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ON SET-VALUED DISCRETE DYNAMICAL SYSTEMS

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Abstract. In this paper, we attempt to study set-valued discrete dynamical systems with the objective of developing a general framework and unifying some results and definitions in the literature. For these purposes, we follow similar ideas to those existing for classical dynamical systems. We focus on discrete dynamical systems in terms of set-valued maps. The solutions associated to our systems are given by sequences of sets. We obtain preliminary results by formulating appropriate notions of set dynamical systems as attractor, stability and invariant sets. For this purpose, we study the ω-limit sets which play an important role for gaining an overall understanding of how the system is behaving, particularly in the long term. We study its properties by using well-known notions from set-valued analysis. We are able to generalize dynamical results in terms of single valued maps by the weaker assumptions on continuity.
Keywords. Discrete-time; limit sets; Lyapunov Stability; Set-valued dynamical systems; Set-valued maps.

1. Introduction

The study of dynamical systems has received more attention in the last twenty years. Many phenomena from social, natural, and economical sciences can be formulated as systems evolving with time in a discrete way. On the other hand, different to a physical model determined by initial conditions, the mathematical models which involve human decisions or biological species require analysis with the help of tools from the theory of set-valued dynamical systems (SVDSs in short).

In the study of dynamical systems, the discrete dynamic has an important role to apply the theoretical results to practical problems. In addition, in terms of real data or numerical results, it is also interesting to know an asymptotic description of its dynamic behaviour. SVDSs allow a multi-valued future and seem more natural in the context of random or control systems. See some applications in [1, 2, 3] in the context of differential inclusions as well as economic flow, and see also recent survey articles [4] and [5] (about other applications in engineering or discretization methods respectively) by Brogliato et al.

A dynamical system is a phase (or state) space X endowed with an evolution map f (dynamical rule) from X to itself. Generally, X is a metric space and f is a continuous function

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$f: X \to X$. In particular, a discrete-time dynamical system is defined by

$$x_{n+1} = f(x_n), \qquad n = 0, 1, 2, \ldots \tag{1.1}$$

It is well-known that several real problems are modelled by (1.1). Such systems describe a relationship between a point in time and a previous point and each state of the system corresponds to a unique point in the state space. The set of properties of all orbits $\{f^n(x)\}_{n\in\mathbb{N}}$ for each $x \in X$ is called the dynamics of the function f. For many practical reasons, it is important to know the behavior of the points (or individuals) of X under f, that is, the behaviour of the system starting with various initial conditions. Thus, we study stability, periodicity or chaos for system (1.1). We emphasize that even very simple functions can have complicated dynamics (see, for instance, [6, 7]).

On the other hand, in many cases such as migration phenomenon in a certain ecosystem, when studying the chaotic dynamics of individual members a natural question that arises is the so-called collective dynamics, that is, how the subsets (not the points) of X move. This leads to consider the set-valued discrete system induced by f and formulated as

$$\bar{f}(A_n) = A_{n+1}, \tag{1.2}$$

where $\bar{f}: \mathscr{K}(X) \to \mathscr{K}(X)$, $\mathscr{K}(X)$ is the family of the nonempty compact subsets of X and $\bar{f}$ is the function induced by f and defined by $\bar{f}(A) = \{f(a): a \in A\}$ in a natural way.

The above approach about of SVDSs is based on the induced set-valued problem associated to a single dynamical system. Its main interest consists in the connection between dynamical properties of the base map f and its induced map $\bar{f}$. In recent years, such topics have attracted many researchers; see, e.g., [8, 9, 10].

There is another approach to study SVDSs which is a natural generalization of (1.1). It is defined as follows:

$$x_{n+1} \in F(x_n), \qquad n = 0, 1, 2, \ldots, \tag{1.3}$$

where $F: X \to \mathscr{K}(X)$ is a set-valued map. In the existing literature the results devoted to (1.3) have been widely explored in the past decades to describe multi-valued differential equations and control systems see [2, 3, 11] and references therein.

In problems (1.2) and (1.3) to represent its collective dynamic, a trajectory can be either a sequence of subsets of X or a sequence of points of X respectively. As far as we know the basic definitions in the existing literature of SVDSs may not be unique since there are different ways to gain an overall understanding of how the system is behaving, particularly in the long term. To study the behaviour of all future orbits or notions of stability, periodicity or chaos in the context of SVDSs it makes sense to present a general framework which extends concepts and results from the classical case. Moreover, due to the complexity in mathematical analysis for set-valued maps, even some basic problems still need to be further clarified. Hence, the dynamics for set valued maps or equivalently the set-valued case in dynamical systems theory are a little problematic and must therefore be studied by specific tools from Set-valued Analysis. See [12] and recently [13], for instance, where conditions via shift maps on inverse limit spaces have been developed from topological or dynamical point of view.

The aim of our paper lies in such a direction. Inspired by some ideas in [14] (devoted to stochastic approximations and differential inclusions) we study discrete dynamical systems and focus on basic results on topological and dynamical properties. Exactly, we present a general framework for discrete dynamical systems in terms of system (1.3) and consider some asymptotic properties which are motivated by the classical theory in the one-dimensional case.

The paper is organized as follows. Firstly, we introduce notations and preliminary results about limit sets and generalized continuity conditions. Section 2 is devoted to establish, in a general context, a set-valued dynamical system which includes or unifies several classical notions in the literature of SVDSs. Then in Section 3, we study invariant sets, ω-limit sets and their relationships. The main results are given in Section 4 where stability notions in the Lyapunov sense are obtained. We prove several new results and improve other ones. Finally, possible future research about set-valued dynamical system is discussed.

2. Notation and Terminology

Throughout this paper, we consider $\mathbb{R}^n$ and d the Euclidean distance. Note that many of the definitions are consistent with general topological spaces.

