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WEAK SEPARATION FUNCTIONS CONSTRUCTED BY GERSTEWITZ AND TOPICAL FUNCTIONS WITH APPLICATIONS IN CONJUGATE DUALITY

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Abstract. This paper aims to construct some nonlinear weak separation functions in image space analysis by virtue of the Gerstewitz and topical functions. Then, applying these separation functions, a framework of conjugate type duality for constrained vector optimization problems is introduced. The primal problem is scalarized and then the separation functions are applied to give a scalar dual problem. Meanwhile, equivalent characterizations of the zero duality gap as well as the strong duality are established via subdifferential calculus, separation properties, and saddle point assertions.

Keywords. Conjugate duality; Image space analysis; Gerstewitz function; Nonlinear weak separation function; Topical function.

1. Introduction

The Gerstewitz function was introduced in [1] by Gerstewitz as separating functional for not necessarily convex sets with the corresponding properties important for applications in vector optimization (see also Gerstewitz and Iwanow in [2]). For assertions in the context of operator theory where a functional of this type is used; see Krasnoselskiı [3] and Rubinov [4]. This function has clear geometric interpretations and abundant properties, such as continuity, sublinearity, convexity, monotonicity, which were studied in [1, 5, 6, 7], and later followed by numerous references, such as [8, 9, 10], under more general assumptions. Lipschitz properties of this function also arouse lots of interests and were widely studied; see [11, 12] and references therein. Gerth and Weidner established general separation theorems for not necessarily convex sets based on the Gerstewitz function in [5], which provides the key instrument for the proof of numerous theories for nonconvex problems. These results and generalizations of the Gerstewitz function make it a powerful tool of scalarization in vector optimization as well as related problems; see [8, 13, 14, 15, 16, 17, 18]. Another nonlinear function, providing a scalarization approach which can be regarded as a dual reformulation of the Gerstewitz function, was introduced in [19] to derive some second order optimality conditions for a general vector optimization problem. The relationships among these two scalarization functions and the so-called oriented distance function (see Zaffaroni [20]) were discussed elaborately in [21] by Crespi, Ginchev and Rocca, by Gutiérrez, Jiménez, Miglierina, and Molho in [8], and by

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Bouza, Quintana, and Tammer in [22]. Though maybe mostly known as a useful scalarization tool, the application scope of Gerstewitz function is way beyond that. In production theory, Gerstewitz type functions are introduced under the name of benefit function and shortage function; see [23, 24], while the coherent risk measure, which is a crucial concept in mathematical finance, can also be formulated by functions of this type; see [25]. Many algorithms for solving different optimization and related problems, like in [26, 27, 28, 29, 30], were constructed by means of Gerstewitz function.

It is worth mentioning that there are close relationships between the Gerstewitz function and the topical function, which is another vital tool exploited in this work. The class of topical functions is an important subclass of the class of Gerstewitz functions. Gerstewitz function carries all the basic features of a topical function, while every topical function can be enveloped by a collection of Gerstewitz functions, forming the foundation of the abstract convex structure of the topical function. Therefore, on one hand, Gerstewitz function can be seen as the support element of the topical function and studied in the context of abstract convex analysis; see [31, 32, 33]. On the other hand, topical functions can also be investigated through Gerstewitz functions. A topical function is a typical abstract convex function, introduced in [34] to model discrete event systems and studied in the scheme of abstract convex analysis in [32, 33]. Now, it has been extended to even more generalized forms with numerous applications in various fields, like optimization, economics and dynamic system; see [35, 36, 37, 38].

In this paper, we want to employ the Gerstewitz function and the topical function to construct weak separation functions, which are essential for image space analysis. Started in [39], the theory of image space analysis has attracted a lot of attention and stimulated considerable research in different kinds of areas during the past few decades. It is an approach that deals with the problems in the image space, namely the space where images of the functions involved in the given problems run, and has been proved to be a powerful tool for investigating lots of mathematical topics. This approach focuses more on the geometric features of the sets corresponding to the problem in the image space, rather than the properties of the related functions, and therefore, furnishes an effective way to study the optimization problems which might be nonconvex, nonsmooth or discontinuous. Using image space analysis, many crucial results of optimization theory, like necessary conditions and constraint qualifications, obtained by the classical way can be rebuilt in a new perception and even led to more general conclusions. Some deep connections among aspects, such as duality, gap functions, error bounds, that might be not evident from other perspective can be revealed by this approach; see [40, 41, 42]. For more details, we refer to [43, 44, 45] and the references therein.

Image space analysis provides a unified framework for studying any problems that can be perceived as the impossibility of a certain parametric system. The impossibility is expressed as the disjointness of two suitable subsets depending on the problem in the image space, which can be clarified by showing that these subsets are contained in different level sets of some separation functions. That is why the separation functions, especially regular weak separation functions, are vital for this approach. Linear separation functions were introduced in [39] by Giannessi, and then extended in [46, 47]. This kind of function is used maybe most frequently, but usually invalid when it comes to the nonconvex case. Hence, some nonlinear ones are also proposed with the help of some augmented Lagrangian functions, and nonlinear scalarization functions; see [48, 49].

The Gerstewitz function has been applied for introducing different separation functions in literatures like [50, 51]. Here, we want to further exploit the capability of the Gerstewitz function for constructing separation functions. The parameter of the collections of separation functions provided in [50] is in the image space, acting as a translation of the original function, while, for those we propose in this work, the parameter is an element in the topological interior of the ordering cone, which is a key factor for defining a Gerstewitz function. This is similar to the situation of [51]. However, the separation functions given there are only regular for investigating weakly efficient solutions, not for efficient ones. Here, besides the weak case, we shall also introduce the groups of separation functions that can be regular for efficient solutions. In addition to the Gerstewitz function, the closely related topical function also offers another way to construct weak separation functions. Starting from monotonicity properties to full topical features, we will make full use of this function in our discussion as well. Then, as application, a framework of conjugate type duality by virtue of the weak separation functions is established. We first give a general form of the dual problem and then investigate equivalent characterizations of the zero duality gap and strong duality via subdifferential calculus, separation and saddle point assertions, under certain additional assumptions.

The rest of this paper is organized as follows. In Section 2, some preliminaries are recalled, including the concepts and properties of Gerstewitz function, and topical function and separation function in image space analysis. In Section 3, we investigate the behavior of the Gerstewitz function when the element in the topological interior of the ordering cone varies. In Section 4, several collections of nonlinear weak separation functions are introduced by means of Gerstewitz and topical functions, and simultaneously, the regularities of them are discussed. In Sections 5, applying these separation functions, together with a scalarization technique, we establish a conjugate type duality framework for a general constrained vector optimization, studying the zero duality gap as well as strong duality statements. In Section 6, the last section, some conclusions are given.

2. Preliminaries

Let X, Y, Z be real Banach spaces equipped with the norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$, while K_X, K_Y and K_Z, inducing the partial order in X, Y and Z, respectively, are closed, convex and pointed cones with nonempty interior. The partial order relation on Y induced by K_Y is denoted by $\succeq_{K_Y}$:

$$y_1 \succeq_{K_Y} y_2 :\Longleftrightarrow y_1 \in y_2 + K_Y.$$

For a subset $A \subset Y$, we denote the topological interior and closure of A by $\operatorname{int} A$ and $\operatorname{cl} A$ respectively, and the closed ball with the center x and the radius ε by $B(x,\varepsilon)$.

In optimization, the following monotonicity properties of a function $\zeta : Y \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ are important.

Definition 2.1. Let $\zeta : Y \to \bar{\mathbb{R}}$.

(i) ζ is called K_Y-monotone if $y_1 \preceq_{K_Y} y_2$ implies $\zeta(y_1) \leq \zeta(y_2)$ for every $y_1, y_2 \in Y$.

(ii) ζ is called strictly $\operatorname{int} K_Y$-monotone if $y_1 \preceq_{\operatorname{int} K_Y} y_2$ implies $\zeta(y_1) < \zeta(y_2)$ for every $y_1, y_2 \in Y$.

Further monotonicity properties are introduced in Definition 4.1.

Recall that an element $\bar{y} \in A$ is said to be a maximal element of A if $(\{\bar{y}\} + K_Y) \cap A = \{\bar{y}\}$, while $\bar{y} \in A$ is said to be a weakly maximal element of A if $(\{\bar{y}\} + \operatorname{int} K_Y) \cap A = \emptyset$. The sets of all the maximal elements and all the weakly maximal elements are denoted by Max$A$ and WMax$A$, respectively.

Next, we introduce the main tools in this paper, namely, the notions of the Gerstewitz and topical functions.

Definition 2.2. ([32]) Let $e \in \operatorname{int} K_Y$. A function $\psi : Y \to \bar{\mathbb{R}}$ is called topical w.r.t. e if it has both of the following properties:

(i) K_Y-monotone, i.e., $y_1 \succeq_{K_Y} y_2 \Rightarrow \psi(y_1) \geq \psi(y_2)$, for all $y_1, y_2 \in Y$.
(ii) translation invariant w.r.t. e, i.e., $\psi(y + \lambda e) = \psi(y) + \lambda$ for all $y \in Y$ and $\lambda \in \mathbb{R}$.

Definition 2.3. ([1]) Let $e \in \operatorname{int} K_Y$ and $\omega \in Y$. The Gerstewitz function is defined as

$$\forall y \in Y: \quad \varphi_{\omega,e}(y) := \sup\{\lambda \in \mathbb{R} : \lambda e \preceq_{K_Y} y + \omega\}.$$

Another function we shall use in this paper, which has the symmetric form of $\varphi_{\omega,e}$, is the function that

$$\forall y \in Y: \quad \xi_{\omega,e}(y) := \inf\{\lambda \in \mathbb{R} : \lambda e \succeq_{K_Y} y + \omega\}. \tag{2.1}$$

Since what is mainly considered here is the case where $\omega = 0_Y$, we shall simply denote $\varphi_{0_Y,e}$ and $\xi_{0_Y,e}$ by φ_e and ξ_e, respectively. The Gerstewitz function and its symmetric form are both very useful in vector optimization and enjoy many good properties. Here we list some of them for later use.

Proposition 2.1. ([6]) *Let $\omega \in Y$ and $\varphi_{\omega,e} : Y \to \mathbb{R}$ be defined as above. Then*

(i) *$\varphi_{\omega,e}$ and $\xi_{\omega,e}$ are finite-valued and continuous.*
(ii) *$\varphi_{\omega,e}$ and $\xi_{\omega,e}$ are translation invariant w.r.t. e.*
(iii) $\{y \in Y : \varphi_{\omega,e}(y) \geq \lambda\} = \lambda e - \omega + K_Y$, $\{y \in Y : \varphi_{\omega,e}(y) > \lambda\} = \lambda e - \omega + \operatorname{int} K_Y$, $\{y \in Y : \varphi_{\omega,e}(y) = \lambda\} = \lambda e - \omega + \operatorname{bd} K_Y$, *for all* $\lambda \in \mathbb{R}$.
(iv) $\{y \in Y : \xi_{\omega,e}(y) \leq \lambda\} = \lambda e - \omega - K_Y$, $\{y \in Y : \xi_{\omega,e}(y) < \lambda\} = \lambda e - \omega - \operatorname{int} K_Y$, $\{y \in Y : \xi_{\omega,e}(y) = \lambda\} = \lambda e - \omega - \operatorname{bd} K_Y$, *for all* $\lambda \in \mathbb{R}$.
(v) $\varphi_{\omega,e}(y) = -\xi_{-\omega,e}(-y)$, *for all* $y \in Y$.
(vi) *$\xi_{\omega,e}$ is strictly* $\operatorname{int} K_Y$*-monotone, i.e.,* $y_1 \preceq_{\operatorname{int} K_Y} y_2$ *implies* $\zeta(y_1) < \zeta(y_2)$ *for every* $y_1, y_2 \in Y$.

As seen in Proposition 2.1(iii) and (iv), the Gerstewitz function is characterized by its lower/upper sublevel sets w.r.t. the level $\lambda \in \mathbb{R}$, given for a function $\zeta : Y \to \bar{\mathbb{R}}$ as

$$\operatorname{lev}_{\leq \lambda} \zeta := \{y \in Y \mid \zeta(y) \leq \lambda\},$$
$$\operatorname{lev}_{\geq \lambda} \zeta := \{y \in Y \mid \zeta(y) \geq \lambda\}$$

as well as the strict lower/upper sublevel sets

$$\operatorname{lev}_{< \lambda} \zeta := \{y \in Y \mid \zeta(y) < \lambda\},$$
$$\operatorname{lev}_{> \lambda} \zeta := \{y \in Y \mid \zeta(y) > \lambda\},$$

respectively.

The class of topical functions is a subclass of the class of Gerstewitz functions, see (iii) and (iv) in the following proposition (shown in [52, Proposition 4.1.1]):

Proposition 2.2. *Let $\zeta : Y \to \bar{\mathbb{R}}$, $e \in Y \setminus \{0\}$, and $\tilde{K} := \mathrm{lev}_{\leq 0}\,\zeta$. Then, the following conditions are equivalent to each other:*

(i) $\mathrm{lev}_{\leq \lambda}\,\zeta = \tilde{K} + \lambda e$,

(ii) $\mathrm{epi}\,\zeta = \{(y,\lambda) \in Y \times \mathbb{R} \mid y \in \tilde{K} + \lambda e\}$.

(iii) $\zeta(y + \lambda e) = \zeta(y) + \lambda$ *for all* $y \in Y$ *and* $\lambda \in \mathbb{R}$.

(iv) $\zeta(y) = \inf\{\lambda \in \mathbb{R} : y \in \lambda e + \tilde{K}\}$, *for all* $y \in Y$.

Proposition 2.3. ([53]) *Let $e \in \mathrm{int}\,K_Y$. For the ordering cone K_Y, one has $y \in K_Y \setminus \{0_Y\}$ if and only if*

$$\forall\, \varepsilon \in (0,1): \quad (1-\varepsilon)\varphi_e(y) + \varepsilon \xi_e(y) > 0.$$

Given a collection of functions $W \subseteq \{w \mid w : Y \to \bar{\mathbb{R}}\}$, a function $h : Y \to \bar{\mathbb{R}}$ is called abstract convex w.r.t. W, or W-convex if h can be enveloped by W, i.e.,

$$\forall\, y \in Y: \quad h(y) = \sup\{w(y) : w \in \mathrm{supp}(h,W)\},$$

where

$$\mathrm{supp}(h,W) := \{w \in W \mid \forall y \in Y:\; w(y) \leq h(y)\}$$

is the support set of h. The corresponding concepts of subdifferential and conjugation, called W-subdifferential (at $\bar{y} \in Y$) and W-conjugate function, are defined by

$$\partial_W h(\bar{y}) = \{w \in W \mid \forall y \in Y:\; w(y) - w(\bar{y}) \leq h(y) - h(\bar{y})\}, \tag{2.2}$$

$$h^{c(W)}(w) = \sup\,\{w(y) - h(y) : y \in Y\} \text{ for all } w \in W, \tag{2.3}$$

respectively.

As shown in [32, Theorem 3.2] by Mohebi and Samet, the topical function is a typical example of abstract convex functions, with Gerstewitz functions as its minorant elements, meaning that a function $f : Y \to \bar{\mathbb{R}}$ is topical w.r.t. e if and only if

$$f(y) = \sup\{\varphi_{\omega,e}(y) : \varphi_{\omega,e} \in \mathrm{supp}(f, W_\varphi^e)\},$$

where $W_\varphi^e = \{\varphi_{\omega,e} : \omega \in Y\}$.

Consider the following constrained vector optimization problem

$$\min\{f(x) :\; g(x) \succeq_{K_Z} 0_Z\}, \tag{VOP}$$

where $f : X \to Y$ and $g : X \to Z$. Set $\mathscr{R} := \{x \in X :\; g(x) \succeq_{K_Z} 0_Z\}$. We say that $x_0 \in \mathscr{R}$ is an efficient solution of (VOP) if $(\{f(x_0)\} - K_Y) \cap f(\mathscr{R}) = \{f(x_0)\}$. Under the assumption $\mathrm{int}\,K_Y \neq \emptyset$, we say that $x_0 \in \mathscr{R}$ is a weakly efficient solution of (VOP) if $(\{f(x_0)\} - \mathrm{int}\,K_Y) \cap f(\mathscr{R}) = \emptyset$.

For an arbitrary $\bar{x} \in X$, define the function $A_{\bar{x}} : X \to Y \times Z$ by $A_{\bar{x}}(x) := (f(\bar{x}) - f(x), g(x))$, and

$$\mathscr{K} := \{(u,v) \in Y \times Z : u = f(\bar{x}) - f(x),\; v = g(x),\; x \in X\},$$

which is actually $A_{\bar{x}}(X)$ and said to be the image of (VOP). It is a subset of space $Y \times Z$, called the image space associated with (VOP). Set

$$\mathscr{H} := \{(u,v) \in Y \times Z : u \in \mathrm{int}\,K_Y,\; v \in K_Z\}, \tag{2.4}$$

$$\mathscr{H}^o := \{(u,v) \in Y \times Z : u \in K_Y \setminus \{0_Y\},\; v \in K_Z\}. \tag{2.5}$$

Then, it is not hard to verify that, for any $\bar{x} \in \mathscr{R}$, it is a weakly efficient solution of (VOP) if and only if $\mathscr{K} \cap \mathscr{H} = \emptyset$, while it is an efficient solution of (VOP) if and only if $\mathscr{K} \cap \mathscr{H}^o = \emptyset$. Such disjointness can be conducted by the separation functions.

Definition 2.4. The class $\mathcal{W}$ of functions $w : Y \times Z \to \bar{\mathbb{R}}$ is called the class of weak separation functions w.r.t. $\mathscr{H}$ given by (2.4) (or w.r.t. $\mathscr{H}^o$ given by (2.5), respectively) if

$$\mathscr{H} \ \ (\mathscr{H}^o) \subset \operatorname{lev}_{\geq 0} w \text{ for all } w \in \mathcal{W} \quad \text{and} \quad \bigcap_{w \in \mathcal{W}} \operatorname{lev}_{>0} w \subset \mathscr{H} \ \ (\mathscr{H}^o), \tag{2.6}$$

while, it is called the class of regular weak separation functions w.r.t. $\mathscr{H}$ ($\mathscr{H}^o$, respectively) if

$$\bigcap_{w \in \mathcal{W}} \operatorname{lev}_{>0} w = \mathscr{H} \ \ (\mathscr{H}^o). \tag{2.7}$$

Here, we denote the upper level set of w w.r.t. the level 0 by

$$\operatorname{lev}_{\geq 0} w := \{(u,v) \in Y \times Z : \ w(u,v) \geq 0\},$$

and the strict upper level set of w w.r.t. the level 0 by

$$\operatorname{lev}_{>0} w := \{(u,v) \in Y \times Z : \ w(u,v) > 0\}.$$

3. Parameter in Gerstewitz Function

We want to make use of the Gerstewitz functions to introduce some nonlinear weak separation functions. From (2.3), it can be seen that the element $e \in \operatorname{int} K_Y$ is essential for obtaining useful properties (like continuity properties, see Proposition 2.1) of the Gerstewitz function, and we shall regard it as a key parameter for the construction of separation functions. Therefore, in this section, we try to figure out how the Gerstewitz function behaves with the variation of $e \in \operatorname{int} K_Y$.

Stability properties of the solution set mapping of the Gerstewitz functional are derived by Penot and Sterna-Karwat in [54, 55] and by Sterna-Karwat in [56]. The assertion in the following proposition was given in a more general form by Sterna-Karwat in [56].

Proposition 3.1. *For any $y \in Y$ fixed, $\varphi_{(\cdot)}(y) : \operatorname{int} K_Y \to \mathbb{R}$ is continuous on $\operatorname{int} K_Y$.*

Proof. First note that Proposition 2.1 (i) guarantees $\varphi_e(y) \in \mathbb{R}$ for all $e \in \operatorname{int} K_Y$ and $y \in Y$, while $\varphi_e(y)e \preceq_{K_Y} y$ as K_Y is closed. Letting $\varepsilon > 0$ be arbitrary, one has $\varphi_e(y) - \varepsilon < \varphi_e(y)$ and hence

$$(\varphi_e(y) - \varepsilon)e \preceq_{\operatorname{int} K_Y} \varphi_e(y)e \preceq_{K_Y} y.$$

Thus there exists some $\delta_1 > 0$ such that $(\varphi_e(y) - \varepsilon)(e+h) \preceq_{K_Y} y$ and $e + h \in \operatorname{int} K_Y$ for all $h \in B(0_Y, \delta_1)$, which indicates

$$\varphi_{e+h}(y) \geq \varphi_e(y) - \varepsilon, \text{ for all } h \in B(0_Y, \delta_1). \tag{3.1}$$

On the other hand, as $\varphi_e(y) < \varphi_e(y) + \varepsilon$, applying Proposition 2.1 (iii), $y \notin (\varphi_e(y) + \varepsilon)e + K_Y$. Then, we can find some $\delta_2 > 0$ satisfying

$$e + h \in \operatorname{int} K_Y \text{ and } y \notin (\varphi_e(y) + \varepsilon)(e+h) + K_Y \text{ for all } h \in B(0_Y, \delta_2).$$

Hence,

$$\varphi_{e+h}(y) < \varphi_e(y) + \varepsilon, \text{ for all } h \in B(0_Y, \delta_2). \tag{3.2}$$

This, together with (3.1), yields $|\varphi_{e+h}(y) - \varphi_e(y)| < \varepsilon$ for all $h \in B(0_Y, \delta)$, where $\delta = \min\{\delta_1, \delta_2\}$. □

Proposition 3.2. *For any $y \in Y$ fixed, one has $\varphi_{\alpha e}(y) = \frac{1}{\alpha}\varphi_e(y)$ for all $\alpha > 0$ and $e \in \operatorname{int} K_Y$.*

Proof. For arbitrary $\alpha > 0$ and $e \in \operatorname{int} K_Y$,

$$\begin{aligned}\varphi_{\alpha e}(y) &= \sup\{\lambda \in \mathbb{R} : \lambda(\alpha e) \preceq_{K_Y} y\} = \sup\{\frac{1}{\alpha}(\lambda\alpha) \in \mathbb{R} : (\lambda\alpha)e \preceq_{K_Y} y\} \\ &= \frac{1}{\alpha}\varphi_e(y).\end{aligned}$$

□

For a given $y \in Y$, we use certain monotonicity properties of $\varphi_{(\cdot)}(y)$ in order to characterize the set to which $y \in Y$ belongs in the next proposition.

