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VECTORIAL PENALISATION IN VECTOR OPTIMISATION IN REAL LINEAR-TOPOLOGICAL SPACES

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Abstract. The aim of this paper is to present a vectorial penalisation approach for vector optimisation problems in which the vector-valued objective function acts between real linear-topological spaces X and Y, where the image space Y is partially ordered by a pointed convex cone. In essence, the approach replaces the original constrained vector optimisation problem (with not necessarily convex feasible set) by two unconstrained vector optimisation problems, where in one of the two problems a penalisation term (function) with respect to the original feasible set is added to the vector objective function. To derive our main results, we use a generalised convexity (quasiconvexity) notion for vector functions in the sense of Jahn. Our results extend/generalise known results in the context of vectorial penalisation in multiobjective/vector optimisation. We put a special emphasis on the construction of appropriate penalisation functions for several popular classes of (vector) optimisation problems (e.g., semidefinite/copositive programming, second-order cone programming, optimisation in function spaces).
Keywords. Generalised Convexity; Pareto Efficiency; Penalisation; Vector Optimisation.

1. INTRODUCTION

In scalar as well as vector optimisation, it is well known that penalisation methods are a very useful tools for dealing with constrained optimisation problems in order to derive useful optimality conditions and corresponding numerical methods.

The essential idea is to reformulate a constrained problem into an unconstrained one and to use the advantages of methods for unconstrained problems also for constrained problems.

The common idea of penalisation methods is to find an optimal solution of a constrained problem as an optimal solution of an unconstrained problem by considering a combination of the objective function with an additional penalty term that incorporates the constraints of the original problem.

In scalar optimisation, there are well-known penalisation techniques: The infinite penalisation, where one works with extended real-valued functions (e.g., with the indicator function), and the exact penalty principle (Clarke type penalisation), which employs the distance to the

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feasible set (or other exact penalty functions w.r.t. the given constraints) under the assumption that the objective function is locally Lipschitz, see Eremin [1], Zangwill [2], Evans, Gould and Tolle [3], Han and Mangasarian [4], Clarke [5] and references therein.

In [6], Ye extended the exact penalty principle to vectorial problems (Clarke-Ye type penalisation), see also Apetrii, Durea and Strugariu [7] where the directional minimal time function is used as penalisation function and the references therein for some extension and more details. Furthermore, a Clarke-Ye type approach to penalisation for vector optimisation problems is derived by Fukuda, Drummond and Raupp in [8], where a vector external penalty function $\vec{v} : \mathbb{R}^n \to \mathbb{R}^m_{\geq}$ (with $\vec{v}$ continuous and $\vec{v}(x) = 0_{\mathbb{R}^m}$ if and only if x belongs to the feasible set $\mathcal{S}$) is used, i.e., the authors study a vector-valued penalisation approach $f + \alpha\vec{v}$.

In general, two main penalisation approaches can be considered for an original objective function f, a penalisation parameter α and a penalisation function v:

- Exact Penalty Principle: $f + \alpha v$.
- Vectorial Penalisation Approach: (f, v).

Notice that the infinite penalisation with $\alpha = 1$ and $v = I_{\mathcal{S}}$ (where $I_{\mathcal{S}}$ denotes the indicator function with respect to the feasible set $\mathcal{S}$), which is well-known in scalar optimisation, is an exception in this context. The advantage of the approach ”Vectorial Penalisation” used in our paper is that we do not need a penalty parameter α.

In our paper, we consider vector optimisation problems with an objective function f acting between real linear-topological spaces X and Y. We suppose that the image space Y is partially ordered by a pointed convex cone $\mathcal{C}$. Vector optimisation in general spaces is a growing up and vibrant field of mathematics with important applications in economics, mathematical finance, risk theory and engineering.

Important relationships between constrained and unconstrained vector optimisation problems are studied by Klamroth and Tind in [9], Günther [10, 11] and by Günther and Tammer in [12, 13], see also Fletcher and Leyffer [14].

In [15] by Durea, Strugariu and Tammer, a penalisation method is discussed where the penalised objective function is composed of the original objective function and the function defining the restrictions. This allows the deduction of suitable optimality conditions for constrained nondifferentiable vector optimisation problems. Recently, Schmölling [16] extended and generalised the vectorial penalisation approach in [12, 13] to problems where X and Y are linear spaces and $\mathcal{C}$ is a convex cone in Y.

The aim of our paper is to extend the results in [10, 11, 12, 13] derived for vector optimisation problems where the objective function takes its values in a finite dimensional Euclidean space $\mathbb{R}^n$ equipped with the natural ordering cone $\mathbb{R}^n_{\geq}$ to problems where X, Y are linear spaces and the ordering cone $\mathcal{C}$ is a convex cone in Y. In order to prove our main results, we assume that certain generalised convexity (quasiconvexity) notions for vector functions are fulfilled.

Let us consider a constrained vector optimisation problem with an objective function that is acting between real linear-topological spaces X (pre-image space) and Y (image space). Furthermore, we are using the following settings:

- $\mathcal{C}$ is a pointed, convex cone in a real linear-topological space Y,
- $\mathcal{S}$ and $\mathcal{D}$ are nonempty sets in a real linear-topological space X with $\mathcal{S} \subseteq \mathcal{D}$.
- The original vector objective function is given by

$$f : \mathcal{D} \to Y$$

and the penalisation function

$$\nu : \mathcal{D} \to \mathbb{R}$$

leads us to an extended vector objective function

$$f^{\otimes} : \mathcal{D} \to Y \times \mathbb{R}, \quad f^{\otimes} := (f, \nu).$$

We deal with penalisation approaches in vector optimisation that aim to replace the original constrained optimisation problem (with not necessarily convex feasible set) by some related problems with an easier structured feasible set (or actually by some unconstrained problems).

In our vectorial penalisation approach, the original constrained vector optimisation problem

$$\operatorname*{argmin}_{x \in \mathcal{S}}{}_{C}\ f(x) \qquad (P_{\mathcal{S}})$$

is replaced by two unconstrained vector optimisation problems

$$\operatorname*{argmin}_{x \in \mathcal{D}}{}_{C}\ f(x) \qquad (P_{\mathcal{D}})$$

and

$$\operatorname*{argmin}_{x \in \mathcal{D}}{}_{C \times \mathbb{R}_{\geq}}\ f^{\otimes}(x). \qquad (P_{\mathcal{D}}^{\otimes})$$

For deriving our main results, we use certain generalised convexity notions for vector functions. We focus on the cone-quasiconvexity concept by Jahn [17]. Further concepts of generalised convexity for vector functions are known from the literature; see, e.g., [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. It is known that generalised convexity assumptions appear in different applications such as economics (see, e.g., Cambini and Martein [28]).

The paper is structured as follows:

In the first part of Section 2, we introduce some basics in linear-topological spaces, we highlight some important examples of convex cones that can be used in our approach (both as ordering cones in the solution concepts and as cones in the constraints) and we introduce some general binary relations based on sets (or cones) in linear-topological spaces. In the second part of Section 2, we formulate our general vector optimisation problem and recall some well-known solution concepts ((weak/strict) efficiency) in vector optimisation and a class of generalised convex (quasiconvex) vector functions in the sense of Jahn [17, Def. 7.11].

In Section 3, we analyse important relationships between constrained and unconstrained vector optimisation problems using generalised convexity concepts for vector functions.

Our main Section 4 contains the vectorial penalisation approach for general vector optimisation problems. After studying basic properties of the penalisation function, which will play a key role in our approach, we investigate special types of penalisation functions for vector problems involving abstract constraints as well as for problems involving explicit generalised inequality and equality constraints. In particular, for the cone-constraint case, we are able to construct appropriate penalisation functions for some popular cones given in spaces of finite and infinite dimension (e.g., Löwner cone, copositive cone, Bishop-Phelps cone, standard cone in the space of Lebesgue-integrable / essentially bounded functions). Furthermore, we analyse the effects of adding/removing a penalisation term to/from the original vector objective function on the nature of the solution sets of the vector optimisation problems (in the spirit of Fliege [29]). We end Section 4 by stating our main results related to our vectorial penalisation approach for general vector optimisation problems, which show profound relationships between the solution sets of the vector problems ($P_{\mathcal{S}}$), ($P_{\mathcal{D}}$), and ($P_{\mathcal{D}}^{\otimes}$).

We conclude our paper with some closing remarks and an outlook for future research in Section 5.

2. Preliminaries

2.1. **Basics in linear-topological spaces.** Throughout the paper, the set of nonnegative real numbers is denoted by $\mathbb{R}_{\geq}$ and the set of positive real numbers by $\mathbb{R}_{>}$. Assume that V is a real linear-topological space with corresponding topological dual space

$$V^* = \{v^* : V \to \mathbb{R} \mid v^* \text{ is linear and continuous}\}.$$

The point 0_V (respectively, 0_{V^*}) represents the origin in V (respectively, V^*). For any set $\Omega \subseteq V$, we denote the topological interior, the topological boundary, the topological closure of Ω by $\mathrm{int}(\Omega)$, $\mathrm{bd}(\Omega)$ and $\mathrm{cl}(\Omega)$, respectively. A set $\Omega \subseteq V$ is called solid if $\mathrm{int}(\Omega) \neq \emptyset$. For any two points $x, y \in V$, the closed, the open, the half-open line segments are defined by

$$[x,y] := \{(1-\lambda)x + \lambda y \mid \lambda \in [0,1]\}, \qquad (x,y) := \{(1-\lambda)x + \lambda y \mid \lambda \in (0,1)\},$$
$$[x,y) := \{(1-\lambda)x + \lambda y \mid \lambda \in [0,1)\}, \qquad (x,y] := \{(1-\lambda)x + \lambda y \mid \lambda \in (0,1]\}.$$

In a normed space setting, $\|\cdot\|_V$ denotes the underlying norm, $\mathbb{S}_V$ the unit sphere, $\mathbb{B}_V$ the closed unit ball, V^* the dual normed space, and $\|\cdot\|_{V^*}$, defined by

$$\|v^*\|_{V^*} := \sup\{v^*(x) \mid x \in \mathbb{B}_V\} = \sup\{v^*(x) \mid x \in \mathbb{S}_V\} \quad \text{for all } v^* \in V^*,$$

the dual norm. For any $\varepsilon > 0$ and $x \in V$, we denote by $B_\varepsilon(x) := \{y \in V \mid \|x-y\|_V < \varepsilon\}$ the open norm ball and by $\overline{B}_\varepsilon(x) := \mathrm{cl}(B_\varepsilon(x))$ the closed norm ball in x of radius ε.

In the finite-dimensional space $\mathbb{R}^n$, we will denote the canonical unit vectors in $\mathbb{R}^n$ by $e^1, \ldots, e^n$, and the Euclidean scalar product by $\langle \cdot, \cdot \rangle$. Moreover, for any $n \in \mathbb{N}$, we introduce the set of indices $I_n := \{1, 2, \ldots, n\}$.

As usual, a set $\Omega \subseteq V$ is convex if $(1-\lambda)x + \lambda y \in \Omega$ for all $x, y \in \Omega$ and $\lambda \in (0,1)$. The convex hull of Ω, i.e., the smallest convex set of V containing Ω, is denoted by $\mathrm{conv}(\Omega)$. It is easy to check that Ω is convex if and only if $\mathrm{conv}(\Omega) = \Omega$.

The following result (cf. [30, Cor. 3.22], [31, Ex. 10.16]) in the product space $V_1 \times V_2$ of two real linear-topological spaces V_1 and V_2 will be later used in our proofs.

Lemma 2.1. *Assume that V_1, V_2 are real linear-topological spaces and $A_1 \subseteq V_1$, $A_2 \subseteq V_2$ are convex sets. Then,*

$$\mathrm{int}(A_1 \times A_2) = \mathrm{int}(A_1) \times \mathrm{int}(A_2).$$

2.2. **Convex cones and order relations.** Cones will play an important role in our work, on the one hand as ordering cones involved in the solution concepts, and on the other hand as cones involved in the constraints of the vector optimisation problems. Recall that a set $C \subseteq V$ with $0_V \in C = \mathbb{R}_{\geq} \cdot C$ $(= C + C)$ is called a (convex) cone. Moreover, we say that the cone C is nontrivial if $\{0_V\} \neq C \neq V$, pointed if $C \cap (-C) = \{0_V\}$, reproducing if $C - C = V$.

Assume that $C \subseteq V$ is a convex cone. In view of Jahn [17, Lem. 1.12 and 1.32], we have

$$\operatorname{int}(C) = C + \operatorname{int}(C). \tag{2.1}$$

The dual cone of C is given by

$$C^+ := \{x^* \in V^* \mid \forall c \in C : x^*(c) \geq 0\},$$

while the quasi-interior of the dual cone is given by

$$C^\# := \{x^* \in V^* \mid \forall c \in C \setminus \{0_V\} : x^*(c) > 0\}.$$

Furthermore, the following assertions are well-known (see Jahn [17, Lem. 1.13 and 1.27]):

- If C is solid, then C is reproducing and C^+ is pointed.
- If $C^\# \neq \emptyset$, then C is pointed.

Assume that $V = \mathbb{R}^n$ is a real normed space. Then, for any nontrivial, convex cone $C \subseteq V$, the following assertions hold true:

- If C is closed and pointed, then $C^\# \neq \emptyset$ (by the Krein-Rutman theorem [17, Th. 3.38]).
- If C is closed, then $C^\# \neq \emptyset$ if and only if $0 \notin \operatorname{cl}(\operatorname{conv}(C \cap \mathbb{S}_V))$ (by [32, Th. 3.6]).
- If C is closed, then $C^\# = \operatorname{int}(C^+)$ (by [32, Th. 2.5, Rem. 2.6]).
- $\operatorname{cl}(C) = (C^+)^+$ (by the bipolar theorem [33, Th. 1.1.9]).
- $\operatorname{int}(C) = (C^+)^\#$ (by the previous two assertions).

To illustrate our general approach, we present some important examples for spaces and cones, and analyse their properties.

Example 2.2 (Standard cone in $\mathbb{R}^n$)**.** Consider the particular framework where $V = \mathbb{R}^n$ is a finite-dimensional Euclidean space and let $C = \mathbb{R}^n_\geq$ be the standard ordering cone. It is well-known that $\mathbb{R}^n_\geq$ is a nontrivial, closed, pointed, solid, reproducing, convex cone with

$$\operatorname{int}\left(\mathbb{R}^n_\geq\right) = \left\{ (v_1, \dots, v_n) \in \mathbb{R}^n \,\middle|\, \forall i \in I_n : v_i > 0 \right\},$$

$$\operatorname{bd}\left(\mathbb{R}^n_\geq\right) = \left\{ (v_1, \dots, v_n) \in \mathbb{R}^n_\geq \,\middle|\, \exists i \in I_n : v_i = 0 \right\}.$$

Example 2.3 (Polyhedral cone in $\mathbb{R}^n$)**.** Consider $V = \mathbb{R}^n$, a nonempty finite set $P \subseteq V \setminus \{0_V\}$ and a polyhedral cone $C := C_P \subseteq V$ given by

$$C_P := \{y \in V \mid \forall p \in P : \; p^T y \leq 0\}.$$

Then, C_P is a closed and convex cone with

$$\operatorname{int}(C_P) = \left\{ y \in V \,\middle|\, \forall p \in P : \; p^T y < 0 \right\},$$

$$\operatorname{bd}(C_P) = \left\{ y \in C_P \,\middle|\, \exists p \in P : p^T y = 0 \right\}.$$

According to Günther and Popovici [34, Sec. 4.2], the following equivalences hold true:

- C_P is nontrivial if and only if $0_V \notin \operatorname{int}(\operatorname{conv}(P))$,
- C_P is pointed if and only if $\{y \in V \mid \forall p \in P : \; p^T y = 0\} = \{0_V\}$,
- C_P is solid if and only if $0_V \notin \operatorname{conv}(P)$.

Example 2.4 (Lexicographic cone in $\mathbb{R}^n$)**.** Let $V = \mathbb{R}^n$ ($n \geq 2$) and let $C = C^n_{\text{lex}}$ be the lexicographic cone, defined as the set of points whose first nonzero coordinate (if any) is positive:

$$C^n_{\text{lex}} := \{0_V\} \cup \left\{ (v_1, \dots, v_n) \in \mathbb{R}^n \,\middle|\, \exists i \in I_n : \; v_i > 0, \nexists j \in I_n, j < i : \; v_j \neq 0 \right\}.$$

It is known that C^n_{lex} is a nontrivial, pointed, solid, reproducing, convex cone with

$$\operatorname{int}(C^n_{\text{lex}}) = \{ (v_1, \dots, v_n) \in \mathbb{R}^n \mid v_1 > 0 \},$$
$$\operatorname{bd}(C^n_{\text{lex}}) = \{ (v_1, \dots, v_n) \in \mathbb{R}^n \mid v_1 = 0 \},$$

so that C^n_{lex} is not closed (see, e.g., Popovici [35]).

Example 2.5 (Löwner cone in $\mathfrak{S}^n$)**.** Consider the space $V = \mathfrak{S}^n$ of all real symmetric $n \times n$ matrices. Then, the Löwner cone is defined by

$$\begin{aligned} C := \mathfrak{S}^n_+ &:= \{M \in V \mid \forall y \in \mathbb{R}^n : \ y^T My \geq 0\} \\ &= \{M \in V \mid M \text{ is positive semidefinite}\}, \end{aligned}$$

while the interior and the boundary of $\mathfrak{S}^n_+$ are given by

$$\begin{aligned} \operatorname{int}(\mathfrak{S}^n_+) &= \{M \in V \mid \forall y \in \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\} : \ y^T My > 0\}, \\ &= \{M \in V \mid M \text{ is positive definite}\}, \\ \operatorname{bd}(\mathfrak{S}^n_+) &= \{M \in \mathfrak{S}^n_+ \mid \exists y \in \mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\} : \ y^T My = 0\}. \end{aligned}$$

It is known that the Löwner cone is a nontrivial, closed, pointed, solid, reproducing, convex cone in V.

Example 2.6 (Copositive cone in $\mathfrak{S}^n$)**.** Consider the space $V = \mathfrak{S}^n$ of all real symmetric $n \times n$ matrices. Then, the copositive cone is defined by

$$\begin{aligned} C := \mathfrak{S}^n_{++} &:= \{M \in V \mid \forall y \in \mathbb{R}^n_+ : \ y^T My \geq 0\} \\ &= \{M \in V \mid M \text{ is copositive}\}, \end{aligned}$$

while the interior and the boundary of $\mathfrak{S}^n_{++}$ are given by

$$\begin{aligned} \operatorname{int}(\mathfrak{S}^n_{++}) &= \{M \in V \mid \forall y \in \mathbb{R}^n_{\geq} \setminus \{0_{\mathbb{R}^n}\} : \ y^T My > 0\}, \\ &= \{M \in V \mid M \text{ is strictly copositive}\}, \\ \operatorname{bd}(\mathfrak{S}^n_{++}) &= \{M \in \mathfrak{S}^n_{++} \mid \exists y \in \mathbb{R}^n_{\geq} \setminus \{0_{\mathbb{R}^n}\} : \ y^T My = 0\}. \end{aligned}$$

It is known that the copositive cone is a nontrivial, closed, pointed, solid, reproducing, convex cone in V which contains the Löwner cone $\mathfrak{S}^n_+$.

The following example in $\mathfrak{S}^n$ generalises the previous two examples.