We define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and denote by $B_\varepsilon(x_0) = \{x \in \mathbb{R}^n : d(x, x_0) < \varepsilon\}$ the ball of radious ε at $x_0 \in \mathbb{R}^n$, by $\wp_0(\mathbb{R}^n)$ the family of nonempty subsets of $\mathbb{R}^n$. A singleton set $\{y\}$ is denoted simply by y. Given a set $A \subset \mathbb{R}^n$, the closure of A is denoted by $\overline{A}$. For arbitrary $A \in \wp_0(\mathbb{R}^n)$ and $x \in X$, the distance of x to A is $d(x,A) = \inf\{d(x,y) : y \in A\}$ and $B_\varepsilon(A) = \{x \in \mathbb{R}^n : d(x,A) < \varepsilon\}$.

We recall that for a sequence of points $\{x_n\}$ the ω-limit set of $\{x_n\}$ is given as the set of accumulation points of the sequence. It is formally defined as:

$$\omega(\{x_n\}) = \bigcap_{n=0}^{\infty} \overline{\{x_k : k \geq n\}}. \tag{2.1}$$

To approximate sets, we recall the following convergence notions for a sequence of nonempty sets $\{A_n\}$ of $\mathbb{R}^n$: $\mathrm{Limsup}_n A_n = \{x : \exists x_{n_k} \in A_{n_k} : x_{n_k} \to x\}$ is its outer limit where $\{n_k\}$ is a subsequence of $\{n\}$, $\mathrm{Liminf}_n A_n = \{x : \exists x_n \in A_n : x_n \to x\}$ is its inner limit. By using the notion of convergence of a sequence of sets by [15], the sequence $\{A_n\}$ converges (in Painlevé-Kuratowski sense) to A, denoted by $A_n \to A$ if $\mathrm{Limsup}_n A_n \subset A \subset \mathrm{Liminf}_n A_n$.
Concerning sequences of singleta, the set limit is not converging or is a singleton made of the limit of the sequence.

The inner and outer limits of a sequence always exist (possibly empty) and are closed. Moreover, taking into account [15, Theorem 4.42], Painlevé-Kuratowski set convergence can be characterized by the integrated set distance defined on the family of nonempty closed subsets of $\mathbb{R}^n$.

We will consider the following results about set convergence.

Theorem 2.1. [16] *Let $\{K_n\}$ be a sequence of subsets of a metric space X and a subset $K \subset X$ satisfying the following property: for any neighborhood U of K, there exists N such that for all $n \geq N$, $K_n \subset U$. Then* $\mathrm{Limsup}_n K_n \subset \overline{K}$. *Conversely, if X is compact, then the upper limit* $\mathrm{Limsup}\, K_n$ *enjoys the above property (is the smallest closed subset satisfying it).*

In a natural way we can define the ω-limit set of $\{A_n\}$ as follows:

$$\omega(\{A_n\}) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{k \geq n} A_k}. \tag{2.2}$$

Furthermore, we can rewrite the ω-limit as the following equivalent formulation via upper limit (see [16, Proposition 1.1.2]).

Lemma 2.1. $\omega(\{A_n\}) = \text{Limsup}_n A_n$.

In the sequel, a set-valued map is a map from a nonempty subset C of $\mathbb{R}^n$ to the family $\wp(X)$ of subsets of $\mathbb{R}^n$. We assume, unless clearly specified otherwise, that the domain of F is C, that is, $\{x \in C\colon F(x) \neq \emptyset\} = C$. Equivalently, $F\colon C \to \wp_0(X)$. We say that F satisfies the union property if $F(A) = \bigcup_{a\in A} F(a)$ for $A \subset C$ (note that this property holds for every single-valued map by definition). Its image, denoted by $\mathscr{F} = \{F(x)\colon x \in C\}$, is a the family of image sets and $\text{Gph}(F) = \{(x,y) \in C \times X\colon y \in F(x)\}$ is its graph.
It is said to be closed-valued (resp. compact-valued, nonempty-valued) if $F(x)$ is closed (resp. compact, nonempty) for every $x \in C$; locally bounded at $\bar{x}$ if for some neighborhood $U \subset \mathbb{R}^n$ of $\bar{x}$ and some $t > 0$ one has $F(C \cap U) \subset t\mathbb{B}$ (where $\mathbb{B}$ is the unit ball) and locally bounded if it is so at every $\bar{x}$.

We recall some notions and results from the literature about semicontinuity (see [17]).

Definition 2.1. F is said to be

- inner semicontinuous (isc) or lower semicontinuous (lsc) if, for any $x \in C$, $y \in F(x)$ and any $\{x_n\}$ with $x_n \in C$ for all n, and $x_n \to x$, there exists $\{y_n\}$ such that $y_n \in F(x_n)$ for all n and $y_n \to y$;
- upper semicontinuous (usc) if, for any $x \in C$ and any open set $V \subset \mathbb{R}^n$ containing $F(x)$, there is an open set $U \subset \mathbb{R}^n$ containing x such that $F(C \cap U) \subset V$;
- outer semicontinuous (osc) if, for any $x \in C$ and any $\{(x_n, y_n)\}$ with $(x_n, y_n) \in \text{Gph}(F)$ for all n and $(x_n, y_n) \to (x, y)$, one has $(x, y) \in \text{Gph}(F)$;
- continuous if it is osc and isc.

From this, we see that if F is osc at $x \in X$, then $F(x)$ is closed. Thus, if F is osc then is closed-valued. Furthermore, F is osc iff $\text{Gph}(F)$ is closed.

Outer and inner semicontinuity of F at $x \in C$ can be expressed in terms of convergent sequences of sets as follows: F is osc (resp. isc) at $x \in C$ iff for any (x_n) with $x_n \in C$ for all n and $x_n \to x$ it holds that $\text{Limsup}_n F(x_n) \subset F(x)$ (resp. $F(x) \subset \text{Liminf}_n F(x_n)$). Thus, a set-valued map $F\colon C \to \wp_0(\mathbb{R}^n)$ is continuous at $\bar{x}$ if and only if $F(x_n) \to F(\bar{x})$ for every $x_n \to \bar{x}$. Or equivalently, F is osc and isc at $\bar{x}$.