Proposition 3.3. *Let $y \in Y$.*

(i) *$y \in \operatorname{int} K_Y$ if and only if*
$e_1 \preceq_{\operatorname{int} K_Y} e_2$ implies $\varphi_{e_1}(y) > \varphi_{e_2}(y)$ for any $e_1,\ e_2 \in \operatorname{int} K_Y$.

(ii) *Given some $e_1 \in \operatorname{int} K_Y$, then*
$y \in \{\lambda e_1 : \lambda > 0\}$ if and only if
$e_1 \preceq_{K_Y \setminus \{0_Y\}} e_2$ implies $\varphi_{e_1}(y) > \varphi_{e_2}(y)$ for any $e_2 \in \operatorname{int} K_Y$.

(iii) *$y \in \operatorname{bd} K_Y$ if and only if*
$\varphi_e(y) = 0$ for every $e \in \operatorname{int} K_Y$.

(iv) *$y \notin K_Y$ if and only if*
$e_1 \preceq_{\operatorname{int} K_Y} e_2$ implies $\varphi_{e_1}(y) < \varphi_{e_2}(y)$ for any $e_1,\ e_2 \in \operatorname{int} K_Y$.

(v) *Given some $e_2 \in \operatorname{int} K_Y$, then*
$y \in \{\lambda e_2 : \lambda < 0\}$ if and only if
$e_1 \preceq_{K_Y \setminus \{0_Y\}} e_2$ implies $\varphi_{e_1}(y) < \varphi_{e_2}(y)$ for any $e_1 \in \operatorname{int} K_Y$.

Proof. (i) Let $y \in \operatorname{int} K_Y$ and pick any $e_1,\ e_2 \in \operatorname{int} K_Y$ such that $e_1 \preceq_{\operatorname{int} K_Y} e_2$. Setting $\lambda_1 = \varphi_{e_1}(y)$ and $\lambda_2 = \varphi_{e_2}(y)$, it follows from Proposition 2.1 (iii) that $\lambda_1 > 0,\ \lambda_2 > 0$, and

$$y = \lambda_1 e_1 + q_1 = \lambda_2 e_2 + q_2$$

for some $q_1,\ q_2 \in \operatorname{bd} K_Y$. Supposing $\lambda_1 \le \lambda_2$, one has

$$q_1 = \lambda_2 e_2 - \lambda_1 e_1 + q_2 \succeq_{K_Y} \lambda_1(e_2 - e_1) + q_2 \succeq_{\operatorname{int} K_Y} 0_Y,$$

contradicting that $q_1 \in \operatorname{bd} K_Y$. Therefore, $\lambda_1 > \lambda_2$, i.e., $\varphi_{e_1}(y) > \varphi_{e_2}(y)$.

Conversely, pick some $e_1 \in \operatorname{int} K_Y$ and let $e_2 = 2e_1$. Then there exists $\varphi_{e_1}(y) > \varphi_{e_2}(y)$ due to $e_1 \preceq_{\operatorname{int} K_Y} 2e_1$, which means, according to Proposition 3.2, $\varphi_{e_1}(y) > \frac{1}{2}\varphi_{e_1}(y)$. This demonstrates that $\varphi_{e_1}(y) > 0$. Hence, with Proposition 2.1 (iii), we can conclude $y \in \operatorname{int} K_Y$.

(ii) Assume that $y = \lambda_1 e_1$ for some $\lambda_1 > 0$ and $e_2 \in \operatorname{int} K_Y$ satisfying $e_1 \preceq_{K_Y \setminus \{0_Y\}} e_2$. Applying Proposition 2.1 (iii), y can also be expressed as $y = \lambda_2 e_2 + q_2$, where $\lambda_2 = \varphi_{e_2}(y) > 0$ and $q_2 \in \operatorname{bd} K_Y$. Supposing $\lambda_1 \le \lambda_2$, one has that $\lambda_1 e_1 = y = \lambda_2 e_2 + q_2$ gives

$$q_2 = \lambda_1 e_1 - \lambda_2 e_2 \preceq_{K_Y} \lambda_2(e_1 - e_2),$$

which is a contradiction. Hence, $\lambda_1 > \lambda_2$, i.e., $\varphi_{e_1}(y) > \varphi_{e_2}(y)$.

Conversely, first according to (i), $y \in \operatorname{int} K_Y$, meaning that $\varphi_e(y) > 0$ for any $e \in \operatorname{int} K_Y$. Supposing $y \notin \{\lambda e_1 : \lambda > 0\}$, one has

$$y = \varphi_{e_1}(y)e_1 + q_1 = \varphi_{e_1}(y)(e_1 + \frac{1}{\varphi_{e_1}(y)}q_1) \tag{3.3}$$

for some $q_1 \in \operatorname{bd} K_Y \setminus \{0_Y\}$. Letting $e_2 = e_1 + \frac{1}{\varphi_{e_1}(y)} q_1$, one has $e_2 \succeq_{K_Y \setminus \{0_Y\}} e_1$, while (3.3) shows $\varphi_{e_1}(y) = \varphi_{e_2}(y)$, which is a contradiction. Thus $y \in \{\lambda e_1 : \lambda > 0\}$.

(iii) Let $y \in \operatorname{bd} K_Y$, which means $y \in 0e + \operatorname{bd} K_Y$ for all $e \in \operatorname{int} K_Y$. According to Proposition 2.1 (iii), this is equivalent to $\varphi_e(y) = 0$ for all $e \in \operatorname{int} K_Y$.

(iv) and (v) can be proved in similar ways as in (i) and (ii), respectively. □

In addition to the monotonicity, there are some other features of $\varphi_{(\cdot)}(y)$ depending on y, and they can be divided into three cases that $y \in \operatorname{int} K_Y$, $y \in \operatorname{bd} K_Y$ or $y \notin K_Y$.

Proposition 3.4. *Let $y \in Y$ and $e_1,\ e_2 \in \operatorname{int} K_Y$.*

(i) $\varphi_{e_1+e_2}(y) < \varphi_{e_1}(y) + \varphi_{e_2}(y)$ *if* $y \in \operatorname{int} K_Y$.

(ii) $\varphi_{e_1+e_2}(y) = \varphi_{e_1}(y) + \varphi_{e_2}(y)$ *if* $y \in \operatorname{bd} K_Y$.

(iii) $\varphi_{e_1+e_2}(y) > \varphi_{e_1}(y) + \varphi_{e_2}(y)$ *if* $y \notin K_Y$.

Proof. (i) Supposing $y \in \operatorname{int} K_Y$, according to Proposition 3.3 (i), one has $e_1 \preceq_{\operatorname{int} K_Y} e_1 + e_2$ implies $\varphi_{e_1+e_2}(y) < \varphi_{e_1}(y)$. Meanwhile, $y \in \operatorname{int} K_Y$ guarantees $\varphi_{e_2}(y) > 0$. Thus $\varphi_{e_1+e_2}(y) < \varphi_{e_1}(y) + \varphi_{e_2}(y)$.

(ii) It straightforwardly follows from Proposition 3.3 (iii).

(iii) Supposing $y \notin K_Y$, according to Proposition 3.3 (iv), one has that $e_1 \preceq_{\operatorname{int} K_Y} e_1 + e_2$ implies $\varphi_{e_1+e_2}(y) > \varphi_{e_1}(y)$. Meanwhile, $y \notin K_Y$ guarantees $\varphi_{e_2}(y) < 0$, which leads to $\varphi_{e_1+e_2}(y) > \varphi_{e_1}(y) + \varphi_{e_2}(y)$. □

Proposition 3.5. *Consider $y \in Y$. Then, if $y \notin \operatorname{int} K_Y$, then $\varphi_{(\cdot)}(y) :\to \mathbb{R}$ is a concave function. Particularly, if $y \in \operatorname{bd} K_Y$, then*

$$\varphi_{\alpha e_1 + (1-\alpha) e_2}(y) = 0 = \alpha \varphi_{e_1}(y) + (1-\alpha)\varphi_{e_2}(y), \textit{ for all } e_1, e_2 \in \operatorname{int} K_Y,\ \alpha \in [0,1].$$

Proof. Suppose $y \notin K_Y$ and

$$y = -\lambda_1 e_1 + q_1,\ y = -\lambda_2 e_2 + q_2,$$

where $\lambda_1, \lambda_2 > 0$, $q_1,\ q_2 \in \operatorname{bd} K_Y$. It follows that

$$e_1 = \frac{1}{\lambda_1}(-y + q_1),\ e_2 = \frac{1}{\lambda_2}(-y + q_2).$$

Then

$$\begin{aligned} \alpha e_1 + (1-\alpha) e_2 &= \frac{\alpha}{\lambda_1}(-y+q_1) + \frac{1-\alpha}{\lambda_2}(-y+q_2) \\ &= -\frac{\alpha\lambda_2 + (1-\alpha)\lambda_1}{\lambda_1\lambda_2} y + \left(\frac{\alpha}{\lambda_1} q_1 + \frac{1-\alpha}{\lambda_2} q_2\right), \end{aligned}$$

and

$$y = -\frac{\lambda_1\lambda_2}{\alpha\lambda_2 + (1-\alpha)\lambda_1}(\alpha e_1 + (1-\alpha) e_2) + \frac{\lambda_1\lambda_2}{\alpha\lambda_2 + (1-\alpha)\lambda_1}\left(\frac{\alpha}{\lambda_1} q_1 + \frac{1-\alpha}{\lambda_2} q_2\right),$$

which imply, according to Proposition 2.1 (iii), that

$$\begin{aligned} \varphi_{\alpha e_1 + (1-\alpha) e_2}(y) &\geq -\frac{\lambda_1\lambda_2}{\alpha\lambda_2 + (1-\alpha)\lambda_1} \\ &\geq -\alpha\lambda_1 - (1-\alpha)\lambda_2 \\ &= \alpha\varphi_{e_1}(y) + (1-\alpha)\varphi_{e_2}(y). \end{aligned}$$

Hence, for any $y \notin K_Y$,

$$\varphi_{\alpha e_1+(1-\alpha)e_2}(y) \geq \alpha\varphi_{e_1}(y)+(1-\alpha)\varphi_{e_2}(y), \text{ for all } e_1,\ e_2 \in \operatorname{int} K_Y,\ \alpha \in [0,1].$$

If $y \in \operatorname{bd} K_Y$, then $\varphi_e(y) = 0$ for any $e \in \operatorname{int} K_Y$. Hence,

$$\varphi_{\alpha e_1+(1-\alpha)e_2}(y) = 0 = \alpha\varphi_{e_1}(y)+(1-\alpha)\varphi_{e_2}(y), \text{ for all } e_1,\ e_2 \in \operatorname{int} K_Y,\ \alpha \in [0,1].$$

□

For the case that $y \in \operatorname{int} K_Y$, we can always find an example with $\varphi_{\alpha e_1+(1-\alpha)e_2}(y) > \alpha\varphi_{e_1}(y)+(1-\alpha)\varphi_{e_2}(y)$ and also an example with

$$\varphi_{\alpha e_1+(1-\alpha)e_2}(y) < \alpha\varphi_{e_1}(y)+(1-\alpha)\varphi_{e_2}(y).$$

Example 3.1. Let $y \in \operatorname{int} K_Y$, $\alpha = \frac{1}{2}$, $e_1 = \frac{1}{2}y$, and $e_2 = y$. Then $\varphi_{e_1}(y) = 2$, $\varphi_{e_2}(y) = 1$. Therefore,

$$\alpha\varphi_{e_1}(y)+(1-\alpha)\varphi_{e_2}(y) = \frac{1}{2}\cdot 2+\frac{1}{2}\cdot 1 = \frac{3}{2},\ \varphi_{\alpha e_1+(1-\alpha)e_2}(y) = \varphi_{\frac{3}{4}y}(y) = \frac{4}{3},$$

satisfying that

$$\varphi_{\alpha e_1+(1-\alpha)e_2}(y) < \alpha\varphi_{e_1}(y)+(1-\alpha)\varphi_{e_2}(y).$$

On the other hand, we can choose some $\alpha \in (\frac{1}{2},1)$ and then pick some $q_1 \in \operatorname{bd} K_Y$ and $e \in \operatorname{int} K_Y$ such that $y-q_1 \in \operatorname{int} K_Y$ and $y-\xi_e(q_1)e \in \operatorname{int} K_Y$. Considering $\xi_e(q_1)$, one sees that there exists some $q_2 \in \operatorname{bd} K_Y$ with $q_2 = \xi_e(q_1)e-q_1$. Let $e_1 = y-q_1$, $e_2 = y-q_2$. Then,

$$\alpha e_1+(1-\alpha)e_2 = y+(1-2\alpha)q_1-(1-\alpha)\xi_e(q_1)e \preceq_{\operatorname{int} K_Y} y,$$

which implies that

$$\varphi_{\alpha e_1+(1-\alpha)e_2}(y) > 1.$$

Meanwhile,

$$\alpha\varphi_{e_1}(y)+(1-\alpha)\varphi_{e_2}(y) = \alpha\varphi_{y-q_1}(y)+(1-\alpha)\varphi_{y-q_2}(y) = 1.$$

4. Weak Separation Functions

In this section, we introduce the collections $\mathcal{W}^{\varphi}_{\Pi_1},\dots,\mathcal{W}^{\varphi}_{\Pi_4}$ of nonlinear weak separation functions by virtue of the Gerstewitz function as well as the collections $\mathcal{W}_1,\dots,\mathcal{W}_5$ of nonlinear weak separation functions by virtue of the topical function. Furthermore, we investigate their properties and study the relationships between them.

Consider the Banach spaces Y and Z with the corresponding ordering cones K_Y and K_Z. Pick $e_Y \in \operatorname{int} K_Y$ and $e_Z \in \operatorname{int} K_Z$, $\alpha > 0$, $\beta \geq 0$, and consider the functions $\varphi_{\alpha e_Y}(\cdot) : Y \to \mathbb{R}$ and $\varphi_{\beta e_Z}(\cdot) : Z \to \mathbb{R}$; see Definition 2.3. Furthermore, define the function φ_{0_Z} on Z by

$$\varphi_{0_Z}(z) := 0 \text{ for all } z \in Z.$$

We now define a collection of functions involving the Gerstewitz function introduced in Definition 2.3 with fixed elements $e_Y \in \operatorname{int} K_Y$ and $e_Z \in \operatorname{int} K_Z$ by

$$\mathcal{W}^{\varphi}_{\Pi_1} := \{\varphi_{\alpha e_Y}(\cdot)+\varphi_{\beta e_Z}(\cdot) : Y \times Z \to \mathbb{R} \mid \alpha > 0,\ \beta \geq 0\}, \tag{4.1}$$

where

$$\Pi_1 := \{(\alpha,\beta) \in \mathbb{R}^2 :\ \alpha > 0,\ \beta \geq 0\}. \tag{4.2}$$

In $\mathcal{W}^{\varphi}_{\Pi_1}$, the parameters α and β are given by Π_1 and the elements used to define the Gerstewitz functions range along e_Y and e_Z, respectively. Note that the Gerstewitz function is defined by any element in the interior of the ordering cone, and for every $(e_Y, e_Z) \in \operatorname{int} K_Y \times \operatorname{int} K_Z$, the functionals φ_{e_Y} and φ_{e_Z} enjoy the properties given in Proposition 2.1.

Let e_Y and e_Z be ranged over the entire $\operatorname{int} K_Y$ and $\operatorname{int} K_Z$, which gives the following collection of functions, involving the Gerstewitz function introduced in Definition 2.3:

$$\mathcal{W}^{\varphi}_{\Pi_2} := \Big\{ \varphi_{e_Y}(\cdot) + \varphi_{e_Z}(\cdot) : Y \times Z \to \mathbb{R} \mid e_Y \in \operatorname{int} K_Y, \quad e_Z \in (\operatorname{int} K_Z \cup \{0_Z\}) \Big\}, \tag{4.3}$$

where

$$\Pi_2 := \operatorname{int} K_Y \times (\operatorname{int} K_Z \cup \{0_Z\}). \tag{4.4}$$

This class of separation functions is actually the class that was proposed in [51].

In the following proposition, we show that collections $\mathcal{W}^{\varphi}_{\Pi_1}$ and $\mathcal{W}^{\varphi}_{\Pi_2}$ are classes of regular weak separation functions w.r.t. $\mathscr{H}$ ($\mathscr{H}$ is given by (2.4)) in the sense of Definition 2.4.

Proposition 4.1. *The collections $\mathcal{W}^{\varphi}_{\Pi_1}$ and $\mathcal{W}^{\varphi}_{\Pi_2}$ are classes of regular weak separation functions w.r.t. $\mathscr{H}$, i.e.,*

$$\bigcap_{\alpha>0,\beta\geq 0} \operatorname{lev}_{>0}(\varphi_{\alpha e_Y}(\cdot) + \varphi_{\beta e_Z}(\cdot)) = \mathscr{H},$$

and

$$\bigcap_{\substack{e_Y \in \operatorname{int} K_Y, \\ e_Z \in (\operatorname{int} K_Z \cup \{0_Z\})}} \operatorname{lev}_{>0}(\varphi_{e_Y}(\cdot) + \varphi_{e_Z}(\cdot)) = \mathscr{H}.$$

Proof. Consider $\mathcal{W}^{\varphi}_{\Pi_1}$ and arbitrary elements $\alpha > 0$, $\beta \geq 0$, and $(u,v) \in \mathscr{H}$. We apply Proposition 2.1 and obtain $\varphi_{\alpha e_Y}(u) > 0$, and $\varphi_{\beta e_Z}(v) \geq 0$. Therefore, $\varphi_{\alpha e_Y}(u) + \varphi_{\beta e_Z}(v) > 0$.

Conversely, suppose $(u,v) \in \bigcap_{\alpha>0,\beta\geq 0} \operatorname{lev}_{>0}(\varphi_{\alpha e_Y}(\cdot) + \varphi_{\beta e_Z}(\cdot))$. Particularly, setting $\alpha = 1$ and $\beta = 0$, we have $\varphi_{e_Y}(u) > 0$, which, according to Proposition 2.1, indicates that $u \in \operatorname{int} K_Y$. If $v \notin K_Z$. From Proposition 2.1, one has $\varphi_{e_Z}(v) < 0$. Using Proposition 3.2, one has

$$\varphi_{\alpha e_Y}(u) + \varphi_{e_Z}(v) = \frac{1}{\alpha}\varphi_{e_Y}(u) + \varphi_{e_Z}(v) \to \varphi_{e_Z}(v),$$

as $\alpha \to +\infty$, which contradicts $(u,v) \in \bigcap_{\alpha>0,\beta\geq 0} \operatorname{lev}_{>0}(\varphi_{\alpha e_Y}(\cdot) + \varphi_{\beta e_Z}(\cdot))$. Hence, $v \in K_Z$. As for $\mathcal{W}^{\varphi}_{\Pi_2}$, it can be proved similarly. □

With regard to $\mathscr{H}^o$, in order to propose the class of separation functions that is regular w.r.t. $\mathscr{H}^o$, we involve a further function $\xi_{\alpha e_Y} : Y \to \mathbb{R}$ (see (2.1)) in the definition of the following collection $\mathcal{W}^{\varphi}_{\Pi_3}$. Set

$$\mathcal{W}^{\varphi}_{\Pi_3} := \Big\{ (\varepsilon \varphi_{\alpha e_Y} + (1-\varepsilon)\xi_{\alpha e_Y})(\cdot) + \varphi_{\beta e_Z}(\cdot) : Y \times Z \to \mathbb{R} \mid \quad \varepsilon \in (0,1), \alpha > 0, \ \beta \geq 0 \Big\}. \tag{4.5}$$

The parameter set for $\mathcal{W}^{\varphi}_{\Pi_3}$ is

$$\Pi_3 := \{(\varepsilon, \alpha, \beta) \in \mathbb{R}^3 : \varepsilon \in (0,1), \alpha > 0, \ \beta \geq 0\}. \tag{4.6}$$

Elements for defining the Gerstewitz functions range along e_Y and e_Z, respectively. Similarly, we can also consider the situation where e_Y and e_Z range over the entire $\operatorname{int} K_Y$ and $\operatorname{int} K_Z$, i.e.,

$$\mathcal{W}^{\varphi}_{\Pi_4} := \Big\{ (\varepsilon\varphi_{e_Y} + (1-\varepsilon)\xi_{e_Y})(\cdot) + \varphi_{e_Z}(\cdot) : Y \times Z \to \mathbb{R} \mid \varepsilon \in (0,1), e_Y \in \operatorname{int} K_Y, e_Z \in (\operatorname{int} K_Z \cup \{0_Z\}) \Big\} \tag{4.7}$$

with the parameter set

$$\Pi_4 := \{\varepsilon \in (0,1)\}. \tag{4.8}$$

In the following proposition, we show that the collections $\mathcal{W}^{\varphi}_{\Pi_3}$ and $\mathcal{W}^{\varphi}_{\Pi_4}$ are classes of regular weak separation functions w.r.t. $\mathscr{H}^o$ in the sense of Definition 2.4, where $\mathscr{H}^o$ is given by (2.5).