Example 2.7 (K-semidefinite cone in $\mathfrak{S}^n$)**.** Consider the space $V = \mathfrak{S}^n$ of all real symmetric $n \times n$ matrices, and a closed, convex cone $K \subseteq \mathbb{R}^n$ with $K \neq \{0_{\mathbb{R}^n}\}$. Then, according to Eichfelder and Jahn [36], the K-semidefinite cone is defined by

$$\begin{aligned} C := \mathfrak{S}^n_K &:= \{M \in V \mid \forall y \in K : \ y^T My \geq 0\}, \\ &= \{M \in V \mid M \text{ is } K\text{-semidefinite}\}, \end{aligned}$$

while the interior and the boundary of $\mathfrak{S}^n_K$ are given by

$$\begin{aligned} \operatorname{int}(\mathfrak{S}^n_K) &= \{M \in V \mid \forall y \in K \setminus \{0_{\mathbb{R}^n}\} : \ y^T My > 0\}, \\ &= \{M \in V \mid M \text{ is strictly } K\text{-semidefinite}\}, \\ \operatorname{bd}(\mathfrak{S}^n_K) &= \{M \in \mathfrak{S}^n_K \mid \exists y \in K \setminus \{0_{\mathbb{R}^n}\} : \ y^T My = 0\}. \end{aligned}$$

It is known that the K-semidefinite cone is a nontrivial, closed, solid, reproducing, convex cone in V, and if K is reproducing (e.g. if K is solid), then $\mathfrak{S}^n_K$ is pointed.

For $K := \mathbb{R}^n$ we have $\mathfrak{S}^n_K = \mathfrak{S}^n_+$, while for $K := \mathbb{R}^n_{\geq}$ we have $\mathfrak{S}^n_K = \mathfrak{S}^n_{++}$.

Example 2.8 (Second-order cone (Lorentz cone) in $\mathbb{R}^n \times \mathbb{R}$)**.** Consider a real normed space $V = \mathbb{R}^n$ with underlying norm $\|\cdot\|_V$. Then, the second-order cone in $V \times \mathbb{R}$ is defined by

$$C := \{(y,t) \in V \times \mathbb{R} \mid \|y\|_V \leq t\},$$

which is a nontrivial, closed, pointed, solid, convex cone with

$$\begin{aligned} \operatorname{int}(C) &= \{(y,t) \in V \times \mathbb{R} \mid \|y\|_V < t\}, \\ \operatorname{bd}(C) &= \{(y,t) \in V \times \mathbb{R} \mid \|y\|_V = t\}. \end{aligned}$$

Let us consider two specific settings:

(1) Consider a norm $\|\cdot\|_{V\times\mathbb{R}}$ on $V \times \mathbb{R}$ that is defined by

$$\|(y,t)\|_{V\times\mathbb{R}} := \|y\|_V + |t| \quad \text{for all } (y,t) \in V \times \mathbb{R}.$$

Then, C is actually a so-called Bishop-Phelps cone in the real normed space $V \times \mathbb{R}$ (for more details, see Example 2.9), since

$$C = \{(y,t) \in V \times \mathbb{R} \mid \|(y,t)\|_{V\times\mathbb{R}} \leq \langle 2e^{n+1}, (y,t)\rangle\}.$$

(2) Assume that the norms $\|\cdot\|_V$ and $\|\cdot\|_{V\times\mathbb{R}}$ are given by the classical l_p norm, $p \in [1,\infty]$, in V and $V \times \mathbb{R}$, respectively. It is known (see Eichfelder [37, Ex. 1.15] and Jahn [38]) that C is a Bishop-Phelps cone in $V \times \mathbb{R}$, since

$$C = \left\{(y,t) \in V \times \mathbb{R} \,\middle|\, \|(y,t)\|_{V\times\mathbb{R}} \leq \left\langle 2^{\frac{1}{p}} e^{n+1}, (y,t)\right\rangle\right\}$$

for $p \in [1,\infty)$, and

$$C = \{(y,t) \in V \times \mathbb{R} \mid \|(y,t)\|_{V\times\mathbb{R}} \leq \langle e^{n+1}, (y,t)\rangle\}$$

for $p = \infty$. Now, consider $p = 2$. Then, C is the so-called Lorentz cone (also known as ice cream cone) and, for any $Q \in \{\mathbb{B}_V, \mathbb{S}_V\}$, admits the representation

$$C = \bigcap_{s\in Q} \left\{(y,t) \in V \times \mathbb{R} \,\middle|\, s^T y \leq t\right\},$$

and further satisfies

$$\operatorname{int}(C) = \bigcap_{s\in Q} \operatorname{int}\left(\left\{(y,t) \in V \times \mathbb{R} \,\middle|\, s^T y \leq t\right\}\right) = \bigcap_{s\in Q} \left\{(y,t) \in V \times \mathbb{R} \,\middle|\, s^T y < t\right\}$$

(for the proof of the latter two equalities one can use the convexity of the function $(y,t) \mapsto \max_{s\in Q} s^T y - t$ and the fact that the strict sublevel set of this function w.r.t. the level zero is included in $\operatorname{int}(C)$, see Zălinescu [33, p.147]). Notice, for the case $p = 2$ we have

$$\left\|\sqrt{2}e^{n+1}\right\|_{(V\times\mathbb{R})^*} = \left\|\sqrt{2}e^{n+1}\right\|_{V\times\mathbb{R}} = \sqrt{2} > 1.$$

Let us present some examples in an infinite dimensional framework.

Example 2.9 (Bishop-Phelps cone in a normed space)**.** Consider a real normed space V. Then, for a given $y^* \in V^*$ (with $\|y^*\|_{V^*} > 1$), the Bishop-Phelps cone is defined by

$$C := C(y^*) := \{y \in V \mid y^*(y) \geq \|y\|_V\}.$$

According to Jahn [39], and Ha and Jahn [40], $C(y^*)$ is a nontrivial, closed, pointed, solid, reproducing, convex cone with

$$\operatorname{int}(C(y^*)) = \{y \in V \mid y^*(y) > \|y\|_V\},$$
$$\operatorname{bd}(C(y^*)) = \{y \in V \mid y^*(y) = \|y\|_V\}.$$

Example 2.10 (Standard cone in the space of real-valued continuous functions)**.** Consider the real normed space $V = C[a,b]$ (with $-\infty < a < b < \infty$) of real-valued continuous functions. Then, the standard cone in $C[a,b]$ is given by

$$C := C_+[a,b] := \{u \in C[a,b] \mid \forall t \in [a,b] : \ u(t) \geq 0\},$$

while the interior and the boundary of $C_+[a,b]$ are given by

$$\operatorname{int}(C_+[a,b]) = \{u \in C[a,b] \mid \forall t \in [a,b] : \ u(t) > 0\},$$
$$\operatorname{bd}(C_+[a,b]) = \{u \in C_+[a,b] \mid \exists t \in [a,b] : \ u(t) = 0\}.$$

Obviously, $C_+[a,b]$ is a nontrivial, closed, pointed, solid, reproducing, convex cone in $C[a,b]$.

Example 2.11 (Standard cone in the space of Lebesgue-integrable / essentially bounded functions)**.** Consider the normed space $V = L^p[a,b]$ with $-\infty < a < b < \infty$, where $L^p[a,b]$ is for $p \in [1,\infty)$ the space of all (equivalence classes of) real-valued p-th power Lebesgue-integrable functions and for $p = \infty$ the space of all (equivalence classes of) essentially bounded functions on $[a,b]$ (i.e., $L^\infty[a,b]$ consists of all measurable functions $u : [a,b] \to \mathbb{R}$ such that the essential supremum (denoted by ess sup) of $|u|$ is finite). The standard cone in $L^p[a,b]$ for $p \in [1,\infty]$ is given by

$$C := L^p_+[a,b] := \{u \in L^p[a,b] \mid u(t) \geq 0 \text{ almost everywhere on } [a,b]\}.$$

It is known that $L^p_+[a,b]$ is a nontrivial, pointed, closed, convex cone in $L^p[a,b]$ which is solid for $p = \infty$ but not solid for $p \in [1,\infty)$ (see Jahn [17, Ex. 1.51]).

Further interesting examples are given in the book by Jahn [17, Sec. 1.4].

Definition 2.12. Consider a real linear-topological space V, points $x, y \in V$ and a nonempty set $A \subseteq V$. Then, we define the following binary relations

$$\begin{aligned} x \leq_A y \quad &:\Longleftrightarrow \quad y \in x + A, \\ x \lneq_A y \quad &:\Longleftrightarrow \quad y \in x + A \setminus \{0_V\}, \\ x <_A y \quad &:\Longleftrightarrow \quad y \in x + \operatorname{int}(A). \end{aligned} \tag{2.2}$$

In the topic of vector optimisation (see Section 2.3), we are interested in the case $A := \mathcal{C}$, where $\mathcal{C}$ is a nontrivial, pointed, convex cone in V. In this case, the space V is partially ordered by $\leq_C$.

2.3. **Vector optimisation.** Throughout the paper, under the following main assumptions

$$\begin{cases} X, Y \text{ real linear-topological spaces,} \\ \emptyset \neq \mathcal{D} \subseteq X,\ f : \mathcal{D} \to Y, \\ \emptyset \neq C \subseteq Y \text{ nontrivial, pointed, convex cone,} \\ \emptyset \neq \mathcal{S} \subseteq \mathcal{D}, \end{cases} \tag{A}$$

we consider the general vector optimisation problem given by

$$\operatorname*{argmin}_{x \in \mathcal{S}}{}_{C}\ f(x) \tag{$P_{\mathcal{S}}$}$$

Solutions of the vector optimisation problem ($P_{\mathcal{S}}$) are defined according to the next definition (see, e.g., Jahn [17]):

Definition 2.13. Assume that (A) is valid. We define the **set of efficient points** of problem ($P_{\mathcal{S}}$) as

$$\operatorname{Eff}(f, \mathcal{S}, C) := \left\{ \bar{x} \in \mathcal{S} \,\middle|\, \nexists x \in \mathcal{S} :\ f(x) \lneq_C f(\bar{x}) \right\},$$

the **set of weakly efficient points** of ($P_{\mathcal{S}}$) as

$$\operatorname{WEff}(f, \mathcal{S}, C) := \left\{ \bar{x} \in \mathcal{S} \,\middle|\, \nexists x \in \mathcal{S} :\ f(x) <_C f(\bar{x}) \right\}$$

as well as the **set of strictly efficient points** of ($P_{\mathcal{S}}$) as

$$\operatorname{SEff}(f, \mathcal{S}, C) := \left\{ \bar{x} \in \mathcal{S} \,\middle|\, \nexists x \in \mathcal{S} \setminus \{\bar{x}\} :\ f(x) \leq_C f(\bar{x}) \right\}.$$

It is easy to check that

$$\operatorname{SEff}(f, \mathcal{S}, C) \subseteq \operatorname{Eff}(f, \mathcal{S}, C) \subseteq \operatorname{WEff}(f, \mathcal{S}, C).$$

Under the validity of (A) consider the assumptions

$$\begin{cases} Z_1, Z_2 \text{ real linear-topological spaces,} \\ g : \mathcal{D} \to Z_1,\ h : \mathcal{D} \to Z_2, \\ \emptyset \neq \mathcal{K} \subsetneq Z_1 \text{ closed, convex cone,} \\ \emptyset \neq \mathcal{U} \subseteq \mathcal{D}. \end{cases} \tag{$A^{g,h}$}$$

Beside the general vector optimisation problem given in ($P_{\mathcal{S}}$) with abstract feasible set $\mathcal{S}$ (as defined in (A)) we are also interested in the case with explicit (generalised inequality and equality) constraints, i.e.,

$$\mathcal{S} := \{x \in \mathcal{U} \mid g(x) \in \mathcal{K},\ h(x) = 0_{Z_2}\}.$$

Example 2.14 (Vectorial Approximation Problems)**.** We assume that X, Y and Z are real Banach spaces, C is a nontrivial, pointed, closed, convex cone in Y. In order to formulate the vectorial approximation problem, let us introduce a vector-valued norm (see [17, Def. 1.35]) $|||\cdot||| : Z \to C$ which for all $z, z_1, z_2 \in Z$ and for all $\lambda \in \mathbb{R}$ satisfies:

$$\begin{array}{ll} (1) & |||z||| = 0_Y \iff z = 0_Z; \\ (2) & |||\lambda\, z\,||| = |\,\lambda\,| \cdot |||z|||; \\ (3) & |||z_1 + z_2|||\ \in\ |||z_1||| + |||z_2||| -\ C. \end{array}$$

Suppose now that the cost function $u : \mathcal{D} \to Y$ is locally Lipschitz continuous, $L_i \in \mathscr{L}(X,Z)$ ($\mathscr{L}(X,Z)$ denotes the set of linear continuous mappings from X to Z) and $\alpha_i \geq 0$ $(i = 1,\dots,n)$. Then, we consider for $x \in \mathcal{S} \subseteq \mathcal{D}$ and $a^i \in Z$ $(i = 1,\dots,n)$ the vectorial approximation problem

$$\underset{x\in\mathcal{S}}{\operatorname{argmin}}_C \; f(x), \tag{2.3}$$

where the objective function $f : \mathcal{D} \to Y$ is given by

$$f(x) := u(x) + \sum_{i=1}^{n} \alpha_i |||L_i x - a^i||| \quad \text{for all } x \in \mathcal{D},$$

see the books [17, Ch. 9] and [41, Sec. 4.1].

Example 2.15. We consider the following problem where a dilating cone is involved. Suppose that X and Z are Banach spaces, Z is equipped with a nontrivial, pointed, closed, convex cone C.

$$\underset{x\in\mathcal{S}}{\operatorname{argmin}}\, f(x) \quad \text{where} \quad \mathcal{S} := \{x \in X \mid Gx \leq_C w\}, \tag{CP}$$

where $G : X \to Z$ is a linear and bounded map, $w \in Z$, the objective map $f : X \to \mathbb{R}$ is continuous, coercive and strictly convex. This problem (CP) is a prototype of a PDE-constrained optimal control problem where the constraint represents the solution map applied to the control variable in linear PDEs. The constraints are described by the cone $C \subseteq Z$. In [42] and [43], a regularisation framework is developed which consists of replacing the ordering cone C by an approximating family of dilating cones, i.e., this conical regularisation approach for problem (CP) consists of constructing a family of approximate optimisation problems. This family of approximating dilating cones $\{C_\varepsilon\}_{\varepsilon\in(0,1)} \subseteq Z$ is a family of pointed, closed, convex cones with nonempty interior such that $C \setminus \{0_Z\} \subseteq \operatorname{int}(C_\varepsilon)$ for all $\varepsilon \in (0,1)$, $C_{\varepsilon_1} \subseteq C_{\varepsilon_2}$ for $\varepsilon_1 \leq \varepsilon_2$ and $C = \bigcap_{0<\varepsilon<1} C_\varepsilon$. The family of regularised problems is then given by replacing the ordering cone C in the optimal control problem (CP) by the dilating cone C_ε:

$$\underset{x\in\mathcal{S}_\varepsilon}{\operatorname{argmin}}\, f(x) \quad \text{where} \quad \mathcal{S}_\varepsilon := \{x \in X \mid Gx \leq_{C_\varepsilon} w\}. \tag{CP$_\varepsilon$}$$

For $\varepsilon \in (0,1)$, the (Henig) dilating cone is defined by

$$C_\varepsilon := \operatorname{cl}\left[\mathbb{R}_{\geq} \cdot \operatorname{conv}(B_C + \varepsilon \mathbb{B}_Z)\right], \tag{2.4}$$

where $B_C := \{z \in C \mid z^*(z) = 1\}$, $z^* \in C^{\#}$, $\|z^*\|_{Z^*} = 1$. It is well known that C_ε in (2.4) is a pointed, closed, solid, reproducing, convex cone with $C = \bigcap_{0<\varepsilon<1} C_\varepsilon$ and $C \setminus \{0_Z\} \subseteq \operatorname{int}(C_\varepsilon)$ for every $\varepsilon \in (0,1)$ (see [44, Th. 1.1] and [43]). In the case $Z = L^2[0,1]$ with the natural ordering cone $L^2_+[0,1]$, for each $\varepsilon \in (0,1)$, $C_\varepsilon = (L^2_+[0,1])_\varepsilon = \operatorname{cl}[\mathbb{R}_{\geq} \cdot \operatorname{conv}(B_C + \varepsilon \mathbb{B}_Z)]$ is a (Henig) dilating cone associated to $C = L^2_+[0,1]$ with the base

$$B_C = \left\{ \varphi \in L^2_+[0,1] \,\middle|\, \int_0^1 \varphi(s)ds = 1 \right\},$$

see [43, Sec. 4.2].

2.4. **Generalised convex vector functions.** In order to derive strong relationships between constrained and unconstrained vector optimisation (in Section 3), we will impose generalised convexity assumptions on the vector objective function f. Concepts of generalised convexity for vector functions are studied by several authors in the literature, for instance by Bagdasar and Popovici [18], Borwein [19], Flores-Bazán [20], Flores-Bazán and Vera [21], Günther and Popovici [22, 23], Jahn [17], Jahn and Sachs [24], Luc [25], Luc and Schaible [26], Popovici [27]. Moreover, we refer the reader to the book by Cambini and Martein [28] for an overview on generalised convexity concepts for scalar functions and corresponding interesting applications in economics.

In our work, we focus on a generalised convexity concept for vector functions proposed by Jahn [17, Def. 7.11], which we recall in the following definition.

Definition 2.16. Assume (A) and let $\mathcal{S}$ be convex and $A \subseteq Y$ be a set. Then, the function f is called A**-quasiconvex** on $\mathcal{S}$ (in the sense of Jahn [17]) if

$$\forall x^1, x^2 \in \mathcal{S},\ x^1 \neq x^2,\ f(x^1) \leq_A f(x^2),\ \exists x^0 \in \mathcal{S} \setminus \{x^2\},\ \forall x \in [x^0, x^2) :\ f(x) \leq_A f(x^2).$$

Remark 2.17. The case $A \in \{\mathcal{C}, \mathcal{C} \setminus \{0_Y\}, \operatorname{int}(\mathcal{C})\}$ will be of special interest. Besides Jahn [17, Ch. 7] also Bagdasar and Popovici [18] studied the A-quasiconvexity concept and its importance in vector optimisation. In particular, the class of A-quasiconvex functions in the sense of Borwein [19]/Luc [25] is for $A = \mathcal{C}$ contained in the class of semistrictly (A)-quasiconvex functions in the sense of Flores-Bazán [20, Def. 2.1], Flores-Bazán and Vera [21, Def. 4.1] (see [18, Def. 3]), which again is contained in our class of A-quasiconvex functions for any of the above mentioned A. It is therefore for $A = \mathcal{C}$ natural to name the concept of Borwein/Luc as strong $\mathcal{C}$-quasiconvexity and the concept by Jahn (from our Definition 2.16 with $A = \mathcal{C}$) as weak $\mathcal{C}$-quasiconvexity. All in all the concept from Jahn is relatively weak and in the case $A = \operatorname{int}(\mathcal{C})$ (and for $A = \mathcal{C}$ if $\mathcal{C}$ closed) further contains the class of explicitly $\mathcal{C}$-quasiconvex functions in the sense of Popovici [27] and for $A = \mathcal{C}$ the classes of naturally $\mathcal{C}$-quasiconvex and scalarly $\mathcal{C}$-quasiconvex functions in the sense of Flores-Bazán [20, Def. 2.4]. For a thorough analysis of the relationships of some generalised convexity concepts for vector functions and their importance in the derivation of local-global type properties in vector optimisation, we refer the reader to the recent paper by Bagdasar and Popovici [18].

