+\eta)}{1-2\varepsilon^p}.$$

On the other hand, we can obtain that

$$\begin{aligned}
\|\mathscr{A}(\boldsymbol{X})\|_p^p \ge& \|\mathscr{A}(\boldsymbol{X}_0)\|_p^p - \|\mathscr{A}(\boldsymbol{X}-\boldsymbol{X}_0)\|_p^p \\
\ge& 2^{-3/2}m(1-4\eta)-2\varepsilon^p\overline{C}m \ge 2^{-3/2}m(1-4\eta)-2\frac{2(1+\eta)}{1-2\varepsilon^p}\cdot\varepsilon^p m.
\end{aligned}$$

Thus, for any $\boldsymbol{X} \in \mathscr{X}$,

$$m\left(2^{-3/2}(1-4\eta)-(1+\eta)\frac{4\varepsilon^p}{1-2\varepsilon^p}\right) \le \|\mathscr{A}(\boldsymbol{X})\|_p^p \le \left(1+\frac{\eta+2\varepsilon^p}{1-2\varepsilon^p}\right)\cdot 2m,$$

with probability at least

$$1-2\left(\frac{9\sqrt{2}en}{\varepsilon L}\right)^{5L}\exp\left(-\frac{m\eta^2}{2cp^{1-2p}}\right).$$

□

5.4. **Proof of Theorem 3.4.**

Proof. According to Theorem 3.1 and Theorem 3.3, we have

$$\overline{C} = \left(1+\frac{\eta+2\varepsilon^p}{1-2\varepsilon^p}\right)\cdot 2m \qquad \text{and} \qquad \underline{C} = m\left(2^{-3/2}(1-4\eta)-(1+\eta)\frac{4\varepsilon^p}{1-2\varepsilon^p}\right).$$

In order to meet

$$\underline{C} > \gamma\overline{C}, \qquad \text{where } \gamma = \frac{1}{a^{2-p}}+\frac{4}{a^{1-p/2}},$$

it is equivalent to obtain that

$$2^{-3/2}(1-4\eta)(1-2\varepsilon^p)-4(1+\eta)\varepsilon^p > 2\gamma(1-2\varepsilon^p)+2\gamma(\eta+2\varepsilon^p). \tag{5.15}$$

The sufficient condition of (5.15) is

$$2^{-3/2}(1-4\eta)-(4(1+\eta)\varepsilon^p+\varepsilon^p) > 2\gamma(1+\eta). \tag{5.16}$$

If

$$2^{-3/2}(1-4\eta) > 4\gamma(1+\eta) \tag{5.17}$$

and

$$4(1+\eta)\varepsilon^p+\varepsilon^p \le (2^{-3/2}(1-4\eta)-2\gamma(1+\eta))/2,$$

then inequality (5.16) holds. Observe $\frac{4}{a}+\frac{1}{a^2}\leq\gamma\leq\frac{4}{a^{1/2}}+\frac{1}{a}$. In order to meet (5.17), η can be taken as some constant that only depends on a. Besides, ε^p can be taken as

$$\varepsilon^p=\frac{2^{-3/2}(1-4\eta)-2\gamma(1+\eta)}{2(1+4(1+\eta))}:=\tau(a)<1.$$

It is enough to prove that

$$2\left(\frac{9\sqrt{2}en}{\varepsilon L}\right)^{5L}\exp\left(-\frac{m\eta^2}{2cp^{1-2p}}\right)\leq\left(\frac{k}{en}\right)^k,$$

as $\left(\frac{k}{en}\right)^k\leq 1/\binom{n}{k}$. This is equivalent to

$$m\gtrsim\frac{p^{1-2p}}{\eta^2}\cdot ak\cdot\left(\log(n/(ak))+\frac{1}{p}\log(1/\tau(a))\right)\gtrsim_a k+pk\log(n/k).$$

The second inequality above is based on $1\leq p^{-p}\leq 2$. Thus it meets the conclusion. □

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