Remark 2.1. We shall employ the outer limit of a set-valued map F at $\bar{x} \in U$ defined as follows (see [15]):

$$\text{Limsup}_{x\to\bar{x}} F(x) \;=\; \Big\{y\colon \exists x_n \in U \to \bar{x}\; \exists y_n \in F(x_n)\colon\; y_n \to y\Big\},$$

where $x \to \bar{x}$ denotes in fact $U \ni x \to \bar{x}$. Clearly, F is osc at $\bar{x} \in X$ iff $\text{Limsup}_{x\to\bar{x}} F(x) \subset F(\bar{x})$ or equivalently $\text{Limsup}_{x\to\bar{x}} F(x) = F(\bar{x})$.

Proposition 2.1. [17, Proposition 2] *The following conditions hold:*

(i) *If F is usc and compact-valued, then it is locally bounded and osc.*
(ii) *F is locally bounded iff $F(B \cap C)$ is bounded for every bounded set B iff whenever $y_n \in F(x_n)$ for all k and $\{x_n\} \subset C$ is bounded, then $\{y_n\}$ is bounded.*
(iii) *If F is usc at $\bar{x} \in X$ and $F(\bar{x})$ is closed, then it is osc at $\bar{x}$. The reverse implication holds if, in addition, F is locally bounded at $\bar{x}$.*

We will consider the following results about the composition of set-valued maps.

Proposition 2.2. *Consider $S\colon \mathbb{R}^n \to \wp_0(\mathbb{R}^n)$ and $T\colon \mathbb{R}^n \to \wp_0(\mathbb{R}^n)$.*

(i) $S(C)$ is closed when S is osc and C is closed and bounded.

(ii) $T \circ S\colon \mathbb{R}^n \to \wp_0(\mathbb{R}^n)$ is osc if S and T are osc.

Proof. (i) and (ii) are particular cases of Exercise 5.26 and Proposition 5.52 in [15] respectively. □

Theorem 2.2. [15, Corollary 5.20] *For any single-valued map $F\colon \mathbb{R}^n \to \mathbb{R}^n$, the following properties are equivalent:*

(i) F is continuous at x;

(ii) F is osc at x and locally bounded at x;

(iii) F is isc at x.

We define the following continuity notions to be used later on.

Definition 2.2. F is said to be

- slightly continuous (slc) at $\bar{x} \in C$ if, for every $x^k \to \bar{x}$, there exists a subsequence $\{x^{k_j}\} \subset \{x^k\}$ such that $F(x^{k_j}) \to F(\bar{x})$. It is said to be slc if it is so at every $\bar{x}$;
- extremely continuous (exc) if $F(A_n) \to F(A)$ for every $A_n \to A$.

Clearly, a exc map is closed-valued and slc. Note that if F is continuous then F is slc and if, in addition, F^{-1} is locally bounded then F is also exc (see [15]).

Proposition 2.3. *Suppose that $A_n \to A$ and F is slc. Then the following holds:*

(i) $F(A) \subset \operatorname{Limsup} F(A_n)$;

(ii) if, in addition, C is compact and $\{A_n\}$ is a sequence of nonempty sets of C,

$$\operatorname{Limsup} F(A_n) \subset F(A).$$

Proof. (i) Let $a \in A$. Since $A_n \to A$, there exists a sequence $\{a_n\}$ with $a_n \in A_n$ for all n converging to a. By the slightly continuity, there exists a subsequence $\{a_{n_k}\} \subset \{a_n\}$ such that $F(a_{n_k}) \to F(a)$. Thus,

$$F(a) \subset \operatorname{Limsup} F(A_n).$$

(ii) Suppose that $y \in \operatorname{Limsup} F(A_n)$, that is, $z_{n_k} \to y$ with $z_{n_k} \in F(A_{n_k})$ for some subsequence $z_{n_k} \in F(A_{n_k})$. Let $a_{n_k} \in A_{n_k}$ such that $z_{n_k} \in F(a_{n_k})$. Since $\{a_{n_k}\} \subset C$ and C is compact, we can assume that there exists a subsequence of $\{a_{n_k}\}$ converging to some $a \in C$. We assume $a_{n_k} \to a$. By $A_n \to A$, we deduce that $a \in A$ and by the slightly continuity we have $F(a_{n'_k}) \to F(a)$ for some subsequence $\{a_{n'_k}\}$ of $\{a_{n_k}\}$. Therefore,

$$y \in F(a) \subset F(A)$$

and we conclude the proof. □

3. Set-Valued Discrete Dynamical Systems

In this section and the following one we denote by X a nonempty compact subset of $\mathbb{R}^n$, by $\wp(X)$ the family of all subsets of X, by $\wp_0(X)$ see above and by $\mathscr{K}(X)$ the family of all nonempty compact subsets of X (note that by the compactness of X if A is closed then A is compact).

In the sequel sections, neighborhoods or limits of sequences are required to lie in X. All definitions considered in the section are valid for any compact metric space.

In this paper a set-valued discrete dynamical system (SVDS) is defined as follows:

$$\Phi\colon \mathbb{N}_0 \times X \to \wp(X)$$

such that

H1 $\Phi(0,x) = \{x\}$ (the intial value condition),
H2 $\Phi(t+s,x) = \Phi(t,\Phi(s,x))$ (semi-group property),
H3 Φ is nonempty-valued (maximal domain),

and satisfies the union property, that is, for $M \subset X$ and $T \subset \mathbb{N}_0$ we have

$$\Phi(n,M) = \bigcup_{x\in M} \Phi(n,x), \quad \Phi(T,x) = \bigcup_{n\in T} \Phi(n,x) \quad \text{and} \quad \Phi(T,M) = \bigcup_{n\in T}\bigcup_{x\in M} \Phi(n,x).$$

The set-valued map Φ is called system evolution map. Generally, in the literature Φ is assumed continuous or closed-valued. Note that we can omit the assumption H3 by redefining $\Phi\colon \mathbb{N}_0 \times X \to \wp_0(X)$. It is clear that the single-valued dynamical system is a particular case of Φ since x is the singleton set $\{x\}$.