Proposition 4.2. *The collections $\mathcal{W}^{\varphi}_{\Pi_3}$ and $\mathcal{W}^{\varphi}_{\Pi_4}$ are classes of regular weak separation functions w.r.t. $\mathscr{H}^o$, i.e.,*

$$\bigcap_{\alpha>0,\beta\geq 0,\varepsilon\in(0,1)} \operatorname*{lev}_{>0}((\varepsilon\varphi_{\alpha e_Y} + (1-\varepsilon)\xi_{\alpha e_Y})(\cdot) + \varphi_{\beta e_Z}(\cdot)) = \mathscr{H}^o,$$

and

$$\bigcap_{\substack{e_Y\in\operatorname{int} K_Y,\\ e_Z\in(\operatorname{int} K_Z\cup\{0_Z\}),\\ \varepsilon\in(0,1)}} \operatorname*{lev}_{>0}((\varepsilon\varphi_{e_Y} + (1-\varepsilon)\xi_{e_Y})(\cdot) + \varphi_{e_Z}(\cdot)) = \mathscr{H}^o.$$

Proof. Considering $\mathcal{W}^{\varphi}_{\Pi_3}$, we pick arbitrary $\varepsilon \in (0,1)$, $\alpha > 0$, $\beta \geq 0$, and $(u,v) \in \mathscr{H}^o$. According to Propositions 2.1 and 2.3, one has $(\varepsilon\varphi_{\alpha e_Y} + (1-\varepsilon)\xi_{\alpha e_Y})(u) > 0$ and $\varphi_{\beta e_Z}(v) \geq 0$. Therefore,

$$(\varepsilon\varphi_{\alpha e_Y} + (1-\varepsilon)\xi_{\alpha e_Y})(u) + \varphi_{\beta e_Z}(v) > 0.$$

Conversely, we assume

$$(u,v) \in \bigcap_{\alpha>0,\beta\geq 0,\varepsilon\in(0,1)} \operatorname*{lev}_{>0}((\varepsilon\varphi_{\alpha e_Y} + (1-\varepsilon)\xi_{\alpha e_Y})(\cdot) + \varphi_{\beta e_Z}(\cdot)). \tag{4.9}$$

Particularly, setting $\alpha = 1$ and $\beta = 0$, we have $(\varepsilon\varphi_{e_Y} + (1-\varepsilon)\xi_{e_Y})(u) > 0$ for every $\varepsilon \in (0,1)$. It follows from Proposition 2.3 that $u \in K_Y \setminus \{0_Y\}$. Supposing $v \notin K_Z$, we find from Proposition 2.1 that $\varphi_{e_Z}(v) < 0$. Using Proposition 3.2, we obtain

$$(\varepsilon\varphi_{\alpha e_Y} + (1-\varepsilon)\xi_{\alpha e_Y})(u) + \varphi_{e_Z}(v) = \frac{1}{\alpha}(\varepsilon\varphi_{e_Y} + (1-\varepsilon)\xi_{e_Y})(u) + \varphi_{e_Z}(v) \to \varphi_{e_Z}(v),$$

as $\alpha \to +\infty$, which contradicts (4.9). Hence, $v \in K_Z$.

For $\mathcal{W}^{\varphi}_{\Pi_4}$, it can be proved similarly. □

Next, we compute some infinmums of the collections of functions introduced in (4.1), (4.3), (4.5), and (4.7). These values can be used to detect whether $(u,v) \in K_Y \times K_Z$ or not, and moreover, they are important conditions for us to investigate the duality theory we shall build in Section 5.

Proposition 4.3. *For the collection $\mathcal{W}^{\varphi}_{\Pi_1}$ in (4.1), one has*

$$\inf_{\beta\geq 0}\{\varphi_{\bar{\alpha}e_Y}(u) + \varphi_{\beta e_Z}(v)\} = \begin{cases} \varphi_{\bar{\alpha}e_Y}(u) & \text{if } v \in K_Z, \\ -\infty & \text{if } v \notin K_Z, \end{cases}$$

for every $\bar{\alpha} > 0$ fixed, while,

$$\inf_{\alpha>0,\beta\geq 0}\{\varphi_{\alpha e_Y}(u)+\varphi_{\beta e_Z}(v)\} = \begin{cases} 0 & \text{if } u \in K_Y \text{ and } v \in K_Z, \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. Let $\bar{\alpha} > 0$ be arbitrary. If $v \in K_Z$, then, by Proposition 2.1, $\varphi_{\beta e_Z}(v) \geq 0$ for any $\beta \geq 0$. Since $\varphi_{0_Z}(v) = 0$, we can deduce $\inf_{\beta\geq 0}\{\varphi_{\bar{\alpha} e_Y}(u)+\varphi_{\beta e_Z}(v)\} = \varphi_{\bar{\alpha} e_Y}(u)$. If $v \notin K_Z$, then, also by Proposition 2.1, $\varphi_{e_Z}(v) < 0$. Then, it follows from Proposition 3.2 that

$$\varphi_{\beta e_Z}(v) = \frac{1}{\beta}\varphi_{e_Z}(v) \to -\infty$$

as $\beta \to 0$. Hence, $\inf_{\beta\geq 0}\{\varphi_{\bar{\alpha} e_Y}(u)+\varphi_{\beta e_Z}(v)\} = -\infty$.

If $u \in K_Y$ and $v \in K_Z$, then, by Proposition 2.1, one has $\varphi_{\alpha e_Y}(u) \geq 0$ and $\varphi_{\beta e_Z}(v) \geq 0$ for any $\alpha > 0$ and $\beta \geq 0$. Note that $\varphi_{0_Z}(v) = 0$. By Proposition 3.2, one has

$$\varphi_{\alpha e_Y}(u) = \frac{1}{\alpha}\varphi_{e_Y}(u) \to 0$$

as $\alpha \to +\infty$, which implies $\inf_{\alpha>0,\beta\geq 0}\{\varphi_{\alpha e_Y}(u)+\varphi_{\beta e_Z}(v)\} = 0$.

If $u \notin K_Y$, then, applying Proposition 2.1, one obtains $\varphi_{e_Y}(u) < 0$. Meanwhile, by Proposition 3.2, one has

$$\varphi_{\alpha e_Y}(u) = \frac{1}{\alpha}\varphi_{e_Y}(u) \to -\infty$$

as $\alpha \to 0$. Hence, $\inf_{\alpha>0,\beta\geq 0}\{\varphi_{\alpha e_Y}(u)+\varphi_{\beta e_Z}(v)\} = -\infty$. Similarly, we can deduce that

$$\inf_{\alpha>0,\beta\geq 0}\{\varphi_{\alpha e_Y}(u)+\varphi_{\beta e_Z}(v)\} = -\infty$$

if $v \notin K_Z$. □

With a similar argument, we can also obtain the following results concerning the collection $\mathcal{W}^{\varphi}_{\Pi_2}$ given by (4.3).

Proposition 4.4. *Let $(u,v) \in Y \times Z$.*

(i) *For the collection $\mathcal{W}^{\varphi}_{\Pi_2}$, one has*

$$\inf_{e_Z\in(\operatorname{int} K_Z\cup\{0_Z\})}\{\varphi_{\bar{e}_Y}(u)+\varphi_{e_Z}(v)\} = \begin{cases} \varphi_{\bar{e}_Y}(u) & \text{if } v \in K_Z, \\ -\infty & \text{if } v \notin K_Z, \end{cases}$$

for every $\bar{e}_Y \in \operatorname{int} K_Y$ fixed, while,

$$\inf_{\substack{e_Y\in\operatorname{int} K_Y,\\ e_Z\in(\operatorname{int} K_Z\cup\{0_Z\})}}\{\varphi_{e_Y}(u)+\varphi_{e_Z}(v)\} = \begin{cases} 0 & \text{if } u \in K_Y \text{ and } v \in K_Z, \\ -\infty & \text{otherwise.} \end{cases}$$

(ii) *For the collection $\mathcal{W}^{\varphi}_{\Pi_3}$, one has*

$$\begin{aligned} &\inf_{\beta\geq 0}\{(\bar{\varepsilon}\varphi_{\bar{\alpha} e_Y}+(1-\bar{\varepsilon})\xi_{\bar{\alpha} e_Y})(u)+\varphi_{\beta e_Z}(v)\} \\ &= \begin{cases} (\bar{\varepsilon}\varphi_{\bar{\alpha} e_Y}+(1-\bar{\varepsilon})\xi_{\bar{\alpha} e_Y})(u) & \text{if } v \in K_Z, \\ -\infty & \text{if } v \notin K_Z, \end{cases} \end{aligned}$$

for every $\bar{\alpha} > 0$ and $\bar{\varepsilon} \in (0,1)$ fixed, while,

$$\inf_{\substack{\alpha>0,\beta\geq 0,\\ \varepsilon\in(0,1)}} \{(\varepsilon\varphi_{\alpha e_Y} + (1-\varepsilon)\xi_{\alpha e_Y})(u) + \varphi_{\beta e_Z}(v)\} = \begin{cases} 0 & \text{if} u \in K_Y \text{ and } v \in K_Z, \\ -\infty & \text{otherwise.} \end{cases}$$

(iii) *For the collection $\mathcal{W}^{\varphi}_{\Pi_4}$, one has*

$$\inf_{e_Z\in(\operatorname{int} K_Z\cup\{0_Z\})} \{(\bar{\varepsilon}\varphi_{\bar{e}_Y} + (1-\bar{\varepsilon})\xi_{\bar{e}_Y})(u) + \varphi_{e_Z}(v)\} = \begin{cases} (\bar{\varepsilon}\varphi_{\bar{e}_Y} + (1-\bar{\varepsilon})\xi_{\bar{e}_Y})(u) & \text{if } v \in K_Z, \\ -\infty & \text{if } v \notin K_Z, \end{cases}$$

for every $\bar{e}_Y \in \operatorname{int} K_Y$ and $\bar{\varepsilon} \in (0,1)$ fixed, while,

$$\inf_{\substack{e_Y\in\operatorname{int} K_Y,\\ e_Z\in(\operatorname{int} K_Z\cup\{0_Z\}),\\ \varepsilon\in(0,1)}} \{(\varepsilon\varphi_{e_Y} + (1-\varepsilon)\xi_{e_Y})(u) + \varphi_{e_Z}(v)\} = \begin{cases} 0 & \text{if } u \in K_Y \text{ and } v \in K_Z, \\ -\infty & \text{otherwise.} \end{cases}$$

The following classes of separation functions come out from the topical function. Here, we consider some further monotonicity properties (compare Definition 2.1).

Definition 4.1. Let $\zeta : Y \to \bar{\mathbb{R}}$ and $\bar{y} \in Y$.

(i) ζ is called strictly $\operatorname{int} K_Y$-monotone at $\bar{y}$ if, for every $y \in Y$, $\bar{y} \preceq_{\operatorname{int} K_Y} y$ implies $\zeta(\bar{y}) < \zeta(y)$, and $y \preceq_{\operatorname{int} K_Y} \bar{y}$ implies $\zeta(y) < \zeta(\bar{y})$.
(ii) ζ is called strongly K_Y-monotone at $\bar{y}$ if, for every $y \in Y$, $\bar{y} \preceq_{K_Y\setminus\{0_Y\}} y$ implies $\zeta(\bar{y}) < \zeta(y)$, and $y \preceq_{K_Y\setminus\{0_Y\}} \bar{y}$ implies $\zeta(y) < \zeta(\bar{y})$.

Furthermore, we introduce collections of functions depending from certain monotonicity properties and the translation invariance as supposed in the definition of topical functions (see Definition 2.2). This means that these collections are related to an approach using abstract convex functions and will be useful for deriving duality statements in Section 5.

Pick some $e_Y \in \operatorname{int} K_Y$ and $e_Z \in \operatorname{int} K_Z$. In the following construction, elements e_Y and e_Z can be fixed, unlike the situation above, where e_Y and e_Z range as parameters. Define the following collections of functions

$$\begin{aligned} \mathcal{W}_1 := \{\zeta : Y \times Z \to \bar{\mathbb{R}} : \quad & (i) \quad \zeta(\cdot, v) \text{ is } K_Y\text{-monotone for every } v \in Z \text{ fixed,} \\ & (ii) \quad \zeta(u, \cdot) \text{ is } K_Z\text{-monotone for every } u \in Y \text{ fixed,} \\ & (iii) \quad \zeta(0_Y, 0_Z) = 0.\} \end{aligned}$$

and

$$
\begin{aligned}
\mathcal{W}_2 := \{\zeta : Y \times Z \to \bar{\mathbb{R}} : \quad & (i) \quad \zeta(\cdot,\cdot) \text{ is } (K_Y \times K_Z)\text{-monotone},\\
& (ii) \quad \zeta(\cdot, v) \text{ is topical w.r.t. } e_Y \text{ for every } v \in Z \text{ fixed},\\
& \qquad \text{or,}\\
& \qquad \zeta(u,\cdot) \text{ is topical w.r.t. } e_Z \text{ for every } u \in Y \text{ fixed},\\
& (iii) \quad \zeta(0_Y, 0_Z) = 0\}.
\end{aligned}
$$

Set

$$
\begin{aligned}
\mathcal{W}_3 := \{\zeta : Y \times Z \to \bar{\mathbb{R}} : \quad & (i) \quad \zeta(\cdot, v) \text{ is topical w.r.t. } e_Y \text{ for every } v \in Z \text{ fixed},\\
& (ii) \quad \zeta(u,\cdot) \text{ is } K_Z\text{-monotone for every } u \in Y \text{ fixed},\\
& (iii) \quad \zeta(0_Y, 0_Z) = 0.\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{W}_4 := \{\zeta : Y \times Z \to \bar{\mathbb{R}} : \quad & (i) \quad \zeta(u, 0_Z) > 0 \text{ for every } u \in K_Y \setminus \{0_Y\},\\
& (ii) \quad \zeta(u,\cdot) \text{ is } K_Z\text{-monotone for every } u \in Y \text{ fixed},\\
& (iii) \quad \zeta(0_Y, 0_Z) = 0.\},
\end{aligned}
$$

while,

$$
\begin{aligned}
\mathcal{W}_5 := \{\zeta : Y \times Z \to \bar{\mathbb{R}} : \quad & (i) \quad \zeta(\cdot, v) \text{ is topical w.r.t. } e_Y, \text{ and strongly}\\
& \qquad K_Y\text{-monotone}\\
& \qquad \text{at any } \bar{u} \in \{\lambda e_Y : \lambda \in \mathbb{R}\} \text{ for every } v \in Z \text{ fixed},\\
& (ii) \quad \zeta(u,\cdot) \text{ is } K_Z \text{ -monotone, for every } u \in Y \text{ fixed},\\
& (iii) \quad \zeta(0_Y, 0_Z) = 0.\}.
\end{aligned}
$$

In the following propositions, (taking into account the notions introduced in Definition 2.4), we claim that $\mathcal{W}_1$ and $\mathcal{W}_2$ are collections of weak separation functions w.r.t. both $\mathscr{H}$ (given by (2.4)) and $\mathscr{H}^o$ (given by (2.5)). $\mathcal{W}_3$ is a collection of regular weak separation functions w.r.t. $\mathscr{H}$, but only a weak one w.r.t. $\mathscr{H}^o$, while, $\mathcal{W}_4$ and $\mathcal{W}_5$ are both collections of regular weak separation functions w.r.t. $\mathscr{H}^o$.

Proposition 4.5. *The collection $\mathcal{W}_1$ and $\mathcal{W}_2$ are classes of weak separation functions w.r.t. both $\mathscr{H}$ and $\mathscr{H}^o$, i.e.,*

$$
\forall \zeta \in \mathcal{W}_i : \ \mathscr{H} \subset \operatorname{lev}_{\geq 0} \zeta, \qquad \bigcap_{\zeta \in \mathcal{W}_i} \operatorname{lev}_{>0} \zeta \subset \mathscr{H} \ (i = 1, 2),
$$

and

$$
\forall \zeta \in \mathcal{W}_i : \ \mathscr{H}^o \subset \operatorname{lev}_{\geq 0} \zeta, \qquad \bigcap_{\zeta \in \mathcal{W}_i} \operatorname{lev}_{>0} \zeta \subset \mathscr{H}^o, \ (i = 1, 2).
$$

Proof. (i) Considering $\mathcal{W}_1$, for every $(u,v) \in \mathscr{H}$ and $\zeta \in \mathcal{W}_1$, we fidn from the increasing property of ζ that $\zeta(u,v) \geq \zeta(0_Y, 0_Z) = 0$, which means that $\mathscr{H} \subset \operatorname{lev}_{\geq 0} \zeta$, $\forall \zeta \in \mathcal{W}_1$. Conversely, Suppose $(u,v) \in \bigcap_{\zeta \in \mathcal{W}_1} \operatorname{lev}_{>0} \zeta$. Since the functions ζ_1 and ζ_2, defined as $\zeta_1(y,z) = \varphi_{e_Y}(y)$ and $\zeta_2(y,z) = \varphi_{e_Z}(z)$, both belong to $\mathcal{W}_1$, we have $\zeta_1(u,v) > 0$ and $\zeta_2(u,v) > 0$, which indicates that $u \in \operatorname{int} K_Y$ and $v \in \operatorname{int} K_Z$. Therefore, $(u,v) \in \mathscr{H}$.

(ii) The assertion for $\mathcal{W}_2$ can be proved similarly as in (i). Just note that the functions ζ_1 and ζ_2 above also belong to $\mathcal{W}_2$.

As for $\mathscr{H}^o$, using the increasing property of ζ in $\mathcal{W}_i$, $i=1,\,2$, we can obtain $\mathscr{H}^o \subset \mathrm{lev}_{\geq 0}\,\zeta$, $\forall \zeta \in \mathcal{W}_i$, $i=1,\,2$. Then, as $\mathscr{H} \subset \mathscr{H}^o$, there is also $\bigcap_{\zeta\in\mathcal{W}_i} \mathrm{lev}_{>0}\,\zeta \subset \mathscr{H}^o$, $i=1,\,2$. □

Proposition 4.6. *The collection $\mathcal{W}_3$ is a class of regular weak separation functions w.r.t. $\mathscr{H}$, i.e.,*

$$\bigcap_{\zeta\in\mathcal{W}_3} \mathrm{lev}_{>0}\,\zeta = \mathscr{H},$$

while it is only a class of weak separation functions w.r.t. $\mathscr{H}^o$, i.e.,

$$\forall \zeta \in \mathcal{W}_3: \ \mathscr{H}^o \subset \mathrm{lev}_{\geq 0}\,\zeta, \qquad \bigcap_{\zeta\in\mathcal{W}_3} \mathrm{lev}_{>0}\,\zeta \subset \mathscr{H}^o.$$

Proof. For every $(u,v)\in\mathscr{H}$, there exists some $\bar\lambda>0$ such that $\bar\lambda e_Y \preceq_{K_Y} u$. Then, for every $\zeta\in\mathcal{W}_3$, we obtain

$$\zeta(u,v) \geq \zeta(\bar\lambda e_Y, 0_Z) = \zeta(0_Y,0_Z)+\bar\lambda = \bar\lambda > 0,$$

as $\zeta(\cdot,v)$ is topical w.r.t. e_Y and $\zeta(0_Y,0_Z)=0$, which means $(u,v)\in\bigcap_{\zeta\in\mathcal{W}_3}\mathrm{lev}_{>0}\,\zeta$.