Definition 2.18. Assume (A) and consider a scalar function $\nu : \mathcal{D} \to \overline{\mathbb{R}}$. For any $r \in \overline{\mathbb{R}}$ and any binary relation $\sim \in \{<, \leq, =, \geq, >\}$ on $\overline{\mathbb{R}}$, the set

$$\operatorname{lev}_{\sim}(\mathcal{S}, \nu, r) := \{x \in \mathcal{S} \mid \nu(x) \sim r\}$$

is called $\sim$**-level set** of the scalar function ν w.r.t. the set $\mathcal{S}$ and the level r.

Definition 2.19. Assume (A) and consider a set $A \subseteq Y$. For any $y \in Y$ and any binary relation $\sim \in \{<_A, \leq_A, \lneq_A, =\}$ on Y (see Definition 2.12), the set

$$\operatorname{lev}_{\sim}(\mathcal{S}, f, y) := \{x \in \mathcal{S} \mid f(x) \sim y\}$$

is called **(generalised)** $\sim$**-level set** of the vector function f w.r.t. the set $\mathcal{S}$ and the level y.

For the class of A-quasiconvex vector functions introduced in Definition 2.16 we state a characterisation in terms of generalised $\leq_A$-level sets of the vector function f.

Lemma 2.20. *Assume (A) and let $\mathcal{S}$ be convex. Then, the function f is A-quasiconvex on $\mathcal{S}$ if and only if*

$$\forall x^2 \in \mathcal{S},\ \forall x^1 \in \mathrm{lev}_{\leq_A}(\mathcal{S} \setminus \{x^2\}, f, f(x^2)),\ \exists x^0 \in \mathcal{S} \setminus \{x^2\} :\ [x^0, x^2) \subseteq \mathrm{lev}_{\leq_A}(\mathcal{S}, f, f(x^2)).$$

3. Relationships between constrained and unconstrained vector optimisation problems

In this section, we analyse important relationships between constrained and unconstrained vector optimisation problems using generalised convexity concepts for vector functions (from our Definition 2.16). Consider the constrained vector optimisation problem

$$\underset{x \in \mathcal{S}}{\mathrm{argmin}_C}\ f(x) \tag{$P_\mathcal{S}$}$$

as given in Section 2.3 under the assumption (A). Beside the vector problem ($P_\mathcal{S}$) we are also interested in the corresponding unconstrained vector problem

$$\underset{x \in \mathcal{D}}{\mathrm{argmin}_C}\ f(x) \tag{$P_\mathcal{D}$}$$

From the definitions of the efficiency concepts (Definition 2.13) we immediately get the following simple relationships between the unconstrained problem ($P_\mathcal{D}$) and the constrained problem ($P_\mathcal{S}$).

Theorem 3.1. *Assume (A). Then, for every set $\mathcal{U} \subseteq X$ with $\mathcal{S} \subseteq \mathcal{U} \subseteq \mathcal{D}$, we have*

$$\mathcal{S} \cap \mathrm{Eff}(f, \mathcal{D}, C) \subseteq \mathrm{Eff}(f, \mathcal{U}, C), \tag{3.1}$$

$$\mathcal{S} \cap \mathrm{WEff}(f, \mathcal{D}, C) \subseteq \mathrm{WEff}(f, \mathcal{U}, C), \tag{3.2}$$

$$\mathcal{S} \cap \mathrm{SEff}(f, \mathcal{D}, C) \subseteq \mathrm{SEff}(f, \mathcal{U}, C). \tag{3.3}$$

The following lemma states useful geometrical characterisations of (weakly, strictly) efficient solutions in terms of generalised $\leq_A$-level sets of the vector function f (see also Ehrgott [45, Th. 2.30]).

Lemma 3.2. *Assume (A) and take any $x \in \mathcal{S}$. Then we have the following equivalences:*

$$x \in \mathrm{Eff}(f, \mathcal{S}, C) \iff \mathrm{lev}_{\leq_C}\big(\mathcal{S}, f, f(x)\big) \subseteq \mathrm{lev}_{=}\big(\mathcal{S}, f, f(x)\big), \tag{3.4}$$

$$x \in \mathrm{WEff}(f, \mathcal{S}, C) \iff \mathrm{lev}_{<_C}\big(\mathcal{S}, f, f(x)\big) = \emptyset, \tag{3.5}$$

$$x \in \mathrm{SEff}(f, \mathcal{S}, C) \iff \mathrm{lev}_{\leq_C}\big(\mathcal{S}, f, f(x)\big) = \{x\}. \tag{3.6}$$

Proof. $x \in \mathrm{Eff}(f, \mathcal{S}, C)$ by definition is the same as saying that there does not exist a $y \in \mathcal{S}$ with $f(y) \lneq_C f(x)$. This again is the same as saying that for every $y \in \mathcal{S}$ with $f(y) \leq_C f(x)$, so $y \in \mathrm{lev}_{\leq_C}\big(\mathcal{S}, f, f(x)\big)$, we have $y \notin \mathrm{lev}_{\lneq_C}\big(\mathcal{S}, f, f(x)\big)$ and therefore $y \in \mathrm{lev}_{=}\big(\mathcal{S}, f, f(x)\big)$.

The equivalency of $x \in \mathrm{WEff}(f, \mathcal{S}, C)$ and $\mathrm{lev}_{<_C}\big(\mathcal{S}, f, f(x)\big)$ is evident.

Even though the last equivalency (3.6) is also evident, we can easily prove it by using that $x \in \mathrm{SEff}(f, \mathcal{S}, C)$ is the same as $x \in \mathrm{Eff}(f, \mathcal{S}, C)$ and $\mathrm{lev}_{=}\big(\mathcal{S}, f, f(x)\big) = \{x\}$ together with (3.4). ■

Using the generalised convexity concept for vector functions from Definition 2.16 (in the sense of Jahn [17]), we are able to derive relationships between the sets of efficient solutions of the constrained vector problem ($P_\mathcal{S}$) and the corresponding unconstrained vector problem ($P_\mathcal{D}$).

Notice, by dealing with the generalised convexity concept for vector functions $f : \mathcal{D} \to Y$ (from Definition 2.16) we have to impose that the domain $\mathcal{D}$ is a convex set.

Theorem 3.3. *Assume (A), let $\mathcal{D}$ be convex and let one of the following conditions be satisfied*

(1) f is C-quasiconvex on $\mathcal{D}$ and non-constant on proper line segments in $\mathcal{S}$, i.e.

$$\forall x^1, x^2 \in \mathcal{S},\ x^1 \neq x^2,\ f(x^1) = f(x^2)\ \exists x \in (x^1, x^2) :\ f(x) \neq f(x^1). \tag{3.7}$$

(2) f is $C \setminus \{0_Y\}$-quasiconvex on $\mathcal{D}$.

Then,

$$\operatorname{int}(\mathcal{S}) \cap \operatorname{Eff}(f, \mathcal{S}, C) \subseteq \operatorname{int}(\mathcal{S}) \cap \operatorname{Eff}(f, \mathcal{D}, C), \tag{3.8}$$

$$\mathcal{S} \cap \operatorname{Eff}(f, \mathcal{D}, C) \subseteq \operatorname{Eff}(f, \mathcal{S}, C) \subseteq \big(\operatorname{int}(\mathcal{S}) \cap \operatorname{Eff}(f, \mathcal{D}, C)\big) \cup \big(\mathcal{S} \cap \operatorname{bd}(\mathcal{S})\big), \tag{3.9}$$

$$\operatorname{Eff}(f, \mathcal{S}, C) \setminus \operatorname{Eff}(f, \mathcal{D}, C) \subseteq \mathcal{S} \cap \operatorname{bd}(\mathcal{S}). \tag{3.10}$$

If additionally $\mathcal{S}$ is open, then $\mathcal{S} \cap \operatorname{Eff}(f, \mathcal{D}, C) = \operatorname{Eff}(f, \mathcal{S}, C)$.

Proof. For $\chi \in \operatorname{int}(\mathcal{S}) \cap \operatorname{Eff}(f, \mathcal{S}, C)$, on the contrary we assume $\chi \notin \operatorname{Eff}(f, \mathcal{D}, C)$ and therefore get the existence of $x^0 \in \mathcal{D}$ such that $f(x^0) \lneq_C f(\chi)$. From the efficiency of χ it follows that $x^0 \in \mathcal{D} \setminus \mathcal{S}$.

Under condition (1) from the C-quasiconvexity of f we get the existence of $\bar{x} \in \mathcal{D} \setminus \{\chi\}$ such that $f(x) \leq_C f(\chi)$ for all $x \in [\bar{x}, \chi)$. Due to $\chi \in \operatorname{int}(\mathcal{S})$ we have $[\bar{x}, \chi) \cap \mathcal{S} \neq \emptyset$. Hence, for any $x \in [\bar{x}, \chi) \cap \mathcal{S}$ we either have $f(x) \lneq_C f(\chi)$ or the condition (3.7) yields the existence of an $\hat{x} \in [x, \chi)$ with this property. In case of (2) we get the existence of $\bar{x} \in \mathcal{D} \setminus \{\chi\}$ such that for all $x \in [\bar{x}, \chi)$ we have $f(x) \lneq_C f(\chi)$. Then, for any $x \in [\bar{x}, \chi) \cap \mathcal{S}$ we get $f(x) \lneq_C f(\chi)$ and $x \in \mathcal{S}$, where again because of $\chi \in \operatorname{int}(\mathcal{S})$ we have $[\bar{x}, \chi) \cap \mathcal{S} \neq \emptyset$. Both cases lead to a contradiction to $\chi \in \operatorname{Eff}(f, \mathcal{S}, C)$, which proves (3.8).

From this we immediately get (3.9) by (3.1) and the fact that $\mathcal{S} = \operatorname{int}(\mathcal{S}) \cup \big(\mathcal{S} \cap \operatorname{bd}(\mathcal{S})\big)$:

$$\begin{aligned}\operatorname{Eff}(f, \mathcal{S}, C) &= \Big(\operatorname{int}(\mathcal{S}) \cup \big(\mathcal{S} \cap \operatorname{bd}(\mathcal{S})\big)\Big) \cap \operatorname{Eff}(f, \mathcal{S}, C) \\ &\subseteq \big(\operatorname{int}(\mathcal{S}) \cap \operatorname{Eff}(f, \mathcal{D}, C)\big) \cup \big(\mathcal{S} \cap \operatorname{bd}(\mathcal{S})\big).\end{aligned}$$

The second inclusion of (3.9) then gives (3.10). The last statement also follows directly from (3.9). ■

We also derive a counterpart to Theorem 3.3 for the concept of weak efficiency.

Theorem 3.4. *Assume (A), let $\mathcal{D}$ be convex and let f be $\operatorname{int}(C)$-quasiconvex on $\mathcal{D}$. Then,*

$$\operatorname{int}(\mathcal{S}) \cap \operatorname{WEff}(f, \mathcal{S}, C) \subseteq \operatorname{int}(\mathcal{S}) \cap \operatorname{WEff}(f, \mathcal{D}, C), \tag{3.11}$$

$$\mathcal{S} \cap \operatorname{WEff}(f, \mathcal{D}, C) \subseteq \operatorname{WEff}(f, \mathcal{S}, C) \subseteq \big(\operatorname{int}(\mathcal{S}) \cap \operatorname{WEff}(f, \mathcal{D}, C)\big) \cup \big(\mathcal{S} \cap \operatorname{bd}(\mathcal{S})\big), \tag{3.12}$$

$$\operatorname{WEff}(f, \mathcal{S}, C) \setminus \operatorname{WEff}(f, \mathcal{D}, C) \subseteq \mathcal{S} \cap \operatorname{bd}(\mathcal{S}). \tag{3.13}$$

If additionally $\mathcal{S}$ is open, then $\mathcal{S} \cap \operatorname{WEff}(f, \mathcal{D}, C) = \operatorname{WEff}(f, \mathcal{S}, C)$.

Proof. Since the proof is similar to the proof of Theorem 3.3, we will make the explanations a bit shorter.

For $\chi \in \operatorname{int}(\mathcal{S}) \cap \operatorname{WEff}(f, \mathcal{S}, C)$ we contrarily assume $\chi \notin \operatorname{WEff}(f, \mathcal{D}, C)$, so that there exists $x^0 \in \mathcal{D} \setminus \mathcal{S}$ with $f(x^0) <_C f(\chi)$. Because of $\chi \in \operatorname{int}(\mathcal{S})$ there exists $\lambda \in \mathbb{R}_>$ such that for $x^\lambda :=$ $\chi + \lambda(x^0 - \chi)$ we have $[\chi, x^\lambda] \subseteq \mathcal{S}$.

From the $\operatorname{int}(\mathcal{C})$-quasiconvexity of f we get the existence of $\bar{x} \in \mathcal{D} \setminus \{\chi\}$ such that for all $x \in [\bar{x}, \chi) \cap \mathcal{S} \neq \emptyset$ we have $f(x) <_C f(\chi)$, which contradicts $\chi \in \mathrm{WEff}(f, \mathcal{S}, \mathcal{C})$ and therefore proves (3.11).

As in Theorem 3.3 the other statements follow from there. ∎

For the concept of strict efficiency we get the following result.

Theorem 3.5. *Assume (A), let $\mathcal{D}$ be convex and f be $\mathcal{C}$-quasiconvex on $\mathcal{D}$. Then,*

$$\operatorname{int}(\mathcal{S}) \cap \mathrm{SEff}(f, \mathcal{S}, \mathcal{C}) \subseteq \operatorname{int}(\mathcal{S}) \cap \mathrm{SEff}(f, \mathcal{D}, \mathcal{C}), \tag{3.14}$$

$$\mathcal{S} \cap \mathrm{SEff}(f, \mathcal{D}, \mathcal{C}) \subseteq \mathrm{SEff}(f, \mathcal{S}, \mathcal{C}) \subseteq \big(\operatorname{int}(\mathcal{S}) \cap \mathrm{SEff}(f, \mathcal{D}, \mathcal{C})\big) \cup \big(\mathcal{S} \cap \operatorname{bd}(\mathcal{S})\big), \tag{3.15}$$

$$\mathrm{SEff}(f, \mathcal{S}, \mathcal{C}) \setminus \mathrm{SEff}(f, \mathcal{D}, \mathcal{C}) \subseteq \mathcal{S} \cap \operatorname{bd}(\mathcal{S}). \tag{3.16}$$

If additionally $\mathcal{S}$ is open, then $\mathcal{S} \cap \mathrm{SEff}(f, \mathcal{D}, \mathcal{C}) = \mathrm{SEff}(f, \mathcal{S}, \mathcal{C})$.

Proof. Also here the proof is similar to the proof of Theorem 3.3.

For $\chi \in \operatorname{int}(\mathcal{S}) \cap \mathrm{SEff}(f, \mathcal{S}, \mathcal{C})$ we in contrast assume $\chi \notin \mathrm{SEff}(f, \mathcal{D}, \mathcal{C})$, so that there exists $x^0 \in \mathcal{D} \setminus \mathcal{S}$, $x^0 \neq \chi$, with $f(x^0) \leq_C f(\chi)$. From the $\mathcal{C}$-quasiconvexity of f on $\mathcal{D}$ we get the existence of $\bar{x} \in \mathcal{D} \setminus \{\chi\}$ such that for all $x \in [\bar{x}, \chi) \cap \mathcal{S} \neq \emptyset$ we have $f(x) \leq_C f(\chi)$. That is a contradiction to $\chi \in \mathrm{SEff}(f, \mathcal{S}, \mathcal{C})$, so we get (3.14).

Again, from this we get the other statements. ∎

Remark 3.6. Similar results to our Theorems 3.3, 3.4 and 3.5 are derived by Günther [10, 11], and Günther and Tammer [12, 13] for the case $Y = \mathbb{R}^m$ and $\mathcal{C} = \mathbb{R}^m_{\geq}$ by using componentwise generalised convexity concepts for vector functions $f : X \to \mathbb{R}^m$. Our Theorems 3.3, 3.4 and 3.5 show that by imposing appropriate generalised convexity assumptions on vector functions $f : X \to Y$ we are able to extend/generalise these results to the case where Y is a real linear-topological space and $\mathcal{C}$ is a nontrivial, pointed, convex cone in Y.

Remark 3.7. Notice that Theorems 3.3, 3.4 and 3.5 provide sufficient conditions (including generalised convexity assumptions on the vector function f) for the validity of the inclusions given in (3.10), (3.13) and (3.16), respectively. Of course, $\operatorname{int}(\mathcal{S}) = \emptyset$ is sufficient for all three inclusions as well. Later in our main theorems in Section 4.3 we will simply assume that the inclusions (3.10), (3.13) and (3.16) are valid, and thus do not require any further generalised convexity assumptions for the vector function f.

4. Vectorial penalisation approach

This section contains the vectorial penalisation approach for general vector optimisation problems. Given a vector function $f : \mathcal{D} \to Y$ (as considered in assumption (A)) and a scalar function $\nu : \mathcal{D} \to \mathbb{R}$, we are interested in the extended (penalised) vector function

$$f^{\otimes} : \mathcal{D} \to Y \times \mathbb{R}, \quad f^{\otimes} := (f, \nu)$$

and the extended (penalised) vector optimisation problem

$$\underset{x \in \mathcal{D}}{\operatorname{argmin}}_{\mathcal{C} \times \mathbb{R}_{\geq}} \; f^{\otimes}(x). \tag{$P^{\otimes}_{\mathcal{D}}$}$$

In our approach, the function $\nu : \mathcal{D} \to \mathbb{R}$ can be seen as a penalisation term w.r.t. the feasible set $\mathcal{S}$ of the vector problem ($P_{\mathcal{S}}$), which is given in (A).

After studying basic properties of the penalisation function ν, we investigate special types of penalisation functions for vector problems involving abstract constraints as well as for problems involving explicit generalised inequality and equality constraints. In particular, for the cone-constraint case we are able to construct appropriate penalisation functions for some popular cones given in spaces of finite and infinite dimension (as given in Section 2.2). Furthermore, we analyse the effects of adding/removing a penalisation term to/from the original vector objective function on the nature of the solution sets of the vector optimisation problems (in the spirit of Fliege [29]). We end this section by stating our main results related to our vectorial penalisation approach for general vector optimisation problems, which show profound relationships between the solution sets of the vector problems ($P_{\mathcal{S}}$), ($P_{\mathcal{D}}$) and ($P^{\otimes}_{\mathcal{D}}$).