Proposition 3.1. *Given a SVDS, there exists a set-valued map $F\colon X \to \wp_0(X)$ such that*

$$\Phi(n,x) = F^n(x) \text{ for all } x \in X,$$

where F^n denotes the n-th composition with the convention $F^0 = Id$ (identity) and satisfying the union property, $F(A) = \bigcup_{x\in A} F(x)$. Conversely, given a map $F\colon X \to \wp_0(X)$ satisfying the union property that we can define a SVDS.

Proof. Given Φ in SVDS, we define F as follows $F(x) = \Phi(1,x)$. Note from H3 that F is a map from X to $\wp_0(X)$ since $\Phi(1,x) \neq \emptyset$ by B3. Moreover, F satisfies the union property by the definition of Φ. We consider the composition $F \circ F$. It follows that

$$F^2(x) = F(F(x)) = F(\Phi(1,x)) = \Phi(1,\Phi(1,x)) = \Phi(2,x)$$

and we can could conclude by induction. In the same manner we can prove that given F we obtain Φ defined in SVDS. □

Using the above result, under some conditions on X, a set-valued discrete dynamical system (SVDS) is equivalent to a pair (X,F) where F is a map (nonempty-valued) from X to $\wp_0(X)$ satisfying the union property.

Remark 3.1. In [14], the set-valued dynamical system is induced by a differential inclusion and Φ is closed (its graph is closed) and compact-valued while in [18] authors consider that Φ as upper semicontinuity and compact valued. By Proposition 2.1, both assumptions are equivalent and, in addition, since closed map is equivalent to osc map, the assumption compact valued in [14] is redundant.

To understand the dynamical properties of a system it is necessary to have information on the behavior of the orbits or trajectories of any point $x \in X$ under the iteration of the space map. Given an initial value $x_0 \in X$, the sequence $\{F^n(x_0)\}$ is uniquely determined and is called trajectory (or solution) of F while, on the other side, an orbit of x_0 (under F) is the following family of sets in $\mathscr{F}$

$$\mathscr{O}(x_0) = \{x_0, F(x_0), F^2(x_0), \dots\}.$$

An orbit is periodic if there exists $m \in \mathbb{N}$ such that $F^i(x) = F^{i+m}(x)$ for any $i \geq 0$.

We also consider $\mathscr{O}(A)$ an orbit (similarly trajectory) with initial condition given by $A \in \wp_0(X)$, in order to study the dynamics of a set of points.

Remark 3.2. In the existing literature, the notation of orbit or trajectory are used indistinctly. Different from the classical case, if we consider a sequence $(x_0, x_1, x_2, \dots)$ where $x_{n+1} \in F(x_n)$ then the trajectories (or orbits) are not uniquely determined by their initial conditions and it may not be desirable.

We begin with a few simple extensions of definitions from the single-valued case.

Definition 3.1. A point $\bar{x} \in X$ is a fixed point of (X, F) if $\bar{x} \in F(\bar{x})$.

Definition 3.2. A set $A \in \wp_0(X)$ is called:

- positive invariant of F if $F(A) \subset A$;
- negative invariant of F if $A \subset F(A)$;
- invariant if $F(A) = A$.

Remark 3.3. Definition 3.1 of fixed point is standard in set-valued analysis theory. However the trajectories at a fixed point x_∞ are not stationary (in general). In addition, if a fixed point $\bar{x}$ satisfies $F(\bar{x}) = \{\bar{x}\}$ then it is called endpoint. In [19], the authors study its existence in terms of set-valued discrete dynamical systems. Note that (only) whenever F is a single-valued map, the definition of fixed point coincides with the classical notion of equilibrium of a dynamical system.

We give the notions presented in Definition 3.2 by analogy. However, they could consider not common in the literature, for instance, in [20] the invariant set is called fixed set.

Since x is identified with a singleton $\{x\}$, we emphasize that in our setting there are different types of sets which need be considered. In this respect, a fixed point could be a negative invariant set.

In dynamical systems theory, a main goal is to understand the existence and structure of invariant sets.

Remark 3.4. Clearly, x is a fixed point if and only if x is a fixed point for F^n for all $n \in \mathbb{N}$. Moreover, A is invariant (resp. positive or negative invariant) for F if and only if A is invariant (resp. positive or negative invariant) for F^n for all $n > 0$. In addition, if F is single-valued or A is singleton, then A is invariant if and only if A a fixed point. Exactly, the fixed points or periodic orbits are examples of invariant sets. Note that the Kakutani fixed point theorem is a generalization of the Brouwer fixed point theorem for set-valued maps and provides sufficient conditions for the existence of invariant sets.

Fixed point theorems provide sufficient conditions for the existence of invariant sets see, for instance, [20] and references therein. Note that if we consider the Hausdorff metric on $\mathscr{K}(X)$, denoted by d_H, we can minimize $d_H(C, F(C))$ subject to $C \subset \mathscr{K}(X)$ to find invariant sets of the set-valued dynamical system (X, F). We emphasize that the invariant sets play a key role in determining the nature of inverse limits (see [21]).

Proposition 3.2. *Let A be a closed positive invariant set of F such that A is minimal (no proper subset of A is positive invariant). Suppose that F is osc. Then A is an invariant limit set.*

Proof. Since A is positive invariant, we have $F^n(A) \subset F^{n-1}(A) \subset A$ for all n. Thus, the sequence $\{F^n(A)\}$ has limit $C = \lim F^n(A) = \bigcap_{n\in\mathbb{N}_0} \overline{F^n(A)}$ (see [16]). Since, $F^n(A)$ is closed (compact) for all n by Proposition 2.2(i) we obtain that $C = \bigcap_{n\in\mathbb{N}_0} F^n(A)$ is a nonempty closed subset of A. It is sufficient to show that C is positive invariant. Indeed,

$$F(C) = F(\bigcap_{n\geq 0} F^n(A)) = \bigcap_{n\geq 0} F(F^n(A)) = \bigcap_{n\geq 0} F^{n+1}(A) = C.$$

Thus, $C = A$ since A is minimal.