On the other hand, we take an arbitrary $(u,v)\in\bigcap_{\zeta\in\mathcal{W}_3}\mathrm{lev}_{>0}\,\zeta$. Define $\psi_{e_Y}: Y\times Z\to\bar{\mathbb{R}}$ by

$$\psi_{e_Y}(y,z) := \sup\,\{\lambda\in\mathbb{R} : (\lambda e_Y, 0_Z) \preceq_{K_Y\times K_Z} (y,z)\}.$$

Then, actually

$$\psi_{e_Y}(y,z) = \begin{cases} \varphi_{e_Y}(y) & \text{if } z\in K_Z,\\ -\infty & \text{if } z\notin K_Z,\end{cases}$$

which indicates $\psi_{e_Y}\in\mathcal{W}_3$. Therefore, we have $\psi_{e_Y}(u,v)>0$, which shows that $u\in\mathrm{int}\,K_Y$ and $v\in K_Z$, i.e., $(u,v)\in\mathscr{H}$. So, we obtain $\bigcap_{\zeta\in\mathcal{W}_3}\mathrm{lev}_{>0}\,\zeta=\mathscr{H}$. Due to the monotonicity property of ζ in $\mathcal{W}_3$, we obtain $\mathscr{H}^o\subset\mathrm{lev}_{\geq 0}\,\zeta$ for all $\zeta\in\mathcal{W}_3$. Then, as $\mathscr{H}\subset\mathscr{H}^o$, there is also $\bigcap_{\zeta\in\mathcal{W}_3}\mathrm{lev}_{>0}\,\zeta=\mathscr{H}\subset\mathscr{H}^o$. □

Proposition 4.7. *The collection $\mathcal{W}_4$ is a class of regular weak separation functions w.r.t. $\mathscr{H}^o$, i.e.,*

$$\bigcap_{\zeta\in\mathcal{W}_4} \mathrm{lev}_{>0}\,\zeta = \mathscr{H}^o.$$

Proof. For every $(u,v)\in\mathscr{H}^o$ and $\zeta\in\mathcal{W}_4$, since $\zeta(u,\cdot)$ is K_Z-monotone, we obtain

$$\zeta(u,v)\geq\zeta(u,0_Y)>0.$$

On the other hand, we take an arbitrary $(u,v)\in\bigcap_{\zeta\in\mathcal{W}_4}\mathrm{lev}_{>0}\,\zeta$. Picking some $\varepsilon\in(0,1)$, we define $\psi^\varepsilon_{e_Y}: Y\times Z\to\bar{\mathbb{R}}$ by

$$\begin{aligned}\psi^\varepsilon_{e_Y}(y,z) &= (1-\varepsilon)\sup\,\{\lambda\in\mathbb{R} : (\lambda e_Y,0_Z)\preceq_{K_Y\times K_Z}(y,z)\}\\ &\quad+\varepsilon\inf\,\{\lambda\in\mathbb{R} : \lambda e_Y \succeq_{K_Y} y\}.\end{aligned}$$

Then, actually

$$\psi^\varepsilon_e(y,z) = \begin{cases} (1-\varepsilon)\varphi_{e_Y}(y)+\varepsilon\xi_{e_Y}(y) & \text{if } z\in K_Z,\\ -\infty & \text{if } z\notin K_Z,\end{cases}$$

indicating $\psi^{\varepsilon}_{e_Y} \in \mathcal{W}_4$ for all $\varepsilon \in (0,1)$. Therefore, we have $\psi^{\varepsilon}_{e_Y}(u,v) > 0$ for all $\varepsilon \in (0,1)$, which shows that $u \in K_Y \setminus \{0_Y\}$ and $v \in K_Z$ according to Proposition 2.3, i.e., $(u,v) \in \mathscr{H}^o$. □

Proposition 4.8. *The collection $\mathcal{W}_5$ is a class of regular weak separation functions w.r.t. $\mathscr{H}^o$, i.e.,*

$$\bigcap_{\zeta \in \mathcal{W}_5} \operatorname{lev}_{>0} \zeta = \mathscr{H}^o.$$

Proof. For every $(u,v) \in \mathscr{H}^o$ and $\zeta \in \mathcal{W}_5$, since $0_Y \in \{\lambda e_Y : \lambda \in \mathbb{R}\}$, we obtain

$$\zeta(u,v) > \zeta(0_Y,v) \geq \zeta(0_Y,0_Z) = 0.$$

On the other hand, take an arbitrary $(u,v) \in \bigcap_{\zeta \in \mathcal{W}_5} \operatorname{lev}_{>0} \zeta$. For any $\varepsilon \in (0,1)$, the function $\psi^{\varepsilon}_{e_Y}$ defined in the proof of Proposition 4.7 also belongs to $\mathcal{W}_5$. Therefore, it follows that $\psi^{\varepsilon}_{e_Y}(u,v) > 0$ for all $\varepsilon \in (0,1)$. By Proposition 2.3, this shows $u \in K_Y \setminus \{0_Y\}$ and $v \in K_Z$, i.e., $(u,v) \in \mathscr{H}^o$. □

For the sake of further applications in duality theory, we also discuss the value of $\inf_{\zeta \in \mathcal{W}_i} \zeta(y,z)$, $i = 1,2,3,4,5$ here. Especially, we derive a description of $\inf_{\zeta \in \mathcal{W}_i} \zeta(y,z)$, $i = 1,2,3,4,5$ in certain terms of the Gerstewitz function.

Proposition 4.9. *For the collection $\mathcal{W}_1$, one has*

$$\inf_{\zeta \in \mathcal{W}_1} \zeta(u,v) = \begin{cases} 0 & \text{if } u \in K_Y \text{ and } v \in K_Z, \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. Since the functions ζ_1 and ζ_2, defined as $\zeta_1(y,z) = \varphi_{e_Y}(y)$ and $\zeta_2(y,z) = \varphi_{e_Z}(z)$, both belong to $\mathcal{W}_1$, we have $\zeta_1(u,v) < 0$ if $u \notin K_Y$, while $\zeta_2(u,v) < 0$ if $v \notin K_Z$. Then, it follows from $\alpha\zeta_1$, $\alpha\zeta_2 \in \mathcal{W}_1$ for all $\alpha > 0$ that $\inf_{\zeta \in \mathcal{W}_1} \zeta(u,v) = -\infty$ if $u \notin K_Y$ or $v \notin K_Z$. For the case that $(u,v) \in K_Y \times K_Z$, since ζ is $(K_Y \times K_Z)$-monotone for every $\zeta \in \mathcal{W}_1$, $\inf_{\zeta \in \mathcal{W}_1} \zeta(u,v) \geq \zeta(0_Y,0_Z) = 0$. Considering the function $\zeta_3 \in \mathcal{W}_1$ defined as $\zeta_3(y,z) \equiv 0$, wesee that there is $\zeta_3(u,v) = 0$. Hence, $\inf_{\zeta \in \mathcal{W}_1} \zeta(u,v) = 0$ when $(u,v) \in K_Y \times K_Z$. □

Proposition 4.10. *For the collection $\mathcal{W}_2$, for $u \in Y$, $v \in Z$, one has*

$$\inf_{\zeta \in \mathcal{W}_2} \zeta(u,v) = \begin{cases} \min\{\varphi_{e_Y}(u), \varphi_{e_Z}(v)\} & \text{if } u \in K_Y \text{ and } v \in K_Z, \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. Considering the functions ζ_1 and ζ_2, defined as $\zeta_1(y,z) = \varphi_{e_Y}(y)$ and $\zeta_2(y,z) = \varphi_{e_Z}(z)$, it is easy to observe that $\zeta_1 + \alpha\zeta_2$ and $\alpha\zeta_1 + \zeta_2$ belong to $\mathcal{W}_2$ for every $\alpha \geq 0$. Applying Proposition 2.1, we have $\zeta_1(u,v) < 0$ if $u \notin K_Y$, while $\zeta_2(u,v) < 0$ if $v \notin K_Z$. Hence, when $\alpha \to +\infty$, $(\zeta_1 + \alpha\zeta_2)(u,v) \to -\infty$ if $v \notin K_Z$, and $(\alpha\zeta_1 + \zeta_2)(u,v) \to -\infty$ if $u \notin K_Y$, implying that $\inf_{\zeta \in \mathcal{W}_2} \zeta(u,v) = -\infty$ if $u \notin K_Y$ or $v \notin K_Z$.

With regard to the case where $(u,v) \in K_Y \times K_Z$, according to Proposition 2.1, we have

$$u = \varphi_{e_Y}(u)e_Y + q_1, \; v = \varphi_{e_Z}(v)e_Z + q_2,$$

for some $q_1 \in \operatorname{bd} K_Y$ and $q_2 \in \operatorname{bd} K_Z$. Then, for any $\zeta \in \mathcal{W}_3$,

$$\zeta(u,v) = \zeta(q_1,v) + \varphi_{e_Y}(u) \geq \zeta(0_Y,0_Z) + \varphi_{e_Y}(u) = \varphi_{e_Y}(u)$$

if ζ is topical w.r.t. e_Y, while,

$$\zeta(u,v) = \zeta(u,q_2) + \varphi_{e_Z}(v) \geq \zeta(0_Y,0_Z) + \varphi_{e_Z}(v) = \varphi_{e_Z}(v)$$

if ζ is topical w.r.t. e_Z. This, together with the fact that $\zeta_1(u,v) = \varphi_{e_Y}(u)$ and $\zeta_2(u,v) = \varphi_{e_Z}(v)$, implies

$$\inf_{\zeta \in \mathcal{W}_2} \zeta(u,v) = \min\{\varphi_{e_Y}(u), \varphi_{e_Z}(v)\},$$

when $(u,v) \in K_Y \times K_Z$. □

Proposition 4.11. *For the collection $\mathcal{W}_3$, for $u \in Y$, $v \in Z$, one has*

$$\inf_{\zeta \in \mathcal{W}_3} \zeta(u,v) = \begin{cases} \varphi_{e_Y}(u) & \text{if } v \in K_Z, \\ -\infty & \text{if } v \notin K_Z. \end{cases}$$

Proof. Since the function

$$\begin{aligned} \psi_{e_Y}(y,z) &= \sup\{\lambda \in \mathbb{R} : (\lambda e_Y, 0_Z) \preceq_{K_Y \times K_Z} (y,z)\} \\ &= \begin{cases} \varphi_{e_Y}(y) & \text{if } z \in K_Z, \\ -\infty & \text{if } z \notin K_Z, \end{cases} \end{aligned}$$

belongs to $\mathcal{W}_3$, we can conclude that $\inf_{\zeta \in \mathcal{W}_3} \zeta(u,v) = -\infty$ when $v \notin K_Z$. If $v \in K_Z$, then, for any $\zeta \in \mathcal{W}_3$, $\zeta(u,v) \geq \zeta(u,0_Z)$ as $\zeta(u,\cdot)$ is K_Z-monotone. According to Proposition 2.1, there exists some $q_u \in \operatorname{bd} K_Y$ such that $u = \varphi_{e_Y}(u)e_Y + q_u$. Then, the topical property guarantees that

$$\zeta(u,0_Z) = \zeta(\varphi_{e_Y}(u)e_Y + q_u, 0_Z) = \varphi_{e_Y}(u) + \zeta(q_u, 0_Z) \geq \varphi_{e_Y}(u).$$

Therefore, $\inf_{\zeta \in \mathcal{W}_3} \zeta(u,v) \geq \varphi_{e_Y}(u)$ in this case. Noting that the function $\psi_{e_Y} \in \mathcal{W}_3$ and $\psi_{e_Y}(u,v) = \varphi_{e_Y}(u)$, we can deduce that $\inf_{\zeta \in \mathcal{W}_3} \zeta(u,v) = \varphi_{e_Y}(u)$. □

Proposition 4.12. *For the collection $\mathcal{W}_4$, for $u \in Y$, $v \in Z$, one has*

$$\inf_{\zeta \in \mathcal{W}_4} \zeta(u,v) = \begin{cases} 0 & \text{if } u \in K_Y \text{ and } v \in K_Z, \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. Considering the function defined as

$$\begin{aligned} \psi_{e_Y,e_Z}(y,z) &:= \sup\{\lambda \in \mathbb{R} : (\lambda e_Y, 0_Z) \preceq_{K_Y \times K_Z} (y,z)\} \\ &\quad + \inf\{\lambda \in \mathbb{R} : \lambda e_Y \succeq_{K_Y} y\} \\ &\quad + \sup\{\lambda \in \mathbb{R} : (0_Y, \lambda e_Z) \preceq_{K_Y \times K_Z} (y,z)\} \\ &= \begin{cases} \varphi_{e_Y}(y) + \xi_{e_Y}(y) + \varphi_{e_Z}(z) & \text{if } y \in K_Y \text{ and } z \in K_Z, \\ -\infty & \text{otherwise,} \end{cases} \end{aligned}$$

we have $\psi_{e_Y,e_Z} \in \mathcal{W}_4$. Hence, we deduce that $\inf_{\zeta \in \mathcal{W}_4} \zeta(u,v) = -\infty$ when $u \notin K_Y$ or $v \notin K_Z$. If $u \in K_Y$ and $v \in K_Z$, then, for any $\zeta \in \mathcal{W}_4$, $\zeta(u,v) \geq \zeta(u,0_Z) \geq 0$ as $\zeta(u,\cdot)$ is K_Z-monotone. Hence, $\inf_{\zeta \in \mathcal{W}_4} \zeta(u,v) \geq 0$ in this case. Noting that the function $\psi_{\alpha e_Y, \alpha e_Z} \in \mathcal{W}_4$ for every $\alpha > 0$ and, according to Proposition 3.2,

$$\psi_{\alpha e_Y, \alpha e_Z}(u,v) = \frac{1}{\alpha} \psi_{e_Y,e_Z}(u,v) \to 0$$

as $\alpha \to= +\infty$, we can conclude that $\inf_{\zeta \in \mathcal{W}_4} \zeta(u,v) = 0$. □

Proposition 4.13. *For the collection $\mathcal{W}_5$, for $u \in Y$, $v \in Z$, one has*

$$\inf_{\zeta \in \mathcal{W}_5} \zeta(u,v) = \begin{cases} \varphi_{e_Y}(u) & \text{if } v \in K_Z, \\ -\infty & \text{if } v \notin K_Z. \end{cases}$$

Proof. Since, for any $\varepsilon \in (0,1)$, function

$$\begin{aligned} \psi^{\varepsilon}_{e_Y}(y,z) &:= (1-\varepsilon)\sup\{\lambda \in \mathbb{R} : (\lambda e_Y, 0_Z) \preceq_{K_Y \times K_Z} (y,z)\} \\ &\quad + \varepsilon \inf\{\lambda \in \mathbb{R} : \lambda e_Y \succeq_{K_Y} y\} \\ &= \begin{cases} (1-\varepsilon)\varphi_{e_Y}(y) + \varepsilon \xi_{e_Y}(y) & \text{if } z \in K_Z, \\ -\infty & \text{if } z \notin K_Z, \end{cases} \end{aligned}$$

belongs to $\mathcal{W}_5$, we can conclude that $\inf_{\zeta \in \mathcal{W}_5} \zeta(u,v) = -\infty$ when $v \notin K_Z$. If $v \in K_Z$, then, for any $\zeta \in \mathcal{W}_5$, $\zeta(u,v) \geq \zeta(u,0_Z)$ as $\zeta(u,\cdot)$ is K_Z-monotone. According to Proposition 2.1, there exists some $q_u \in \operatorname{bd} K_Y$ such that $u = \varphi_{e_Y}(u)e_Y + q_u$. Then, the topical property shows that

$$\zeta(u,0_Z) = \zeta(\varphi_{e_Y}(u)e_Y + q_u, 0_Z) = \varphi_{e_Y}(u) + \zeta(q_u,0_Z) \geq \varphi_{e_Y}(u).$$

Therefore, $\inf_{\zeta \in \mathcal{W}_5} \zeta(u,v) \geq \varphi_{e_Y}(u)$ in this case. Meanwhile, since $\psi^{\varepsilon}_{e_Y} \in \mathcal{W}_5$ for every $\varepsilon \in (0,1)$ and $\psi^{\varepsilon}_{e_Y}(u,v) \to \varphi_{e_Y}(u)$ when $\varepsilon \to 1$, we can deduce that $\inf_{\zeta \in \mathcal{W}_5} \zeta(u,v) = \varphi_{e_Y}(u)$. □

Remark 4.1. According to Propositions 4.11 and 4.13, given some $(u,v) \in Y \times Z$, the infimum values $\inf_{\zeta \in \mathcal{W}_3} \zeta(u,v)$ and $\inf_{\zeta \in \mathcal{W}_5} \zeta(u,v)$ not only give a detection of whether $(u,v) \in K_Y \times K_Z$ or not, but also present the scalarization of u by φ_{e_Y}.

We will use the properties of the collections $\mathcal{W}_1, \dots, \mathcal{W}_5$ of (regular) weak separation functions derived in this section for the discussion of useful dual problems in the examples of Section 5.

5. Framework of Conjugate Duality with Scalarization

In this section, we consider the general constrained vector optimization that

$$\min\{f(x) : \; g(x) \succeq_{K_Z} 0_Z\}, \tag{VOP}$$

where $f : X \to Y$ and $g : X \to Z$. Denote the feasible set $\{x \in X : g(x) \succeq_{K_Z} 0_Z\}$ by $\mathscr{R}$. We shall give a framework of conjugate duality for this problem with the help of the separation functions we constructed in the previous section.

One way to give the dual problem for a (VOP) is to scalarize the objective function of (VOP) and to construct a scalar dual problem to the scalarized problem. Then, one is looking for certain relationships between the dual problem and the original (VOP).

Duality assertions for vector optimization problems based on a scalarization by means of nonlinear scalarizing functions are given in the books by Göpfert, Riahi, Tammer, Zălinescu [6, Section 3.7] and Boţ, Grad, and Wanka [57]. The main tools in the proofs of the duality statements in these books are the monotonicity and the translation invariance of the scalarizing functions. These properties are essential for the definition of the collections of (regular) weak separation functions introduced and discussed in Section 4.

Given a scalarization function $p : Y \to \mathbb{R}$, we have the following scalar problem

$$\min\{p(f(x)) : \; g(x) \succeq_{K_Z} 0_Z\}. \tag{SP}$$

Then, in general,

(i) If $\bar{x}$ is a solution to (SP), it must be at least a weakly efficient solution for (VOP);

(ii) However, the inverse is not necessarily true. That is to say, even if $\bar{x}$ is an efficient solution to (VOP), it might not be a solution to (SP); see Example 5.1.

Example 5.1. Suppose $X = \mathbb{R}$ and $(Y,K_Y) = (Z,K_Z) = (\mathbb{R}^2,\mathbb{R}^2_+)$. Take ξ_{e_Y} as p, where $e_Y = (1,1)$. Then, $\xi_{e_Y}(y) = \max\{y_1,y_2\}$ for every $y = (y_1,y_2) \in Y$. Set $f(x) = (x,-x+1)$, $g(x) = (x,-x+1)$. It is not hard to verify that the whole segment $\overline{AB} := \{(y_1,y_2) : y_2 = -y_1+1,\ y_1 \geq 0,\ y_2 \geq 0\}$ is the efficient solution set for (VOP). However, for (SP), $(\frac{1}{2},\frac{1}{2})$ is the only solution with optimal value

$$\xi_e(f(\frac{1}{2},\frac{1}{2})) = \max\{\frac{1}{2},-\frac{1}{2}+1\} = \frac{1}{2},$$

while, any point $(y_1,y_2) \in \overline{AB}$, when $y_1 \neq \frac{1}{2}$, is not a solution of (SP). Using ξ_{e_Y} to scalarize the (VOP) fails to detect these efficient solutions.

For the sake of reaching the whole solution set, we try to consider a collection of scalarization functions $\mathcal{P}$. If, for every (weakly) efficient solution $\bar{x}$ of (VOP), there exists some $p_{\bar{x}} \in \mathcal{P}$ such that $\bar{x}$ is a solution to the corresponding scalar problem given by $p_{\bar{x}}$, i.e.,

$$\min\{p_{\bar{x}}(f(x)) :\ g(x) \succeq_{K_Z} 0_Z\}, \tag{SP-$p_{\bar{x}}$}$$

then we can detect all the (weakly) efficient solutions of (VOP) with the whole collection $\mathcal{P}$.

To be more general, we study a collection of scalar optimization problems corresponding to (VOP)

$$\{(\mathrm{SP}-\pi)\ :\ \pi \in \Pi\},$$

where Π is a certain parameter set, which means that (SP-π) represents a certain way of scalarization, where $\pi \in \Pi$ is the parameter involved in it. The feasible set of $(\mathrm{SP}-\pi)$ is denoted by $\mathscr{R}_\pi$. As we want to use this family of problems to detect all the (weakly) efficient solutions of (VOP), $(\mathrm{SP}-\pi)$, $\pi \in \Pi$, is required to satisfy the following assumptions.

Assumption $\mathcal{S}$:

(i) For every $\pi \in \Pi$, if $\bar{x}$ is a solution to $(\mathrm{SP}-\pi)$, then it is also a (weakly) efficient solution to (VOP).

(ii) If $\bar{x}$ is an (weakly) efficient solution to (VOP), then there exists some $\bar{\pi} \in \Pi$ such that $\bar{x}$ is a solution to $(\mathrm{SP}-\bar{\pi})$.

We provides some examples for scalarized problems corresponding to (VOP) next.

Example 5.2. Picking some $e_Y \in \operatorname{int} K_Y$, we consider the scalarized problem

$$\min\{\xi_{e_Y}(f(x)-f(\hat{x})) :\ g(x) \succeq_{K_Z} 0_Z\}, \tag{SP-$\hat{x}$}$$

where $\hat{x} \in \mathscr{R}$ is the parameter and the scalarizing functional ξ_{e_Y} is given by (2.1).

(i) For any parameter $\hat{x} \in \mathscr{R}$, if $\bar{x}$ is a solution to $(\mathrm{SP}-\hat{x})$, but not a weak efficient solution to (VOP), which means that there exists some $\tilde{x} \in \mathscr{R}$ such that $f(\tilde{x}) \preceq_{\operatorname{int} K_Y} f(\bar{x})$, then we obtain $f(\tilde{x}) - f(\hat{x}) \preceq_{\operatorname{int} K_Y} f(\bar{x}) - f(\hat{x})$. Applying Proposition 2.1, we have $\xi_{e_Y}(f(\tilde{x}) - f(\hat{x})) < \xi_{e_Y}(f(\bar{x}) - f(\hat{x}))$, contradicting that $\bar{x}$ is a solution to $(\mathrm{SP}-\hat{x})$.