Throughout this section, it is convenient to introduce the following assumptions:

$$\begin{cases} \text{Assume } (A), \\ \nu : \mathcal{D} \to \mathbb{R}, \\ f^{\otimes} := (f, \nu). \end{cases} \tag{$A^{\otimes}$}$$

4.1. **Penalisation functions.** Assume ($A^{\otimes}$) and consider a set $\mathcal{U} \subseteq \mathcal{D}$ with $\emptyset \neq \mathcal{S} \subseteq \mathcal{U}$. For the penalisation function $\nu : \mathcal{D} \to \mathbb{R}$, we are interested in properties of the following type:

$$\forall x^0 \in \mathrm{bd}(\mathcal{S}):\ \mathrm{lev}_{\leq}\big(\mathcal{U}, \nu, \nu(x^0)\big) = \mathcal{S} \tag{$\mathcal{A}1$}$$

$$\forall x^0 \in \mathrm{bd}(\mathcal{S}):\ \mathrm{lev}_{=}\big(\mathcal{U}, \nu, \nu(x^0)\big) = \mathrm{bd}(\mathcal{S}) \tag{$\mathcal{A}2$}$$

$$\forall x^0 \in \mathcal{S}:\ \mathrm{lev}_{=}\big(\mathcal{U}, \nu, \nu(x^0)\big) = \mathrm{lev}_{\leq}\big(\mathcal{U}, \nu, \nu(x^0)\big) = \mathcal{S} \tag{$\mathcal{A}3$}$$

$$\forall x^0 \in \mathcal{S}:\ \mathrm{lev}_{\leq}\big(\mathcal{U}, \nu, \nu(x^0)\big) \subseteq \mathcal{S} \tag{$\mathcal{A}4$}$$

$$\mathrm{lev}_{\leq}(\mathcal{U}, \nu, 0) = \mathcal{S} \tag{$\mathcal{A}5$}$$

$$\mathrm{lev}_{=}(\mathcal{U}, \nu, 0) = \mathrm{bd}(\mathcal{S}) \tag{$\mathcal{A}6$}$$

$$\forall x^1 \in \mathrm{bd}(\mathcal{S})\ \exists x^2 \in \mathrm{int}(\mathcal{S}):\ [x^2, x^1) \subseteq \mathrm{lev}_{<}\big(\mathcal{U}, \nu, \nu(x^1)\big) \tag{$\mathcal{A}7$}$$

Dealing with the properties ($\mathcal{A}1$), ($\mathcal{A}2$), ($\mathcal{A}6$) and ($\mathcal{A}7$) we impose the condition $\mathrm{bd}(\mathcal{S}) \subseteq \mathcal{U}$ and when considering ($\mathcal{A}7$) we assume $\mathcal{U}$ to be convex.

For simplicity, if ν satisfies one of the properties ($\mathcal{A}1$) - ($\mathcal{A}7$) based on some sets $\mathcal{U}$ and $\mathcal{S}$, we briefly use the notation

$$\nu \in \mathcal{A}_i(\mathcal{S}, \mathcal{U}) \quad (\text{for } i \in \{1, \ldots, 7\}).$$

Proposition 4.1. *For any set $\mathcal{U} \subseteq \mathcal{D}$ with $\emptyset \neq \mathcal{S} \subseteq \mathcal{U}$, the following assertions hold true:*

(1) If $\mathrm{bd}(\mathcal{S}) \subseteq \mathcal{U}$ and $\nu \in \mathcal{A}_1(\mathcal{S}, \mathcal{U})$, then $\mathcal{S}$ is closed and

$$\forall x^0 \in \mathrm{bd}(\mathcal{S}):\ \mathrm{bd}(\mathcal{S}) \subseteq \mathrm{lev}_{=}\big(\mathcal{U}, \nu, \nu(x^0)\big). \tag{4.1}$$

(2) If $\nu \in \mathcal{A}_1(\mathcal{S}, \mathcal{U}) \cap \mathcal{A}_2(\mathcal{S}, \mathcal{U})$, then

$$\forall x^0 \in \mathrm{bd}(\mathcal{S}):\ \mathrm{lev}_{<}\big(\mathcal{U}, \nu, \nu(x^0)\big) = \mathrm{int}(\mathcal{S}), \tag{4.2}$$

(3) If $\nu \in \mathcal{A}_3(\mathcal{S}, \mathcal{U})$, then

$$\forall x^0 \in \mathcal{S}:\ \mathrm{lev}_{<}\big(\mathcal{U}, \nu, \nu(x^0)\big) = \emptyset. \tag{4.3}$$

(4) $\nu \in \mathcal{A}_3(\mathcal{S}, \mathcal{U})$ is equivalent to

$$\forall x^0 \in \mathcal{S}:\ \mathrm{lev}_{\leq}\big(\mathcal{U}, \nu, \nu(x^0)\big) = \mathcal{S}.$$

(5) If $\mathcal{S}$ is closed, then $\mathcal{A}_3(\mathcal{S},\mathcal{U}) = \mathcal{A}_1(\mathcal{S},\mathcal{U}) \cap \mathcal{A}_2(\mathcal{S},\mathcal{U})$ is equivalent to $\operatorname{int}(\mathcal{S}) = \emptyset$.

Proof. For (1) consider any $x^1, x^2 \in \operatorname{bd}(\mathcal{S})$. Because of $x^i \in \operatorname{lev}_{\leq}\left(\mathcal{U}, \nu, \nu(x^i)\right)$ for $i \in \{1,2\}$ by ($\mathcal{A}1$) these satisfy $x^1, x^2 \in \mathcal{S}$, which proves the closedness of $\mathcal{S}$, and therefore for $j \in \{1,2\} \setminus \{i\}$ we get $x^j \in \operatorname{lev}_{\leq}\left(\mathcal{U}, \nu, \nu(x^i)\right)$, which proves (4.1). (2) and (3) are obvious. (4) follows by the fact that for any $x^0, x^1 \in \mathcal{S}$ the conditions $x^1 \in \operatorname{lev}_{\leq}\left(\mathcal{U}, \nu, \nu(x^0)\right)$ and $x^0 \in \operatorname{lev}_{\leq}\left(\mathcal{U}, \nu, \nu(x^1)\right)$ imply $x^1 \in \operatorname{lev}_{=}\left(\mathcal{U}, \nu, \nu(x^0)\right)$. Let us prove (5). Under $\operatorname{int}(\mathcal{S}) = \emptyset$ and the closedness of $\mathcal{S}$, the equality $\mathcal{A}_3(\mathcal{S},\mathcal{U}) = \mathcal{A}_1(\mathcal{S},\mathcal{U}) \cap \mathcal{A}_2(\mathcal{S},\mathcal{U})$ follows immediately. Assume now that $\mathcal{A}_3(\mathcal{S},\mathcal{U}) = \mathcal{A}_1(\mathcal{S},\mathcal{U}) \cap \mathcal{A}_2(\mathcal{S},\mathcal{U})$. It is easy to check that $\mathcal{A}_3(\mathcal{S},\mathcal{U}) \neq \emptyset$ (put $\nu \equiv 0$ in $\mathcal{S}$ and $\nu \equiv 1$ in $\mathcal{U} \setminus \mathcal{S}$). Thus, (2) and (3) yield $\emptyset = \operatorname{lev}_{<}\left(\mathcal{U}, \nu, \nu(x^0)\right) = \operatorname{int}(\mathcal{S})$ for $x^0 \in \operatorname{bd}(\mathcal{S})$. ■

Now, we study some relationships between some of the properties ($\mathcal{A}1$) - ($\mathcal{A}7$).

Proposition 4.2. *For any set $\mathcal{U} \subseteq \mathcal{D}$ with $\emptyset \neq \mathcal{S} \subseteq \mathcal{U}$ and* $\operatorname{bd}(\mathcal{S}) \subseteq \mathcal{U}$, *the following assertions hold true:*

(1) $\bigcup_{i \in \{1,3,5\}} \mathcal{A}_i(\mathcal{S},\mathcal{U}) \subseteq \mathcal{A}_4(\mathcal{S},\mathcal{U})$.
(2) $\mathcal{A}_5(\mathcal{S},\mathcal{U}) \cap \mathcal{A}_6(\mathcal{S},\mathcal{U}) \subseteq \mathcal{A}_1(\mathcal{S},\mathcal{U}) \cap \mathcal{A}_2(\mathcal{S},\mathcal{U})$.
(3) If $\nu \in \mathcal{A}_1(\mathcal{S},\mathcal{U}) \cap \mathcal{A}_2(\mathcal{S},\mathcal{U})$, then $\nu - \nu(x^0) \in \mathcal{A}_5(\mathcal{S},\mathcal{U}) \cap \mathcal{A}_6(\mathcal{S},\mathcal{U})$ for all $x^0 \in \operatorname{bd}(\mathcal{S})$.

Proof. All three assertions follow immediately from the definitions of $\mathcal{A}_i(\mathcal{S},\mathcal{U})$, $i \in \{1,2,\dots,6\}$. ■

We are primarily interested in a penalisation function $\nu : \mathcal{U} \to \mathbb{R}$ that satisfies $\nu \in \mathcal{A}_1(\mathcal{S},\mathcal{U}) \cap \mathcal{A}_2(\mathcal{S},\mathcal{U})$ (called **type 1 representation** for simplicity) or $\nu \in \mathcal{A}_3(\mathcal{S},\mathcal{U})$ (called **type 2 representation**), which is most often used with the concrete value $\nu\big|_{\mathcal{S}} \equiv 0$.

Remark 4.3. Notice, for a given function $\nu : \mathcal{D} \to \mathbb{R}$ with $\emptyset \neq \mathcal{S} \subseteq \mathcal{U} \subseteq \mathcal{D}$, the following assertions are equivalent:

- $\nu \in \mathcal{A}_3(\mathcal{S},\mathcal{U})$ and $\nu\big|_{\mathcal{S}} \equiv 0$.
- $\operatorname{lev}_{\leq}(\mathcal{U}, \nu, 0) = \operatorname{lev}_{=}(\mathcal{U}, \nu, 0) = \mathcal{S}$.
- ν is nonnegative on $\mathcal{U}$ and satisfies $\nu \in \mathcal{A}_5(\mathcal{S},\mathcal{U})$.

If additionally $\mathcal{S} = \operatorname{bd}(\mathcal{S})$ (e.g., sets of isolated points), then also the following statement is equivalent:

- $\nu \in \mathcal{A}_5(\mathcal{S},\mathcal{U}) \cap \mathcal{A}_6(\mathcal{S},\mathcal{U})$.

We will use these alternative formulations of the type 2 representation in the upcoming examples in Section 4.1.1 and throughout Section 4.1.2.

4.1.1. *Abstract constraints.* In this section, let us assume that the feasible set $\mathcal{S}$ of the vector optimisation problem ($P_{\mathcal{S}}$) (as defined in (A)) is given by an abstract set in X, which satisfies certain properties (e.g., closedness, convexity, solidness, polyhedrality). Our aim is to provide some popular functions that satisfy a type 1/type 2 representation.

Example 4.4 (Minkowski functional)**.** Given a nonempty, closed set $\mathcal{S}$ in a real linear-topological space X, the **Minkowski functional** $p_{\mathcal{S}} : X \to [0, +\infty]$ associated to the set $\mathcal{S}$ is defined by

$$p_{\mathcal{S}}(x) := \inf\left\{ t \in \mathbb{R}_{>} \,\middle|\, x \in t\mathcal{S} \right\} \quad \text{for all } x \in X.$$

Assume that $\mathcal{S} \subseteq X$ is convex with $0_X \in \operatorname{int}(\mathcal{S})$ and $s \in \mathcal{S}$. Then, the following properties of $p_{\mathcal{S}}$ are valid:

- $p_{\mathcal{S}}$ is a real-valued, continuous and sublinear function.
- If $\nu : X \to \mathbb{R}$ is given by $\nu(x) := p_{\mathcal{S}-s}(x-s)$ for all $x \in X$, then $\nu \in \mathcal{A}_i(\mathcal{S}, \mathcal{U})$ for all $i \in \{1,2,4\}$.
- If $\nu : X \to \mathbb{R}$ is given by $\nu(x) := p_{\mathcal{S}-s}(x-s) - 1$ for all $x \in X$, then $\nu \in \mathcal{A}_i(\mathcal{S}, \mathcal{U})$ for all $i \in \{1,2,4,5,6\}$.

Example 4.5 (Distance function)**.** For any nonempty, closed set $\mathcal{S}$ in a normed space $(X, \|\cdot\|)$, we consider the **distance to the set** $\mathcal{S}$ given by the function $d_{\mathcal{S}} : X \to \mathbb{R}$, where

$$d_{\mathcal{S}}(x) := \inf\{\|x-s\| \mid s \in \mathcal{S}\} \quad \text{for all } x \in X.$$

The following properties of $d_{\mathcal{S}}$ are important:

- $d_{\mathcal{S}}$ is Lipschitz continuous with constant 1.
- $d_{\mathcal{S}}$ is convex if and only if $\mathcal{S}$ is convex.
- If $\mathcal{S}$ is a convex cone, then $d_{\mathcal{S}}$ is sublinear.
- $d_{\mathcal{S}}$ satisfies the property $d_{\mathcal{S}} \in \mathcal{A}_i(\mathcal{S}, \mathcal{U})$ for all $i \in \{3,4,5\}$ and $d_{\mathcal{S}}\big|_{\mathcal{S}} \equiv 0$.

Example 4.6 (Oriented/signed distance function)**.** Let $\mathcal{S}$ be a nonempty, proper, closed set in a normed space $(X, \|\cdot\|)$. Then, the **oriented/signed distance function** (by Hiriart-Urruty [46]) $\triangle_{\mathcal{S}} : X \to \mathbb{R}$ is given by

$$\triangle_{\mathcal{S}}(x) := d_{\mathcal{S}}(x) - d_{X\setminus\mathcal{S}}(x) = \begin{cases} d_{\mathcal{S}}(x) & \text{for } x \in X \setminus \mathcal{S}, \\ -d_{X\setminus\mathcal{S}}(x) & \text{for } x \in \mathcal{S}. \end{cases}$$

Let us recall some well-known properties of $\triangle_{\mathcal{S}}$ (see Hiriart-Urruty [46] and Zaffaroni [47]):

- $\triangle_{\mathcal{S}}$ is real-valued and Lipschitz continuous with constant 1.
- $\mathcal{S}$ is convex if and only if $\triangle_{\mathcal{S}}$ is convex.
- If $\mathcal{S}$ is a convex cone, then $\triangle_{\mathcal{S}}$ is sublinear.
- $\triangle_{\mathcal{S}}$ fulfills the property $\triangle_{\mathcal{S}} \in \mathcal{A}_i(\mathcal{S}, \mathcal{U})$ for all $i \in \{1,2,4,5,6\}$.

Example 4.7 (Gerstewitz functional)**.** Suppose that $\mathcal{S} \subsetneq X$ is a nonempty, proper, closed subset of a linear-topological space X and $C \subsetneq X$ is a proper, closed, convex cone such that the free-disposal assumption $\mathcal{S} - C = \mathcal{S}$. Let $k \in C \setminus (-C)$ be a direction. Then, the **Gerstewitz functional** $\phi_{\mathcal{S},k} : X \to \overline{\mathbb{R}}$ associated to the pair $(\mathcal{S}, k)$ is defined by

$$\phi_{\mathcal{S},k}(x) := \inf\left\{ t \in \mathbb{R} \,\middle|\, x \in \mathcal{S} + tk \right\} \quad \text{for all } x \in X.$$

For the case that C is solid and $k \in \text{int}(C)$, let us recall some well-known properties of the Gerstewitz functional $\phi_{\mathcal{S},k}$ (see Gerstewitz [48], Göpfert, Riahi, Tammer and Zălinescu [41, Sec. 2.3]) and Khan, Tammer and Zălinescu [49, Sec. 5.2]:

- $\phi_{\mathcal{S},k}$ is real-valued and continuous.
- $\phi_{\mathcal{S},k}$ is convex if and only if $\mathcal{S}$ is convex.
- If $\mathcal{S} = -C$, then $\phi_{\mathcal{S},k}$ is sublinear.
- $\phi_{\mathcal{S},k}$ fulfills the property $\phi_{\mathcal{S},k} \in \mathcal{A}_i(\mathcal{S}, \mathcal{U})$ for all $i \in \{1,2,4,5,6\}$.

If $\mathcal{S}$ is given by a polyhedral set, then some more specific formulations for the values of $\phi_{\mathcal{S},k}$ are known (see, e.g., Günther and Popovici [22, 34], and Tammer and Winkler [50]).

4.1.2. *Explicit generalised inequality and equality constraints.* Consider the vector optimisation problem ($P_{\mathcal{S}}$) with explicit generalised inequality and equality constraints (i.e., the feasible set $\mathcal{S}$ is given by

$$\mathcal{S} = \{x \in \mathcal{U} \mid g(x) \in \mathcal{K}\,,\ h(x) = 0_{Z_2}\}) \tag{4.4}$$

and assume that ($A^{g,h}$) is valid. As already mentioned, we are primarily interested in a penalisation function $\nu : \mathcal{U} \to \mathbb{R}$ that satisfies a type 1 representation (i.e., $\nu \in \mathcal{A}_1(\mathcal{S},\mathcal{U}) \cap \mathcal{A}_2(\mathcal{S},\mathcal{U})$) or a type 2 representation (i.e., $\nu \in \mathcal{A}_3(\mathcal{S},\mathcal{U})$).

In a first step, we focus on the construction of a function $\phi : Z_1 \to \mathbb{R}$ that possesses the (cone representation; $\mathcal{K}$-representing) property $\phi \in \mathcal{A}_5(\mathcal{K}, Z_1) \cap \mathcal{A}_6(\mathcal{K}, Z_1)$ or the property $\phi \in \mathcal{A}_3(\mathcal{K}, Z_1)$.

Remark 4.8. It is easy to see that some abstract functions from Section 4.1.1 enjoy such properties. More precisely, the distance function $d_{\mathcal{K}}$ (as given in Example 4.5 with $S := \mathcal{K}$) satisfies $d_{\mathcal{K}} \in \mathcal{A}_3(\mathcal{K}, Z_1)$ while the oriented/signed distance function $\triangle_{\mathcal{K}}$ (as given in Example 4.6 with $S := \mathcal{K}$) and the Gerstewitz functional $\phi_{\mathcal{K},k}$ with $k \in -\operatorname{int}(\mathcal{K})$ (as given in Example 4.7 with $S := \mathcal{K}$ and $C := -\mathcal{K}$) satisfy the property $\triangle_{\mathcal{K}},\, \phi_{\mathcal{K},k} \in \mathcal{A}_5(\mathcal{K}, Z_1) \cap \mathcal{A}_6(\mathcal{K}, Z_1)$. By Jahn [39, Prop 3.6] we know that any sublinear, upper semi-continuous function $\phi \in \mathcal{A}_5(\mathcal{K}, Z_1)$ with $\{z \in Z_1 \mid \phi(z) < 0\} \neq \emptyset$ belongs to $\mathcal{A}_6(\mathcal{K}, Z_1)$. Thus, also for other functions such properties are valid (see the references Eichfelder and Jahn [51], Jahn [39], and our upcoming Example 4.10).

The next theorem shows us a general way to construct a function ϕ satisfying the $\mathcal{K}$-representing property $\phi \in \mathcal{A}_5(\mathcal{K}, Z_1) \cap \mathcal{A}_6(\mathcal{K}, Z_1)$ or $\phi \in \mathcal{A}_3(\mathcal{K}, Z_1)$ with $\phi\big|_{\mathcal{K}} \equiv 0$ (see Remark 4.3).

Theorem 4.9. *Consider a real linear-topological space Z, a finite-dimensional Euclidean space $V = \mathbb{R}^n$, a nonempty set $Q \subseteq V$, a function $\eta : \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$ that satisfies $\eta \in \mathcal{A}_5(\{0\}, \mathbb{R}_{\geq})$, and a function $\psi : Z \times Q \to \mathbb{R}$. Define the set*

$$C := \{z \in Z \mid \forall s \in Q:\ \psi(z,s) \leq 0\} = \bigcap_{s \in Q} \operatorname{lev}_{\leq}(Z, \psi(\cdot,s), 0).$$

Then, the following assertions hold true:

(1) If $\psi(\cdot,s)$ is positively homogeneous for all $s \in Q$, and $\psi(0_Z,\cdot) \equiv 0$, then C is a cone.
(2) If $\psi(\cdot,s)$ is lower semi-continuous (respectively, quasiconvex) for all $s \in Q$, then C is closed (respectively, convex).
(3) If any of the following assumptions is satisfied
- *Q is compact and $\psi(z,\cdot) : Q \to \mathbb{R}$ is upper semi-continuous for all $z \in Z$;*
- *Q is bounded and $\psi(z,\cdot) : Q \to \mathbb{R}$ is uniformly continuous for all $z \in Z$;*
- *Q is bounded, ψ is defined on $Z \times \operatorname{cl}(Q)$, and $\psi(z,\cdot) : \operatorname{cl}(Q) \to \mathbb{R}$ is upper semi-continuous for all $z \in Z$,*

then the supremum of $\{\psi(z,s) \mid s \in Q\}$ (for a given $z \in Z$) exists, and so $\phi : Z \to \mathbb{R}$, defined by

$$\phi(z) := \sup\{\psi(z,s) \mid s \in Q\} \quad \textit{for all } z \in Z,$$

satisfies the property $\phi \in \mathcal{A}_5(C,Z)$. If, in addition, for any $c \in C$ there is $s \in Q$ with $\psi(c,s) = 0$, then $\phi \in \mathcal{A}_3(C,Z)$.