To finish the proof we only need to prove that $A \subset F(A)$. If $y \in A$ then $y \in \operatorname{Limsup} F^n(A)$ which follows from the definition of C. Thus, there exists $z_{n_k} \in F^{n_k}(A)$ with $z_{n_k} \to y$. In particular, we can rewrite z_{n_k} as follows $z_{n_k} \in F(F^{n_k-1}(A))$. Therefore, there exists a sequence $\{a_{n_k}\}$ with $a_{n_k} \in F^{n_k-1}(A) \subset A$. Since A is compact, there exists $a \in A$ such that $a_{n_k} \to a$, up to subsequences. Then $(a_{n_k}, z_{n_k}) \in \operatorname{Gph}(F)$ and

$$(a_{n_k}, z_{n_k}) \to (a, y).$$

By the outer continuity assumption $y \in F(a)$ and we conclude. □

We point out that taking into account Proposition 2.1(i) the condition osc can be replaced by usc if we assume that F is compact-valued.

The existence of minimal invariant sets is always guaranteed under the existence of positive invariant sets but not the uniqueness.

Proposition 3.3. *Let A be a compact positive invariant set of F. Then there exists at least a subset of A which is minimal positive invariant for F.*

Proof. Consider $\mathscr{A}$ the family of all nonempty compact and positive invariant subsets of A, that is, $\mathscr{A} = \{C \in \mathscr{K}(X) \colon\ C \subset A \text{ and } F(C) \subset C\}$. It is clear that $\mathscr{A}$ is partially ordered by the set inclusion. In addition, given a totally ordered subset of $\mathscr{A}$, $\mathscr{A}'$, the set $\bigcap_{A\in\mathscr{A}'} A$ is nonempty, compact and belongs to $\mathscr{A}$. Thus, by Zorn's Lemma the proof is finished. □

Note that $\{x \in A \colon F(x) \subset A\}$ is the maximal invariant set in A and it contains all invariant sets which are contained in A.

Remark 3.5. Compare the above propositions with [1, Proposition 3.1 and 3.2] where the continuity of the set-valued map $\Phi(\cdot,\cdot)$ is assumed.

Similarly to the classical case, it is possible to connect positive invariant sets and set limits under mild continuity assumptions. Exactly, we obtain sufficient conditions for (positive) invariant sets.

Theorem 3.1. *Suppose that for a subset $M \subset X$ the sequence $\{F^n(M)\}$ (or trajectory of M under F) converges to A and $A \subset X$. The following conditions hold:*

(i) If F is isc, then A is positive invariant.
(ii) If F is slc, then A is invariant.
(iii) If F is exc, then A is invariant.

Proof. (i) We have to prove that $F(A) \subset A$. Let $a \in A$. Since $F^n(M) \to A$ there exists a sequence $\{y_n\}$ with $y_n \in F^n(M)$ such that $y_n \to a$. By the inner semicontinuity of F, we obtain

$$F(a) \subset \operatorname{Liminf} F(y_n).$$

Since $F(y_n) \subset F(F^n(M)) = F^{n+1}(M)$, from above we deduce

$$F(a) \subset \operatorname{Liminf}_n F^{n+1}(M) = A.$$

(ii) By proposition 2.3(i), $F(A) \subset \operatorname{Limsup} F(F^n(M)) = \operatorname{Limsup} F^{n+1}(M) = A$, and A is positive invariant.

On the other hand, by Proposition 2.3(ii), we obtain $\operatorname{Limsup} F(F^n(M)) \subset F(A)$. Since $F^n(M) \to A$ and $\operatorname{Limsup} F^{n+1}(M) = \operatorname{Limsup} F^n(M)$, we deduce $A \subset F(A)$.

(iii) Since $F^n(M) \to A$, then $F(F^n(M)) \to F(A)$ or equivalently $F^{n+1}(M) \to F(A)$. Thus, $F(A) = \lim_n F^{n+1}(M) = \lim_n F^n(M) = A$. Note that (iii) is a particular case of (ii). □

Remark 3.6. Note that if F is single-valued and M and A are singleton, we obtain the sufficient condition of fixed points under generalized continuity assumption. In particular, taking into account Theorem 2.2, Theorem 3.1 states that the well-known sufficient condition (via limits) of fixed points of real functions can be extended to set-valued maps in terms of limit sets.

Corollary 3.1. *Suppose that F is a single-valued map and F is continuous. If for some $x \in X$ there exists $l = \lim F^n(x)$, then l is a fixed point of F.*

From above, one way to localizate invariants sets is to consider limits of trajectories, in particular, to study the ω-limit of such sequences of sets. On the other hand, since A is a limit set A then is also closed (or equivalent compact for being $A \subset X$). Therefore, in this context of limit sets, it seems natural to replace $\wp_0(X)$ by $\mathscr{K}(X)$ or equivalently to assume, in addition, that Φ is nonempty compact-valued in property H3.

Motivated by [14] and taking into account (2.1) and (2.2), we define the ω-limit of F as follows:

Definition 3.3. Let $A \in \wp_0(X)$. A map $\omega_F : \wp_0(X) \to \mathscr{K}(X)$ defined by the ω-limit of the trajectories of F, that is,

$$\omega_F(A) = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} F^m(A)}$$

is called the ω-limit map of F.

In the sequel, we omit F in ω_F. Note that ω is well-defined. Indeed, $\omega(A)$ is a closed of a compact X and by compactness it is nonempty since the sets $\bigcup_{m \geq n} F^m(A)$ are nested.