(ii) Suppose that $\bar{x}$ is a weakly efficient solution to (VOP). Setting $\hat{x} = \bar{x}$, we have the scalar problem

$$\min\{\xi_{e_Y}(f(x)-f(\bar{x})) :\ g(x) \succeq_{K_Z} 0_Z\}. \tag{SP-$\bar{x}$}$$

We claim that $\bar{x}$ is a solution to $(\mathrm{SP}-\bar{x})$. Otherwise, there exists some $\tilde{x}$ with $g(\tilde{x}) \succeq_{K_Z} 0_Z$ such that

$$\xi_{e_Y}(f(\tilde{x})-f(\bar{x})) < \xi_{e_Y}(f(\bar{x})-f(\bar{x})) = 0,$$

which, by Proposition 2.1, implies $f(\tilde{x})-f(\bar{x}) \in -\operatorname{int} K_Y$, i.e. $f(\tilde{x}) \preceq_{\operatorname{int} K_Y} f(\bar{x})$, which contradicts that $\bar{x}$ is a weakly efficient solution to (VOP).

In the example above, parameter $\hat{x}$ ranges in the feasible set of (VOP). With a similar argument, it is easy to observe that it still works if $\hat{x}$ ranges in the whole space X. Besides, we can also see e_Y as a parameter by letting it range in $\operatorname{int} K_Y$. That is to say, the group of problems

$$\{\min\{\xi_{e_Y}(f(x)-f(\hat{x})) : g(x) \succeq_{K_Z} 0_Z\} : \ \hat{x} \in X\},$$

and

$$\{\min\{\xi_{e_Y}(f(x)-f(\hat{x})) : g(x) \succeq_{K_Z} 0_Z\} : \ (\hat{x}, e_Y) \in X \times \operatorname{int} K_Y\},$$

both satisfy Assumption $\mathcal{S}$ w.r.t. weakly efficient solution.

Example 5.3. For the case where Y and Z are finite dimensional spaces, while $f = (f_1, f_2, \dots, f_n) : X \to \mathbb{R}^n$ and $g = (g_1, g_2, \dots, g_m) : X \to \mathbb{R}^m$ are both convex, the (VOP) is a convex problem. Then, the collection

$$\{\min\{\sum_{i=1}^{n} \alpha_i f_i(x) : \ g(x) \succeq_{\mathbb{R}^m_+} 0_{\mathbb{R}^m}\} : \ \sum_{i=1}^{n} \alpha_i = 1, \alpha \geq 0\},$$

satisfies Assumption $\mathcal{S}$ w.r.t. weakly efficient solution.

Next, we give some examples for efficient solutions.

Example 5.4. Fixing some $e_Y \in \operatorname{int} K_Y$, consider the problems that

$$\min\{\alpha\varphi_{e_Y}(f(x)-f(\hat{x})) + (1-\alpha)\xi_{e_Y}(f(x)-f(\hat{x})) : \ g(x) \succeq_{K_Z} 0_Z\}, \qquad (\mathrm{SP}_e\text{-}(\hat{x},\alpha))$$

where $\hat{x} \in \mathscr{R}$, $\alpha \in (0,1)$ are the parameters, φ_{e_Y} is given by Definition 2.3, and ξ_{e_Y} is given by (2.1).

(i) We claim that, for any $\alpha \in (0,1)$, if $\bar{x}$ is a solution to $(\mathrm{SP}-(\bar{x},\alpha))$, i.e., $\bar{x}$ is a fixed point of the set-valued function

$$\begin{array}{rcl} \Psi^e_\alpha : X & \rightrightarrows & X \\ \hat{x} & \longmapsto & \mathrm{sol}(\mathrm{SP}-(\hat{x},\alpha)), \end{array}$$

then $\bar{x}$ is an efficient solution to (VOP), where $\mathrm{sol}(\mathrm{SP}-(\hat{x},\alpha))$ denotes the solution set of $(\mathrm{SP}-(\hat{x},\alpha))$.

Otherwise, there exists some $\tilde{x} \in \mathscr{R}$ such that $f(\tilde{x}) \preceq_{K_Y \setminus \{0_Y\}} f(\bar{x})$, i.e., $f(\tilde{x}) - f(\bar{x}) \preceq_{K_Y \setminus \{0_Y\}} 0_Y$. Then, according to Proposition 2.3,

$$\begin{array}{rl} & \alpha\varphi_{e_Y}(f(\tilde{x})-f(\bar{x})) + (1-\alpha)\xi_{e_Y}(f(\tilde{x})-f(\bar{x})) \\ < & 0 \\ = & \alpha\varphi_{e_Y}(f(\bar{x})-f(\bar{x})) + (1-\alpha)\xi_{e_Y}(f(\bar{x})-f(\bar{x})), \end{array}$$

which contradicts that $\bar{x}$ is a solution of $\mathrm{SP}-(\bar{x},\alpha)$.

(ii) Conversely, we claim that if $\bar{x}$ is an efficient solution of (VOP), then it is a solution of the scalar problem that

$$\min\{\sup_{\alpha\in(0,1)}\{\alpha\varphi_{e_Y}(f(x)-f(\bar{x}))+(1-\alpha)\xi_{e_Y}(f(x)-f(\bar{x}))\}:\ g(x)\succeq_{K_Z}0_Z\}. \qquad (\mathrm{SP}_e\text{-}\sup_\alpha)$$

If $\bar{x}$ is an efficient solution of (VOP), then, for any point $x\in\mathscr{R}$, there is $f(x)\not\preceq_{K_Y\setminus\{0_Y\}} f(\bar{x})$. If $f(x)=f(\bar{x})$, then

$$\alpha\varphi_{e_Y}(f(x)-f(\bar{x}))+(1-\alpha)\xi_{e_Y}(f(x)-f(\bar{x}))=0,\ \forall\alpha\in(0,1).$$

If $f(x)\neq f(\bar{x})$, then $f(x)-f(\bar{x})\notin -K_Y$, and therefore, by Proposition 2.1, there is some $\bar{\alpha}\in(0,1)$ such that $\bar{\alpha}\varphi_{e_Y}(f(x)-f(\bar{x}))+(1-\bar{\alpha})\xi_{e_Y}(f(x)-f(\bar{x}))>0$, which shows that

$$\sup_{\alpha\in(0,1)}\{\alpha\varphi_{e_Y}(f(x)-f(\bar{x}))+(1-\alpha)\xi_{e_Y}(f(x)-f(\bar{x}))\}>0.$$

Hence,

$$\begin{aligned}&\sup_{\alpha\in(0,1)}\{\alpha\varphi_{e_Y}(f(\bar{x})-f(\bar{x}))+(1-\alpha)\xi_{e_Y}(f(\bar{x})-f(\bar{x}))\}\\ =\ &0\\ \leq\ &\sup_{\alpha\in(0,1)}\{\alpha\varphi_{e_Y}(f(x)-f(\bar{x}))+(1-\alpha)\xi_{e_Y}(f(x)-f(\bar{x}))\},\end{aligned}$$

for any $x\in\mathscr{R}$.

Similarly, let $\hat{x}$ range in the whole space X and e_Y range in $\operatorname{int}K_Y$. The class of problems

$$\min\{\alpha\varphi_{e_Y}(f(x)-f(\hat{x}))+(1-\alpha)\xi_{e_Y}(f(x)-f(\hat{x})):\ g(x)\succeq_{K_Z}0_Z\}, \qquad (\mathrm{SP}_e\text{-}(\hat{x},\alpha,e_Y))$$

where $(\hat{x},\alpha,e_Y)\in X\times(0,1)\times\operatorname{int}K_Y$ is the parameter, enjoys similar results. Although this example is not consistent with Assumption $\mathbb{S}$ exactly, the collection of scalar problem $\{(\mathrm{SP}-(\hat{x},\alpha)):\hat{x}\in\mathscr{R},\ \alpha\in(0,1)\}$ still offers a way to detect all the efficient solutions of (VOP).

Example 5.5. Fixing some $l\in K_Y^{\#}:=\{y^*\in Y^*:y^*(y)>0$ for all $y\in K_Y\setminus\{0_Y\}\}$ ($K_Y^{\#}$ denotes the quasi-interior of the dual cone of K_Y), or some scalarization function $l:Y\to\mathbb{R}$ that is strongly K_Y-monotone (i.e., for all $y_1,\ y_2\in Y$, $y_1\preceq_{K_Y\setminus\{0_Y\}}y_2$ implies $l(y_1)<l(y_2)$), consider the problem

$$\min\{l(f(x)):\ f(x)\preceq_{K_Y}f(\hat{x}),\ g(x)\succeq_{K_Z}0_Z\}, \qquad (\mathrm{SP}_e\text{-}\hat{x})$$

where $\hat{x}\in\mathscr{R}$ is the parameter.

We have proved, in [58] that, for any parameter $\hat{x}\in\mathscr{R}$, if $\bar{x}$ is a solution to $(\mathrm{SP}-\hat{x})$, then it is also an efficient solution to (VOP). Conversely, if $\bar{x}$ is an efficient solution of (VOP), then it is a solution to the scalar problem with the parameter itself, i.e., the problem that

$$\min\{l(f(x)):\ f(x)\preceq_{K_Y}f(\bar{x}),\ g(x)\succeq_{K_Z}0_Z\}, \qquad (\mathrm{SP}_e\text{-}\bar{x})$$

which means that $\bar{x}$ is an efficient solution to (VOP) if and only if it is a fixed point of the set-valued map that

$$\begin{aligned}\Psi^e:X&\ \rightrightarrows\ X\\ \hat{x}&\ \longmapsto\ \mathrm{sol}(\mathrm{SP}_e-\hat{x}).\end{aligned}$$

Thus, the collection

$$\{\min\{l(f(x)):\ f(x)\preceq_{K_Y}f(\bar{x}),\ g(x)\succeq_{K_Z}0_Z\}:\ \hat{x}\in\mathscr{R}\}$$

satisfies Assumption $\mathcal{S}$ w.r.t. efficient solution. Furthermore, if we also let l range as a parameter, we consider

$$\min\{\hat{l}(f(x)) : \ f(x) \preceq_{K_Y} f(\bar{x}), \ g(x) \succeq_{K_Z} 0_Z\}, \qquad (\text{SP}_e\text{-}(\hat{x},\hat{l}))$$

where $\hat{x} \in \mathscr{R}$, while, $\hat{l} \in L_{K_Y \setminus \{0_Y\}}$ or $\hat{l} \in L_{K_Y \setminus \{0_Y\}-\text{mono}}$. Here, $L_{K_Y \setminus \{0_Y\}}$ is defined as a subset of

$$\{y^* \in Y^* : y^*(y) > 0, \text{ for all } y \in K_Y \setminus \{0_Y\}\}$$

and $L_{K_Y \setminus \{0_Y\}-\text{mono}}$ is defined as a subset of

$$\{l : Y \to \mathbb{R} : l \text{ is strongly } K_Y\text{-monotone}\}.$$

Then, the collection

$$\{\min\{\hat{l}(f(x)) : \ f(x) \preceq_{K_Y} f(\hat{x}), \ g(x) \succeq_{K_Z} 0_Z\} : \ \hat{x} \in \mathscr{R}, \ \hat{l} \in L_{K_Y \setminus \{0_Y\}}\}$$

or

$$\{\min\{\hat{l}(f(x)) : \ f(x) \preceq_{K_Y} f(\hat{x}), \ g(x) \succeq_{K_Z} 0_Z\} : \ \hat{x} \in \mathscr{R}, \ \hat{l} \in L_{K_Y \setminus \{0_Y\}-\text{mono}}\}$$

still satisfies Assumption $\mathcal{S}$ w.r.t. efficient solution. Similarly, fixing some $l \in \{y^* \in Y^* : y^*(y) > 0, \forall y \in \operatorname{int} K_Y\}$, or some scalarization function $l : Y \to \mathbb{R}$ that is strictly $\operatorname{int} K_Y$-monotone (i.e., $y_1 \preceq_{\operatorname{int} K_Y} y_2$ implies $l(y_1) < l(y_2)$), for instance, the Gerstewitz function φ_{e_Y}. Consider the problems

$$\min\{l(f(x)) : \ f(x) \preceq_{K_Y} f(\hat{x}), \ g(x) \succeq_{K_Z} 0_Z\}, \qquad (\text{SP-}\hat{x})$$

where $\hat{x} \in \mathscr{R}$ is the parameter. Then, $\bar{x}$ is a weakly efficient solution of (VOP) if and only if it is a fixed point of the set-valued map that

$$\begin{aligned} \Psi : X \quad &\rightrightarrows \quad X \\ \hat{x} \quad &\longmapsto \quad \operatorname{sol}(\text{SP} - \hat{x}). \end{aligned}$$

Thus, the collection

$$\{\min\{l(f(x)) : \ f(x) \preceq_{K_Y} f(\bar{x}), \ g(x) \succeq_{K_Z} 0_Z\} : \ \hat{x} \in \mathscr{R}\}$$

satisfies Assumption $\mathcal{S}$ w.r.t. weakly efficient solution. Also, if we let l range as a parameter, we consider

$$\min\{\hat{l}(f(x)) : \ f(x) \preceq_{K_Y} f(\bar{x}), \ g(x) \succeq_{K_Z} 0_Z\}, \qquad (\text{SP-}(\hat{x},\hat{l}))$$

where $\hat{x} \in \mathscr{R}$, while,

$$\hat{l} \in L_{\operatorname{int} K_Y} \subset \{y^* \in Y^* : y^*(y) > 0, \ \forall y \in \operatorname{int} K_Y\}$$

or

$$\hat{l} \in L_{\operatorname{int} K_Y-\text{mono}} \subset \{l : Y \to \mathbb{R} : l \text{ is strictly } \operatorname{int} K_Y\text{-monotone}\}.$$

Then, the collection

$$\{\min\{\hat{l}(f(x)) : \ f(x) \preceq_{K_Y} f(\hat{x}), \ g(x) \succeq_{K_Z} 0_Z\} : \ \hat{x} \in \mathscr{R}, \ \hat{l} \in L_{\operatorname{int} K_Y}\}$$

or

$$\{\min\{\hat{l}(f(x)) : \ f(x) \preceq_{K_Y} f(\hat{x}), \ g(x) \succeq_{K_Z} 0_Z\} : \ \hat{x} \in \mathscr{R}, \ \hat{l} \in L_{\operatorname{int} K_Y-\text{mono}}\}$$

still satisfies Assumption $\mathcal{S}$ w.r.t. weakly efficient solution.

We come back to these examples for discussing special constructions of the dual problem at the end of this section. In order to formulate the following scalarized problem depending from parameters $\pi \in \Pi$, we use the notations g_π, Z_Π and K_{Z_Π} to denote the corresponding terms in (VOP) after scalarization. Note that the collection $\{(\mathrm{SP}-\pi) : \pi \in \Pi\}$ is a family of problems that scalarize the original (VOP) in a similar way. Hence, for every $\pi \in \Pi$, $(\mathrm{SP}-\pi)$ has a uniform constrained space (Z_Π, K_{Z_Π}). This is the reason that we use notation (Z_Π, K_{Z_Π}) rather than (Z_π, K_{Z_π}). However, for different parameters, the constrained map might be different. Hence, we use g_π instead of g_Π.

Consider the scalarized primal problem

$$\min\{p(f(x);\pi) : \ g_\pi(x) \succeq_{(K_Z)_\Pi} 0_{Z_\Pi}\}, \tag{SP-π}$$

where Π is a parameter set, $\pi \in \Pi$ is a fixed parameter, $p(\cdot;\pi) : Y \to \mathbb{R}$, $g_\pi : X \to Z_\Pi$, $(K_Z)_\Pi$ is the ordering cone of Z_Π, and 0_{Z_Π} is the zero element in Z_Π. We suppose that this collection of problems $\{(\mathrm{SP}-\pi) : \pi \in \Pi\}$ satisfies Assumption $\mathcal{S}$ w.r.t. (weakly) efficient solution and $\inf\{p(f(x);\pi) : x \in X\} > -\infty$ for every $\pi \in \Pi$.

In order to formulate the dual problem to (SP-π), $\pi \in \Pi$, we consider another collection of functions

$$W_\Gamma := \{w(\cdot;\gamma) : Z_\Pi \to \mathbb{R} \cup \{-\infty\} : \ \gamma \in \Gamma\}$$

such that $w(0_{Z_\Pi};\gamma) \in \mathbb{R}$ for every $\gamma \in \Gamma$. Here, Γ is the parameter set of W_Γ.

Remark 5.1. We explain these notions using some examples studied before. For the case of Example 5.2, the scalarized problem is

$$\min\{\xi_{e_Y}(f(x) - f(\hat{x})) : \ g(x) \succeq_{K_Z} 0_Z\}. \tag{SP-$\hat{x}$}$$

So, $\hat{x} \in \mathscr{R} = \{x \in X : \ g(x) \succeq_{K_Z} 0_Z\}$ are the parameters. Hence, $\Pi = \mathscr{R}$, $\pi = \hat{x}$, while, $g_\pi = g$, $(Z_\Pi, K_{Z_\Pi}) = (Z, K_Z)$.

In Example 5.5, (VOP) is scalarized as

$$\min\{l(f(x)) : \ f(x) \preceq_{K_Y} f(\hat{x}), \ g(x) \succeq_{K_Z} 0_Z\}, \tag{SP$_e$-$\hat{x}$}$$

i.e.,

$$\min\{l(f(x)) : \ f(\hat{x}) - f(x) \succeq_{K_Y} 0_Y, \ g(x) \succeq_{K_Z} 0_Z\}, \tag{SP$_e$-$\hat{x}$}$$

where $l \in \{y^* \in Y^* : y^*(y) > 0, \ \text{ for all } y \in K_Y \setminus \{0_Y\}\}$, or l is some scalarization function that is strongly K_Y-monotone. In this situation, $\Pi = \mathscr{R} = \{x \in X : \ g(x) \succeq_{K_Z} 0_Z\}$ and $\pi = \hat{x}$, while, $g_\pi(\cdot) = (f(\hat{x}) - f(\cdot), g(\cdot))$, $Z_\Pi = Y \times Z$, and $K_{Z_\Pi} = K_Y \times K_Z$.

For $(\mathrm{SP}-\pi)$, we define the function $F_\pi : X \times Z_\Pi \to \bar{\mathbb{R}}$ by

$$F_\pi(x, z_\Pi) := \begin{cases} p(f(x);\pi) & \text{if } g_\pi(x) \succeq_{K_{Z_\Pi}} z_\Pi, \\ +\infty & \text{if } g_\pi(x) \not\succeq_{K_{Z_\Pi}} z_\Pi. \end{cases}$$

We now consider the unconstrained scalarized primal problem

$$\min\{F_\pi(x, z_\Pi) : \ x \in X\}. \tag{SP-(π, z_Π)}$$

The corresponding optimal value function $\Phi_\pi : Z_\Pi \to \bar{\mathbb{R}}$ is defined as

$$\Phi_\pi(z_\Pi) := \inf\{F_\pi(x, z_\Pi) : \ x \in X\}.$$

The Lagrange function is given corresponding to the scalarized primal objective function F_π of the unconstrained problem and to the collection of functions W_Γ with $w(\cdot;\gamma) \in W_\Gamma$, i.e.,

$$L_\pi(x,\gamma) := \inf\{F_\pi(x,z_\Pi) - w(z_\Pi;\gamma) + w(0_{Z_\Pi};\gamma) : \; z_\Pi \in Z_\Pi\},$$

for every $x \in X$ and $\gamma \in \Gamma$. It is not hard to observe that

$$\begin{aligned} L_\pi(x,\gamma) &= \inf\{F_\pi(x,z_\Pi) - w(z_\Pi;\gamma) + w(0_{Z_\Pi};\gamma) : \; z_\Pi \in Z_\Pi\} \\ &= \inf\{p(f(x);\pi) - w(z_\Pi;\gamma) + w(0_{Z_\Pi};\gamma) : \; g_\pi(x) \succeq_{K_{Z_\Pi}} z_\Pi\} \\ &= p(f(x);\pi) - w(g_\pi(x);\gamma) + w(0_{Z_\Pi};\gamma), \end{aligned}$$

if $w(\cdot;\gamma)$ is K_{Z_Π}-monotone (i.e., $z_\Pi^1 \preceq_{K_{Z_\Pi}} z_\Pi^2$ implies $w(z_\Pi^1;\gamma) \leq w(z_\Pi^2;\gamma)$), for every $\gamma \in \Gamma$. Getting rid of the primal variable x, we obtain the objective function of the dual problem as

$$\begin{aligned} & \inf\{L_\pi(x,\gamma) : \; x \in X\} \\ = \; & \inf\{\inf\{F_\pi(x,z_\Pi) - w(z_\Pi;\gamma) + w(0_{Z_\Pi};\gamma) : \; z_\Pi \in Z_\Pi\} : \; x \in X\} \\ = \; & \inf\{\inf\{F_\pi(x,z_\Pi) : \; x \in X\} - w(z_\Pi;\gamma) + w(0_{Z_\Pi};\gamma) : \; z_\Pi \in Z_\Pi\} \\ = \; & -\sup\{-\Phi_\pi(z_\Pi) + w(z_\Pi;\gamma) : \; z_\Pi \in Z_\Pi\} + w(0_{Z_\Pi};\gamma) \\ = \; & -\Phi_\pi^{c(W_\Gamma)}(\gamma) + w(0_{Z_\Pi};\gamma), \end{aligned}$$

where $\Phi_\pi^{c(W_\Gamma)}$ is the W_Γ-conjugate function of Φ_π given by (2.3).