(4) If

$$\operatorname{int}(C) = \{z \in Z \mid \forall s \in Q:\ \psi(z,s) < 0\} = \bigcap_{s\in Q} \operatorname{lev}_{<}(Z, \psi(\cdot,s), 0), \tag{4.5}$$

Q is compact, and $\psi(z,\cdot)$ is upper semi-continuous for all $z \in Z$, then $\phi : Z \to \mathbb{R}$, defined by

$$\phi(z) := \max\{\psi(z,s) \mid s \in Q\} \quad \textit{for all } z \in Z,$$

satisfies the property $\phi \in \mathcal{A}_5(C,Z) \cap \mathcal{A}_6(C,Z)$.

(5) If Q is finite, then $\phi : Z \to \mathbb{R}$, defined by

$$\phi(z) := \sum_{s\in Q} \eta(\max\{\psi(z,s), 0\}) \quad \textit{for all } z \in Z,$$

satisfies the property $\phi \in \mathcal{A}_3(C,Z) \cap \mathcal{A}_5(C,Z)$

(6) If Q is compact and solid with $\operatorname{cl}(\operatorname{int}(Q)) = \operatorname{cl}(Q)$ (e.g., if Q is convex), $\psi(z,\cdot)$ is continuous for all $z \in Z$, and η is continuous on $\mathbb{R}_{\geq}$, then $\phi : Z \to \mathbb{R}$, defined by

$$\phi(z) := \int_{s\in Q} \eta(\max\{\psi(z,s), 0\})\, ds \quad \textit{for all } z \in Z,$$

satisfies the property $\phi \in \mathcal{A}_3(C,Z) \cap \mathcal{A}_5(C,Z)$.

(7) Let $K \subseteq V$ be the cone generated by Q, i.e., $K = \mathbb{R}_{\geq} \cdot Q$. If $\psi(c,\cdot)$ is positively homogeneous for all $c \in C$, ψ is defined on $Z \times K$, and $\psi(c, 0_V) \leq 0$ for all $c \in C$, then

$$C = \{z \in Z \mid \forall k \in K:\ \psi(z,k) \leq 0\} = \bigcap_{k\in K} \operatorname{lev}_{\leq}(Z, \psi(\cdot,k), 0),$$

and if further (4.5) is valid and $0_V \notin Q$, then

$$\operatorname{int}(C) = \{z \in Z \mid \forall k \in K \setminus \{0_V\}:\ \psi(z,k) < 0\} = \bigcap_{k\in K\setminus\{0_V\}} \operatorname{lev}_{<}(Z, \psi(\cdot,k), 0).$$

Proof. Assertions (1) and (2) are easy to check.

(3) The existence of the supremum of $\{\psi(z,s) \mid s \in Q\}$ (for a given $z \in Z$) under one of the first two assumptions is well-known.

If Q is bounded, then $\operatorname{cl}(Q)$ is bounded as well, hence $\operatorname{cl}(Q)$ is compact (in the finite dimensional setting). Therefore, the upper semi-continuous function $\psi(z,\cdot) : \operatorname{cl}(Q) \to \mathbb{R}$ attains its maximum by the Weierstrass theorem. Consequently, for all $z \in Z$

$$\sup\{\psi(z,s) \mid s \in Q\} \leq \max\{\psi(z,s) \mid s \in \operatorname{cl}(Q)\} < \infty.$$

Since

$$C = \bigcap_{s\in Q} \operatorname{lev}_{\leq}(Z, \psi(\cdot,s), 0) = \operatorname{lev}_{\leq}(Z, \phi, 0),$$

we infer $\phi \in \mathcal{A}_5(C,Z)$. If, in addition, for any $c \in C$ there is $s \in Q$ with $\psi(c,s) = 0$, then $\phi(c) = \sup\{\psi(c,s) \mid s \in Q\} = 0$ for all $c \in C$, hence $\phi \in \mathcal{A}_3(C,Z)$.

(4) By (3) we get that $\phi \in \mathcal{A}_5(C,Z)$, where the supremum is actually a maximum. Moreover, since (4.5) is valid, we get

$$\operatorname{int}(C) = \bigcap_{s\in Q} \operatorname{lev}_{<}(Z, \psi(\cdot,s), 0) = \operatorname{lev}_{<}(Z, \phi, 0),$$

i.e., $\phi \in \mathcal{A}_6(C,Z)$.

(5) If Q is finite, then ϕ is well-defined and nonnegative, and we have $\phi(z) = 0$ if and only if $z \in C$ (i.e., $\psi(z,s) \leq 0$ for all $s \in Q$), taking into account that $\eta \in \mathcal{A}_5(\{0\}, \mathbb{R}_{\geq})$. This shows that $\phi \in \mathcal{A}_3(C,Z) \cap \mathcal{A}_5(C,Z)$.

(6) Define the function $h : Z \times Q \to \mathbb{R}$ by $h(z,s) := \eta(\max\{\psi(z,s), 0\})$ for all $(z,s) \in Z \times Q$. Under our assumptions, $h(z,\cdot)$ is continuous and nonnegative for all $z \in Z$. By the famous Weierstrass theorem, minimum and maximum of $h(z,\cdot)$ are attained on the compact set Q. Denote by $\mu(Q)$ the Lebesgue measure of Q. Clearly, $\mu(Q) \in [0,\infty)$ by the compactness of Q. Hence, for any $z \in Z$,

$$\infty > \mu(Q) \max_{s \in Q} h(z,s) \geq \phi(z) = \int_{s \in Q} h(z,s)\, ds \geq \mu(Q) \min_{s \in Q} h(z,s) \geq 0.$$

In particular, we see that ϕ is well-defined. It remains to show that $\phi \in \mathcal{A}_3(C,Z) \cap \mathcal{A}_5(C,Z)$. It is easy to check that $\phi(c) = 0$ for all $c \in C$ (i.e., $\psi(c,s) \leq 0$ for all $s \in Q$). Take now some $z \notin C$, i.e., $\psi(z,s) > 0$ for some $s \in Q$. Due to the (lower-semi)continuity of $\psi(z,\cdot)$ (i.e., all strict upper level sets of $\psi(z,\cdot)$ are open in Q), there is $\varepsilon > 0$ such that $\psi(z,w) > 0$ for all $w \in Q \cap B_\varepsilon(s)$. By our assumption $\mathrm{cl}(\mathrm{int}(Q)) = \mathrm{cl}(Q)$, there are $s^0 \in \mathrm{int}(Q)$ and $\delta \in (0,\varepsilon)$ with $\bar{B}^0 := \overline{B}_\delta(s^0) \subseteq Q \cap B_\varepsilon(s)$. Clearly, $\psi(z,w) > 0$ for all $w \in \bar{B}^0$, hence $\min_{w \in \bar{B}^0} h(z,w) = \min_{w \in \bar{B}^0} \eta(\psi(z,w)) > 0$, taking into account that $h(z,\cdot)$ is continuous on $Q (\supseteq \bar{B}^0)$ and $\bar{B}^0$ is compact. Moreover, since $\bar{B}^0$ is solid and compact, we have $\mu(\bar{B}^0) \in (0,\infty)$. Consequently, we conclude

$$\phi(z) = \int_{s \in Q} h(z,s)\; ds \geq \int_{w \in \bar{B}^0} h(z,w)\; dw \geq \mu(\bar{B}^0) \min_{w \in \bar{B}^0} h(z,w) > 0.$$

The facts that $\phi(c) = 0$ for all $c \in C$ and $\phi(z) > 0$ for all $z \notin C$ ensure $\phi \in \mathcal{A}_3(C,Z) \cap \mathcal{A}_5(C,Z)$.

(7) If $K = \mathbb{R}_{\geq} \cdot Q$, $\psi(c,\cdot)$ is positively homogeneous for all $c \in C$, and $\psi(c, 0_V) \leq 0$ for all $c \in C$, then

$$C = \{z \in Z \mid \forall t \geq 0, s \in Q :\; \psi(z,ts) = t\psi(z,s) \leq 0\} = \bigcap_{k \in K} \mathrm{lev}_{\leq}(Z, \psi(\cdot,k), 0).$$

If further (4.5) is valid and $0_V \notin Q$, then $K \setminus \{0_V\} = \mathbb{R}_{>} \cdot Q$. Hence

$$\mathrm{int}(C) = \{z \in Z \mid \forall t > 0, s \in Q :\; \psi(z,ts) = t\psi(z,s) < 0\} = \bigcap_{k \in K \setminus \{0_V\}} \mathrm{lev}_{<}(Z, \psi(\cdot,k), 0). \quad \blacksquare$$

Thanks to Theorem 4.9, we are able to construct functions $\phi : Z_1 \to \mathbb{R}$ with the desired cone representation properties for some important examples of spaces and cones considered in Section 2.2.

Example 4.10. We list some important examples of spaces and cones:

- **Standard cone in $\mathbb{R}^n$:** Consider $Z := V := \mathbb{R}^n$, $C := -\mathbb{R}^n_{\geq}$, $Q := \{e^1, \ldots, e^n\}$, $\psi(z,s) := s^T z$. Then, the function ϕ, defined by

$$\phi(z) := \max\{z_1, \ldots, z_n\} \quad \text{for all } z = (z_1, \ldots, z_n) \in Z,$$

satisfies the property $\phi \in \mathcal{A}_5(C,Z) \cap \mathcal{A}_6(C,Z)$. Notice that the function ϕ is continuous but nonsmooth.

In contrast, the function ϕ, defined by

$$\phi(z) := \sum_{i=1}^{n} (\max\{z_i, 0\})^2 \quad \text{for all } z = (z_1, \ldots, z_n) \in Z,$$

satisfies the property $\phi \in \mathcal{A}_3(C,Z) \cap \mathcal{A}_5(C,Z)$ and is continuously differentiable so that $\phi\big|_C \equiv 0$. From numerical optimisation, it is well-known that ϕ is a penalty function w.r.t. $-\mathbb{R}^n_{\geq}$ (see the recent paper by Jahn [39]).

Due to Theorem 4.9 (4) (applied for $Q := \{e^1\}$, $\psi(z,s) = \max\{z_1, \ldots, z_n\}$), the function ϕ, defined by

$$\phi(z) := (\max\{\max\{z_1, \ldots, z_n\}, 0\})^2 = (\max\{z_1, \ldots, z_n, 0\})^2 \quad \text{for all } z = (z_1, \ldots, z_n) \in Z,$$

also possesses the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$.

- **Polyhedral cone in $\mathbb{R}^n$:** Consider $Z := V := \mathbb{R}^n$, a nonempty finite set $P \subseteq Z \setminus \{0_Z\}$ with $0_Z \notin \operatorname{conv}(P)$, and a polyhedral cone $C := C_P \subseteq Z$ given by

$$C_P := \{z \in Z \mid \forall p \in P: \; p^T z \leq 0\}.$$

Let $\psi : Z \times P \to \mathbb{R}$ be defined by $\psi(z,p) = p^T z$. Then, the function ϕ, defined by

$$\phi(z) := \max\{p^T z \mid p \in P\} \quad \text{for all } z \in Z,$$

satisfies the property $\phi \in \mathcal{A}_5(C,Z) \cap \mathcal{A}_6(C,Z)$.

Moreover, the function ϕ, defined by

$$\phi(z) := (\max\{p^T z \mid p \in P \cup \{0_Z\}\})^2 \quad \text{for all } z \in Z,$$

respectively, by

$$\phi(z) := \sum_{p \in P} (\max\{p^T z, 0\})^2 \quad \text{for all } z \in Z,$$

satisfies the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$.

- ***K*-semidefinite cone (Löwner cone, copositive cone) in $\mathfrak{S}^n$:** Consider the space $Z = \mathfrak{S}^n$ of all real symmetric $n \times n$ matrices, $V = \mathbb{R}^n$, a nontrivial, closed, convex cone $K \subseteq V$ (for instance $K = \mathbb{R}^n$ or $K = \mathbb{R}^n_{\geq}$), and the negative K-semidefinite cone $C := -\mathfrak{S}^n_K$ in Z. Let $\psi : Z \times Q \to \mathbb{R}$ be defined by $\psi(M,s) = s^T M s$.
  (a) For the compact set $Q := K \cap \mathbb{S}_V$, the function ϕ, defined by

$$\phi(M) := \max\{s^T M s \mid s \in Q\} \quad \text{for all } M \in Z, \tag{4.6}$$

  satisfies the property $\phi \in \mathcal{A}_5(C,Z) \cap \mathcal{A}_6(C,Z)$.
  (b) For the compact, convex set $Q := \operatorname{cl}(\operatorname{conv}(K \cap \mathbb{S}_V))$, if $K^\# \neq \emptyset$, then the function ϕ, defined by (4.6), satisfies $\phi \in \mathcal{A}_5(C,Z) \cap \mathcal{A}_6(C,Z)$ as well. Notice, in view of Section 2.2 we have $K^\# \neq \emptyset$ if and only if $0_V \notin \operatorname{cl}(\operatorname{conv}(K \cap \mathbb{S}_V))$.
  Furthermore, assuming Q is solid, then the function ϕ, defined by

$$\phi(M) := \int_{s \in Q} (\max\{s^T M s, 0\})^2 \, ds \quad \text{for all } M \in Z, \tag{4.7}$$

  satisfies the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$.

(c) For the compact, convex set $Q := K \cap \mathbb{B}_V$, it follows that the function ϕ, defined by (4.6), possesses the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$ (since $0_V \in Q$).
Moreover, assuming K is solid, then the function ϕ, defined by (4.7) satisfies the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$.

(d) For any choice $Q \in \{K \cap \mathbb{S}_V, \operatorname{cl}(\operatorname{conv}(K \cap \mathbb{S}_V))\}$ one has that ϕ, defined by

$$\phi(M) := (\max\{s^T M s \mid s \in Q \cup \{0_V\}\})^2 \quad \text{for all } M \in Z,$$

has the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$.

- **Löwner cone in $\mathfrak{S}^n$:** Consider the space $Z = \mathfrak{S}^n$ of all real symmetric $n \times n$ matrices, $V = \mathbb{R}^n$, the finite set $Q := \{1, \dots, n\}$, and the negative Löwner cone $C := -\mathfrak{S}^n_+$. Let $\psi : Z \times Q \to \mathbb{R}$ be defined by $\psi(M,s) = \operatorname{eig}_s(M)$ (where $\operatorname{eig}(M)$ is a vector of the n eigenvalues of M and $\operatorname{eig}_s(M)$ is its s^{th} component). Then, the function ϕ, defined by

$$\phi(M) := \max\{\operatorname{eig}_s(M) \mid s \in Q\} \quad \text{for all } M \in Z$$

satisfies the property $\phi \in \mathcal{A}_5(C,Z) \cap \mathcal{A}_6(C,Z)$. Notice, by the famous Rayleigh-Ritz theorem we have (for $M \in Z$)

$$\phi(M) = \max\{\operatorname{eig}_s(M) \mid s \in Q\} = \max\{s^T M s \mid s \in \mathbb{S}_V\},$$

i.e., ϕ is exactly the function (4.6) from the previous example (case (a) with $K = \mathbb{R}^n$).
Moreover, the function ϕ, defined by

$$\phi(M) := \sum_{s \in Q} (\max\{\operatorname{eig}_s(M), 0\})^2 \quad \text{for all } M \in Z$$

satisfies the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$.

- **Second-order cone (Lorentz cone) in $\mathbb{R}^n \times \mathbb{R}$:** Consider a real normed space $V = \mathbb{R}^n$, the space $Z = V \times \mathbb{R}$, and the negative second-order cone $C := \{(y,t) \in Z \mid t + \|y\|_V \le 0\}$. Then, the function ϕ, defined by

$$\phi(y,t) := t + \|y\|_V \quad \text{for all } (y,t) \in Z,$$

satisfies the property $\phi \in \mathcal{A}_5(C,Z) \cap \mathcal{A}_6(C,Z)$.
Moreover, the function ϕ, defined by

$$\phi(y,t) := (\max\{t + \|y\|_V, 0\})^2 \quad \text{for all } (y,t) \in Z,$$

possesses the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$.
Consider the space $(V, \|\cdot\|_2)$. According to Section 2.2, the negative Lorentz cone admits the representation

$$C = \bigcap_{s \in Q} \{(y,t) \in V \times \mathbb{R} \mid s^T y \ge t\},$$

where $Q \in \{\mathbb{B}_V, \mathbb{S}_V\}$. Then, the function ϕ, defined by

$$\phi(y,t) := \max\{t - s^T y \mid s \in Q\} \quad \text{for all } (y,t) \in Z,$$

ensures $\phi \in \mathcal{A}_5(C,Z)$, and since

$$\operatorname{int}(C) = \bigcap_{s \in Q} \{(y,t) \in V \times \mathbb{R} \mid s^T y > t\},$$

we have $\phi \in \mathcal{A}_6(C,Z)$. In contrast, the function ϕ, defined by

$$\phi(y,t) := \int_{s\in\mathbb{B}_V} (\max\{t - s^T y, 0\})^2\, ds \quad \text{for all } (y,t) \in Z,$$

has the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$.

- **Bishop-Phelps cone in a normed space:** Consider a real normed space Z, and the Bishop-Phelps cone $C := -C(y^*) = C(-y^*)$ for some $y^* \in Z^*$ with $\|y^*\|_{Z^*} > 1$. Then, the function ϕ, defined by

$$\phi(y) := y^*(y) + \|y\|_Z \quad \text{for all } y \in Z,$$

satisfies the property $\phi \in \mathcal{A}_5(C,Z) \cap \mathcal{A}_6(C,Z)$. Notice that functions of this type are known as Bishop-Phelps type (normlinear) functionals (see, e.g., Jahn [39]).

Moreover, the function ϕ, defined by

$$\phi(y) := (\max\{y^*(y) + \|y\|_Z\,, 0\})^2 \quad \text{for all } y \in Z,$$

possesses the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$.

- **Standard cone in $C[a,b]$:** Consider the space $Z = C[a,b]$ (with $-\infty < a < b < \infty$) of real-valued continuous functions, $V = \mathbb{R}$, the solid, compact, convex set $Q = [a,b] \subseteq V$, and the negative standard cone $C := -C_+[a,b]$ in Z. Let $\psi : Z \times Q \to \mathbb{R}$ be defined by $\psi(u,s) = u(s)$. Then, the function ϕ, defined by

$$\phi(u) := \max\{u(s) \mid s \in Q\} \quad \text{for all } u \in Z,$$

owns the property $\phi \in \mathcal{A}_5(C,Z) \cap \mathcal{A}_6(C,Z)$.