Note that a singleton $\{x\}$ is identified with x. Consequently, for a point $x \in X$, we have

$$\omega(x) = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} F^m(x)}. \tag{3.1}$$

We emphasize that, in general,

$$\omega(A) \supsetneq \bigcup_{a \in A} \omega(a).$$

Consequently, the union property is not satisfied by the set-valued map $x \to \omega(x)$. Moreover, any periodic orbit of F is an ω-limit set of any of the points in the orbit since $\omega(x)$ is the set of the cluster points of the trajectory with initial value x (see Proposition 3.4 below). Thus, the structure of ω-limit sets gives us a way of understanding how the dynamical system (X,F) is behaving. Therefore, the ω-limit map provides an asymptotic description of the dynamics of the system.

Remark 3.7. In [22] the author deals the classical dynamical systems on continua to study equicontinuity and continuity of the induced map ω defined in (3.1) being F continuous single-valued map. The ω-limit map is rarely continuous as we can deduce from [23] where the map ω is studied (being F a real function and $X=[0,1]$). Exactly, its continuity structure is related to forms of chaos.

Lemma 3.1. *Let $A,B \in \wp_0(X)$. The following conditions hold:*

(i) $\omega(A) = \operatorname{Limsup}_n F^n(A)$.
(ii) *If $A \subset B$ then $\omega(A) \subset \omega(B)$.*
(iii) $\omega(A\cup B) = \omega(A)\cup\omega(B)$.
(iv) $\omega(A\cap B) \subset \omega(A)\cap\omega(B)$.
(v) $\omega \circ F = \omega$

Proof. Its follows from Definition 3.3, Lemma 2.1 or properties of the upper limit (see [16, Proposition 1.2.1]). □

The invariance property via F implies the invariance property via w as the following result states.

Proposition 3.4. *Let $A \in \mathscr{K}(X)$.*

(i) *If A is positive invariant, then $\omega(A) \subset A$.*
(ii) *If A is negative invariant, then $A \subset \omega(A)$.*
(iii) *If A is invariant, then $\omega(A) = A$.*

Proof. It directly follows from Lemma 3.1(i) and (ii). □

Theorem 3.2. *Let $A \in \wp_0(X)$.*

(i) *If F is osc, then $\omega(A)$ is negative invariant. In particular, $\omega\circ F(A) \subset F\circ\omega(A)$.*
(ii) *If F is isc, then $\omega(A)$ is positive invariant. In particular, $F\circ\omega(A) \subset \omega\circ F(A)$.*

Proof. (i) Suppose that $y \in \omega(A)$. Then, by Lemma 3.1(i), there exists $y_{n_k} \in F^{n_k}(A)$ such that $y = \lim y_{n_k}$. Since $F^{n_k}(A) = F(F^{n_k-1}(A))$, we can assume that there exists $z_{n_k} \in F^{n_k-1}(A)$ such that $y_{n_k} \in F(z_{n_k})$. Thus, choosing a subsequence (if necessary), we may assume that z_{n_k} converges to some $z \in X$ (by the compactness of the space). Applying the outer semicontinuity of F, we deduce that $(z_{n_k}, y_{n_k}) \in \operatorname{Gph}(F)$ converges to (z,y) being $y \in F(z)$. Since

$$z \in \operatorname{Limsup} F^{n_k-1}(A) = \omega(A),$$

we deduce $y \in F(\omega(A))$.

Consequently, $\omega(A) \subset F(\omega(A))$. By replacing A by $F(A)$ we have $\omega(F(A)) \subset F(\omega(F(A)))$ and by Lemma 3.1(v) we obtain

$$\omega(F(A)) \subset F(\omega(A)).$$

(ii) We have to prove that $F(\omega(A)) \subset \omega(A)$. Let $z \in F(y)$ with $y \in \omega(A)$. Thus, there exists a subsequence $y_{n_k} \in F^{n_k}(A)$ such that $y_{n_k} \to y$. By the inner semicontinuity of F we have $F(y) \subset \operatorname{Liminf} F(y_{n_k})$. Since

$$\operatorname{Liminf} F(y_{nk}) \subset \operatorname{Limsup} F(y_{n_k}) \subset \operatorname{Limsup} F^n(A) = \omega(A),$$

we finish the proof.

A similar reasoning as the first can be done to prove $F \circ \omega(A) \subset \omega \circ F(A)$. □

Corollary 3.2. *Let $A \in \wp_0(X)$ and let F be continuous. Then the following conditions hold:*

(i) *$\omega(A)$ is invariant;*
(ii) *$F \circ \omega = \omega$;*
(iii) *provided A is compact, $\omega(A) = A$ if and only if A is invariant;*
(iv) *$\omega = F^n \circ \omega = \omega \circ F^n$ for all $n \in \mathbb{N}_0$.*

Proof. (i) and (ii) can be deduced from Theorem 3.2.

(iii) If A is invariant, by Proposition 3.4, $\omega(A) = A$. Conversely, if $\omega(A) = A$, taking image by F, we have $F(\omega(A)) = F(A)$. By (ii), $\omega(A) = F(A)$, and we obtain $F(A) = A$.

(iv) From (ii) and Lemma 3.1(v).

(v) It follows from (iv) and (ii). □

According to Corollary 3.2, the invariant sets and the ω-limit map play an important role in dynamical systems. Thus, the properties of the ω-limit map allows to study the sets of dynamical systems with possible intricate structures.

The above properties of ω-limit sets complete those presented in [18] for continuous-time.

4. ATTRACTORS AND STABILITY

The central subject of ω-limit sets is the concept of an attractor set. In this section, we introduce new generalizations from the classical case. We note that the following notions may not be equivalent to those existing in the literature.

Similarly to the previous section, we consider (X,F) a SVDS where $X \subset \mathbb{R}^n$ is nonempty compact and F is a set-valued map $X \to \wp_0(X)$ satisfying the union property.