So, we consider the following dual problem to (SP-π):

$$\max\{-\Phi_\pi^{c(W_\Gamma)}(\gamma) + w(0_{Z_\Pi};\gamma) : \; \gamma \in \Gamma\}, \tag{SD-π}$$

with the optimal value

$$\begin{aligned} \mathrm{val}(\mathrm{SD}-\pi) &= \sup\{-\Phi_\pi^{c(W_\Gamma)}(\gamma) + w(0_{Z_\Pi};\gamma) : \; \gamma \in \Gamma\} \\ &= \Phi_\pi^{c(W_\Gamma)c(W_\Gamma)}(0_{Z_\Pi}). \end{aligned}$$

In the following theorem, we show a weak duality statement for (SP-π) and (SD-π) in scalarized form.

Theorem 5.1 (Weak duality for (SP-π) and (SD-π))**.** *For all $\bar{x} \in \{x \in X : \; g_\pi(x) \succeq_{K_{Z_\Pi}} 0_{Z_\Pi}\}$ and $\bar{\gamma} \in \Gamma$, it holds that*

$$-\Phi_\pi^{c(W_\Gamma)}(\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}) \leq p(f(\bar{x});\pi).$$

Proof. For any $\bar{\gamma} \in \Gamma$ and $\bar{x} \in \{x \in X : \; g_\pi(x) \succeq_{K_{Z_\Pi}} 0_{Z_\Pi}\}$, we obtain

$$\begin{aligned} & -\Phi_\pi^{c(W_\Gamma)}(\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}) \\ = \; & \inf\{\inf\{F_\pi(x,z_\Pi) - w(z_\Pi;\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}) : \; z_\Pi \in Z_\Pi\} : \; x \in X\} \\ \leq \; & F_\pi(\bar{x},z_\Pi) - w(z_\Pi;\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}), \end{aligned}$$

for any $z_\Pi \in Z_\Pi$. Particularly, setting $z_\Pi = 0_{Z_\Pi}$, we obtain

$$\begin{aligned} -\Phi_\pi^{c(W_\Gamma)}(\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}) &\leq F_\pi(\bar{x},z_\Pi) - w(0_{Z_\Pi};\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}) \\ &= p(f(\bar{x});\pi). \end{aligned}$$

□

Next, we discuss the zero duality gap and strong duality, for which we first consider the relationship to a separation result. Consider a parameter set Θ and assume that there exists another collection of functions

$$W_\Theta := \{w(\cdot;\theta) : Y \to \mathbb{R} \cup \{-\infty\} : \ \theta \in \Theta\},$$

such that

$$\{w(\cdot;\theta) + w(\cdot;\gamma) : Y \times Z_\Pi \to \mathbb{R} \cup \{-\infty\} : \ \theta \in \Theta, \ \gamma \in \Gamma\} \tag{5.1}$$

forms a collection of weak separation functions for (VOP) w.r.t. $\mathscr{H}_\Pi$ (or $\mathscr{H}^o_\Pi$), where

$$\mathscr{H}_\Pi := \operatorname{int} K_Y \times K_{Z_\Pi} \text{ and } \mathscr{H}^o_\Pi := (K_Y \setminus \{0_Y\}) \times K_{Z_\Pi}.$$

For the case that parameter π is only involved in the objective function, meaning that $g_\pi = g$ and $(Z_\Pi, K_{Z_\Pi}) = (Z, K_Z)$, there is the following result. This separation result can be considered as a conclusion from strong duality.

Theorem 5.2. *Let Π, Γ, and Θ be parameter sets such that (5.1) forms a collection of weak separation functions for (VOP). Consider the pair (SP-π) and (SD-π). Suppose that, for every $\pi \in \Pi$, there exists some $\theta_\pi \in \Theta$ and a function $w(\cdot;\theta_\pi) \in W_\Theta$ such that*

$$\forall y_1 \ y_2 \in Y : \quad w(y_1 - y_2;\theta_\pi) \le p(y_1;\pi) - p(y_2;\pi). \tag{5.2}$$

Let $\pi \in \Pi$ be arbitrary. Assume that $\bar{x}$ is a solution of $(SP - \pi)$ and $\bar{\gamma}$ is a solution of $(SD - \pi)$ with $val(SP - \pi) = val(SD - \pi)$. Then, there are separation functions $w(\cdot;\theta_\pi) \in W_\Theta$ and $w(\cdot;\bar{\gamma}) \in W_\Gamma$ such that

$$\forall x \in X : \quad w(f(\bar{x}) - f(x);\theta_\pi) + w(g(x);\bar{\gamma}) - w(0_Z;\bar{\gamma}) \le 0.$$

Proof. It follows from $\operatorname{val}(\mathrm{SP} - \pi) = \operatorname{val}(\mathrm{SD} - \pi)$ with solutions $\bar{x}$ and $\bar{\gamma}$ that

$$p(f(\bar{x});\pi) = -\Phi_\pi^{c(W_\Gamma)}(\bar{\gamma}) + w(0_Z;\bar{\gamma}).$$

Then,

$$\forall z \in Z : \quad -p(f(\bar{x});\pi) + w(0_Z;\bar{\gamma}) = \Phi_\pi^{c(W_\Gamma)}(\bar{\gamma}) \ge w(z;\bar{\gamma}) - \Phi_\pi(z).$$

Particularly, for any $x \in X$, setting $z = g(x)$, it gives that

$$-p(f(\bar{x});\pi) + w(0_Z;\bar{\gamma}) \ge w(g(x);\bar{\gamma}) - \Phi_\pi(g(x)) \ge w(g(x);\bar{\gamma}) - p(f(x);\pi),$$

i.e.,

$$\forall x \in X : p(f(\bar{x});\pi) - p(f(x);\pi) + w(g(x);\bar{\gamma}) - w(0_Z;\bar{\gamma}) \le 0.$$

Taking into account (5.2), for $\pi \in \Pi$, we see that there exists some $\theta_\pi \in \Theta$ such that

$$w(f(\bar{x}) - f(x);\theta_\pi) \le p(f(\bar{x});\pi) - p(f(x);\pi).$$

Then, the following separation result holds true

$$\forall x \in X : w(f(\bar{x}) - f(x);\theta_\pi) + w(g(x);\bar{\gamma}) - w(0_Z;\bar{\gamma}) \le 0.$$

□

Conversely, we obtain a strong duality statement from a separation result in the following theorem.

Theorem 5.3. *Let Π, Γ and Θ be parameter sets such that (5.1) forms a collection of weak separation functions for (VOP). Suppose, for every $\theta \in \Theta$, a function $w(\cdot;\theta) \in W_\Theta$ and $\bar{y} \in$ range f, there exists some $\pi_\theta \in \Pi$ such that*

$$\forall y \in Y: \quad w(\bar{y}-y;\theta) \geq p(\bar{y};\pi_\theta) - p(y;\pi_\theta),$$

while, for every $\gamma \in \Gamma$, $w(\cdot;\gamma) \in W_\Gamma$ is K_Z-monotone. Assume that there exist some $\bar{x}$ with $g(\bar{x}) \succeq_{K_Z} 0_Z$ and $\bar{\gamma} \in \Gamma$, $\bar{\theta} \in \Theta$ such that

$$\forall x \in X: \quad w(f(\bar{x})-f(x);\bar{\theta}) + w(g(x);\bar{\gamma}) - w(0_Z;\bar{\gamma}) \leq 0 \tag{5.3}$$

for $w(\cdot;\bar{\theta}) \in W_\Theta$ and $w(\cdot;\bar{\gamma}) \in W_\Gamma$. Then, there exists some $\pi_{\bar{\theta}} \in \Pi$ for which

$$val(SP-\pi_{\bar{\theta}}) = val(SD-\pi_{\bar{\theta}})$$

and $\bar{x} \in sol(SP-\pi_{\bar{\theta}})$, $\bar{\gamma} \in sol(SD-\pi_{\bar{\theta}})$.

Proof. For an arbitrary $z \in Z$, consider the $x \in X$ with $g(x) \succeq_{K_Z} z$. It follows from (5.3) and the monotonicity property of $w(\cdot;\bar{\gamma})$ that

$$w(z;\bar{\gamma}) - w(0_Z;\bar{\gamma}) \leq w(g(x);\bar{\gamma}) - w(0_Z;\bar{\gamma}) \leq -w(f(\bar{x})-f(x);\bar{\theta}).$$

Since, for $\bar{\theta} \in \Theta$ and $f(\bar{x}) \in$ range f, there exists some $\pi_{\bar{\theta}} \in \Pi$ such that

$$w(f(\bar{x})-f(x);\bar{\theta}) \geq p(f(\bar{x});\pi_{\bar{\theta}}) - p(f(x);\pi_{\bar{\theta}}),$$

we obtain

$$w(z;\bar{\gamma}) - w(0_Z;\bar{\gamma}) + p(f(\bar{x});\pi_{\bar{\theta}}) \leq p(f(x);\pi_{\bar{\theta}}).$$

This holds for all $z \in Z$ and $x \in X$ with $g(x) \succeq_{K_Z} z$, which indicates that

$$\begin{aligned} \forall z \in Z: \ w(z;\bar{\gamma}) - w(0_Z;\bar{\gamma}) + p(f(\bar{x});\pi_{\bar{\theta}}) &\leq \inf\{p(f(x);\pi_{\bar{\theta}}) : g(x) \succeq_{K_Z} z\} \\ &= \Phi_{\pi_{\bar{\theta}}}(z). \end{aligned}$$

Hence,

$$\forall z \in Z: \quad w(z;\bar{\gamma}) - \Phi_{\pi_{\bar{\theta}}}(z) \leq w(0_Z;\bar{\gamma}) - p(f(\bar{x});\pi_{\bar{\theta}}),$$

which shows that

$$p(f(\bar{x});\pi_{\bar{\theta}}) \leq -\Phi_{\pi_{\bar{\theta}}}^{c(W_\Gamma)}(\bar{\gamma}) + w(0_Z;\bar{\gamma}).$$

Then, due to the weak duality assertion in Theorem 5.1, this means $\mathrm{val}(\mathrm{SP}-\pi_{\bar{\theta}}) = \mathrm{val}(\mathrm{SD}-\pi_\theta)$, while, $\bar{x}$ and $\bar{\gamma}$ are solutions of $(\mathrm{SP}-\pi_{\bar{\theta}})$ and $(\mathrm{SD}-\pi_{\bar{\theta}})$, respectively. □

Remark 5.2. Theorem 5.3 works if we give the stronger assumptions that for every $\theta \in \Theta$, there exists some $\pi_\theta \in \Pi$ such that

$$\forall y_1,\ y_2\ \in Y: \quad w(y_1-y_2;\theta) \geq p(y_1;\pi_\theta) - p(y_2;\pi_\theta),$$

and, for every $\gamma \in \Gamma$, $w(\cdot;\gamma)$ is K_Z-monotone. These conditions only depend on the choice of $\{p(\cdot;\pi) : \pi \in \Pi\}$, $\{w(\cdot;\theta) : \theta \in \Theta\}$ and $\{w(\cdot;\gamma) : \gamma \in \Gamma\}$. It is independent of the specific (VOP).

In the following remarks, we show that the conditions of Theorems 5.2, 5.3 are fulfilled for the collections of separation functions $\mathcal{W}_{\Pi_2}^{\varphi}$ and $\mathcal{W}_{\Pi_3}^{\varphi}$.

Remark 5.3. Taking Example 5.2, for instance, the scalar problem is

$$\{\min\{\xi_{e_Y}(f(x)-f(\hat{x})) : g(x) \succeq_{K_Z} 0_Z\} : \ (\hat{x}, e_Y) \in X \times \operatorname{int} K_Y\}, \qquad (\text{SP-}(\hat{x}, e_Y))$$

with parameter $(\hat{x}, e_Y) \in X \times \operatorname{int} K_Y$, meaning that $\Pi = X \times \operatorname{int} K_Y$, $\pi = (\hat{x}, e_Y)$, while,

$$p(y;\pi) = p(y;\hat{x}, e_Y) = \xi_{e_Y}(y - f(\hat{x})), \ g_\pi = g, \ (Z_\Pi, K_{Z_\Pi}) = (Z, K_Z).$$

Consider the collection of separation functions given by (4.3) (this is a class of regular weak separation functions; see Proposition 4.1):

$$\mathcal{W}^{\varphi}_{\Pi_2} = \{\varphi_{e_Y}(\cdot) + \varphi_{e_Z}(\cdot) : Y \times Z \to \mathbb{R} : e_Y \in \operatorname{int} K_Y, \ e_Z \in (\operatorname{int} K_Z \cup \{0_Z\})\},$$

meaning that $\Theta = \operatorname{int} K_Y$, $\Gamma = (\operatorname{int} K_Z \cup \{0_Z\})$, and $w(\cdot, \theta) = \varphi_{e_Y}(\cdot)$, $w(\cdot, \gamma) = \varphi_{e_Z}(\cdot)$ (see the collection of functions in (5.1)). Picking an arbitrary $\pi = (\hat{x}, e_Y) \in \Pi$,

$$\begin{aligned} p(y_1;\hat{x}, e_Y) - p(y_2;\hat{x}, e_Y) &= \xi_{e_Y}(y_1 - f(\hat{x})) - \xi_{e_Y}(y_2 - f(\hat{x})) \\ &= -\varphi_{e_Y}(f(\hat{x}) - y_1) + \varphi_{e_Y}(f(\hat{x}) - y_2) \\ &\geq \varphi_{e_Y}(y_1 - y_2), \end{aligned}$$

for all $y_1, \ y_2 \in Y$. Hence, for every $\pi = (\hat{x}, e_Y) \in \Pi$, set $\theta_{(\hat{x}, e_Y)} = e_Y$, then

$$\forall y_1 \ y_2 \in Y : \quad w(y_1 - y_2; \theta_\pi) \leq p(y_1;\pi) - p(y_2;\pi).$$

Conversely, pick an arbitrary $\theta \in \Theta$, i.e., $e_Y \in \operatorname{int} K_Y$, and $\bar{y} \in \operatorname{range} f$, assuming that $f(\bar{x}) = \bar{y}$. For any $y \in Y$, there is

$$\begin{aligned} w(\bar{y} - y; \theta) &= \varphi_{e_Y}(\bar{y} - y) \\ &= -\xi_{e_Y}(y - \bar{y}) \\ &= \xi_{e_Y}(\bar{y} - f(\bar{x})) - \xi_{e_Y}(y - f(\bar{x})) \\ &= p(\bar{y};\bar{x}, e_Y) - p(y;\bar{x}, e_Y). \end{aligned}$$

Hence, for arbitrary $e_Y \in \operatorname{int} K_Y$, and $\bar{y} \in \operatorname{range} f$, picking some $\bar{x} \in f^{-1}(\bar{y})$ and setting $\pi_\theta = (\bar{x}, e_Y)$, one has

$$\forall y \in Y : \quad w(\bar{y} - y; \theta) \geq p(\bar{y}; \pi_\theta) - p(y; \pi_\theta).$$

Meanwhile, for every $\gamma \in \Gamma$, i.e., $e_Z \in \operatorname{int} K_Z \cup \{0_Z\}$, according to Proposition 2.1, $w(\cdot;\gamma) = \varphi_{e_Z}(\cdot)$ is K_Z-monotone. Thus, the assumptions in Theorems 5.2 and 5.3 are all satisfied.

Remark 5.4. As for the scalar problem

$$\min\{\alpha\varphi_{e_Y}(f(x) - f(\hat{x})) + (1-\alpha)\xi_{e_Y}(f(x) - f(\hat{x})) : \ g(x) \succeq_{K_Z} 0_Z\}, \qquad (\text{SP}_e\text{-}(\hat{x}, \alpha, e_Y))$$

given in Example 5.4, in which $(\hat{x}, \alpha, e_Y) \in X \times (0,1) \times \operatorname{int} K_Y$ is the parameter, meaning that $\Pi = X \times (0,1) \times \operatorname{int} K_Y$, $\pi = (\hat{x}, \alpha, e_Y)$ and

$$p(y;\pi) = p(y;\hat{x}, \alpha, e_Y) = \alpha\varphi_{e_Y}(f(x) - f(\hat{x})) + (1-\alpha)\xi_{e_Y}(f(x) - f(\hat{x})),$$

$g_\pi = g$, $(Z_\Pi, K_{Z_\Pi}) = (Z, K_Z)$.

If we consider the collection of separation functions given by (4.3) (this is a class of regular weak separation functions, see Proposition 4.1):

$$\mathcal{W}^{\varphi}_{\Pi_2} = \{\varphi_{e_Y}(\cdot) + \varphi_{e_Z}(\cdot) : Y \times Z \to \mathbb{R} : e_Y \in \operatorname{int} K_Y, \ e_Z \in (\operatorname{int} K_Z \cup \{0_Z\})\},$$

then $\Theta = \operatorname{int} K_Y$, $\Gamma = (\operatorname{int} K_Z \cup \{0_Z\})$, and $w(\cdot;\theta) = \varphi_{e_Y}(\cdot)$, $w(\cdot;\gamma) = \varphi_{e_Z}(\cdot)$. Picking an arbitrary $\pi = (\hat{x}, \alpha, e_Y) \in \Pi$, one has

$$\begin{aligned} p(y_1;\hat{x},\alpha,e_Y) - p(y_2;\hat{x},\alpha,e_Y) &= \alpha\varphi_{e_Y}(y_1 - f(\hat{x})) + (1-\alpha)\xi_{e_Y}(y_1 - f(\hat{x})) \\ &\quad -\alpha\varphi_{e_Y}(y_2 - f(\hat{x})) - (1-\alpha)\xi_{e_Y}(y_2 - f(\hat{x})) \\ &= \alpha\varphi_{e_Y}(y_1 - f(\hat{x})) - \alpha\varphi_{e_Y}(y_2 - f(\hat{x})) \\ &\quad +(1-\alpha)\xi_{e_Y}(y_1 - f(\hat{x})) - (1-\alpha)\xi_{e_Y}(y_2 - f(\hat{x})) \\ &\geq \alpha\varphi_{e_Y}(y_1 - y_2) + (1-\alpha)\varphi_{e_Y}(y_1 - y_2) \\ &= \varphi_{e_Y}(y_1 - y_2), \end{aligned}$$

for all $y_1,\ y_2 \in Y$. Hence, for every $\pi = (\hat{x}, \alpha, e_Y) \in \Pi$, setting $\theta_{(\hat{x},\alpha,e_Y)} = e_Y$, one has

$$\forall y_1\ y_2 \in Y: \quad w(y_1 - y_2;\theta_\pi) \leq p(y_1;\pi) - p(y_2;\pi), i.e.,$$

the condition (5.2) in Theorem 5.2 is satisfied.

If what we consider is the collection of separation functions given by (4.5) (this is a class of regular weak separation functions; see Proposition 4.2):

$$\begin{aligned} \mathcal{W}^{\varphi}_{\Pi_3} = \{(\alpha\varphi_{e_Y} + (1-\alpha)\xi_{e_Y})(\cdot) + \varphi_{e_Z}(\cdot) : Y \times Z \to \mathbb{R} \mid \\ \alpha \in (0,1), e_Y \in \operatorname{int} K_Y, e_Z \in (\operatorname{int} K_Z \cup \{0_Z\})\}, \end{aligned}$$

then $\Theta = (0,1) \times \operatorname{int} K_Y$, $\Gamma = (\operatorname{int} K_Z \cup \{0_Z\})$, and $w(\cdot;\theta) = (\alpha\varphi_{e_Y} + (1-\alpha)\xi_{e_Y})(\cdot)$, $w(\cdot;\gamma) = \varphi_{e_Z}(\cdot)$ (see the collection of functions in (5.1)). Picking an arbitrary $\theta \in \Theta$, i.e., $(\alpha, e_Y) \in (0,1) \times \operatorname{int} K_Y$, and $\bar{y} \in \operatorname{range} f$ with $f(\bar{x}) = \bar{y}$, for any $y \in Y$,

$$\begin{aligned} w(\bar{y} - y;\theta) &= \alpha\varphi_{e_Y}(\bar{y} - y) + (1-\alpha)\xi_{e_Y}(\bar{y} - y) \\ &= -\alpha\xi_{e_Y}(y - \bar{y}) - (1-\alpha)\varphi_{e_Y}(y - \bar{y}) \\ &= (1-\alpha)\varphi_{e_Y}(\bar{y} - f(\bar{x})) + \alpha\xi_{e_Y}(\bar{y} - f(\bar{x})) \\ &\quad -(1-\alpha)\varphi_{e_Y}(y - f(\bar{x})) - \alpha\xi_{e_Y}(y - f(\bar{x})) \\ &= p(\bar{y};\bar{x},1-\alpha,e_Y) - p(y;\bar{x},1-\alpha,e_Y). \end{aligned}$$

Hence, for arbitrary $(\alpha, e_Y) \in (0,1) \times \operatorname{int} K_Y$, and $\bar{y} \in \operatorname{range} f$, taking some $\bar{x} \in f^{-1}(\bar{y})$ and setting $\pi_\theta = (\bar{x}, 1-\alpha, e_Y)$, one has

$$w(\bar{y} - y;\theta) \geq p(\bar{y};\pi_\theta) - p(y;\pi_\theta),\ \forall y \in Y.$$

Besides, applying Proposition 2.1, $w(\cdot;\gamma) = \varphi_{e_Z}(\cdot)$ is K_Z-monotone for every $\gamma \in \Gamma$. Thus, the conditions in Theorem 5.3 are all satisfied.