Moreover, the function ϕ, defined by

$$\phi(u) := (\max\{\max\{u(s) \mid s \in Q\}, 0\})^2 \quad \text{for all } u \in Z,$$

respectively, by

$$\phi(u) := \int_{s\in Q} (\max\{u(s), 0\})^2\, ds \quad \text{for all } u \in Z,$$

satisfies the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$.

- **Standard cone in $L^p[a,b]$:** Consider the space $Z = L^p[a,b]$ (with $-\infty < a < b < \infty$ and $p \in [1,\infty]$) of all real-valued p-th power Lebesgue-integrable functions (essentially bounded functions, respectively), the negative standard cone $C := -L^p_+[a,b]$ in Z, and a positive real number δ. Then, the function ϕ, defined by

$$\phi(u) := \phi_\delta(u) := \min\{\operatorname{ess\,sup}\{u(s) \mid s \in [a,b]\}, \delta\} \quad \text{for all } u \in Z,$$

owns the property $\phi \in \mathcal{A}_5(C,Z)$, while the function ϕ, defined by

$$\phi(u) := (\max\{\phi_\delta(u), 0\})^2 \quad \text{for all } u \in Z,$$

satisfies the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$. Notice that for $u \in L^p[a,b] \setminus L^\infty[a,b]$ with $p \in [1,\infty)$ one may has $\operatorname{ess\,sup}\{u(s) \mid s \in [a,b]\} = \infty$ but $\phi_\delta(u) = \delta < \infty$. For $p = \infty$ it is allowed to put $\delta = \infty$ such that the minimum given in the function ϕ_δ is always attained in the first argument.

- **Closed convex cone in $\mathbb{R}^n$ (construction based on the dual cone):** Consider $Z = V = \mathbb{R}^n$ (hence $V^* = V$), a nontrivial, closed, convex cone $C \subseteq V$, and a compact set $Q \subseteq V$ such that $C^+ = \mathbb{R}_{\geq} \cdot Q$ (for instance
$$Q \in \{C^+ \cap \mathbb{B}_{V^*}, C^+ \cap \mathbb{S}_{V^*}, \mathrm{cl}(\mathrm{conv}(C^+ \cap \mathbb{S}_{V^*}))\}).$$
Let $\psi : Z \times Q \to \mathbb{R}$ be defined by $\psi(z,s) = s^T z$. By Section 2.2, we have
$$C = \{z \in Z \mid \forall c \in C^+ : c^T z \geq 0\},$$
and
$$\mathrm{int}(C) = \{z \in Z \mid \forall c \in C^+ \setminus \{0_V\} : c^T z > 0\}.$$
  (a) The function ϕ, defined by
$$\phi(z) := \max\{-s^T z \mid s \in Q\} \quad \text{for all } z \in Z,$$
  owns the property $\phi \in \mathcal{A}_5(C,Z)$. If C is solid and $0_V \notin Q$, then $\phi \in \mathcal{A}_6(C,Z)$. If $0_V \in Q$, then $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$. Notice, the function ϕ is used by Drummond and Svaiter [52] for deriving a steepest decent method in vector optimisation.
  (b) The function ϕ, defined by
$$\phi(z) := (\max\{-s^T z \mid s \in Q \cup \{0_V\}\})^2 \quad \text{for all } z \in Z,$$
  satisfies the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$.
  (c) If Q is solid and convex, then the function ϕ, defined by
$$\phi(z) := \int_{s \in Q} (\max\{-s^T z, 0\})^2 \, ds \quad \text{for all } z \in Z,$$
  satisfies the property $\phi \in \mathcal{A}_3(C,Z)$ with $\phi\big|_C \equiv 0$ as well.

Finally under the validity of assumption ($A^{g,h}$), we are able to show a way to construct a penalisation function $\nu : \mathcal{U} \to \mathbb{R}$ that satisfies either the type 1 representation (Remark 4.11) or the type 2 representation (Remark 4.12).

Remark 4.11 (Type 1 representation)**.** Given functions $\phi \in \mathcal{A}_5(\mathcal{K}, Z_1) \cap \mathcal{A}_6(\mathcal{K}, Z_1)$ and $\zeta \in \mathcal{A}_5(\{0_{Z_2}\}, Z_2) \cap \mathcal{A}_6(\{0_{Z_2}\}, Z_2)$, the penalisation function
$$\nu := \max\{\phi \circ g, \zeta \circ h\}$$
satisfies
$$\begin{aligned} \mathcal{S} &= \{x \in \mathcal{U} \mid \phi(g(x)) \leq 0, \ \zeta(h(x)) \leq 0\} \\ &= \{x \in \mathcal{U} \mid \nu(x) = \max\{\phi(g(x)), \zeta(h(x))\} \leq 0\} \\ &= \mathrm{lev}_{\leq}(\mathcal{U}, \nu, 0), \end{aligned}$$
i.e., $\nu \in \mathcal{A}_5(\mathcal{S}, \mathcal{U})$ is valid. If $h \equiv 0_{Z_2}$ (i.e., only a cone constraint is considered) and
$$\mathrm{int}(\mathcal{S}) = \{x \in \mathcal{U} \mid \nu(x) = \phi(g(x)) < 0\} = \mathrm{lev}_{<}(\mathcal{U}, \nu, 0),$$
then $\nu \in \mathcal{A}_6(\mathcal{S}, \mathcal{U})$ is valid. Conditions for ensuring $\mathrm{int}(\mathcal{S}) = \mathrm{lev}_{<}(\mathcal{U}, \nu, 0)$ are studied by Günther [10, 11], and Günther and Tammer [12, 13] (see also Zălinescu [33, p.147]).

Remark 4.12 (Type 2 representation)**.** Given functions $\phi \in \mathcal{A}_3(\mathscr{K}, Z_1)$ and $\zeta \in \mathcal{A}_5(\{0_{Z_2}\}, Z_2) \cap \mathcal{A}_6(\{0_{Z_2}\}, Z_2)$, the penalisation function

$$\nu := \phi \circ g + \zeta \circ h$$

satisfies

$$\mathcal{S} = \{x \in \mathcal{U} \mid \nu(x) = \phi(g(x)) + \zeta(h(x)) = 0\} = \text{lev}_{=}(\mathcal{U}, \nu, 0) = \text{lev}_{\leq}(\mathcal{U}, \nu, 0),$$

i.e., $\nu \in \mathcal{A}_3(\mathcal{S}, \mathcal{U})$ with $\nu\big|_{\mathcal{S}} \equiv 0$.

Let us pick out two examples in which we fully describe the structure of the penalisation function for the feasible set given in (4.4).

Example 4.13. Consider the Euclidean spaces $Z_1 := \mathbb{R}^n$ and $Z_2 := \mathbb{R}^m$, and the negative standard cone $\mathscr{K} := -\mathbb{R}^n_{\geq}$. As we know from Example 4.10, the function $\phi : Z_1 \to \mathbb{R}$, defined by

$$\phi(y) := \sum_{i=1}^{n} (\max\{y_i, 0\})^2 \quad \text{for all } y = (y_1, \ldots, y_n) \in Z_1,$$

satisfies the property $\phi \in \mathcal{A}_3(\mathscr{K}, Z_1)$ with $\phi\big|_{\mathscr{K}} \equiv 0$. Moreover, the function $\zeta(\cdot) := \|\cdot\|_{Z_2}^2$ has the property $\zeta \in \mathcal{A}_5(\{0_{Z_2}\}, Z_2) \cap \mathcal{A}_6(\{0_{Z_2}\}, Z_2)$. Hence, the penalisation function $\nu : \mathcal{U} \to \mathbb{R}$, given by

$$\nu(x) = \sum_{i=1}^{n} (\max\{g_i(x), 0\})^2 + \sum_{i=1}^{m} (h_i(x))^2 \quad \text{for all } x \in \mathcal{U},$$

possesses the property $\nu \in \mathcal{A}_3(\mathcal{S}, \mathcal{U})$ and also $\nu\big|_{\mathcal{S}} \equiv 0$. Notice that such a type of penalisation function is also used by Han and Mangasarian [4] in the framework of the exact penalty principle for scalar optimisation problems with inequality and equality constraints.

Example 4.14. Consider the data given in Example 2.15. Using the given nontrivial, pointed, closed, (possibly not solid) convex cone $C \subseteq Z$ and the family of approximating dilating cones $\{C_\varepsilon\}_{\varepsilon \in (0,1)} \subseteq Z$ (which contains only pointed, closed, solid, convex cones), the feasible sets of problems (CP) and (CP$_\varepsilon$) are given by

$$\mathcal{S} = \{x \in X \mid Gx \leq_C w\} = \{x \in X \mid Gx - w \in -C\}$$

and

$$\mathcal{S}_\varepsilon = \{x \in X \mid Gx \leq_{C_\varepsilon} w\} = \{x \in X \mid Gx - w \in -C_\varepsilon\}.$$

Define $C_0 := C$. For any $\varepsilon \in [0, 1)$,

- the distance function $\phi_\varepsilon := d_{-C_\varepsilon}$ satisfies $\phi_\varepsilon \in \mathcal{A}_3(-C_\varepsilon, Z)$ with $\phi_\varepsilon\big|_{-C_\varepsilon} \equiv 0$,
- the oriented/signed distance $\phi_\varepsilon := \triangle_{-C_\varepsilon}$ satisfies the property $\phi_\varepsilon \in \mathcal{A}_5(-C_\varepsilon, Z) \cap \mathcal{A}_6(-C_\varepsilon, Z)$,
- the Gerstewitz functional $\phi_\varepsilon := \phi_{-C_\varepsilon, k}$ with $k \in C \setminus \{0_Z\}$ ($\subseteq \text{int}(C_\varepsilon)$ for every $\varepsilon \in (0, 1)$) satisfies the property $\phi_\varepsilon \in \mathcal{A}_5(-C_\varepsilon, Z) \cap \mathcal{A}_6(-C_\varepsilon, Z)$ if $\varepsilon > 0$.

Hence, for any $\varepsilon \in [0, 1)$, the penalisation function $\nu_\varepsilon : X \to \mathbb{R}$, given by

- $\nu_\varepsilon(x) := d_{-C_\varepsilon}(Gx - w)$, satisfies $\nu_\varepsilon \in \mathcal{A}_3(\mathcal{S}_\varepsilon, X)$ with $\nu\big|_{\mathcal{S}_\varepsilon} \equiv 0$.
- $\nu_\varepsilon(x) := \triangle_{-C_\varepsilon}(Gx - w)$, satisfies $\nu_\varepsilon \in \mathcal{A}_5(\mathcal{S}_\varepsilon, X)$.
- $\nu_\varepsilon(x) := \phi_{-C_\varepsilon, k}(Gx - w)$, satisfies $\nu_\varepsilon \in \mathcal{A}_5(\mathcal{S}_\varepsilon, X)$ if $\varepsilon > 0$.

Consider a convex function $\phi_\varepsilon \in \mathcal{A}_5(-C_\varepsilon, Z) \cap \mathcal{A}_6(-C_\varepsilon, Z)$ and define $x \mapsto \nu_\varepsilon(x) := \phi_\varepsilon(Gx - w)$. Under $\nu_\varepsilon \in \mathcal{A}_5(\mathcal{S}_\varepsilon, X)$, i.e.,

$$\mathcal{S}_\varepsilon = \{x \in X \mid Gx - w \in -C_\varepsilon\} = \{x \in X \mid \phi_\varepsilon(Gx - w) \le 0\},$$

the condition $\nu_\varepsilon \in \mathcal{A}_6(\mathcal{S}_\varepsilon, X)$ means

$$\mathrm{int}(\mathcal{S}_\varepsilon) = \{x \in X \mid Gx - w \in -\mathrm{int}(C_\varepsilon)\} = \{x \in X \mid \phi_\varepsilon(Gx - w) < 0\}.$$

The function $x \mapsto \nu_\varepsilon(x) = \phi_\varepsilon(Gx - w)$ is convex (as a composition of the convex function ϕ_ε with the affine linear function $x \mapsto Gx - w$). Thus, if there is $x \in X$ with $\nu_\varepsilon(x) = \phi_\varepsilon(Gx - w) < 0$ (which is a Slater-type condition), then $\nu_\varepsilon \in \mathcal{A}_6(\mathcal{S}_\varepsilon, X)$ is valid (see Zălinescu [33, p.147]). Notice that this Slater-type condition implies that both $\mathrm{int}(\mathcal{C}_\varepsilon)$ and $\mathrm{int}(\mathcal{S}_\varepsilon)$ are nonempty. Due to $\mathrm{int}(C_{\varepsilon_1}) \subseteq \mathrm{int}(C_{\varepsilon_2})$ for $\varepsilon_1 \le \varepsilon_2$, if the Slater-type condition for $\varepsilon_1 \ge 0$ is valid, then it holds true also for $\varepsilon_2 \ge \varepsilon_1$.

4.2. **The effects of adding/removing a penalisation term.** In this section, we analyse the effects of adding/removing a penalisation term to/from the original vector objective function on the nature of the solution sets of the vector optimisation problems.

Remark 4.15. Notice that Fliege [29] (see also Mäkelä and Nikulin [53]) initiated an analysis of the effects of adding/removing objectives to/from a multiobjective optimisation problem (based on the standard ordering cone) on the solution sets. Our upcoming results generalise some results derived by Fliege [29] and Mäkelä and Nikulin [53].

Consider the (not-penalised) vector optimisation problem

$$\operatorname*{argmin}_{x \in \mathcal{S}}{}_{\mathcal{C}}\ f(x) \tag{$P_\mathcal{S}$}$$

as given in Section 2.3 under the assumption (A). Beside the vector optimisation problem ($P_\mathcal{S}$) we are interested in the extended (penalised) vector optimisation problem

$$\operatorname*{argmin}_{x \in \mathcal{S}}{}_{\mathcal{C} \times \mathbb{R}_\ge}\ f^\otimes(x). \tag{$P_\mathcal{S}^\otimes$}$$

In the following part of the section, we will look at relationships between the solution sets of the non-penalised and the penalised vector problem. Although we will use $\mathcal{S}$ as the feasible set, we want to point out, that $\mathcal{S}$ can be any arbitrary set with $\mathcal{S} \subseteq \mathcal{D}$.

For the concept of weak efficiency, we have the following results (cf. [29, Sec. 2]), [53, Sec. 3.2], and [54, Cor. 3.3]).

Proposition 4.16. *Assume* (A). *Then,*

$$\mathrm{WEff}(f, \mathcal{S}, \mathcal{C}) \subseteq \mathrm{WEff}(f^\otimes, \mathcal{S}, \mathcal{C} \times \mathbb{R}_\ge), \tag{4.8}$$

and

$$\begin{aligned} &\mathrm{WEff}(f^\otimes, \mathcal{S}, \mathcal{C} \times \mathbb{R}_\ge) \setminus \mathrm{WEff}(f, \mathcal{S}, \mathcal{C}) \\ &= \left\{ x \in \mathrm{WEff}(f^\otimes, \mathcal{S}, \mathcal{C} \times \mathbb{R}_\ge) \,\middle|\, \exists y \in \mathcal{S} :\ f(y) <_\mathcal{C} f(x),\ \nu(y) \ge \nu(x) \right\}. \end{aligned} \tag{4.9}$$

If $\mathrm{WEff}(f, \mathcal{S}, \mathcal{C}) = \mathrm{WEff}(f^\otimes, \mathcal{S}, \mathcal{C} \times \mathbb{R}_\ge)$, *then* $\operatorname*{argmin}_{x \in \mathcal{S}} \nu(x) \subseteq \mathrm{WEff}(f, \mathcal{S}, \mathcal{C})$.

Proof. From Lemma 2.1 we have $\operatorname{int}(\mathcal{C}\times\mathbb{R}_{\geq}) = \operatorname{int}(\mathcal{C})\times\mathbb{R}_{>}$, which gives (4.8).

Because of

$$\begin{aligned}&\mathrm{WEff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\setminus\mathrm{WEff}(f,\mathcal{S},\mathcal{C})\\&=\big\{\,\chi\in\mathcal{S}\,\big|\,\forall x\in\mathcal{S}:\ f^{\otimes}(x)\not<_{\mathcal{C}\times\mathbb{R}_{\geq}} f^{\otimes}(\chi),\ \exists y\in\mathcal{S}:\ f(y)<_{\mathcal{C}} f(\chi)\,\big\}\end{aligned}$$

we get (4.9). Lastly, for $\chi\in\operatorname*{argmin}_{x\in\mathcal{S}} v(x)$, we have $v(x)\notin v(\chi)-\mathbb{R}_{>}$ for every $x\in\mathcal{S}$. Therefore, we have $f^{\otimes}(x)\notin f^{\otimes}(\chi)-\operatorname{int}(\mathcal{C}\times\mathbb{R}_{\geq})$ and with $\mathrm{WEff}(f,\mathcal{S},\mathcal{C})=\mathrm{WEff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})$ we get the claim. ∎

Similar relationships are also valid for the concept of strict efficiency (cf. [53, Cor. 2]).

Proposition 4.17. *Assume (A). Then,*

$$\mathrm{SEff}(f,\mathcal{S},\mathcal{C})\subseteq\mathrm{SEff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq}) \tag{4.10}$$

and

$$\begin{aligned}&\mathrm{SEff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\setminus\mathrm{SEff}(f,\mathcal{S},\mathcal{C})\\&=\big\{\,\chi\in\mathrm{SEff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\,\big|\,\exists y\in\mathcal{S}:\ f(y)\leq_{\mathcal{C}} f(\chi),\ v(y)>v(\chi)\,\big\}.\end{aligned}\tag{4.11}$$

Proof. For $\chi\in\mathrm{SEff}(f,\mathcal{S},\mathcal{C})$ and every $x\in\mathcal{S}\setminus\{\chi\}$ we have $f(x)\notin f(\chi)-\mathcal{C}$ from which directly follows $f^{\otimes}(x)\notin f^{\otimes}(\chi)-\mathcal{C}\times\mathbb{R}_{\geq}$ and with this it is $\chi\in\mathrm{SEff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})$.

The other claim follows from

$$\begin{aligned}&\mathrm{SEff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\setminus\mathrm{SEff}(f,\mathcal{S},\mathcal{C})\\&=\big\{\,\chi\in\mathcal{S}\,\big|\,\forall x\in\mathcal{S}\setminus\{\chi\}:\ f^{\otimes}(x)\not\leq_{\mathcal{C}\times\mathbb{R}_{\geq}} f^{\otimes}(\chi),\ \exists y\in\mathcal{S}\setminus\{\chi\}:\ f(y)\leq_{\mathcal{C}} f(\chi)\,\big\}.\end{aligned}$$ ∎

In contrast, for the concept of efficiency we have in general

$$\mathrm{Eff}(f,\mathcal{S},\mathcal{C})\nsubseteq\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\quad\text{and}\quad\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\nsubseteq\mathrm{Eff}(f,\mathcal{S},\mathcal{C}). \tag{4.12}$$

The following theorem characterises the intersection and both relative complements (set differences) of the sets of efficient solutions of $(P_{\mathcal{S}})$ and $(P_{\mathcal{S}}^{\otimes})$.