Given $A \in \wp_0(X)$, the domain of attraction of A is defined by

$$\mathscr{A}(A) = \{x \in X : F^n(x) \to A\},$$

and defines all trajectories of (X,F) such that they are attracted by A. Note that $\mathscr{A}(A) \neq \emptyset$ implies that A is closed (then compact). On the other hand, according to Theorem 3.1, under continuity assumptions, it seems natural to assume that A is (positive) invariant. Moreover, if $X = \mathscr{A}(A)$ then A is a global attractor of SVDS.

Now, we introduce the following notions of local attractor.

Definition 4.1. $A \in \wp_0(X)$ is said to be:

- an upper local attractor if there exists a neighborhood U of A such that each $x \in U$ satisfies
$$\operatorname{Limsup}_n F^n(x) \subset A;$$
- a lower local attractor if there exists a neighborhood U of A such that each $x \in U$ satisfies
$$A \subset \operatorname{Liminf}_n F^n(x);$$

- a local attractor if there exists a neighborhood U of A such that each $x \in U$ satisfies
$$F^n(x) \to A;$$
- a strong upper local attractor if there exists a neighborhood U of A such that
$$\mathrm{Limsup}_n F^n(U) \subset A.$$

Remark 4.1. Note that A is local attractor if and only if A is upper and lower local attractor. A local attractor attracts all trajectories nearby while that an upper local attractor contains the ω-limit sets of each trajectory nearby. The property of lower local attractor is typically harder to verify than that of upper local attractor and it means that A is contained in the limits of trajectories around A. Whenever A is a local attractor, A is compact (A is a limit set in a compact set). Moreover, by Theorem 3.1, under certain continuity assumptions, a local attractor is always invariant or positive invariant. Note that if A is a singleton $\{a\}$, then the notion of strong upper local attractor coincides with upper local attractor since we have $\omega(U) = \omega(x) = \{a\}$ for all $x \in U$.

Remark 4.2. The above notion of local attractor is consistent with that given for equilibrium points in the classical case. The concepts of attractor defined in [14] and [18] (both devoted to continuous-time) are different from the above ones. Exactly, in [14] the notion of attractor is defined for invariant sets and is stronger than upper local attractor. The notion of attractive set defined in [18], by Proposition 4.1(i) (see below), coincides with upper local attractor. Compare also with that given in [1] where the notion of attractor is defined for invariant sets via the Hausdorff semi-distance.

Lemma 4.1. *Let $A \in \wp_0(X)$. If A is a strong upper local attractor for a neighborhood U of A, then, for any neighborhood V of A, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $F^n(U) \subset V$.*

Proof. It follows from Theorem 2.1 by taking $K_n = F^n(U)$ (since $\mathrm{Limsup}_n F^n(U) \subset \overline{A}$). □

Now, we rewrite the above notions via the ω-limit map by using Lemma 2.1.

Proposition 4.1. *Let $A \in \wp_0(X)$. The following properties hold.*

(i) *A is upper local attractor if and only if there exists U a neighborhood of A such that*
$$\bigcup_{x \in U} \omega(x) \subset A$$

(ii) *If A is lower local attractor then there exists U a neighborhood of A such that*
$$A \subset \bigcup_{x \in U} \omega(x)$$

(iii) *A is local attractor if and only if there exists a neighborhood U of A such that $U \subset \mathscr{A}(A)$.*

(iv) *If A is strong upper local attractor if and only if there exists a neighborhood U of A such that*
$$\omega(U) \subset A.$$

Definition 4.2. A set $A \in \wp_0(X)$ is stable (resp. weak stable) in the Lyapunov sense if for any neighborhood V of A, there exists a neighborhood U of A such that
$$F^n(U) \subset V \text{ for all } n \in \mathbb{N}_0 \text{ (resp. for all } n > N \text{ and some } N \in \mathbb{N}\text{)}.$$

Roughly speaking, a set A is stable when taking x near to A as initial value, the trajectory will be near to A for all future time.

Lemma 4.2. *Let $A \in \mathscr{K}(X)$. If A is stable, then A is positive invariant.*

Proof. By stability, we have $F^n(A) \subset V$ for all $n \in \mathbb{N}_0$ and for any neighborhood V of A. Taking $V = B_\varepsilon(A)$ with $\varepsilon > 0$ we conclude

$$F^n(A) \subset \bigcap_{\varepsilon>0} B_\varepsilon(A) = \overline{A} = A \qquad \text{for all } n \in \mathbb{N}_0.$$

□

Theorem 4.1. *The following conditions hold:*

(i) If $A \in \wp_0(X)$ is strong upper local attractor, then A is weak stable.
(ii) Suppose that F is osc and locally bounded.
If $A \in \mathscr{K}(X)$ is strong upper local attractor and positive invariant, then A is stable.

Proof. (i) Let V a neighborhood of A. By Proposition 4.1, there exists U' a neighborhood of A such that $\operatorname{Limsup} F^n(U') \subset A$. Taking into account Lemma 4.1, we have that

$$F^n(U') \subset V \text{ for all } \quad n > N \text{ and some } \quad N > 0. \tag{4.1}$$

Therefore, A is weak stable.

(ii) It suffices to show that the above inclusion holds for all $n \leq N$. Indeed, since A is positive invariant we obtain $F^n(A) \subset A$ for all $n \in \mathbb{N}_0$. Thus, V is a neighborhood of $F^n(A)$ for all $n \in \mathbb{N}_0$.

On the other hand, according to Proposition 2.2(ii), F^n is osc for all $n \in \mathbb{N}_0$. Moreover, since F is locally bounded it is easy to check that F^2 is also locally bounded (by Proposition 2.2(i) and Proposition 2.1(ii)). Thus, by induction, we have that F^n is locally bounded for all $n \in \mathbb{N}_0$.