Now, we try to consider the collection of scalar problems like what is given in Example 5.5,

$$\min\{\hat{l}(f(x)) :\ f(x) \preceq_{K_Y} f(\hat{x}),\ g(x) \succeq_{K_Z} 0_Z\}, \qquad \text{(SP-}(\hat{x},\hat{l}))$$

for which the parameters are not only given in objective function for scalarization, but also involved in constraint as level sets. Assuming that $\hat{x} \in \mathscr{R} = \{x \in X : g(x) \succeq_{K_Z} 0_Z\}$ and $\hat{l} \in \hat{L}$, where $\hat{L}$ is a group of scalarization functions. Then, $\Pi = \mathscr{R} \times \hat{L}$ and $\pi = (\hat{x}, \hat{l})$. Meanwhile, $g_\pi(\cdot) = (f(\hat{x}) - f(\cdot), g(\cdot))$, $Z_\Pi = Y \times Z$, and $K_{Z_\Pi} = K_Y \times K_Z$. The corresponding collection of weak separation functions has the following form

$$\{w(y;\theta) + w(y', z;\gamma) : Y \times Y \times Z \to \mathbb{R} \cup \{-\infty\} : \theta \in \Theta, \gamma \in \Gamma\}. \qquad (5.4)$$

In this case, we have the following assertions concerning a conclusion from strong duality.

Theorem 5.4. *Consider a group $\hat{L}$ of scalarization functions and $\Pi = \mathscr{R} \times \hat{L}$. Let Γ and Θ be parameter sets such that (5.4) forms a collection of weak separation functions for (VOP). Assume that for every $\hat{l} \in \hat{L}$, there exists some $\theta_{\hat{l}} \in \Theta$ such that for $w(\cdot;\theta_{\hat{l}}) \in W_\Theta$ holds*

$$\forall y_1\ y_2 \in Y: \quad w(y_1 - y_2;\theta_{\hat{l}}) \le \hat{l}(y_1) - \hat{l}(y_2),$$

while, for every $\gamma \in \Gamma$, $w(\cdot,\cdot;\gamma)$ is $(K_Y \times K_Z)$-monotone. Let $\pi = (\hat{x},\hat{l}) \in \Pi$ be arbitrary. Suppose that $\bar{x}$ is a solution of $(SP-(\hat{x},\hat{l}))$ and $\bar{\gamma}$ is a solution of $(SD-(\hat{x},\hat{l}))$ satisfying $val(SP-(\hat{x},\hat{l})) = val(SD-(\hat{x},\hat{l}))$. Then, there are separation functions $w(\cdot;\theta_{\hat{l}})$ and $w(\cdot,\cdot;\bar{\gamma})$ corresponding to (5.4) such that

$$\forall x \in X: \quad w(f(\bar{x}) - f(x);\theta_{\hat{l}}) + w(f(\bar{x}) - f(x), g(x);\bar{\gamma}) - w(0_Y, 0_Z;\bar{\gamma}) \le 0.$$

Proof. $\mathrm{val}(\mathrm{SP}-(\hat{x},\hat{l})) = \mathrm{val}(\mathrm{SD}-(\hat{x},\hat{l}))$ with solutions $\bar{x}$ and $\bar{\gamma}$ means

$$\hat{l}(f(\bar{x})) = -\Phi_\pi^{c(W_\Gamma)}(\bar{\gamma}) + w(0_Y, 0_Z;\bar{\gamma}),$$

where $\pi = (\hat{x},\hat{l})$. Besides, there are $g(\bar{x}) \succeq_{K_Z} 0_Z$ and $f(\bar{x}) \preceq_{K_Z} f(\hat{x})$. Then,

$$\forall (y,z) \in Y \times Z: \ -\hat{l}(f(\bar{x})) + w(0_Y, 0_Z;\bar{\gamma}) = \Phi_\pi^{c(W_\Gamma)}(\bar{\gamma}) \ge w(y,z;\bar{\gamma}) - \Phi_\pi(y,z).$$

Particularly, for any $x \in X$, setting $y = f(\hat{x}) - f(x)$ and $z = g(x)$, it can be deduced that

$$\begin{aligned} & -\hat{l}(f(\bar{x})) + w(0_Y, 0_Z;\bar{\gamma}) \\ \ge\ & w(f(\hat{x}) - f(x), g(x);\bar{\gamma}) - \Phi_\pi(f(\hat{x}) - f(x), g(x)) \\ \ge\ & w(f(\hat{x}) - f(x), g(x);\bar{\gamma}) - \hat{l}(f(x)), \end{aligned}$$

i.e.,

$$\hat{l}(f(\bar{x})) - \hat{l}(f(x)) + w(f(\hat{x}) - f(x), g(x);\bar{\gamma}) - w(0_Y, 0_Z;\bar{\gamma}) \le 0.$$

For $\hat{l} \in \hat{L}$, there exists some $\theta_{\hat{l}} \in \Theta$ such that

$$w(f(\bar{x}) - f(x);\theta_{\hat{l}}) \le \hat{l}(f(\bar{x})) - \hat{l}(f(x)).$$

Then, together with the monotonicity property of $w(\cdot,\cdot;\bar{\gamma})$ and $f(\bar{x}) \preceq_{K_Y} f(\hat{x})$, we can deduce

$$w(f(\bar{x}) - f(x);\theta_{\hat{l}}) + w(f(\bar{x}) - f(x), g(x);\bar{\gamma}) - w(0_Y, 0_Z;\bar{\gamma}) \le 0.$$

□

The inverse is given as follows.

Theorem 5.5. *Consider a group $\hat{L}$ of scalarization functions and $\Pi = \mathscr{R} \times \hat{L}$. Let Γ and Θ be parameter sets such that (5.4) forms a collection of weak separation functions for (VOP). Suppose for every $\theta \in \Theta$ and $\bar{y} \in$ range f, there exists some $\hat{l}_\theta \in \hat{L}$ such that*

$$\forall y \in Y: \quad w(\bar{y} - y;\theta) \ge \hat{l}_\theta(\bar{y}) - \hat{l}_\theta(y),$$

while, for every $\gamma \in \Gamma$, $w(\cdot,\cdot;\gamma)$ is $(K_Y \times K_Z)$-monotone. Assume that there exist some $\bar{x}$ with $g(\bar{x}) \succeq_{K_Z} 0_Z$ and $\bar{\gamma} \in \Gamma$, $\bar{\theta} \in \Theta$ such that

$$\forall x \in X: \quad w(f(\bar{x}) - f(x);\bar{\theta}) + w(f(\bar{x}) - f(x), g(x);\bar{\gamma}) - w(0_Y, 0_Z;\bar{\gamma}) \le 0. \tag{5.5}$$

Then, there exists some $\pi_{\bar{\theta}} = (\hat{x}_{\bar{\theta}},\hat{l}_{\bar{\theta}}) \in \Pi$ for which

$$val(SP-(\hat{x}_{\bar{\theta}},\hat{l}_{\bar{\theta}})) = val(SD-(\hat{x}_{\bar{\theta}},\hat{l}_{\bar{\theta}}))$$

and $\bar{x} \in sol(SP-(\hat{x}_{\bar{\theta}},\hat{l}_{\bar{\theta}}))$, $\bar{\gamma} \in sol(SD-(\hat{x}_{\bar{\theta}},\hat{l}_{\bar{\theta}}))$.

Proof. Picking an arbitrary $(y,z) \in Y \times Z$ for the $x \in X$ satisfying $g(x) \succeq_{K_Z} z$ and $f(\bar{x}) - f(x) \succeq_{K_Y} y$, (5.5) and the monotonicity property of $w(\cdot,\cdot;\bar{\gamma})$ imply that

$$\begin{aligned} w(y,z;\bar{\gamma}) - w(0_Y,0_Z;\bar{\gamma}) &\leq w(f(\bar{x}) - f(x), g(x);\bar{\gamma}) - w(0_Y,0_Z;\bar{\gamma}) \\ &\leq -w(f(\bar{x}) - f(x);\bar{\theta}). \end{aligned}$$

Since, for $\bar{\theta} \in \Theta$ and $f(\bar{x}) \in$ range f, there exists some $\hat{l}_{\bar{\theta}} \in \hat{L}$ such that

$$w(f(\bar{x}) - f(x);\bar{\theta}) \geq \hat{l}_{\bar{\theta}}(f(\bar{x})) - \hat{l}_{\bar{\theta}}(f(x)),$$

we obtain

$$w(y,z;\bar{\gamma}) - w(0_Y,0_Z;\bar{\gamma}) + \hat{l}_{\bar{\theta}}(f(\bar{x})) \leq \hat{l}_{\bar{\theta}}(f(x)).$$

This holds for all $(y,z) \in Y \times Z$ and $x \in X$ with $g(x) \succeq_{K_Z} z$, $f(\bar{x}) - f(x) \succeq_{K_Y} y$, showing that

$$\forall (y,z) \in Y \times Z: \quad w(y,z;\bar{\gamma}) - w(0_Y,0_Z;\bar{\gamma}) + \hat{l}_{\bar{\theta}}(f(\bar{x})) \leq \Phi_{\pi_{\bar{\theta}}}(y,z),$$

where $\pi_{\bar{\theta}} = (\hat{x}_{\bar{\theta}}, \hat{l}_{\bar{\theta}}) = (\bar{x}, \hat{l}_{\bar{\theta}})$. Therefore,

$$\forall (y,z) \in Y \times Z: \quad w(y,z;\bar{\gamma}) - \Phi_{\pi_{\bar{\theta}}}(y,z) \leq w(0_Y,0_Z;\bar{\gamma}) - \hat{l}_{\bar{\theta}}(f(\bar{x})),$$

which implies

$$\hat{l}_{\bar{\theta}}(f(\bar{x})) \leq -\Phi_{\pi_{\bar{\theta}}}^{c(W_\Gamma)}(\bar{\gamma}) + w(0_Y,0_Z;\bar{\gamma}).$$

Then, due to the weak duality in Theorem 5.1, this means $\text{val}(\text{SP} - \pi_{\bar{\theta}}) = \text{val}(\text{SD} - \pi_{\bar{\theta}})$, while, $\bar{x}$ and $\bar{\gamma}$ are solutions of $(\text{SP} - \pi_{\bar{\theta}})$ and $(\text{SD} - \pi_{\bar{\theta}})$, respectively. Taking into account $\pi_{\bar{\theta}} = (\hat{x}_{\bar{\theta}}, \hat{l}_{\bar{\theta}}) = (\bar{x}, \hat{l}_{\bar{\theta}})$, we get $\text{val}(\text{SP} - (\hat{x}_{\bar{\theta}}, \hat{l}_{\bar{\theta}})) = \text{val}(\text{SD} - (\hat{x}_{\bar{\theta}}, \hat{l}_{\bar{\theta}}))$ and $\bar{x} \in \text{sol}(\text{SP} - (\hat{x}_{\bar{\theta}}, \hat{l}_{\bar{\theta}}))$, $\bar{\gamma} \in \text{sol}(\text{SD} - (\hat{x}_{\bar{\theta}}, \hat{l}_{\bar{\theta}}))$. □

Remark 5.5. In Example 5.5, for the class of scalar problems

$$\min\{\hat{l}(f(x)) : \ f(x) \preceq_{K_Y} f(\hat{x}), \ g(x) \succeq_{K_Z} 0_Z\}, \qquad (\text{SP-}(\hat{x},\hat{l}))$$

where $\hat{x} \in \mathscr{R}$ and $\hat{l} \in L_{K_Y \setminus \{0_Y\}} \subset \{y^* \in Y^* : y^*(y) > 0, \ \forall y \in K_Y \setminus \{0_Y\}\}$. Hence, $\Pi = \mathscr{R} \times L_{K_Y \setminus \{0_Y\}}$ and $\pi = (\hat{x}, \hat{l})$. Meanwhile, $g_\pi(\cdot) = (f(\hat{x}) - f(\cdot), g(\cdot))$, $Z_\Pi = Y \times Z$ and $K_{Z_\Pi} = K_Y \times K_Z$. Consider the collection of functions

$$\begin{aligned} \mathcal{W}^{\hat{l},\varphi} = \{&\hat{l}(y) + \varphi_{(e_Y,e_Z)}(y',z) : Y \times Y \times Z \to \mathbb{R} \mid \\ &\hat{l} \in L_{K_Y \setminus \{0_Y\}}, (e_Y,e_Z) \in (\text{int}\, K_Y \times \text{int}\, K_Z) \cup \{(0_Y,0_Z)\}\}. \end{aligned}$$

In this situation, we have $\Theta = L_{K_Y \setminus \{0_Y\}}$, $\Gamma = (\text{int}\, K_Y \times \text{int}\, K_Z) \cup \{(0_Y,0_Z)\}$, and $w(\cdot;\theta) = \hat{l}(\cdot)$, $w(\cdot,\cdot;\gamma) = \varphi_{(e_Y,e_Z)}(\cdot,\cdot)$ involved in the collection of functions in (5.4). It is not hard to observe that this is a collection of regular weak separation functions w.r.t. $\mathscr{H}_\Pi^o$ if

$$\cap_{l \in L_{K_Y \setminus \{0_Y\}}} \text{lev}_{>0}\, l \subset K_Y \setminus \{0_Y\}.$$

Applying Proposition 2.1, for every $(e_Y,e_Z) \in (\text{int}\, K_Y \times \text{int}\, K_Z) \cup \{(0_Y,0_Z)\}$, $\varphi_{(e_Y,e_Z)}(\cdot,\cdot)$ is $(K_Y \times K_Z)$-monotone, while, every $\hat{l} \in L_{K_Y \setminus \{0_Y\}}$ is linear. Thus, the conditions in Theorems 5.2 and 5.3 are all satisfied. Then, for any $\pi = (\hat{x}, \hat{l}) \in \Pi$, if $\bar{x}$ is a solution of $(\text{SP} - (\hat{x}, \hat{l}))$ and $\bar{\gamma} = (\bar{e}_Y, \bar{e}_Z)$ is a solution of $(\text{SD} - (\hat{x}, \hat{l}))$, with $\text{val}(\text{SP} - (\hat{x}, \hat{l})) = \text{val}(\text{SD} - (\hat{x}, \hat{l}))$, then for $\theta_{\hat{l}} = \hat{l} \in L_{K_Y \setminus \{0_Y\}}$, one has

$$\forall x \in X: \quad \hat{l}(f(\bar{x}) - f(x)) + \varphi_{(\bar{e}_Y,\bar{e}_Z)}(f(\bar{x}) - f(x), g(x)) \leq \ 0.$$

Conversely, if there exist some $\bar{x}$ with $g(\bar{x}) \succeq_{K_Z} 0_Z$ and $\bar{\gamma} = (\bar{e}_Y, \bar{e}_Z) \in \Gamma$, $\bar{l} \in L_{K_Y \setminus \{0_Y\}}$, such that

$$\forall x \in X: \quad \bar{l}(f(\bar{x}) - f(x)) + \varphi_{(\bar{e}_Y, \bar{e}_Z)}(f(\bar{x}) - f(x), g(x)) \leq \quad 0.$$

then, setting $\pi_{\bar{l}} = (\bar{x}, \bar{l})$, we have $\mathrm{val}(\mathrm{SP} - (\bar{x}, \bar{l})) = \mathrm{val}(\mathrm{SD} - (\bar{x}, \bar{l}))$ and $\bar{x} \in \mathrm{sol}(\mathrm{SP} - (\bar{x}, \bar{l}))$, $(\bar{e}_Y, \bar{e}_Z) \in \mathrm{sol}(\mathrm{SD} - (\bar{x}, \bar{l}))$.

Next, we consider the correlation between the strong duality and the saddle points. We consider the general situation, i.e., the scalarized problem

$$\min\{p(f(x); \pi) : \ g_\pi(x) \succeq_{K_{Z_\Pi}} 0_{Z_\Pi}\}, \tag{SP-π}$$

with $\mathscr{R}_\pi \neq \emptyset$ for every $\pi \in \Pi$, where $\pi \in \Pi$ is the parameter, and the collection of functions

$$W_\Gamma = \{w(\cdot; \gamma) : Z_\Pi \to \bar{\mathbb{R}} : \ \gamma \in \Gamma\}.$$

Recall that the corresponding Lagrange function for $(\mathrm{SP} - \pi)$ is

$$L_\pi(x, \gamma) = \inf\{F_\pi(x, z_\Pi) - w(z_\Pi; \gamma) + w(0_{Z_\Pi}; \gamma) : \ z_\Pi \in Z_\Pi\},$$

for every $x \in X$ and $\gamma \in \Gamma$. If, in addition, $w(\cdot; \gamma)$ is K_{Z_Π}-monotone for every $\gamma \in \Gamma$, then

$$L_\pi(x, \gamma) = p(f(x); \pi) - w(g_\pi(x); \gamma) + w(0_{Z_\Pi}; \gamma).$$

For a pair $(\bar{x}, \bar{\gamma}) \in X \times \Gamma$, we say it is a saddle point of $(\mathrm{SP} - \pi)$ if

$$\forall x \in X, \ \gamma \in \Gamma: \quad L_\pi(\bar{x}, \gamma) \leq L_\pi(\bar{x}, \bar{\gamma}) \leq L_\pi(x, \bar{\gamma}). \tag{5.6}$$

In the next theorem, we show the relationship between strong duality and the existence of a saddle point.

Theorem 5.6. *Let Π and Γ be parameter sets and W_Γ be a collection of functions. Suppose that $w(\cdot; \gamma) \in W_\Gamma$ is K_{Z_Π}-monotone for every $\gamma \in \Gamma$, and*

$$\inf\{w(z_\Pi; \gamma) - w(0_{Z_\Pi}; \gamma) : \gamma \in \Gamma\} = \begin{cases} 0 & \text{if } z_\Pi \succeq_{K_{Z_\Pi}} 0_{Z_\Pi}, \\ -\infty & \text{if } z_\Pi \not\succeq_{K_{Z_\Pi}} 0_{Z_\Pi}. \end{cases} \tag{5.7}$$

Let $\pi \in \Pi$ be an arbitrary element. Then, a pair $(\bar{x}, \bar{\gamma}) \in X \times \Gamma$ is a saddle point of $(SP - \pi)$ if and only if

$$val(SP - \pi) = val(SD - \pi),$$

while, $\bar{x}$ and $\bar{\gamma}$ are solutions of $(SP - \pi)$ and $(SD - \pi)$, respectively.