Theorem 4.18. *Under condition (A) we have*

$$\begin{aligned}&\mathrm{Eff}(f,\mathcal{S},\mathcal{C})\setminus\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\\&=\big\{\,\chi\in\mathrm{Eff}(f,\mathcal{S},\mathcal{C})\,\big|\,\exists y\in\mathcal{S}:\ f(y)=f(\chi),\ v(y)<v(\chi)\,\big\},\end{aligned}\tag{4.13}$$

$$\begin{aligned}&\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\setminus\mathrm{Eff}(f,\mathcal{S},\mathcal{C})\\&=\big\{\,\chi\in\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\,\big|\,\exists y\in\mathcal{S}:\ f(y)\lneq_{\mathcal{C}} f(\chi),\ v(y)>v(\chi)\,\big\},\end{aligned}\tag{4.14}$$

$$\begin{aligned}&\mathrm{Eff}(f,\mathcal{S},\mathcal{C})\cap\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\\&=\big\{\,\chi\in\mathcal{S}\,\big|\,\forall x\in\mathcal{S}:\ \big(f(x)=f(\chi)\text{ and }v(x)\geq v(\chi)\big)\text{ or }f(x)\not\lneq_{\mathcal{C}} f(\chi)\,\big\}.\end{aligned}\tag{4.15}$$

Proof. Under condition (A), it is easy to check that

$$(\mathcal{C}\times\mathbb{R}_{\geq})\setminus\{0_Y\times 0\}=\Big(\big(\mathcal{C}\setminus\{0_Y\}\big)\times\mathbb{R}_{\geq}\Big)\cup(\mathcal{C}\times\mathbb{R}_{>}). \tag{4.16}$$

Hence,

$$
\begin{aligned}
&\mathrm{Eff}(f,\mathcal{S},\mathcal{C}) \setminus \mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\\
&\overset{(4.16)}{=} \left\{ \chi \in \mathcal{S} \;\middle|\; \begin{array}{l} \forall x \in \mathcal{S}:\ f(x) \notin f(\chi) - \mathcal{C}\setminus\{0_Y\}, \\ \exists y \in \mathcal{S}:\ f^{\otimes}(y) \in f^{\otimes}(\chi) - (\mathcal{C}\setminus\{0_Y\})\times\mathbb{R}_{\geq} \text{ or} \\ \qquad\qquad f^{\otimes}(y) \in f^{\otimes}(\chi) - \mathcal{C}\times\mathbb{R}_{>} \end{array} \right\}\\
&= \left\{ \chi \in \mathcal{S} \;\middle|\; \begin{array}{l} \forall x \in \mathcal{S}:\ f(x) \notin f(\chi) - \mathcal{C}\setminus\{0_Y\}, \\ \exists y \in \mathcal{S}:\ f^{\otimes}(y) \in f^{\otimes}(\chi) - \mathcal{C}\times\mathbb{R}_{>} \end{array} \right\}\\
&= \left\{ \chi \in \mathcal{S} \;\middle|\; \begin{array}{l} \forall x \in \mathcal{S}:\ f(x) \notin f(\chi) - \mathcal{C}\setminus\{0_Y\}, \\ \exists y \in \mathcal{S}:\ f(y) = f(\chi),\ \nu(y) \in \nu(\chi) - \mathbb{R}_{>} \end{array} \right\}\\
&= \left\{ \chi \in \mathrm{Eff}(f,\mathcal{S},\mathcal{C}) \;\middle|\; \exists y \in \mathcal{S}:\ f(y) = f(\chi),\ \nu(y) < \nu(\chi) \right\}.
\end{aligned}
$$

Similarly, for (4.14), we get

$$
\begin{aligned}
&\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq}) \setminus \mathrm{Eff}(f,\mathcal{S},\mathcal{C})\\
&= \left\{ \chi \in \mathcal{S} \;\middle|\; \begin{array}{l} \forall x \in \mathcal{S}:\ f^{\otimes}(x) \notin f^{\otimes}(\chi) - (\mathcal{C}\times\mathbb{R}_{\geq})\setminus\{0_Y\times 0\}, \\ \exists y \in \mathcal{S}:\ f(y) \in f(\chi) - \mathcal{C}\setminus\{0_Y\} \end{array} \right\}\\
&\overset{(4.16)}{=} \left\{ \chi \in \mathcal{S} \;\middle|\; \begin{array}{l} \forall x \in \mathcal{S}:\ f^{\otimes}(x) \notin f^{\otimes}(\chi) - \big(\mathcal{C}\setminus\{0_Y\}\big)\times\mathbb{R}_{\geq} \text{ and} \\ \qquad\qquad f^{\otimes}(x) \notin f^{\otimes}(\chi) - \mathcal{C}\times\mathbb{R}_{>}, \\ \exists y \in \mathcal{S}:\ f(y) \in f(\chi) - \mathcal{C}\setminus\{0_Y\} \end{array} \right\}\\
&= \left\{ \chi \in \mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq}) \;\middle|\; \exists y \in \mathcal{S}:\ f(y) \in f(\chi) - \mathcal{C}\setminus\{0_Y\} \text{ and } \nu(y) \notin \nu(\chi) - \mathbb{R}_{\geq} \right\}.
\end{aligned}
$$

Lastly, for (4.15), we have

$$
\begin{aligned}
&\mathrm{Eff}(f,\mathcal{S},\mathcal{C}) \cap \mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\\
&\overset{(4.16)}{=} \left\{ \chi \in \mathcal{S} \;\middle|\; \begin{array}{l} \forall x \in \mathcal{S}:\ f(x) \notin f(\chi) - \mathcal{C}\setminus\{0_Y\}, \\ \forall x \in \mathcal{S}:\ f^{\otimes}(x) \notin f^{\otimes}(\chi) - (\mathcal{C}\setminus\{0_Y\})\times\mathbb{R}_{\geq}, \\ \forall x \in \mathcal{S}:\ f^{\otimes}(x) \notin f^{\otimes}(\chi) - \mathcal{C}\times\mathbb{R}_{>} \end{array} \right\}\\
&= \left\{ \chi \in \mathcal{S} \;\middle|\; \begin{array}{l} \forall x \in \mathcal{S}:\ f(x) \notin f(\chi) - \mathcal{C} \text{ or } f(x) = f(\chi), \\ \forall x \in \mathcal{S}:\ f(x) \notin f(\chi) - \mathcal{C} \text{ or } \nu(x) \notin \nu(\chi) - \mathbb{R}_{>} \end{array} \right\}\\
&= \left\{ \chi \in \mathcal{S} \;\middle|\; \begin{array}{l} \forall x \in \mathcal{S}:\ f(x) \notin f(\chi) - \mathcal{C} \text{ or} \\ \qquad\qquad \big(f(x) = f(\chi),\ \nu(x) \geq \nu(\chi)\big) \end{array} \right\},
\end{aligned}
$$

where the second equivalency follows from the fact that because of $f(x) \notin f(\chi) - \mathcal{C}\setminus\{0_Y\}$ the statement $f^{\otimes}(x) \notin f^{\otimes}(\chi) - (\mathcal{C}\setminus\{0_Y\})\times\mathbb{R}_{\geq}$ is superfluous. ■

We are primarily interested in the case that the contrary statement of (4.12) is valid, i.e., all efficient solutions of ($P_{\mathcal{S}}$) are efficient for ($P_{\mathcal{S}}^{\otimes}$) as well, or vice versa. The following result characterises such inclusion properties for the sets of efficient solutions (cf. [29, Cor. 3], [53, Th. 1 and 2]).

Corollary 4.19. *Assume condition (A). Then the following statements are equivalent*

(1) $\mathrm{Eff}(f,\mathcal{S},\mathcal{C}) \subseteq \mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})$

(2) $\forall \chi \in \mathrm{Eff}(f,\mathcal{S},\mathcal{C}),\ \forall x \in \mathcal{S}:\ f(x) \neq f(\chi)$ *or* $\nu(x) \geq \nu(\chi)$

(3) $\forall \chi \in \mathrm{Eff}(f,\mathcal{S},\mathcal{C}),\ \forall x \in \mathcal{S}:\ f(x) \neq f(\chi)$ *or* $\nu(x) = \nu(\chi)$.

Moreover, we have the following equivalencies

(4) $\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\subseteq\mathrm{Eff}(f,\mathcal{S},\mathcal{C})$
(5) $\forall x\in\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq}),\ \forall x\in\mathcal{S}:\ f(x)=f(x)$ *or* $f(x)\not\leq_{\mathcal{C}} f(x)$ *or* $\nu(x)\leq\nu(x)$
(6) $\forall x\in\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq}),\ \forall x\in\mathcal{S}:\ f(x)=f(x)$ *or* $f(x)\not\leq_{\mathcal{C}} f(x)$.

Proof. For the first part, $\mathrm{Eff}(f,\mathcal{S},\mathcal{C})\subseteq\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})$ and $\mathrm{Eff}(f,\mathcal{S},\mathcal{C})\setminus\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})=\emptyset$ are equivalent, which by (4.13) from Theorem 4.18 is equivalent to statement (2). For $x\in\mathrm{Eff}(f,\mathcal{S},\mathcal{C})$ and $x\in\mathcal{S}$ with $f(x)=f(x)$, we have $x\in\mathrm{Eff}(f,\mathcal{S},\mathcal{C})$, so that by (2) we actually get $\nu(x)\leq\nu(x)\leq\nu(x)$ and therefore $\nu(x)=\nu(x)$, which proves the equivalency to (3).

Similarly, $\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\subseteq\mathrm{Eff}(f,\mathcal{S},\mathcal{C})$ is equivalent to $\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})\setminus\mathrm{Eff}(f,\mathcal{S},\mathcal{C})=\emptyset$, which by (4.14) is the same as

$$\forall x\in\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq}),\ \forall x\in\mathcal{S}:\ f(x)\not\lneq_{\mathcal{C}} f(x)\text{ or }\nu(x)\leq\nu(x).$$

It is $f(x)\not\lneq_{\mathcal{C}} f(x)$ exactly then, when $f(x)=f(x)$ or $f(x)\not\leq_{\mathcal{C}} f(x)$, which shows (5). For $x\in\mathrm{Eff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})$ and $x\in\mathcal{S}$ with $f(x)\neq f(x)$ we get $f^{\otimes}(x)\not\lneq_{\mathcal{C}\times\mathbb{R}_{\geq}} f(x)$. Therefore, $f(x)\lneq_{\mathcal{C}} f(x)$ implies $\nu(x)>\nu(x)$, which makes the condition $\nu(x)\leq\nu(x)$ in (5) obsolete and gives (6). ∎

Remark 4.20. Mäkelä and Nikulin [53] and Malinowska [54] pointed out some disagreements in the paper [29] from Fliege by providing counterexamples to certain statements, corrected these and improved some results. By comparing our results to the ones by Fliege these differences can be seen in Corollary 4.19.

4.3. **Vectorial penalisation - relationships between the solution sets.** We end Section 4 by stating our main results related to our vectorial penalisation approach for general vector optimisation problems, which show profound relationships between the solution sets of the vector problems ($P_{\mathcal{S}}$), ($P_{\mathcal{D}}$) and ($P^{\otimes}_{\mathcal{D}}$). These results (in combination with our results from Section 3) extend/generalise known results from the literature in the topic of vectorial penalisation in multiobjective/vector optimisation. Let us take a more detailed look at this in the following remark:

Remark 4.21. Günther and Tammer [12, 13] (see also Günther [10, 11]) proposed a vectorial penalisation approach for multiobjective optimisation problems (X is a real linear-topological space, $Y=\mathbb{R}^m$, and $\mathcal{C}=\mathbb{R}^m_{\geq}$), where the main results are derived for componentwise generalised convex vector functions and the concepts of (strict, weak) efficiency. In the same setting, Günther, Tammer and Yao [55] derived necessary optimality conditions for constrained multiobjective optimisation problems.

Durea, Strugariu and Tammer [15] studied a vectorial penalisation approach to derive necessary optimality conditions for constrained vector optimisation problems (X is a real normed space, Y is a linear-topological space, and $\mathcal{C}$ is a nontrivial, closed, pointed, convex cone). The authors considered only penalisation functions satisfying a ($\mathcal{A}3$)-property and did not involve any generalised convexity assumptions on the vector objective function.

Recently Schmölling [16] extended/generalised the vectorial penalisation approach (X and Y are linear spaces, and $\mathcal{C}$ is a convex cone in Y).

4.3.1. *Efficiency.* In this section, we derive our main relationships between the sets of efficient solutions of the vector problems ($P_{\mathcal{S}}$), ($P_{\mathcal{D}}$) and ($P^{\otimes}_{\mathcal{D}}$).

Theorem 4.22. *Assume (A).*

(1) If $\nu \in \mathcal{A}_4(\mathcal{S},\mathcal{D})$ and for some $\chi \in \mathrm{Eff}(f,\mathcal{S},\mathcal{C})$ it is

$$\mathrm{lev}_=\big(\mathcal{S},f,f(\chi)\big) \subseteq \mathrm{lev}_=\big(\mathcal{D},\nu,\nu(\chi)\big), \tag{4.17}$$

then we have $\chi \in \mathcal{S} \cap \mathrm{Eff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})$.

(2) Let $\mathcal{S}$ be closed and suppose $\nu \in \mathcal{A}_1(\mathcal{S},\mathcal{D})$. Then we have

$$\big(\mathcal{S}\cap \mathrm{Eff}(f,\mathcal{D},\mathcal{C})\big) \cup \big(\mathrm{bd}(\mathcal{S})\cap \mathrm{Eff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big) \subseteq \mathrm{Eff}(f,\mathcal{S},\mathcal{C}). \tag{4.18}$$

Furthermore, assume $\nu \in \mathcal{A}_2(\mathcal{S},\mathcal{D})$ and (3.10) in the form

$$\mathrm{Eff}(f,\mathcal{S},\mathcal{C}) \setminus \mathrm{Eff}(f,\mathcal{D},\mathcal{C}) \subseteq \mathrm{bd}(\mathcal{S}). \tag{4.19}$$

Then we get the equality

$$\big(\mathcal{S}\cap \mathrm{Eff}(f,\mathcal{D},\mathcal{C})\big) \cup \big(\mathrm{bd}(\mathcal{S})\cap \mathrm{Eff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big) = \mathrm{Eff}(f,\mathcal{S},\mathcal{C}). \tag{4.20}$$

(3) Assume $\nu \in \mathcal{A}_3(\mathcal{S},\mathcal{D})$. Then we have

$$\mathrm{Eff}(f,\mathcal{S},\mathcal{C}) = \mathcal{S}\cap \mathrm{Eff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq}). \tag{4.21}$$

Furthermore, let $\mathcal{S}$ be closed. Then we have

$$\big(\mathcal{S}\cap \mathrm{Eff}(f,\mathcal{D},\mathcal{C})\big) \cup \big(\mathrm{bd}(\mathcal{S})\cap \mathrm{Eff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big) \subseteq \mathrm{Eff}(f,\mathcal{S},\mathcal{C}). \tag{4.18}$$

If additionally (4.19) is valid then we get (4.20), more precisely we have

$$\begin{aligned}\mathrm{Eff}(f,\mathcal{S},\mathcal{C}) &= \big(\mathcal{S}\cap \mathrm{Eff}(f,\mathcal{D},\mathcal{C})\big) \cup \big(\mathrm{bd}(\mathcal{S})\cap \mathrm{Eff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big)\\ &= \big(\mathrm{int}(\mathcal{S})\cap \mathrm{Eff}(f,\mathcal{D},\mathcal{C})\big) \cup \big(\mathrm{bd}(\mathcal{S})\cap \mathrm{Eff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big).\end{aligned}$$

Proof. (1) Assume $\chi \in \mathrm{Eff}(f,\mathcal{S},\mathcal{C}) \setminus \mathrm{Eff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})$. Then by (4.16) we get

$$\exists x \in \mathcal{D}:\ (f,\nu)(x) \in (f,\nu)(\chi) - \big((\mathcal{C}\setminus\{0_Y\})\times\mathbb{R}_{\geq}\big) \cup (\mathcal{C}\times\mathbb{R}_{>}).$$

Now ($\mathcal{A}4$) gives $x \in \mathcal{S}$, so that from the efficiency of χ we get $f(x) = f(\chi)$ and $\nu(x) < \nu(\chi)$, which contradicts (4.17).

(2) By (3.1) we have $\mathcal{S}\cap\mathrm{Eff}(f,\mathcal{D},\mathcal{C}) \subseteq \mathrm{Eff}(f,\mathcal{S},\mathcal{C})$. Now consider $\chi \in \mathrm{bd}(\mathcal{S})\cap\mathrm{Eff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})$. For any $x\in\mathcal{S}$ from ($\mathcal{A}1$) we get $\nu(x) \leq \nu(\chi)$ whereof (4.14) implies $\chi \in \mathrm{Eff}(f,\mathcal{S},\mathcal{C})$, so we have (4.18).

For the other inclusion consider $\chi \in \mathrm{Eff}(f,\mathcal{S},\mathcal{C})$. If not already $\chi \in \mathrm{Eff}(f,\mathcal{D},\mathcal{C})$ by (4.19) we get $\chi \in \mathrm{bd}(\mathcal{S})$. Assume $\chi \notin \mathrm{Eff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})$. Then we get

$$\exists x \in \mathcal{D}:\ \big(f(x) \leq_{\mathcal{C}} f(\chi),\ \nu(x) < \nu(\chi)\big) \quad \text{or} \quad \big(f(x) \lneq_{\mathcal{C}} f(\chi),\ \nu(x) \leq \nu(\chi)\big). \tag{4.22}$$

Reminding that $\chi \in \mathrm{bd}(\mathcal{S})$, condition ($\mathcal{A}1$) implies $x \in \mathcal{S}$, such that from $\chi \in \mathrm{Eff}(f,\mathcal{S},\mathcal{C})$ the second case in (4.22) can be ruled out. As in (4.13) we get $f(x) = f(\chi)$ and $\nu(x) < \nu(\chi)$. From the first one we can deduce $x \in \mathrm{Eff}(f,\mathcal{S},\mathcal{C})$ and as with χ we get $x \in \mathrm{bd}(\mathcal{S})$ whereas together with (4.2) the second one gives $x \in \mathrm{int}(\mathcal{S})$. Therefore, we have (4.20).

(3) By (4.3) we get the following equivalences, which prove (4.21)

$$\begin{aligned} &\chi \in \mathrm{Eff}(f, \mathcal{S}, \mathcal{C}) \\ \iff &\quad \nexists x \in \mathcal{S}:\ f(x) \lneq_{\mathcal{C}} f(\chi) \\ \overset{(\mathcal{A}3)}{\iff} &\quad \nexists x \in \mathcal{D}:\ \big(f(x) \lneq_{\mathcal{C}} f(\chi),\ \nu(x) \le \nu(\chi)\big) \text{ or } \nu(x) < \nu(\chi) \\ \iff &\quad \chi \in \mathcal{S} \cap \mathrm{Eff}(f^{\otimes}, \mathcal{D}, \mathcal{C} \times \mathbb{R}_{\geq}). \end{aligned}$$

By (3.1) we have $\mathcal{S} \cap \mathrm{Eff}(f, \mathcal{D}, \mathcal{C}) \subseteq \mathrm{Eff}(f, \mathcal{S}, \mathcal{C})$ and with (4.21) from $\mathrm{bd}(\mathcal{S}) \subseteq \mathcal{S}$ we get (4.18).

For (4.20) we consider $\chi \in \mathrm{Eff}(f, \mathcal{S}, \mathcal{C})$. If $\chi \in \mathcal{S} \cap \mathrm{Eff}(f, \mathcal{D}, \mathcal{C})$, we are done. Otherwise by (4.19) we get $\chi \in \mathrm{bd}(\mathcal{S})$ where (4.21) proves the claim. The last claim follows from the fact that $\mathrm{bd}(\mathcal{S}) \cap \mathrm{Eff}(f, \mathcal{D}, \mathcal{C}) \subseteq \mathrm{Eff}(f^{\otimes}, \mathcal{D}, \mathcal{C} \times \mathbb{R}_{\geq})$ by (4.21). ■

4.3.2. *Weak efficiency.* Next, we derive similar relationships between the sets of weakly efficient solutions of the vector problems ($P_{\mathcal{S}}$), ($P_{\mathcal{D}}$) and ($P^{\otimes}_{\mathcal{D}}$). In order to prepare our upcoming theorem, we recall an appropriate upper semicontinuity concept for vector functions (see [41, Def. 2.5.25]).