Fix any $k \leq N$. Then, by Proposition 2.1(iii), F^k is usc. Thus, since $F^k(x) \subset F^k(A) \subset V$ for each $x \in A$, there exists U_x a neighborhood of x such that

$$F^k(U_x) \subset V.$$

Therefore, the open sets U_x cover the compact set A, so a finite family of them cover it as well, say for $x_1, \ldots, x_r$ with corresponding $U_{x_1}, \ldots, U_{x_r}$. Likewise, $A \subset U_k = U_{x_1} \cup \cdots \cup U_{x_r}$ and $F^k(U_k) \subset V$.

By applying an identical reasoning we can find $U_N, \ldots, U_1$ such that

$$F^i(U_i) \subset V$$

for each $i \in \{N, N-1, \ldots, 1\}$. Let $U = U' \cap (U_N \cap \cdots \cap U_1)$. Then $A \subset U$ and from (4.1) we obtain

$$F^n(U) \subset V \qquad \text{for all} \qquad n \in \mathbb{N}_0$$

which completes the proof. □

From theorem established above and Lemma 4.2 we obtain the following characterization.

Corollary 4.1. *Suppose that F is osc and locally bounded and $A \in \mathscr{K}(X)$ is a strong upper local attractor. The following conditions are equivalent*

- *A is stable;*
- *A is positive invariant.*

In particular, if F is a continuous single-valued and a singleton $\{a\}$ is upper local attractor (see Remark 4.1) then a is a stable point if and only if is a fixed point.

Definition 4.3. We say that $A \in \mathscr{K}(X)$ is

(i) upper locally asymptotically stable (ULAS) if A is stable and upper local attractor;
(ii) lower locally asymptotically stable (LLAS) if A is stable and lower local attractor;
(iii) locally asymptotically stable (LAS) if A is stable and local attractor;
(iv) strong upper locally asymptotically stable (s-ULAS) if A is stable and strong upper local attractor.

Now we characterize the notion of LAS under certain conditions of generalized continuity of F and invariance for A.

Theorem 4.2. *Suppose that $A \in \mathscr{K}(X)$ is negative invariant and F is isc. A is LAS if and only if there exists a compact neighborhood U of A small enough and positive invariant such that*

$$F^n(x) \to A$$

for any $x \in U$.

Proof. Since A is local attractor there exists a neighborhood U' in X with $A \subset U'$ such that

$$F^n(x) \to A$$

for each $x \in U'$. We must show that also there exists a compact subset U'' of U' that is small enough, U'' contains A and is positive invariant.
Let V a neighborhood of A small enough such that

$$A \subset \overline{V} \subset U'. \tag{4.2}$$

Thus, by the stability of A, there exists W a neighborhood of A such that

$$F^n(W) \subset V \text{ for all } \quad n \in \mathbb{N}_0. \tag{4.3}$$

Denoting $C = \bigcup_n F^n(W)$, from (4.3) we have $C \subset V$. In addition, since A is negative invariant we also have $A \subset C$.

On the other hand, taking into account that F satisfies the union property we deduce

$$F(C) = F(\bigcup_n F^n(W)) = \bigcup_n F(F^n(W)) \subset C.$$

Hence, C is positive invariant. Consider

$$U = \overline{C}. \tag{4.4}$$

We obtain that U is compact and small enough satisfying

$$A \subset U \subset \overline{V} \subset U' \tag{4.5}$$

due to (4.2).

The proof is completed by showing that U is also positive invariant. Indeed, if $y \in F(x)$ and $x \in U$, from (4.4) there exists a sequence $x_n \in C$ such that $x_n \to x$. By the inner semicontinuity of F, there exists a sequence $y_n \in F(x_n)$ such that $y_n \to y$. Since $F(C) \subset C$ we have $y \in \overline{C} = U$. Hence, $F(U) \subset U$ and we conclude.

We emphasize that the proof above is constructive since given U' (any neighborhood of A) we obtain U which is small enough according to (4.5).

For the converse, it is clear that A is a local attractor. To finish the proof it is sufficient to show that A is stable. Suppose that V is a neighborhood of A. If necessary, we can take U as long as it is small enough to satisfy $A \subset U \subset V$. Therefore, since U is positive invariant we have

$$F^n(U) \subset U \subset V$$

for all $n \in \mathbb{N}_0$ as required. □

Corollary 4.2. *Suppose that F is a continuous single valued map and $\bar{x}$ is a fixed point. Then $\bar{x}$ is LAS if and only if there exists a compact neighborhood U of $\bar{x}$ small enough and positive invariant such that*

$$\lim F^n(x) = \bar{x}$$

for any $x \in U$.

Proof. It follows from Theorem 4.2 and Remark 4.2. □

The above result can be considered an extension of the familiar characterization of fixed points in the framework of real functions defined on an interval. Compare with stability conditions established in [14] and [18].

5. Conclusions

The purpose of this paper is to show a general approach to study set-valued dynamical systems by using certain tools from the set-valued analysis theory. The desirable situation that state map is continuous puts considerable restrictions on the problem. We prove results which are not dependent on the continuity assumptions. In special, we obtain as applications or particular cases improvements or generalizations of real functions. Our contributions include basic notions motivated from the classical results and those given in [14]. Our framework is more general since we relax the continuity assumptions. However, we note that their consideration of basic notions (for a continuous dynamical system) is quite similar to ours here. We show that ω-limit sets play a fundamental role to give an asymptotic description of the dynamics.

Future research directions could be to study Lyapunov maps or the family of all ω-limit sets of a set-valued dynamical system similar to [24] where a geometric characterization of ω-limit sets are given for a real function. Research on nonexpansive or contractive maps would be wellcome to give new stability conditions. An alternative way would be to analyze the structure of ω-limits, see [23].

On the other hand, an important subject is to consider practical problems, for instance, models about population dynamics in which, according to [25], it would be appropriate to induce set-valued dynamical systems. Likewise, we emphasize that set-valued maps from an interval into the closed subsets of an interval do arise in various areas of science and mathematical modeling, most notably economics and game theory.

Another future research is to consider a dynamical systems approach for studying certain optimization problems or conversely to approach a vector optimization problem from a dynamical point of view, taking advantage of methods of global analysis; see [19, 26].

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