Proof. Suppose that $(\bar{x}, \bar{\gamma}) \in X \times \Gamma$ is a saddle point of $(\mathrm{SP} - \pi)$, meaning that

$$\forall x \in X, \ \gamma \in \Gamma: \ L_\pi(\bar{x}, \gamma) \leq L_\pi(\bar{x}, \bar{\gamma}) \leq L_\pi(x, \bar{\gamma}),$$

i.e.,

$$\forall \gamma \in \Gamma: \ p(f(\bar{x}); \pi) - w(g_\pi(\bar{x}); \gamma) + w(0_{Z_\Pi}; \gamma) \leq p(f(\bar{x}); \pi) - w(g_\pi(\bar{x}); \bar{\gamma}) + w(0_{Z_\Pi}; \bar{\gamma}), \tag{5.8}$$

and

$$\forall x \in X: \ p(f(\bar{x}); \pi) - w(g_\pi(\bar{x}); \bar{\gamma}) + w(0_{Z_\Pi}; \bar{\gamma}) \leq p(f(x); \pi) - w(g_\pi(x); \bar{\gamma}) + w(0_{Z_\Pi}; \bar{\gamma}). \tag{5.9}$$

It follows from (5.9) that $w(g_\pi(\bar{x}); \bar{\gamma}) - w(0_{Z_\Pi}; \bar{\gamma}) \neq -\infty$, while, with (5.8), we obtain

$$\forall \gamma \in \Gamma: \quad w(g_\pi(\bar{x}); \gamma) - w(0_{Z_\Pi}; \gamma) \geq w(g_\pi(\bar{x}); \bar{\gamma}) - w(0_{Z_\Pi}; \bar{\gamma}).$$

Together with (5.7), we can deduce that $g(\bar{x}) \in K_{Z_\Pi}$ and

$$w(g_\pi(\bar{x});\bar{\gamma}) - w(0_{Z_\Pi};\bar{\gamma}) = \inf\{w(g_\pi(\bar{x});\gamma) - w(0_{Z_\Pi};\gamma) : \ \gamma \in \Gamma\} = 0.$$

Then, (5.9) becomes

$$\forall x \in X : \quad p(f(\bar{x});\pi) + w(g_\pi(x);\bar{\gamma}) - w(0_{Z_\Pi};\bar{\gamma}) \leq p(f(x);\pi).$$

Then, for an arbitrary $z_\Pi \in Z_\Pi$, considering those $x \in X$ with $g_\pi(x) \succeq_{K_{Z_\Pi}} z_\Pi$, as $w(\cdot;\bar{\gamma})$ is K_{Z_Π}-monotone, there is

$$p(f(\bar{x});\pi) + w(z_\Pi;\bar{\gamma}) - w(0_{Z_\Pi};\bar{\gamma}) \leq p(f(x);\pi).$$

This implies

$$p(f(\bar{x});\pi) + w(z_\Pi;\bar{\gamma}) - w(0_{Z_\Pi};\bar{\gamma}) \leq \Phi_\pi(z_\Pi),$$

i.e.,

$$w(z_\Pi;\bar{\gamma}) - \Phi_\pi(z_\Pi) \leq -p(f(\bar{x});\pi) + w(0_{Z_\Pi};\bar{\gamma}).$$

Due to the arbitrariness of z_Π, we can get

$$p(f(\bar{x});\pi) \leq -\Phi_\pi^{c(W_\Gamma)}(\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}).$$

With the weak duality assertion in Theorem 5.1, we can conclude that

$$\mathrm{val}(\mathrm{SP}-\pi) = p(f(\bar{x});\pi) = -\Phi_\pi^{c(W_\Gamma)}(\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}) = \mathrm{val}(\mathrm{SD}-\pi),$$

while, $\bar{x} \in \mathrm{sol}(\mathrm{SP}-\pi)$ and $\bar{\gamma} \in \mathrm{sol}(\mathrm{SD}-\pi)$. Conversely, suppose that $\bar{x}$ is feasible for $(\mathrm{SP}-\pi)$, and

$$p(f(\bar{x});\pi) = -\Phi_\pi^{c(W_\Gamma)}(\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}).$$

Then,

$$\begin{aligned} \forall z_\Pi \in Z_\Pi : \quad -p(f(\bar{x});\pi) + w(0_{Z_\Pi};\bar{\gamma}) &= \Phi_\pi^{c(W_\Gamma)}(\bar{\gamma}) \\ &\geq w(z_\Pi;\bar{\gamma}) - \Phi_\pi(z_\Pi). \end{aligned} \tag{5.10}$$

Pick an arbitrary $x \in X$, setting $z_\Pi = g_\pi(x)$. Then, obviously $g_\pi(x) \succeq_{K_{Z_\Pi}} z_\Pi$, and therefore, $\Phi_\pi(z_\Pi) = \Phi_\pi(g_\pi(x)) \leq p(f(x);\pi)$. With (5.10), we obtain

$$\forall x \in X : \quad -p(f(\bar{x});\pi) + w(0_{Z_\Pi};\bar{\gamma}) \geq w(g_\pi(x);\bar{\gamma}) - p(f(x);\pi),$$

i.e.,

$$\forall x \in X : \quad p(f(\bar{x});\pi) \leq p(f(x);\pi) - w(g_\pi(x);\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}).$$

Particularly, for the case where $x = \bar{x}$, there is

$$w(g_\pi(\bar{x});\bar{\gamma}) - w(0_{Z_\Pi};\bar{\gamma}) \leq 0.$$

On the other hand, since $g_\pi(\bar{x}) \succeq_{K_{Z_\Pi}} 0_{Z_\Pi}$, we have

$$\begin{aligned} w(g_\pi(\bar{x});\bar{\gamma}) - w(0_{Z_\Pi};\bar{\gamma}) &\geq \inf\{w(g_\pi(\bar{x});\gamma) - w(0_{Z_\Pi};\gamma) : \gamma \in \Gamma\} \\ &= 0. \end{aligned}$$

Hence,

$$w(g_\pi(\bar{x});\bar{\gamma}) - w(0_{Z_\Pi};\bar{\gamma}) = 0.$$

Then, from (5.11), we obtain

$$\forall x \in X : \ p(f(\bar{x});\pi) - w(g_\pi(\bar{x});\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}) \leq p(f(x);\pi) - w(g_\pi(x);\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}),$$

i.e.,

$$\forall x \in X : \ L_\pi(\bar{x},\bar{\gamma}) \leq L_\pi(x,\bar{\gamma}).$$

Also, as

$$w(g_\pi(\bar{x});\bar{\gamma}) - w(0_{Z_\Pi};\bar{\gamma}) = 0 = \inf\{w(g_\pi(\bar{x});\gamma) - w(0_{Z_\Pi};\gamma) : \gamma \in \Gamma\},$$

we have

$$\forall \gamma \in \Gamma : \quad w(g_\pi(\bar{x});\bar{\gamma}) - w(0_{Z_\Pi};\bar{\gamma}) \leq w(g_\pi(\bar{x});\gamma) - w(0_{Z_\Pi};\gamma).$$

Thus, for all $\gamma \in \Gamma$,

$$p(f(\bar{x});\pi) - w(g_\pi(\bar{x});\gamma) + w(0_{Z_\Pi};\gamma) \leq p(f(\bar{x});\pi) - w(g_\pi(\bar{x});\bar{\gamma}) + w(0_{Z_\Pi};\bar{\gamma}),$$

meaning that, for all $\gamma \in \Gamma$, $L_\pi(\bar{x},\gamma) \leq L_\pi(\bar{x},\bar{\gamma})$. □

Remark 5.6. Taking Example 5.2, for instance, the scalarized problem is

$$\min\{\xi_{e_Y}(f(x) - f(\hat{x})) : g(x) \succeq_{K_Z} 0_Z\}, \qquad \text{(SP-}(\hat{x},e_Y))$$

for which $\pi = (\hat{x},e_Y)$ and $\Pi = X \times \operatorname{int} K_Y$, while, $p(y;\pi) = p(y;\hat{x},e_Y) = \xi_{e_Y}(y - f(\hat{x}))$, $g_\pi = g$, $(Z_\Pi, K_{Z_\Pi}) = (Z, K_Z)$. Regard the collection of functions

$$\{\varphi_{e_Z}(\cdot) : Z \to \mathbb{R} : \ e_Z \in \operatorname{int} K_Z \cup \{0_Z\}\}$$

as W_Γ, i.e., $\Gamma = \operatorname{int} K_Z \cup \{0_Z\}$, $w(\cdot;\gamma) = w(\cdot;e_Z) = \varphi_{e_Z}(\cdot)$. Since $\varphi_{e_Z}(\cdot)$ is K_Z-monotone for every $e_Z \in \operatorname{int} K_Z \cup \{0_Z\}$, the corresponding Lagrange function for $(\mathrm{SP} - (\hat{x},e_Y))$ is

$$\begin{aligned} L_{(\hat{x},e_Y)}(x,e_Z) &= \xi_{e_Y}(f(x) - f(\hat{x})) - \varphi_{e_Z}(g(x)) + \varphi_{e_Z}(0_Z) \\ &= \xi_{e_Y}(f(x) - f(\hat{x})) - \varphi_{e_Z}(g(x)), \end{aligned}$$

for every $x \in X$ and $e_Z \in \operatorname{int} K_Z \cup \{0_Z\}$. Applying Proposition 4.4, there is also

$$\inf\{\varphi_{e_Z}(z) : \ e_Z \in \operatorname{int} K_Z \cup \{0_Z\}\} = \begin{cases} 0 & \text{if } z \succeq_{K_Z} 0_Z, \\ -\infty & \text{if } z \not\succeq_{K_Z} 0_Z. \end{cases}$$

All conditions of Theorem 5.6 are satisfied. Therefore, the strong duality w.r.t. parameter $(\hat{x},e_Y)$ holds, meaning that

$$\xi_{e_Y}(f(\bar{x}) - f(\hat{x})) = -\Phi^{c(W_\Gamma)}_{(\hat{x},e_Y)}(\bar{e}_Z),$$

for some $\bar{x} \in \mathscr{R}$ and $\bar{e}_Z \in \operatorname{int} K_Z \cup \{0_Z\}$, if and only if $(\bar{x},\bar{e}_Z)$ is a saddle point of $(\mathrm{SP} - (\hat{x},e_Y))$.

Remark 5.7. With regard to Example 5.5, for the collection of scalar problems

$$\min\{\hat{l}(f(x)) : \ f(x) \preceq_{K_Y} f(\hat{x}), \ g(x) \succeq_{K_Z} 0_Z\}, \qquad \text{(SP-}(\hat{x},\hat{l}))$$

where $\hat{x} \in \mathscr{R}$ and $\hat{l} \in \hat{L} \subset \{l : Y \to \mathbb{R}\}$, where $\hat{L}$ is a group of scalarization functions, there are $\Pi = \mathscr{R} \times \hat{L}$ and $\pi = (\hat{x},\hat{l})$, while, $g_\pi(\cdot) = (f(\hat{x}) - f(\cdot), g(\cdot))$, $Z_\Pi = Y \times Z$, and $K_{Z_\Pi} = K_Y \times K_Z$. Take the collection of functions

$$\{\varphi_{(e_Y,e_Z)}(y,z) : Y \times Z \to \mathbb{R} : \ (e_Y,e_Z) \in (\operatorname{int} K_Y \times \operatorname{int} K_Z) \cup \{(0_Y,0_Z)\}\}$$

as W_Γ, meaning that $\Gamma = (\operatorname{int} K_Y \times \operatorname{int} K_Z) \cup \{(0_Y, 0_Z)\}$. Since $\varphi_{(e_Y,e_Z)}$ is $(K_Y \times K_Z)$-monotone for every $(e_Y, e_Z) \in (\operatorname{int} K_Y \times \operatorname{int} K_Z) \cup \{(0_Y, 0_Z)\}$, the corresponding Lagrange function for $(\mathrm{SP} - (\hat{x}, \hat{l}))$ is

$$\begin{aligned} L_{(\hat{x},\hat{l})}(x, e_Y, e_Z) &= \hat{l}(f(x)) - \varphi_{(e_Y,e_Z)}(g_{(\hat{x},\hat{l})}(x)) + \varphi_{(e_Y,e_Z)}(0_Y, 0_Z) \\ &= \hat{l}(f(x)) - \varphi_{(e_Y,e_Z)}(f(\hat{x}) - f(x), g(x)), \end{aligned}$$

for every $x \in X$ and $(e_Y, e_Z) \in (\operatorname{int} K_Y \times \operatorname{int} K_Z) \cup \{(0_Y, 0_Z)\}$.

Besides, it follows from Proposition 4.4 that

$$\begin{aligned} &\inf\{\varphi_{(e_Y,e_Z)}(y,z) : (e_Y, e_Z) \in (\operatorname{int} K_Y \times \operatorname{int} K_Z) \cup \{(0_Y, 0_Z)\}\} \\ &\qquad = \begin{cases} 0 & \text{if } (y,z) \succeq_{K_Y \times K_Z} (0_Y, 0_Z), \\ -\infty & \text{if } (y,z) \not\succeq_{K_Y \times K_Z} (0_Y, 0_Z). \end{cases} \end{aligned}$$

The conditions of Theorem 5.6 are also satisfied. Therefore, the strong duality w.r.t. parameter $(\hat{x}, \hat{l})$ holds, i.e.,

$$\hat{l}(f(\bar{x})) = -\Phi^{c(W_\Gamma)}_{(\hat{x},\hat{l})}(\bar{e}_Y, \bar{e}_Z),$$

for some $\bar{x} \in \mathscr{R}_{(\hat{x},\hat{l})}$ and $(\bar{e}_Y, \bar{e}_Z) \in (\operatorname{int} K_Y \times \operatorname{int} K_Z) \cup \{(0_Y, 0_Z)\}$, if and only if $(\bar{x}, \bar{e}_Z)$ is a saddle point of $(\mathrm{SP} - (\hat{x}, \hat{l}))$.

Next, we investigate the characterization of zero dual gap property by virtue of subdifferentials (see (2.2)). We also consider the general situation, the scalarized problem

$$\min\{p(f(x);\pi) : \ g_\pi(x) \succeq_{K_{Z_\Pi}} 0_{Z_\Pi}\}, \tag{SP-π}$$

where $\pi \in \Pi$ is the parameter.

In the next theorem, we derive a characterization of the strong duality assertion by the W_Γ-subdifferential given in (2.2).

Theorem 5.7. *Let Π and Γ be parameter sets and W_Γ a collection of functions. Let $\pi \in \Pi$ be arbitrary and consider the problems $(SP - \pi)$ and $(SD - \pi)$. Then, for the primal problem $(SP - \pi)$ with the corresponding optimal value function Φ_π,*

$$\bar{\gamma} \in \partial_{W_\Gamma} \Phi_\pi(0_{Z_\Pi})$$

if and only if

$$val(SP - \pi) = val(SD - \pi),$$

and $\bar{\gamma}$ is a solution of $(SD - \pi)$.

Proof. Suppose $\bar{\gamma} \in \partial_{W_\Gamma} \Phi_\pi(0_{Z_\Pi})$, meaning that

$$\forall z_\Pi \in Z_\Pi : \quad \Phi_\pi(z_\Pi) - \Phi_\pi(0_{Z_\Pi}) \geq w(z_\Pi, \bar{\gamma}) - w(0_{Z_\Pi}, \bar{\gamma}),$$

i.e.,

$$\forall z_\Pi \in Z_\Pi : \quad w(z_\Pi, \bar{\gamma}) - \Phi_\pi(z_\Pi) \leq w(0_{Z_\Pi}, \bar{\gamma}) - \Phi_\pi(0_{Z_\Pi}).$$

This shows

$$\Phi^{c(W_\Gamma)}_\pi(\bar{\gamma}) \leq w(0_{Z_\Pi}, \bar{\gamma}) - \Phi_\pi(0_{Z_\Pi}).$$

Therefore,

$$\Phi_\pi(0_{Z_\Pi}) \leq w(0_{Z_\Pi}, \bar{\gamma}) - \Phi^{c(W_\Gamma)}_\pi(\bar{\gamma}),$$

which, according to the weak duality assertion in Theorem 5.1, indicates that

$$\mathrm{val}(\mathrm{SP}-\pi)=\mathrm{val}(\mathrm{SD}-\pi),$$

and $\bar{\gamma}$ is a solution of $(\mathrm{SD}-\pi)$.

Conversely, $\mathrm{val}(\mathrm{SP}-\pi)=\mathrm{val}(\mathrm{SD}-\pi)$ and $\bar{\gamma}\in\mathrm{sol}(\mathrm{SD}-\pi)$ imply

$$\Phi_\pi(0_{Z_\Pi})=w(0_{Z_\Pi},\bar{\gamma})-\Phi_\pi^{c(W_\Gamma)}(\bar{\gamma}).$$

Then, it follows that

$$\forall z_\Pi\in Z_\Pi:\quad w(0_{Z_\Pi},\bar{\gamma})-\Phi_\pi(0_{Z_\Pi})=\Phi_\pi^{c(W_\Gamma)}(\bar{\gamma})\geq w(z_\Pi,\bar{\gamma})-\Phi_\pi(z_\Pi),$$

i.e.,

$$\forall z_\Pi\in Z_\Pi:\quad \Phi_\pi(z_\Pi)-\Phi_\pi(0_{Z_\Pi})\geq w(z_\Pi,\bar{\gamma})-w(0_{Z_\Pi},\bar{\gamma}).$$

Hence, $\bar{\gamma}\in\partial_{W_\Gamma}\Phi_\pi(0_{Z_\Pi})$.

□

This is a result straightforward from the definition of subdifferential in the sense of abstract convexity. As the notion of subgradient only involves the information about the optimal value of the primal problem, when using the subdifferential to discuss the duality, we can only get the zero dual gap with a solution of dual problem. It will not give us a solution of the primal problem. However, if we could somehow compute the value of $-\Phi_\pi^{c(W_\Gamma)}(\bar{\gamma})+w(0_{Z_\Pi};\bar{\gamma})$ for some $\bar{\gamma}\in\partial_{W_\Gamma}\Phi_\pi(0_{Z_\Pi})$, then, for any $\bar{x}\in\mathscr{R}_\pi$ satisfying $p(f(\bar{x});\pi)\leq-\Phi_\pi^{c(W_\Gamma)}(\bar{\gamma})+w(0_{Z_\Pi};\bar{\gamma})$, we deduce $\bar{x}\in\mathrm{sol}(\mathrm{SP}-\pi)$ by the weak duality assertion in Theorem 5.1.

Remark 5.8. Take, for instance, Example 5.2, where the scalar problem is

$$\min\{\xi_{e_Y}(f(x)-f(\hat{x})):g(x)\succeq_{K_Z}0_Z\},\qquad(\text{SP-}(\hat{x},e_Y))$$

with $\Pi=X\times\mathrm{int}K_Y$, $\pi=(\hat{x},e_Y)$ and $p(y;\pi)=p(y;\hat{x},e_Y)=\xi_{e_Y}(y-f(\hat{x}))$, $g_\pi=g$, $(Z_\Pi,K_{Z_\Pi})=(Z,K_Z)$.

The collection of functions involved in the dual problem is given by

$$W_\Gamma=\{\varphi_{e_Z}(\cdot):Z\to\mathbb{R}:\ e_Z\in\mathrm{int}K_Z\cup\{0_Z\}\},$$

where $\Gamma=\mathrm{int}K_Z\cup\{0_Z\}$. For any $(\hat{x},e_Y)\in X\times\mathrm{int}K_Y$, $\bar{e}_Z\in\partial_{W_\Gamma}\Phi_{(\hat{x},e_Y)}(0_Z)$ means

$$\forall z\in Z:\quad \Phi_{(\hat{x},e_Y)}(z)-\Phi_{(\hat{x},e_Y)}(0_Z)\geq\varphi_{\bar{e}_Z}(z)-\varphi_{\bar{e}_Z}(0_Z).$$

Suppose that we find a parameter $(\hat{x},e_Y)$, for which the corresponding optimal value map $\Phi_{(\hat{x},e_Y)}$ is subdifferentiable at 0_Z. If there exists some $\bar{e}_Z\in\partial_{W_\Gamma}\Phi_{(\hat{x},e_Y)}(0_Z)$ such that $-\Phi_{(\hat{x},e_Y)}^{c(W_\Gamma)}(\bar{e}_Z)\geq 0$, then we can conclude that $\hat{x}$ is a solution of $(\mathrm{SP}-(\hat{x},e_Y))$ as long as $\hat{x}$ is feasible.

Remark 5.9. Similar thing happens in Example 5.5, in which the collection of scalar problems is

$$\min\{\hat{l}(f(x)):\ f(x)\preceq_{K_Y}f(\hat{x}),\ g(x)\succeq_{K_Z}0_Z\},\qquad(\text{SP-}(\hat{x},\hat{l}))$$

with parameters $\hat{x}\in\mathscr{R}$ and $\hat{l}\in\hat{L}\subset\{l:Y\to\mathbb{R}\}$. There are $\Pi=\mathscr{R}\times\hat{L}$ and $\pi=(\hat{x},\hat{l})$, while, $g_\pi(\cdot)=(f(\hat{x})-f(\cdot),g(\cdot))$, $Z_\Pi=Y\times Z$, and $K_{Z_\Pi}=K_Y\times K_Z$. The collection of functions involved in the dual problem is given by

$$W_\Gamma=\{\varphi_{(e_Y,e_Z)}(y,z):Y\times Z\to\mathbb{R}:\ (e_Y,e_Z)\in(\mathrm{int}K_Y\times\mathrm{int}K_Z)\cup\{(0_Y,0_Z)\}\},$$

where $\Gamma = (\operatorname{int} K_Y \times \operatorname{int} K_Z) \cup \{(0_Y, 0_Z)\}$. For any $(\hat{x}, \hat{l}) \in \mathscr{R} \times \hat{L}$, $(\bar{e}_Y, \bar{e}_Z) \in \partial_{W_\Gamma} \Phi_{(\hat{x},\hat{l})}(0_Y, 0_Z)$ means

$$\forall (y,z) \in Y \times Z: \ \Phi_{(\hat{x},\hat{l})}(y,z) - \Phi_{(\hat{x},\hat{l})}(0_Y, 0_Z) \geq \varphi_{(\bar{e}_Y,\bar{e}_Z)}(y,z) - \varphi_{\bar{e}_Y,\bar{e}_Z}(0_Y, 0_Z).$$

Suppose that we find a parameter $(\hat{x}, \hat{l})$ such that the corresponding optimal value map $\Phi_{(\hat{x},\hat{l})}$ is subdifferentiable at $(0_Y, 0_Z)$. If there exists some $(\bar{e}_Y, \bar{e}_Z) \in \partial_{W_\Gamma} \Phi_{(\hat{x},\hat{l})}(0_Y, 0_Z)$ with

$$-\Phi^{c(W_\Gamma)}_{(\hat{x},\hat{l})}(\bar{e}_Y, \bar{e}_Z) \geq \hat{l}(f(\hat{x})),$$

then we deduce that $\hat{x}$ is a solution of $(\mathrm{SP} - (\hat{x}, \hat{l}))$.

6. Conclusions

In this paper, collections of nonlinear weak separation functions were constructed via the Gerstewitz and topical functions. Simultaneously, the properties of Gerstewitz function with respect to the parameters were investigated. Then, with the aid of these separation functions, we established a conjugate dual problem for a general constrained vector optimization problem. Moreover, equivalent characterizations of the zero duality gap property and strong duality by means of subdifferentials, separation properties, and the saddle point assertions were obtained.

For further research it is of interest to show sufficient conditions for the existence of saddle points, the existence of optimal solutions of the primal and dual problem such that strong duality statements hold as well as for the existence of elements of the subdifferential of the primal optimal value function. Furthermore, it is of interest to derive duality statements based on our general approach for special classes of vector optimization problems in order to get manageable dual problems taking advantage of the special structure of the primal problem. This can be useful for the development of numerical methods.

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