Definition 4.23. Assume (A) and let $x^0 \in \mathcal{S}$. The function f is called **$\mathcal{C}$-upper semicontinuous** in x^0 if for any neighbourhood $V \subseteq Y$ of $f(x^0)$ there exists a neighbourhood $U \subseteq X$ of x^0 such that $f[U \cap \mathcal{S}] \subseteq V - \mathcal{C}$. Moreover, f is called **$\mathcal{C}$-upper semicontinuous** on $\mathcal{S}$ if it is so in every $x^0 \in \mathcal{S}$.

Now, we are able to state our main relationships between the sets of weakly efficient solutions of ($P_{\mathcal{S}}$), ($P_{\mathcal{D}}$) and ($P^{\otimes}_{\mathcal{D}}$).

Theorem 4.24. *Assume ($A^{\otimes}$). Then we have*

$$\mathcal{S} \cap \mathrm{bd}(\mathcal{S}) \cap \mathrm{WEff}(f, \mathcal{D}, \mathcal{C}) \subseteq \mathcal{S} \cap \mathrm{bd}(\mathcal{S}) \cap \mathrm{WEff}(f^{\otimes}, \mathcal{D}, \mathcal{C} \times \mathbb{R}_{\geq}), \tag{4.23}$$

$$\begin{aligned} &\big(\mathrm{int}(\mathcal{S}) \cap \mathrm{WEff}(f, \mathcal{D}, \mathcal{C})\big) \cup \big(\mathcal{S} \cap \mathrm{bd}(\mathcal{S}) \cap \mathrm{WEff}(f^{\otimes}, \mathcal{D}, \mathcal{C} \times \mathbb{R}_{\geq})\big) \\ &= \big(\mathcal{S} \cap \mathrm{WEff}(f, \mathcal{D}, \mathcal{C})\big) \cup \big(\mathcal{S} \cap \mathrm{bd}(\mathcal{S}) \cap \mathrm{WEff}(f^{\otimes}, \mathcal{D}, \mathcal{C} \times \mathbb{R}_{\geq})\big). \end{aligned} \tag{4.24}$$

(1) If ν additionally satisfies ($\mathcal{A}4$) with $\nu \in \mathcal{A}_4(\mathcal{S}, \mathcal{D})$, we have the barrier

$$\mathcal{S} \cap \mathrm{WEff}(f, \mathcal{D}, \mathcal{C}) \subseteq \mathrm{WEff}(f, \mathcal{S}, \mathcal{C}) \subseteq \mathcal{S} \cap \mathrm{WEff}(f^{\otimes}, \mathcal{D}, \mathcal{C} \times \mathbb{R}_{\geq}). \tag{4.25}$$

(2) Now consider the case that $\mathrm{bd}(\mathcal{S}) \subseteq \mathcal{D}$ and $\nu \in \mathcal{A}_1(\mathcal{S}, \mathcal{D}) \cap \mathcal{A}_2(\mathcal{S}, \mathcal{D}) \cap \mathcal{A}_7(\mathcal{S}, \mathcal{D})$ and let f be $\mathcal{C}$-upper semicontinuous on $\mathcal{S}$. Then we have (4.25) and

$$\mathrm{WEff}(f, \mathcal{S}, \mathcal{C})$$

$$\supseteq \big(\mathrm{int}(\mathcal{S}) \cap \mathrm{WEff}(f, \mathcal{D}, \mathcal{C})\big) \cup \big(\mathcal{S} \cap \mathrm{bd}(\mathcal{S}) \cap \mathrm{WEff}(f^{\otimes}, \mathcal{D}, \mathcal{C} \times \mathbb{R}_{\geq})\big) \tag{4.26}$$

$$= \big(\mathcal{S} \cap \mathrm{WEff}(f, \mathcal{D}, \mathcal{C})\big) \cup \big(\mathcal{S} \cap \mathrm{bd}(\mathcal{S}) \cap \mathrm{WEff}(f^{\otimes}, \mathcal{D}, \mathcal{C} \times \mathbb{R}_{\geq})\big). \tag{4.27}$$

(3) If we instead assume $\nu \in \mathcal{A}_3(\mathcal{S}, \mathcal{D})$, we have

$$\mathcal{S} \cap \mathrm{WEff}(f^{\otimes}, \mathcal{D}, \mathcal{C} \times \mathbb{R}_{\geq}) = \mathcal{S}.$$

(4) If besides $\nu \in \mathcal{A}_4(\mathcal{S}, \mathcal{D})$ we further assume (3.13), so

$$\mathrm{WEff}(f, \mathcal{S}, \mathcal{C}) \setminus \mathrm{WEff}(f, \mathcal{D}, \mathcal{C}) \subseteq \mathcal{S} \cap \mathrm{bd}(\mathcal{S}), \tag{3.13}$$

then we also get

$$\begin{aligned}&\mathrm{WEff}(f,\mathcal{S},\mathcal{C})\\&\subseteq \big(\mathrm{int}(\mathcal{S})\cap \mathrm{WEff}(f,\mathcal{D},\mathcal{C})\big)\cup\big(\mathcal{S}\cap \mathrm{bd}(\mathcal{S})\cap \mathrm{WEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big) \quad (4.28)\\&=\big(\mathcal{S}\cap \mathrm{WEff}(f,\mathcal{D},\mathcal{C})\big)\cup\big(\mathcal{S}\cap \mathrm{bd}(\mathcal{S})\cap \mathrm{WEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big). \quad (4.29)\end{aligned}$$

Proof. By Proposition 4.16 (4.8), we have (4.23) and with this also (4.24).

(1) By (3.2), we obtain the first inclusion of (4.25) and for any $x\in \mathrm{WEff}(f,\mathcal{S},\mathcal{C})$ by Lemma 2.1 we get

$$\begin{aligned}&\mathrm{lev}_{<_{\mathcal{C}\times\mathbb{R}_{\geq}}}\big(\mathcal{D},f^{\otimes},f^{\otimes}(x)\big)\\&=\mathrm{lev}_{<_{\mathcal{C}}}\big(\mathcal{D},f,f(x)\big)\cap \mathrm{lev}_{<}\big(\mathcal{D},\nu,\nu(x)\big)\\&\overset{(\mathcal{A}4)}{\subseteq}\mathrm{lev}_{<_{\mathcal{C}}}\big(\mathcal{D},f,f(x)\big)\cap\mathcal{S}=\emptyset,\end{aligned}$$

which proves $x\in \mathrm{WEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})$ and therefore completes (4.25).

(2) Due to Proposition 4.2 (1) we get (4.25) from point (1). If $\mathrm{int}(\mathcal{C})=\emptyset$ we immediately get $\mathrm{WEff}(f,\mathcal{D},\mathcal{C})=\mathrm{WEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})=\mathcal{D}$ and with this the claim. Therefore, from now on we assume $\mathrm{int}(\mathcal{C})\neq\emptyset$. Because of (4.24) we only need to prove (4.26). From (3.2) we get the inclusion $\mathrm{int}(\mathcal{S})\cap \mathrm{WEff}(f,\mathcal{D},\mathcal{C})\subseteq \mathrm{WEff}(f,\mathcal{S},\mathcal{C})$ and for any $x\in \mathcal{S}\cap\mathrm{bd}(\mathcal{S})\cap \mathrm{WEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})$ we get $x\in \mathrm{WEff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})$. If we assume $x\notin \mathrm{WEff}(f,\mathcal{S},\mathcal{C})$ by (4.9) we get the existence of $y\in\mathcal{S}$ with $f(y)<_{\mathcal{C}} f(x)$ and $\nu(y)\geq\nu(x)$. From ($\mathcal{A}1$) we now get $\nu(y)=\nu(x)$ and ($\mathcal{A}2$) gives $y\in\mathrm{bd}(\mathcal{S})$ wherefore ($\mathcal{A}7$) yields the existence of $z\in\mathrm{int}(\mathcal{S})$ such that $[z,y)\subseteq \mathrm{lev}_{<}\big(\mathcal{D},\nu,\nu(x)\big)$. Since $f(x)-f(y)\in\mathrm{int}(\mathcal{C})$ there exists a neighbourhood V of $f(x)-f(y)$ such that all $v\in V$ satisfy $v\in\mathrm{int}(\mathcal{C})$. Then, $f(x)-V$ is a neighbourhood of $f(y)$. By the $\mathcal{C}$-upper semicontinuity of f we find a neighbourhood U of y such that for all $x\in U$ with the help of (2.1) we have $f(x)\in f(x)-V-\mathcal{C}\subseteq f(x)-\mathrm{int}(\mathcal{C})$, so $f(x)<_{\mathcal{C}} f(x)$. For any $x\in U\cap[z,y)\neq\emptyset$ this gives $f^{\otimes}(x)<_{\mathcal{C}\times\mathbb{R}_{\geq}} f^{\otimes}(x)$ which contradicts the weak efficiency of x.

(3) For any $x\in\mathcal{S}$ because of ($\mathcal{A}3$) by (4.3) we have $\mathrm{lev}_{<}(\mathcal{D},\nu,\nu(x))=\emptyset$. Now we get the claim with the help of Lemma 2.1.

(4) For any $x\in \mathrm{WEff}(f,\mathcal{S},\mathcal{C})$ we either have $x\in\mathrm{WEff}(f,\mathcal{D},\mathcal{C})$ or by (3.13) and (4.25) get $x\in\mathcal{S}\cap\mathrm{bd}(\mathcal{S})\cap\mathrm{WEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})$. ■

4.3.3. *Strict efficiency.* Finally, we state our main relationships between the sets of strictly efficient solutions of the vector problems ($P_{\mathcal{S}}$), ($P_{\mathcal{D}}$) and ($P^{\otimes}_{\mathcal{D}}$).

Theorem 4.25. *Assume* ($A^{\otimes}$). *Then we have*

$$\mathcal{S}\cap\mathrm{bd}(\mathcal{S})\cap\mathrm{SEff}(f,\mathcal{D},\mathcal{C})\subseteq\mathcal{S}\cap\mathrm{bd}(\mathcal{S})\cap\mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq}), \quad (4.30)$$

$$\begin{aligned}&\big(\mathrm{int}(\mathcal{S})\cap\mathrm{SEff}(f,\mathcal{D},\mathcal{C})\big)\cup\big(\mathcal{S}\cap\mathrm{bd}(\mathcal{S})\cap\mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big)\\&=\big(\mathcal{S}\cap\mathrm{SEff}(f,\mathcal{D},\mathcal{C})\big)\cup\big(\mathcal{S}\cap\mathrm{bd}(\mathcal{S})\cap\mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big).\end{aligned} \quad (4.31)$$

(1) If ν additionally satisfies ($\mathcal{A}4$) with $\nu\in\mathcal{A}_4(\mathcal{S},\mathcal{D})$, we get the barrier

$$\mathcal{S}\cap\mathrm{SEff}(f,\mathcal{D},\mathcal{C})\subseteq\mathrm{SEff}(f,\mathcal{S},\mathcal{C})\subseteq\mathcal{S}\cap\mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq}). \quad (4.32)$$

(2) Moreover, assume now $\mathrm{bd}(\mathcal{S}) \subseteq \mathcal{D}$ *and* $\nu \in \mathcal{A}_1(\mathcal{S},\mathcal{D})$. *Then we have* (4.32) *and*

$$\begin{aligned} &\mathrm{SEff}(f,\mathcal{S},\mathcal{C}) \\ &\supseteq \big(\mathrm{int}(\mathcal{S}) \cap \mathrm{SEff}(f,\mathcal{D},\mathcal{C})\big) \cup \big(\mathcal{S} \cap \mathrm{bd}(\mathcal{S}) \cap \mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big) &(4.33)\\ &= \big(\mathcal{S} \cap \mathrm{SEff}(f,\mathcal{D},\mathcal{C})\big) \cup \big(\mathcal{S} \cap \mathrm{bd}(\mathcal{S}) \cap \mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big). &(4.34)\end{aligned}$$

(3) If we instead assume $\nu \in \mathcal{A}_3(\mathcal{S},\mathcal{D})$ *we also get* (4.32), (4.33) *and* (4.34). *Furthermore, we get*

$$\mathcal{S} \cap \mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq}) \subseteq \mathrm{SEff}(f,\mathcal{S},\mathcal{C}) \tag{4.35}$$

and with this also

$$\mathcal{S} \cap \mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq}) = \mathrm{SEff}(f,\mathcal{S},\mathcal{C}). \tag{4.36}$$

(4) If besides $\nu \in \mathcal{A}_4(\mathcal{S},\mathcal{D})$ *we further assume* (3.16)*, so*

$$\mathrm{SEff}(f,\mathcal{S},\mathcal{C}) \setminus \mathrm{SEff}(f,\mathcal{D},\mathcal{C}) \subseteq \mathcal{S} \cap \mathrm{bd}(\mathcal{S}), \tag{3.16}$$

then we also get

$$\begin{aligned} &\mathrm{SEff}(f,\mathcal{S},\mathcal{C}) \\ &\subseteq \big(\mathrm{int}(\mathcal{S}) \cap \mathrm{SEff}(f,\mathcal{D},\mathcal{C})\big) \cup \big(\mathcal{S} \cap \mathrm{bd}(\mathcal{S}) \cap \mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big) &(4.37)\\ &= \big(\mathcal{S} \cap \mathrm{SEff}(f,\mathcal{D},\mathcal{C})\big) \cup \big(\mathcal{S} \cap \mathrm{bd}(\mathcal{S}) \cap \mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})\big). &(4.38)\end{aligned}$$

Proof. By Proposition 4.17 (4.10) we immediately get (4.30) and with this we get (4.31).

(1) From (3.3) we get the first inclusion of (4.32). For any $x \in \mathrm{SEff}(f,\mathcal{S},\mathcal{C})$ by (3.3) and ($\mathcal{A}4$) we get the second one through

$$\begin{aligned} x &\in \mathrm{lev}_{\leq}\big(\mathcal{D},\nu,\nu(x)\big) \cap \mathrm{SEff}(f,\mathcal{S},\mathcal{C}) \\ &\subseteq \mathrm{SEff}\Big(f,\mathcal{D} \cap \mathrm{lev}_{\leq}\big(\mathcal{D},\nu,\nu(x)\big),\mathcal{C}\Big) \\ &\subseteq \mathcal{S} \cap \mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq}). \end{aligned}$$

(2) Due to Proposition 4.2 (1) we get (4.32). By proving the inclusion of (4.34) we also get (4.33), (4.31) even gives the equality of the right sides. Because of (3.3) we have $\mathcal{S} \cap \mathrm{SEff}(f,\mathcal{D},\mathcal{C}) \subseteq \mathrm{SEff}(f,\mathcal{S},\mathcal{C})$. For $x \in \mathcal{S} \cap \mathrm{bd}(\mathcal{S}) \cap \mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq})$ from (3.3) we get $x \in \mathrm{SEff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})$ and by ($\mathcal{A}1$) and (4.11) we get $x \in \mathrm{SEff}(f,\mathcal{S},\mathcal{C})$ and with this (4.34).

(3) Again, by Proposition 4.2 (1) we get (4.32). Now, (3.3) gives $\mathcal{S} \cap \mathrm{SEff}(f^{\otimes},\mathcal{D},\mathcal{C}\times\mathbb{R}_{\geq}) \subseteq \mathrm{SEff}(f^{\otimes},\mathcal{S},\mathcal{C}\times\mathbb{R}_{\geq})$ and by ($\mathcal{A}3$) and (4.11) we get (4.35). With (4.32) we get (4.36). Lastly, from (3.3) and (4.35) we get (4.33) and by (4.31) also (4.34).

(4) For any $x \in \mathrm{SEff}(f,\mathcal{S},\mathcal{C})$ we either have $x \in \mathrm{SEff}(f,\mathcal{D},\mathcal{C})$ or by (3.16) get $x \in \mathrm{bd}(\mathcal{S})$ and again with (4.32) and (4.31) acquire the desired result. ∎

5. Conclusions

In this paper, we presented a vectorial penalisation approach for general vector optimisation problems in real linear-topological spaces where the involved ordering cone $\mathcal{C}$ is assumed to be a pointed, convex cone. With the aid of a generalised convexity concept ($\mathcal{C}$-quasiconvexity) by Jahn [17], some generalisations of results by Günther and Tammer [12, 13] (see Section 3) as well as by Fliege [29] and Mäkelä and Nikulin [53] (see Section 4.2), we were able to extend/generalise the vectorial penalisation approach for multiobjective (vector) optimisation problems studied in [10, 11, 12, 13, 15]. More precisely, in our main Theorems 4.22, 4.24 and 4.25 we derived profound relationships between the solution sets of the original constrained vector optimisation problem (with not necessarily convex feasible set) and two unconstrained vector optimisation problems, where in one of the two problems a penalisation term w.r.t. the original feasible set is added to the vector objective function.

In Section 4.1.2, we put a special emphasis on the construction of appropriate penalisation functions for several popular classes of (vector) optimisation problems (e.g., semidefinite/copositive programming, second-order cone programming, optimisation in function spaces).

Beside the solution concepts studied in this paper, one can also consider so-called proper efficiency solution concepts for the vector optimisation problem ($P_{\mathcal{S}}$), which are known to be relevant in practice. The set of properly efficient points (in the sense of Henig [56]) can be defined as follows

$$\mathrm{PEff}(f,\mathcal{S},C) := \bigcup_{D\in\mathscr{F}(C)} \mathrm{WEff}(f,\mathcal{S},D),$$

where $\mathscr{F}(C)$ is a family of (nontrivial, pointed, solid, convex) Henig-type dilating cones (see, e.g., [42, 43, 57]). For the set $\mathrm{WEff}(f,\mathcal{S},D)$ ($\subseteq \mathrm{Eff}(f,\mathcal{S},C)$) with $D\in\mathscr{F}(C)$ one can apply our derived penalisation results in the paper. As far as we know, in the literature on vectorial penalisation techniques, no results are known for the set $\mathrm{PEff}(f,\mathcal{S},C)$ (in particular, for $Y=\mathbb{R}^m$ and the standard cone $C=\mathbb{R}^m_{\geq}$).

The topic of penalisation in vector optimisation in real linear spaces based on algebraic notions (such as algebraic/vectorial closure, algebraic interior, relative algebraic interior) could be interesting for further extensions of our results (see, e.g., Günther, Khazayel and Tammer [57], Novo and Zălinescu [58], and Schmölling [16]).

Our derived results can be used to establish (necessary, sufficient) optimality conditions for general vector optimisation problems (extending the results derived for multiobjective optimisation problems by Günther, Tammer and Yao [55]). The advantage is that such optimality conditions will have a simple structure because the normal cone w.r.t. the constraints $\mathcal{S}$ is not involved. This is due to the fact that one exploits the inclusion (respectively, the corresponding equality)

$$\mathrm{WEff}(f,\mathcal{S},C) \subseteq \big(\mathrm{int}(\mathcal{S})\cap \mathrm{WEff}(f,X,C)\big) \cup \big(\mathcal{S}\cap \mathrm{bd}(\mathcal{S})\cap \mathrm{WEff}(f^{\otimes},X,C\times\mathbb{R}_{\geq})\big),$$

which is ensured by our Theorem 4.24.

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